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A class of C^1 rational interpolation splines in one and two dimensions with region control

Yuanpeng Zhu[1] · Meng Wang[2]

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Abstract
In this work, we use a kind of C^1 rational interpolation splines in one and two dimensions to generate curves and surfaces with region control. Simple data-dependent sufficient constraints are derived on the local control parameters to generate C^1 interpolation curves lying strictly between two given piecewise linear curves and C^1 interpolation surfaces lying strictly between two given piecewise bi-cubic blending linear interpolation surfaces.

Keywords Interpolation curve · Interpolation surface · C^1 continuity · Region control

Mathematics Subject Classification 65D05 · 65D17

1 Introduction

The problem of modeling interpolation curves and surfaces to given data has been studied with various requirements, such as the preservation of the shape features of the data, the smoothness of the interpolation curves and surfaces, the computational complexity, and so on. Generally speaking, for most applications, C^1 continuity is sufficient. In this paper, we are particularly concerned with the construction of C^1 interpolation curves and surfaces with region control.

There are some C^1 interpolation curve models based on rational cubic splines. In Sarfraz et al. (2010), a kind of positivity-preserving interpolation curves was developed based on a class of rational cubic/quadratic interpolation spline with two local free parameters. However,

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✉ Yuanpeng Zhu
ypzhu@scut.edu.cn

Meng Wang
1833995496@qq.com

1 School of Mathematics, South China University of Technology, Guangzhou 510641, People's Republic of China

2 School of Business Administration, South China University of Technology, Guangzhou 510641, People's Republic of China

the conditions for the interpolation spline preserving positivity developed in Sarfraz et al. (2010) were not sufficient. In Qin et al. (2016), this drawback was overcome by introducing a new local tension parameter τ_i into the rational cubic/quadratic interpolation spline. In Walther and Schmidt (1999), range restricted interpolation using Gregory's rational cubic splines was proposed. In Duan et al. (1999), the authors discussed constrained interpolation problems by means of rational cubic spline interpolation with linear denominators, but there are still some cases in which the constrained interpolation cannot be solved, which means that there are no such positive parameters to make the rational cubic spline curve defined to lie between the given piecewise lines in some cases. Later, in Duan et al. (2000) and Duan et al. (2005), the problems of generating interpolation curves lying strictly between two given piecewise linear curves were solved using weighted rational cubic/linear interpolation splines. In Duan et al. (2006), based on the idea of adding more parameters into the interpolating spline to enhance the constraining ability, a weighted rational cubic spline interpolation was constructed using two kinds of rational cubic spline with quadratic denominator. However, the conditions for interpolation curves lying strictly between two given piecewise linear curves given in Duan et al. (1999, 2000, 2005) and Duan et al. (2006) were non-explicit. Therefore, these conditions are inconvenient in pratical application. Recently, an automatic algorithm for generating C^2 constrained interpolation curves was developed in Zhu (2018).

For visualizing data given on a rectangular grid, some C^1 interpolation surfaces with shape control have been proposed, see for example (Hussain and Sarfraz 2008; Hussain et al. 2014; Abbas et al. 2014) and Qin et al. (2017). In Hussain and Sarfraz (2008), by replacing the classical cubic Hermite interpolation basis for the classical bi-cubic Coons surface with a kind of rational cubic Hermite-type interpolation basis, a class of C^1 rational bi-cubic functions was presented. And constraints concerning the local control parameters were given for visualizing 3D positive data on a rectangular grid. However, like the classical bi-cubic Coons surface technique, the given interpolation scheme needs to provide the cross-boundary derivatives or the twists on a rectangular grid for generating interpolation surfaces. In Hussain et al. (2014), based upon the Boolean sum of cubic interpolating operators, by blending a kind of rational cubic interpolation splines as the boundary functions, simpler scheme without using the twists for constructing C^1 shape-preserving interpolation surfaces was given. This transfinite interpolation method was of great convenience for it was possible to control the shape of the interpolation surfaces using the boundary functions, though it had to pay the price that the generated interpolation surfaces had zero twist vectors at the data points. However, as pointed out in Abbas et al. (2014), this method did not depict the positive or monotonic surfaces because they conserved the shape of data only on the boundaries of patch. Thus one asks if it is possible to generate positivity and/or monotonicity-preserving interpolation surfaces by controlling the four boundary curves of each local interpolation surface patch. In Qin et al. (2017), using a class of C^1 bi-cubic partially blended rational quartic/linear interpolation splines and imposing new constrains on the four boundary curves of each local interpolation surface patch, simple sufficient data-dependent conditions were derived for the local control parameters to generate C^1 positivity and/or monotonicity-preserving interpolation surfaces for positive and/or monotonic data on rectangular grids. There are some references about the construction of range restricted interpolants to scattered data, see for example (Chan and Ong 1999; Brodlie et al. 2005) and the references therein. In Chan and Ong (1999), a local construction of a C^1 interpolating surface subject to range restrictions which included constant, linear, quadratic or cubic polynomial surfaces on the triangulation of the data sites as upper and lower bounds were developed. In Brodlie et al. (2005), the modified quadratic Shepard method was developed for interpolation of scattered data with constraining the interpolant within [0, 1] limits. And it was shown that the [0, 1] constraints can be generalised to any arbitrary functions as lower and upper bounds. However, in Chan and Ong (1999) and Brodlie et al. (2005), they used constant, linear, quadratic or cubic polynomial functions as bounds. In this work, we use piecewise bi-cubic blending linear interpolation surfaces as bounds, which has been rarely discussed in the references.

The purpose of this paper is to further study the region control properties of the C^1 rational interpolation spline curves and surfaces given in Qin et al. (2016). Simple explicit schemes are developed for constructing C^1 interpolation curves lying strictly between two given piecewise linear curves and C^1 interpolation surfaces lying strictly between two given piecewise bi-cubic blending linear interpolation surfaces.

The rest of this paper is organized as follows. Section 2 recalls the rational cubic/quadratic interpolation spline with three local free parameters given in Qin et al. (2016). In Sect. 3, simple sufficient data-dependent constraints are derived for constructing C^1 interpolation curves lying strictly between two given piecewise linear curves. In Sect. 4, simple sufficient explicit conditions for constructing C^1 interpolation surfaces lying strictly between two given piecewise bi-cubic blending linear interpolation surfaces are discussed in detail. Conclusions are provided in Sect. 5.

2 C^1 rational interpolation spline curves and surfaces

In this section, we recall the C^1 rational cubic/quadratic interpolation spline with three local free parameters and bi-cubic partially blended rational cubic/quadratic interpolation surfaces given in Qin et al. (2016).

2.1 C^1 rational cubic/quadratic interpolation spline curves

Let $f_i \in R$, $i = 1, 2, \ldots, n$, be data given at the distinct knots $x_i \in R$, $i = 1, 2, \ldots, n$, with interval spacing $h_i = x_{i+1} - x_i > 0$, and let $d_i \in R$ be denoted the first derivative values defined at the knots. In Qin et al. (2016), for $x \in [x_i, x_{i+1}]$, a piecewise C^1 rational cubic/quadratic interpolation spline with three local free parameters u_i, τ_i and v_i is defined over each subinterval $I_i = [x_i, x_{i+1}]$ as follows

$$R\left(x\right) = \frac{\sum_{k=0}^{3} C_{ik}(1-t)^{3-k}t^k}{u_i(1-t)^2 + \tau_i(1-t)t + v_i t^2}, \tag{1}$$

where $t = \left(x - x_i\right)/h_i$, $u_i, \tau_i, v_i \in (0, +\infty)$, $i = 1, 2, \ldots, n-1$, and

$$\begin{aligned} C_{i0} &= u_i f_i, \\ C_{i1} &= \tau_i f_i + u_i \left(f_i + h_i d_i\right), \\ C_{i2} &= \tau_i f_{i+1} + v_i \left(f_{i+1} - h_i d_{i+1}\right), \\ C_{i3} &= v_i f_{i+1}. \end{aligned}$$

In applications, the first derivative values d_i, $i = 1, 2, \ldots, n$ are not known and should be specified in advance. In this paper, they are computed using the following arithmetic mean method

$$\begin{cases} d_1 = \Delta_1 - \dfrac{h_1}{h_1 + h_2}\left(\Delta_2 - \Delta_1\right), \\ d_i = \dfrac{\Delta_{i-1} + \Delta_i}{2}, i = 2, 3, \ldots, n-1, \\ d_n = \Delta_{n-1} + \dfrac{h_{n-1}}{h_{n-2} + h_{n-1}}\left(\Delta_{n-1} - \Delta_{n-2}\right), \end{cases} \tag{2}$$

where $\Delta_i := (f_{i+1} - f_i)/h_i$. This arithmetic mean method is the three-point difference approximation based on arithmetic calculation, which is computationally economical and suitable for visualization of shaped data, see for example Sarfraz et al. (2010). In the next discussion, we will also denote $R(x)$ as $R(x;\, f_i,\, f_{i+1};\, d_i,\, d_{i+1};\, u_i,\, \tau_i,\, v_i)$ for $x \in [x_i, x_{i+1}]$.

2.2 C^1 bi-cubic partially blended rational cubic/quadratic interpolation surfaces

Let $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m\}$ be a given set of data points defined over the rectangular domain $[a, b] \times [c, d]$, where $\pi_x : a = x_1 < x_2 < \cdots < x_n = b$ is the partition of $[a, b]$ and $\pi_y : c = y_1 < y_2 < \cdots < y_m = d$ is the partition of $[c, d]$. We use the notation $\pi_{i,j} := [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. For $(x, y) \in \pi_{i,j}$, $1 \le i \le n-1$, $1 \le j \le m-1$, using the boolean sum of cubic interpolating operators to blend together four rational cubic/quadratic interpolation splines given in (1) as the boundary functions, a bi-cubic partially blended rational cubic/quadratic interpolation surface is constructed as follows

$$\begin{aligned} S(x, y) &:= ((Q_1 \oplus Q_2)\, F)\,(x, y) \\ &= (Q_1 F)\,(x, y) + (Q_2 F)\,(x, y) - (Q_1 Q_2 F)\,(x, y), \end{aligned} \tag{3}$$

where

$$\begin{aligned} (Q_1 F)\,(x, y) &:= \begin{bmatrix} F(x_i, y) & F(x_{i+1}, y) \end{bmatrix} \begin{bmatrix} b_0(t) \\ b_1(t) \end{bmatrix}, \\ (Q_2 F)\,(x, y) &:= \begin{bmatrix} F(x, y_j) & F(x, y_{j+1}) \end{bmatrix} \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \\ (Q_1 Q_2 F)\,(x, y) &:= \begin{bmatrix} b_0(t) & b_1(t) \end{bmatrix} \begin{bmatrix} F_{i,j} & F_{i,j+1} \\ F_{i+1,j} & F_{i+1,j+1} \end{bmatrix} \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \end{aligned}$$

with $h_i^x = x_{i+1} - x_i$, $h_j^y = y_{j+1} - y_j$, $t = (x - x_i)/h_i^x$, $s = (y - y_j)/h_j^y$, and

$$\begin{aligned} b_0(w) &:= (1 - w)^2 (1 + 2w), \qquad b_1(w) := w^2\,[1 + 2(1 - w)], \\ F(x, y_j) &:= R(x;\, F_{i,j},\, F_{i+1,j};\, D_{i,j}^x,\, D_{i+1,j}^x;\, u_{i,j}^x,\, \tau_{i,j}^x,\, v_{i,j}^x), \\ F(x, y_{j+1}) &:= R(x;\, F_{i,j+1},\, F_{i+1,j+1};\, D_{i,j+1}^x,\, D_{i+1,j+1}^x;\, u_{i,j+1}^x,\, \tau_{i,j+1}^x,\, v_{i,j+1}^x), \\ F(x_i, y) &:= R(y;\, F_{i,j},\, F_{i,j+1};\, D_{i,j}^y,\, D_{i,j+1}^y;\, u_{i,j}^y,\, \tau_{i,j}^y,\, v_{i,j}^y), \\ F(x_{i+1}, y) &:= R(y;\, F_{i+1,j},\, F_{i+1,j+1};\, D_{i+1,j}^y,\, D_{i+1,j+1}^y;\, u_{i+1,j}^y,\, \tau_{i+1,j}^y,\, v_{i+1,j}^y). \end{aligned}$$

Here, $D_{i,j}^x, D_{i,j}^y$ are known as the first partial derivatives and $(u_{i,j}^x)_{(n-1)\times m}$, $(u_{i,j}^y)_{n\times(m-1)}$, $(\tau_{i,j}^x)_{(n-1)\times m}$, $(\tau_{i,j}^y)_{n\times(m-1)}$, $(v_{i,j}^x)_{(n-1)\times m}$, $(v_{i,j}^y)_{n\times(m-1)}$ serve as local control parameters. For $2 \le i \le n-1$, $2 \le j \le m-1$, we set the first partial derivatives $D_{i,j}^x, D_{i,j}^y$ as follows

$$\begin{cases} D_{i,j}^x = \dfrac{\Delta_{i-1,j}^x + \Delta_{i,j}^x}{2}, \\ D_{i,j}^y = \dfrac{\Delta_{i,j-1}^y + \Delta_{i,j}^y}{2}, \end{cases} \tag{4}$$

where $\Delta_{i,j}^x = (F_{i+1,j} - F_{i,j})/h_i^x$, $\Delta_{i,j}^y = (F_{i,j+1} - F_{i,j})/h_j^y$.

And for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, we set the derivative values $D^x_{1,j}$, $D^y_{i,1}$, $D^x_{n,j}$ and $D^y_{i,m}$ as follows

$$\begin{cases} D^x_{1,j} = \Delta^x_{1,j} - \dfrac{h^x_1}{h^x_1 + h^x_2}\left(\Delta^x_{2,j} - \Delta^x_{1,j}\right), \\ D^y_{i,1} = \Delta^y_{i,1} - \dfrac{h^y_1}{h^y_1 + h^y_2}\left(\Delta^y_{i,2} - \Delta^y_{i,1}\right), \\ D^x_{n,j} = \Delta^x_{n-1,j} + \dfrac{h^x_{n-1}}{h^x_{n-2} + h^x_{n-1}}\left(\Delta^x_{n-1,j} - \Delta^x_{n-2,j}\right), \\ D^y_{i,m} = \Delta^y_{i,m-1} + \dfrac{h^y_{n-1}}{h^y_{m-2} + h^y_{m-1}}\left(\Delta^y_{i,m-1} - \Delta^y_{i,m-2}\right). \end{cases} \tag{5}$$

3 Region control of the interpolation curves

In this section, we shall deal with the problem of constraining the C^1 interpolant $R(x)$ given in Qin et al. (2016) to lie strictly between two given piecewise linear interpolation curves.

For $x \in [x_i, x_{i+1}]$, $t = (x - x_i)/h_i$, $i = 1, 2, \ldots, n-1$, given two piecewise linear interpolants $g(x) = (1-t)g_i + tg_{i+1}$, $g^*(x) = (1-t)g^*_i + tg^*_{i+1}$ and a data set (x_i, f_i) with $g(x_i) = g_i < f_i < g^*_i = g^*(x_i)$, if the interpolant $R(x)$ of the data set (x_i, f_i), $i = 1, 2, \ldots, n$ satisfies

$$g(x) < R(x) < g^*(x)$$

for any $x \in [x_1, x_n]$, then $R(x)$ is called the constrained interpolant lying strictly between two given piecewise linear interpolants $g(x)$ and $g^*(x)$.

For $x \in [x_i, x_{i+1}]$, $1 \le i \le n-1$, the interpolant $R(x)$ lies strictly above the piecewise linear interpolant $g(x)$ if $R(x) > g(x)$, which is equivalent to

$$\begin{aligned} R(x) - g(x) &= \frac{\sum_{k=0}^{3} C_{ik}(1-t)^{3-k}t^k}{u_i(1-t)^2 + \tau_i(1-t)t + v_i t^2} - \left[(1-t)\, g_i + t g_{i+1}\right] \\ &= \frac{\sum_{k=0}^{3} A_{ik}(1-t)^{3-k}t^k}{u_i(1-t)^2 + \tau_i(1-t)t + v_i t^2} > 0, \end{aligned} \tag{6}$$

where A_{ik}, $k = 0, 1, 2, 3$ are as follows

$$\begin{aligned} A_{i0} &= u_i\left(f_i - g_i\right), \\ A_{i1} &= \tau_i\left(f_i - g_i\right) + u_i\left(f_i - g_{i+1} + h_i d_i\right), \\ A_{i2} &= \tau_i\left(f_{i+1} - g_{i+1}\right) + v_i\left(f_{i+1} - g_i - h_i d_{i+1}\right), \\ A_{i3} &= v_i\left(f_{i+1} - g_{i+1}\right). \end{aligned}$$

Since $f_i - g_i > 0$, $\forall i$, we can see that $A_{i0} > 0$ and $A_{i3} > 0$. Thus, the conditions $A_{i1} \ge 0$ and $A_{i2} \ge 0$ are sufficient to ensure $R(x) - g(x) > 0$ ($\forall x \in [x_i, x_{i+1}]$), from which we can get the following sufficient conditions for $R(x) - g(x) > 0$ ($\forall x \in [x_i, x_{i+1}]$)

$$\begin{cases} u_i > 0, \quad v_i > 0, \\ \tau_i = \max\left\{\dfrac{-u_i\left[(f_i - g_{i+1}) + h_i d_i\right]}{f_i - g_i}, \dfrac{-v_i\left[(f_{i+1} - g_i) - h_i d_{i+1}\right]}{f_{i+1} - g_{i+1}}, 0\right\} + \rho_i, \end{cases} \tag{7}$$

where $1 \le i \le n-1$, and $\rho_i \ge 0$ serve as free control parameters.

And for $x \in [x_i, x_{i+1}]$, $1 \le i \le n-1$, the interpolant $R(x)$ lies strictly below the piecewise linear interpolant $g^*(x)$ if $R(x) < g^*(x)$, which is equivalent to

$$\begin{aligned} g^*(x) - R(x) &= \left[(1-t)\, g_i^* + t g_i^*\right] - \frac{\sum_{k=0}^{3} C_{ik}(1-t)^{3-k} t^k}{u_i(1-t)^2 + \tau_i(1-t)t + v_i t^2} \\ &= \frac{\sum_{k=0}^{3} A_{ik}^*(1-t)^{3-k} t^k}{u_i(1-t)^2 + \tau_i(1-t)t + v_i t^2} > 0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} A_{i0}^* &= u_i\left(g_i^* - f_i\right), \\ A_{i1}^* &= \tau_i\left(g_i^* - f_i\right) + u_i\left(g_{i+1}^* - f_i - h_i d_i\right), \\ A_{i2}^* &= \tau_i\left(g_{i+1}^* - f_{i+1}\right) + v_i\left(g_i^* - f_{i+1} + h_i d_{i+1}\right), \\ A_{i3}^* &= v_i\left(g_{i+1}^* - f_{i+1}\right). \end{aligned}$$

In a similar manner, we can also obtain the following sufficient conditions that can ensure $g^*(x) - R(x) > 0$ for any $x \in [x_i, x_{i+1}]$

$$\begin{cases} u_i > 0, \quad v_i > 0, \\ \tau_i = \max\left\{ \dfrac{-u_i\left[\left(g_{i+1}^* - f_i\right) - h_i d_i\right]}{g_i^* - f_i}, \dfrac{-v_i\left[\left(g_i^* - f_{i+1}\right) + h_i d_{i+1}\right]}{g_{i+1}^* - f_{i+1}}, 0 \right\} + \rho_i, \end{cases} \tag{9}$$

where $1 \le i \le n-1$, and $\rho_i \ge 0$ serve as free control parameters.

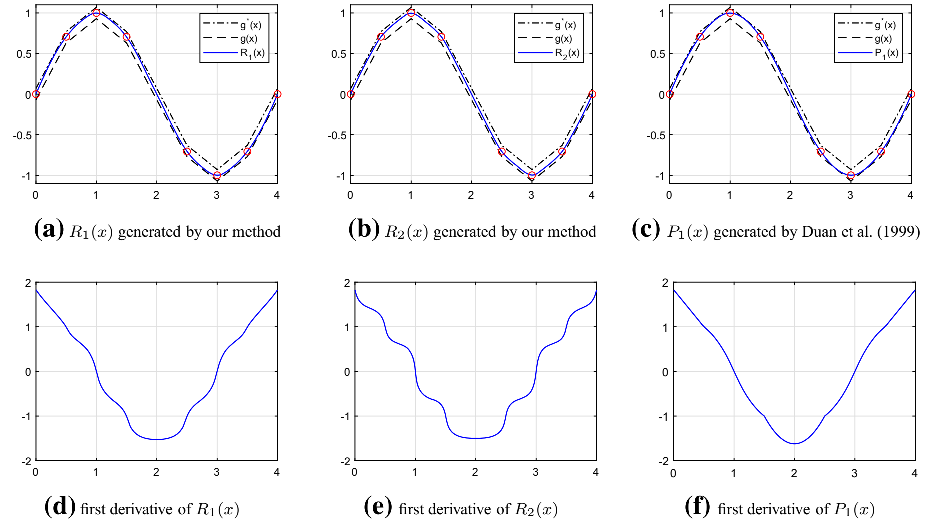
In summary, for $x \in [x_1, x_n]$, the following simple explicit data-dependent conditions are sufficient to ensure $g(x) < R(x) < g^*(x)$

$$\begin{cases} u_i > 0, \quad v_i > 0, \\ \tau_i = \max\left\{ \dfrac{-v_i\left[(f_{i+1} - g_i) - h_i d_{i+1}\right]}{f_{i+1} - g_{i+1}}, \dfrac{-u_i\left[(f_i - g_{i+1}) + h_i d_i\right]}{f_i - g_i}, \right. \\ \qquad \left. \dfrac{-u_i\left[\left(g_{i+1}^* - f_i\right) - h_i d_i\right]}{g_i^* - f_i}, \dfrac{-v_i\left[\left(g_i^* - f_{i+1}\right) + h_i d_{i+1}\right]}{g_{i+1}^* - f_{i+1}}, 0 \right\} + \rho_i, \end{cases} \tag{10}$$

where $i = 1, 2, \ldots, n-1$, and $\rho_i \ge 0$ serve as free control parameters.

Figure 1a, b shows the constrained interpolation curves $R_1(x)$ and $R_2(x)$ for the 2D data set given in Table 1 and the corresponding graphics of their first derivatives. The free control parameters for generating $R_1(x)$ are set with all $\rho_i = 0$. And $R_2(x)$ is generated by changing all ρ_i from 0 to 2.5. As a comparison, Fig. 1c shows the constrained interpolation curve $P_1(x)$ generated by the method given in Duan et al. (1999), where all the parameters α_i and β_i in $P_1(x)$ take the same value of 1. It can be seen that all the three interpolation spline curves $R_1(x)$, $R_2(x)$ and $P_1(x)$ lie strictly between the two given piecewise linear curves $g(x)$ and $g^*(x)$ and all of them attain the expected C^1 continuity.

Figure 2a, b shows the constrained interpolation curves $R_3(x)$ and $R_4(x)$ for the 2D data set given in Table 2 and the corresponding graphics of their first derivatives. The free control parameters for generating $R_3(x)$ are set with all $\rho_i = 0$. And $R_4(x)$ is generated by changing all ρ_i from 0 to 1.5. Besides, Fig. 2c shows the curve $P_2(x)$ generated by the method given in Duan et al. (1999) will all $\alpha_i = \beta_i = 5$. It can be seen that both the interpolation spline curves



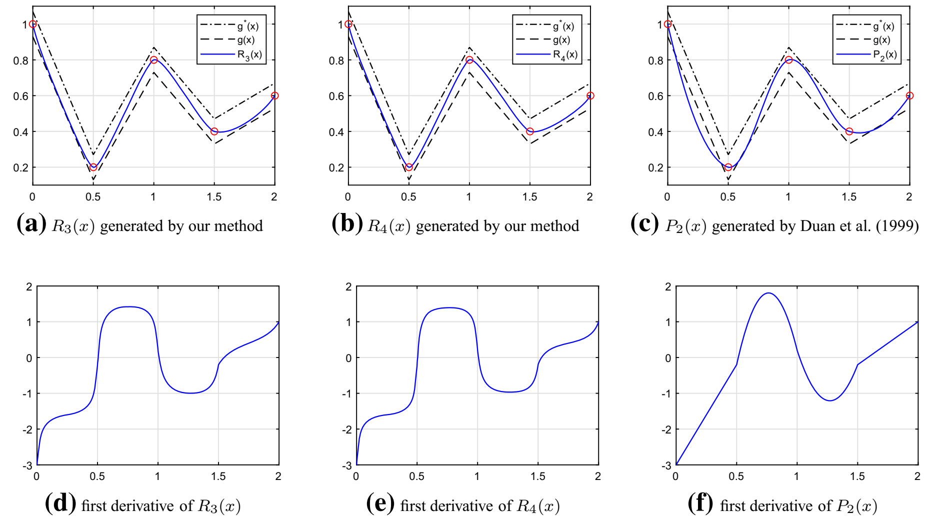
(a) $R_1(x)$ generated by our method **(b)** $R_2(x)$ generated by our method **(c)** $P_1(x)$ generated by Duan et al. (1999)

(d) first derivative of $R_1(x)$ **(e)** first derivative of $R_2(x)$ **(f)** first derivative of $P_1(x)$

Fig. 1 Constrained interpolation curves for the 2D data set given in Table 1 and the corresponding graphics of their first derivatives

Table 1 The 2D data set given in Duan et al. (2000)

i	1	2	3	4	5	6	7	8
x_i	0.0000	0.5000	1.0000	1.5000	2.5000	3.0000	3.5000	4.0000
g_i	−0.0700	0.6300	0.9300	0.6300	−0.7700	−1.0700	−0.7700	−0.0700
f_i	0.0000	0.7071	1.0000	0.7071	−0.7071	−1.0000	−0.7071	0.0000
g_i^*	0.0700	0.7700	1.0700	0.7700	−0.6300	−0.9300	−0.6300	0.0700



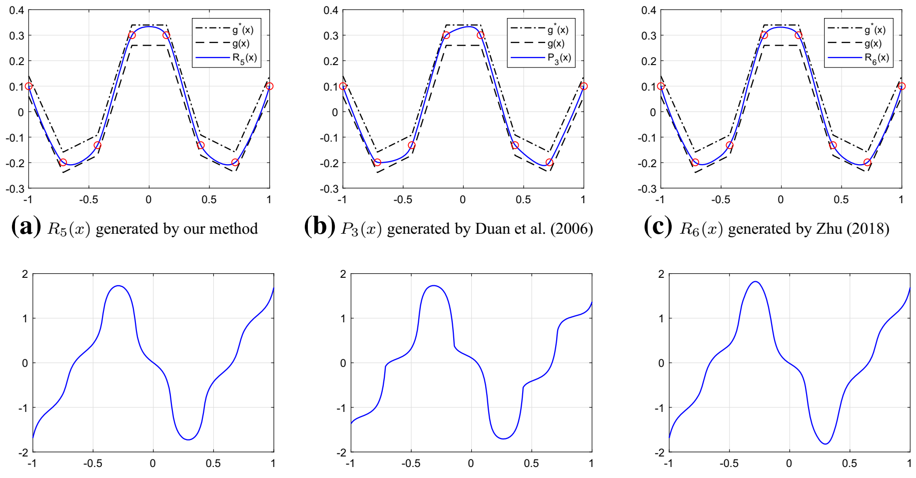
(a) $R_3(x)$ generated by our method **(b)** $R_4(x)$ generated by our method **(c)** $P_2(x)$ generated by Duan et al. (1999)

(d) first derivative of $R_3(x)$ **(e)** first derivative of $R_4(x)$ **(f)** first derivative of $P_2(x)$

Fig. 2 Constrained interpolation curves for the 2D data set given in Table 2 and the corresponding graphics of their first derivatives

Table 2 The 2D data set given in Duan et al. (2005)

i	1	2	3	4	5
x_i	0.00	0.50	1.00	1.50	2.00
g_i	0.93	0.13	0.73	0.33	0.53
f_i	1.00	0.20	0.80	0.40	0.60
g_i^*	1.07	0.27	0.87	0.47	0.67



(a) $R_5(x)$ generated by our method **(b)** $P_3(x)$ generated by Duan et al. (2006) **(c)** $R_6(x)$ generated by Zhu (2018)

(d) first derivative of $R_5(x)$ **(e)** first derivative of $P_3(x)$ **(f)** first derivative of $R_6(x)$

Fig. 3 Constrained interpolation curves for the 2D data set given in Table 3 and the corresponding graphics of their first derivatives

Table 3 A 2D data set

i	1	2	3	4	5	6	7	8
x_i	−1.0000	−0.7143	−0.4286	−0.1429	0.1429	0.4286	0.7143	1.0000
g_i	0.0600	−0.2389	−0.1716	0.2600	0.2600	−0.1716	−0.2389	0.0600
f_i	0.1000	−0.1989	−0.1316	0.3000	0.3000	−0.1316	−0.1989	0.1000
g_i^*	0.1400	−0.1589	−0.0916	0.3400	0.3400	−0.0916	−0.1589	0.1400

$R_3(x)$ and $R_4(x)$ lie strictly between the two piecewise linear interpolants $g(x)$ and $g^*(x)$ and both of them reach the desired C^1 continuity. However, it is obvious that $P_2(x)$ does not lie strictly between the two given piecewise linear curves $g(x)$ and $g^*(x)$. The reason is that the constraint conditions of inequality group given in Duan et al. (1999) have no solution for the data set given in Table 2, so that there are no positive parameters meeting the conditions to make the curve $P_2(x)$ lie between the two given piecewise lines in this case, which implies that the constraint conditions given in Duan et al. (1999) have particular requirements for given data set.

Figure 3 shows the comparisons among the constrained interpolation curve $R_5(x)$ generated by our method, the constrained interpolation curve $P_3(x)$ generated by the method given in Duan et al. (2006) and the constrained interpolation curve $R_6(x)$ generated by the method given in Zhu (2018) for the same 2D data set given in Table 3. The free control parameters for generating $R_5(x)$ are set with all $\rho_i = 0$. And the parameters for generating $P_3(x)$ are set with all $\alpha_i = \beta_i = 0.4$ and $\lambda = 0.5$. The curve $R_6(x)$ is constructed using the automatic algorithm for generating C^2 constrained interpolation curves developed in Zhu (2018). From the result, we can see that all the three curves $R_5(x)$, $P_3(x)$ and $R_6(x)$ lie strictly between the two given piecewise linear curves $g(x)$ and $g^*(x)$. It is obvious that the constrained interpolation curves $R_5(x)$ and $R_6(x)$ are slightly different in their shapes. And for the same symmetry data set given in Table 3, the resulting constrained interpolation curves $R_5(x)$ and $R_6(x)$ are more symmetrical about the ordinate axis than the constrained interpolation curve $P_3(x)$.

4 Region control of the interpolation surfaces

In this section, we want to develop simple schemes so that the C^1 interpolation surface $S(x, y)$ given in (3) can lie strictly between two piecewise bi-cubic blending linear interpolation surfaces.

Let $\{(x_i, y_i, G_{i,j}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$, $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ and $\{(x_i, y_i, G^*_{i,j}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ be three given sets of data points defined over the rectangular domain $[a, b] \times [c, d]$ and satisfy $G_{i,j} < F_{i,j} < G^*_{i,j}, \forall i, j$.

We say that the interpolation surface $S(x, y)$ of the data set $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ lies strictly between two piecewise bi-cubic blending linear interpolation surfaces $G(x, y)$ and $G^*(x, y)$ if for any $(x, y) \in \pi_{i,j}, i = 1, 2, \ldots, n-1, j = 1, 2, \ldots, m-1$, the following constrains hold

$$G(x, y) < S(x, y) < G^*(x, y), \tag{11}$$

where

$$\begin{aligned} G(x, y) &:= ((Q_1 \oplus Q_2) G)(x, y) \\ &= (Q_1 G)(x, y) + (Q_2 G)(x, y) - (Q_1 Q_2 G)(x, y), \\ G^*(x, y) &:= \left((Q_1 \oplus Q_2) G^*\right)(x, y) \\ &= \left(Q_1 G^*\right)(x, y) + \left(Q_2 G^*\right)(x, y) - \left(Q_1 Q_2 G^*\right)(x, y), \end{aligned}$$

and

$$\begin{aligned} (Q_1 G)(x, y) &:= \left[G(x_i, y) \ G(x_{i+1}, y) \right] \begin{bmatrix} b_0(t) \\ b_1(t) \end{bmatrix}, \\ (Q_2 G)(x, y) &:= \left[G(x, y_j) \ G(x, y_{j+1}) \right] \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \\ (Q_1 Q_2 G)(x, y) &:= \left[b_0(t) \ b_1(t) \right] \begin{bmatrix} G_{i,j} & G_{i,j+1} \\ G_{i+1,j} & G_{i+1,j+1} \end{bmatrix} \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \\ \left(Q_1 G^*\right)(x, y) &:= \left[G^*(x_i, y) \ G^*(x_{i+1}, y) \right] \begin{bmatrix} b_0(t) \\ b_1(t) \end{bmatrix}, \\ \left(Q_2 G^*\right)(x, y) &:= \left[G^*(x, y_j) \ G^*(x, y_{j+1}) \right] \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \\ \left(Q_1 Q_2 G^*\right)(x, y) &:= \left[b_0(t) \ b_1(t) \right] \begin{bmatrix} G^*_{i,j} & G^*_{i,j+1} \\ G^*_{i+1,j} & G^*_{i+1,j+1} \end{bmatrix} \begin{bmatrix} b_0(s) \\ b_1(s) \end{bmatrix}, \end{aligned}$$

with $h_i^x = x_{i+1} - x_i$, $h_j^y = y_{j+1} - y_j$, $t = (x - x_i)/h_i^x$, $s = (y - y_j)/h_j^y$, and

$$\begin{aligned}
G(x, y_j) &:= (1-t)\, G_{i,j} + tG_{i+1,j}, \quad G(x, y_{j+1}) := (1-t)\, G_{i,j+1} + tG_{i+1,j+1},\\
G(x_i, y) &:= (1-s)\, G_{i,j} + sG_{i,j+1}, \quad G(x_{i+1}, y) := (1-s)\, G_{i+1,j} + sG_{i+1,j+1},\\
G^*(x, y_j) &:= (1-t)\, G^*_{i,j} + tG^*_{i+1,j}, \quad G^*(x, y_{j+1}) := (1-t)\, G^*_{i,j+1} + tG^*_{i+1,j+1},\\
G^*(x_i, y) &:= (1-s)\, G^*_{i,j} + sG^*_{i,j+1}, \quad G^*(x_{i+1}, y) := (1-s)\, G^*_{i+1,j} + sG^*_{i+1,j+1}.
\end{aligned}$$

To develop simple sufficient data-dependent constraints on the local control parameters so that the generated interpolation surface $S(x, y)$ can lie strictly above $G(x, y)$, without loss of generality, for any $(x, y) \in \pi_{i,j}$, $1 \le i \le n-1$, $1 \le j \le m-1$, we rewrite the expression of $S(x, y) - G(x, y)$ as the following form

$$\begin{aligned}
&S(x, y) - G(x, y)\\
&\quad = b_0(t)\left[F(x_i, y) - G(x_i, y)\right] + b_1(t)\left[F(x_{i+1}, y) - G(x_{i+1}, y)\right]\\
&\qquad + b_0(s)\left[F(x, y_j) - G(x, y_j)\right] + b_1(s)\left[F(x, y_{j+1}) - G(x, y_{j+1})\right]\\
&\qquad - b_0(t)b_0(s)\left(F_{i,j} - G_{i,j}\right) - b_0(t)b_1(s)\left(F_{i,j+1} - G_{i,j+1}\right)\\
&\qquad - b_1(t)b_0(s)\left(F_{i+1,j} - G_{i+1,j}\right) - b_1(t)b_1(s)\left(F_{i+1,j+1} - G_{i+1,j+1}\right)\\
&\quad = b_0(s)\left[F(x, y_j) - G(x, y_j) - \frac{1}{2}b_0(t)\left(F_{i,j} - G_{i,j}\right) - \frac{1}{2}b_1(t)\left(F_{i+1,j} - G_{i+1,j}\right)\right]\\
&\qquad + b_1(s)\left[F(x, y_{j+1}) - G(x, y_{j+1}) - \frac{1}{2}b_0(t)\left(F_{i,j+1} - G_{i,j+1}\right)\right.\\
&\qquad \left. - \frac{1}{2}b_1(t)\left(F_{i+1,j+1} - G_{i+1,j+1}\right)\right]\\
&\qquad + b_0(t)\left[F(x_i, y) - G(x_i, y) - \frac{1}{2}b_0(s)\left(F_{i,j} - G_{i,j}\right) - \frac{1}{2}b_1(s)\left(F_{i,j+1} - G_{i,j+1}\right)\right]\\
&\qquad + b_1(t)\left[F(x_{i+1}, y) - G(x_{i+1}, y) - \frac{1}{2}b_0(s)\left(F_{i+1,j} - G_{i+1,j}\right)\right.\\
&\qquad \left. - \frac{1}{2}b_1(s)\left(F_{i+1,j+1} - G_{i+1,j+1}\right)\right].
\end{aligned}$$

Since both of the two functions $b_0(w)$ and $b_1(w)$ are strictly positive for any $w \in (0, 1)$, we can see that $S(x, y) - G(x, y) > 0$ for any $(x, y) \in \pi_{i,j}$ if the following constraints hold

$$\begin{cases}
F(x, y_j) - G(x, y_j) - \frac{1}{2}b_0(t)\left(F_{i,j} - G_{i,j}\right) - \frac{1}{2}b_1(t)\left(F_{i+1,j} - G_{i+1,j}\right) > 0,\\
F(x, y_{j+1}) - G(x, y_{j+1}) - \frac{1}{2}b_0(t)\left(F_{i,j+1} - G_{i,j+1}\right) - \frac{1}{2}b_1(t)\left(F_{i+1,j+1} - G_{i+1,j+1}\right) > 0,\\
F(x_i, y) - G(x_i, y) - \frac{1}{2}b_0(s)\left(F_{i,j} - G_{i,j}\right) - \frac{1}{2}b_1(s)\left(F_{i,j+1} - G_{i,j+1}\right) > 0,\\
F(x_{i+1}, y) - G(x_{i+1}, y) - \frac{1}{2}b_0(s)\left(F_{i+1,j} - G_{i+1,j}\right) - \frac{1}{2}b_1(s)\left(F_{i+1,j+1} - G_{i+1,j+1}\right) > 0.
\end{cases} \tag{12}$$

For $F(x, y_j) - G(x, y_j) - \frac{1}{2}b_0(t)\left(F_{i,j} - G_{i,j}\right) - \frac{1}{2}b_1(t)\left(F_{i+1,j} - G_{i+1,j}\right)$, we have

$$
F(x,y_j)-G(x,y_j)-\frac{1}{2}b_0(t)\left(F_{i,j}-G_{i,j}\right)-\frac{1}{2}b_1(t)\left(F_{i+1,j}-G_{i+1,j}\right)
$$
$$
=\frac{\sum_{k=0}^{5}B_{ik}(1-t)^{5-k}t^k}{u_{i,j}^x(1-t)^2+\tau_{i,j}^x(1-t)t+v_{i,j}^x t^2},
$$

where

$$
\begin{aligned}
B_{i0}&=\frac{1}{2}u_{i,j}^x\left(F_{i,j}-G_{i,j}\right),\\
B_{i1}&=\frac{1}{2}\left[\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+2u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)\right]+\frac{1}{2}u_{i,j}^x\left(F_{i,j}-G_{i,j}\right),\\
B_{i2}&=\frac{1}{2}\left[\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+4u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)-v_{i,j}^x\left(F_{i,j}-G_{i,j}\right)\right]\\
&\quad+\left[\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)\right.\\
&\quad\left.-\frac{3}{2}u_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)\right]+u_{i,j}^x\left(F_{i,j}-G_{i,j}\right),\\
B_{i3}&=\left[\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)-\frac{3}{2}v_{i,j}^x\left(F_{i,j}-G_{i,j}\right)\right]\\
&\quad+\frac{1}{2}\left[\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+4v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)\right.\\
&\quad\left.-u_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)\right]+v_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right),\\
B_{i4}&=\frac{1}{2}\left[\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+2v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)\right]\\
&\quad+\frac{1}{2}v_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right),\\
B_{i5}&=\frac{1}{2}v_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right).
\end{aligned}
$$

Since $F_{i,j}-G_{i,j}>0, \forall i,j$, from the above expressions, it is obvious that the conditions $\tau_{i,j}^x>0, u_{i,j}^x>0, v_{i,j}^x>0$ together with $B_{ir}\geq 0, r=1,2,3,4$ are sufficient to ensure $F(x,y_j)-G(x,y_j)-\frac{1}{2}b_0(t)\left(F_{i,j}-G_{i,j}\right)-\frac{1}{2}b_1(t)\left(F_{i+1,j}-G_{i+1,j}\right)>0$.

It is clear that $\tau_{i,j}^x>0, u_{i,j}^x>0, v_{i,j}^x>0$ together with the following conditions are sufficient to ensure $B_{ir}\geq 0, r=1,2,3,4$,

$$
\begin{cases}
\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+2u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)\geq 0,\\
\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+4u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)-v_{i,j}^x\left(F_{i,j}-G_{i,j}\right)\geq 0,\\
\tau_{i,j}^x\left(F_{i,j}-G_{i,j}\right)+u_{i,j}^x\left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)-\frac{3}{2}v_{i,j}^x\left(F_{i,j}-G_{i,j}\right)\geq 0,\\
\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)-\frac{3}{2}u_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)\geq 0,\\
\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+4v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)-u_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)\geq 0,\\
\tau_{i,j}^x\left(F_{i+1,j}-G_{i+1,j}\right)+2v_{i,j}^x\left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)\geq 0.
\end{cases}
\tag{13}
$$

Further, we can see that if $F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x < 0$ and $F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x < 0$, the following conditions are sufficient to ensure the constrains (13) hold

$$\begin{cases} \tau_{i,j}^x \left(F_{i,j}-G_{i,j}\right)+4u_{i,j}^x \left(F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x\right)-\dfrac{3}{2}v_{i,j}^x \left(F_{i,j}-G_{i,j}\right) \geq 0, \\ \tau_{i,j}^x \left(F_{i+1,j}-G_{i+1,j}\right)+4v_{i,j}^x \left(F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x\right)-\dfrac{3}{2}u_{i,j}^x \left(F_{i+1,j}-G_{i+1,j}\right) \geq 0. \end{cases} \tag{14}$$

Moreover, for $F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x \geq 0$ and $F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x \geq 0$, it is easy to check that the following conditions are sufficient to ensure the constrains (13) hold

$$\begin{cases} \tau_{i,j}^x \left(F_{i,j}-G_{i,j}\right)-\dfrac{3}{2}v_{i,j}^x \left(F_{i,j}-G_{i,j}\right) \geq 0, \\ \tau_{i,j}^x \left(F_{i+1,j}-G_{i+1,j}\right)-\dfrac{3}{2}u_{i,j}^x \left(F_{i+1,j}-G_{i+1,j}\right) \geq 0. \end{cases} \tag{15}$$

From the above analysis, we can immediately obtain the following sufficient conditions to ensure the positivity of $F(x,y_j)-G(x,y_j)-\frac{1}{2}b_0(t)\left(F_{i,j}-G_{i,j}\right)-\frac{1}{2}b_1(t)\left(F_{i+1,j}-G_{i+1,j}\right)$

$$\begin{cases} u_{i,j}^x > 0, \quad v_{i,j}^x > 0, \\ \tau_{i,j}^x = \max\left\{-4u_{i,j}^x \left(\dfrac{F_{i,j}-G_{i+1,j}+h_i^x D_{i,j}^x}{F_{i,j}-G_{i,j}}\right)+\dfrac{3}{2}v_{i,j}^x, \right. \\ \qquad \left. -4v_{i,j}^x \left(\dfrac{F_{i+1,j}-G_{i,j}-h_i^x D_{i+1,j}^x}{F_{i+1,j}-G_{i+1,j}}\right)+\dfrac{3}{2}u_{i,j}^x, \dfrac{3}{2}u_{i,j}^x, \dfrac{3}{2}v_{i,j}^x\right\}+\rho_{i,j}^x, \end{cases} \tag{16}$$

where $1 \leq i \leq n-1$, $1 \leq j \leq m$, and $\rho_{i,j}^x, \rho_{i,j}^y \geq 0$ serve as free control parameters.

In the same way, we can also derive similar conditions to ensure $F(x,y_{j+1})-G(x,y_{j+1})-\frac{1}{2}b_0(t)\left(F_{i,j+1}-G_{i,j+1}\right)-\frac{1}{2}b_1(t)\left(F_{i+1,j+1}-G_{i+1,j+1}\right) > 0$, $F(x_i,y)-G(x_i,y)-\frac{1}{2}b_0(s)\left(F_{i,j}-G_{i,j}\right)-\frac{1}{2}b_1(s)\left(F_{i,j+1}-G_{i,j+1}\right) > 0$ and $F(x_{i+1},y)-G(x_{i+1},y)-\frac{1}{2}b_0(s)\left(F_{i+1,j}-G_{i+1,j}\right)-\frac{1}{2}b_1(s)\left(F_{i+1,j+1}-G_{i+1,j+1}\right) > 0$.

In conclusion, the following conditions are sufficient to ensure $S(x,y)-G(x,y) > 0$ $(\forall (x,y) \in \pi_{i,j})$

$$\begin{cases} u_{i,k}^x > 0, \quad v_{i,k}^x > 0, \\ \tau_{i,k}^x = \max\left\{-4u_{i,k}^x \left(\dfrac{F_{i,k}-G_{i+1,k}+h_i^x D_{i,k}^x}{F_{i,k}-G_{i,k}}\right)+\dfrac{3}{2}v_{i,k}^x, \right. \\ \qquad \left. -4v_{i,k}^x \left(\dfrac{F_{i+1,k}-G_{i,k}-h_i^x D_{i+1,k}^x}{F_{i+1,k}-G_{i+1,k}}\right)+\dfrac{3}{2}u_{i,k}^x, \dfrac{3}{2}u_{i,k}^x, \dfrac{3}{2}v_{i,k}^x\right\}+\rho_{i,k}^x, \\ u_{l,j}^y > 0, \quad v_{l,j}^y > 0, \\ \tau_{l,j}^y = \max\left\{-4u_{l,j}^y \left(\dfrac{F_{l,j}-G_{l,j+1}+h_j^y D_{l,j}^y}{F_{l,j}-G_{l,j}}\right)+\dfrac{3}{2}v_{l,j}^y, \right. \\ \qquad \left. -4v_{l,j}^y \left(\dfrac{F_{l,j+1}-G_{l,j}-h_j^y D_{l,j+1}^y}{F_{l,j+1}-G_{l,j+1}}\right)+\dfrac{3}{2}u_{l,j}^y, \dfrac{3}{2}u_{l,j}^y, \dfrac{3}{2}v_{l,j}^y\right\}+\rho_{l,j}^y, \end{cases} \tag{17}$$

where $k = j, j+1$, $l = i, i+1$, and $\rho_{i,k}^x, \rho_{l,j}^y \geq 0$ serve as free control parameters.

For any $(x, y) \in \pi_{i,j}$, we further rewrite the expression of $G^*(x, y) - S(x, y)$ as follows

$$
\begin{aligned}
G^*(x, y) - S(x, y) &= b_0(t)\left[G^*(x_i, y) - F(x_i, y)\right] + b_1(t)\left[G^*(x_{i+1}, y) - F(x_{i+1}, y)\right] \\
&\quad + b_0(s)\left[G^*(x, y_j) - F(x, y_j)\right] + b_1(s)\left[G^*(x, y_{j+1}) - F(x, y_{j+1})\right] \\
&\quad - b_0(t)b_0(s)\left(G^*_{i,j} - F_{i,j}\right) - b_0(t)b_1(s)\left(G^*_{i,j+1} - F_{i,j+1}\right) \\
&\quad - b_1(t)b_0(s)\left(G^*_{i+1,j} - F_{i+1,j}\right) - b_1(t)b_1(s)\left(G^*_{i+1,j+1} - F_{i+1,j+1}\right) \\
&= b_0(s)\left[G^*(x, y_j) - F(x, y_j) - \frac{1}{2}b_0(t)\left(G^*_{i,j} - F_{i,j}\right)\right. \\
&\quad \left. - \frac{1}{2}b_1(t)\left(G^*_{i+1,j} - F_{i+1,j}\right)\right] \\
&\quad + b_1(s)\left[G^*(x, y_{j+1}) - F(x, y_{j+1}) - \frac{1}{2}b_0(t)\left(G^*_{i,j+1} - F_{i,j+1}\right)\right. \\
&\quad \left. - \frac{1}{2}b_1(t)\left(G^*_{i+1,j+1} - F_{i+1,j+1}\right)\right] \\
&\quad + b_0(t)\left[G^*(x_i, y) - F(x_i, y) - \frac{1}{2}b_0(s)\left(G^*_{i,j} - F_{i,j}\right) - \frac{1}{2}b_1(s)\left(G^*_{i,j+1} - F_{i,j+1}\right)\right] \\
&\quad + b_1(t)\left[G^*(x_{i+1}, y) - F(x_{i+1}, y) - \frac{1}{2}b_0(s)\left(G^*_{i+1,j} - F_{i+1,j}\right)\right. \\
&\quad \left. - \frac{1}{2}b_1(s)\left(G^*_{i+1,j+1} - F_{i+1,j+1}\right)\right].
\end{aligned}
$$

Similarly, since both of the two functions $b_0(w)$ and $b_1(w)$ are strictly positive for any $w \in (0, 1)$, we can see that for any $(x, y) \in \pi_{i,j}$, $G^*(x, y) - S(x, y) > 0$ if the following constraints hold

$$
\begin{cases}
G^*(x, y_j) - F(x, y_j) - \frac{1}{2}b_0(t)\left(G^*_{i,j} - F_{i,j}\right) - \frac{1}{2}b_1(t)\left(G^*_{i+1,j} - F_{i+1,j}\right) > 0, \\
G^*(x, y_{j+1}) - F(x, y_{j+1}) - \frac{1}{2}b_0(t)\left(G^*_{i,j+1} - F_{i,j+1}\right) - \frac{1}{2}b_1(t)\left(G^*_{i+1,j+1} - F_{i+1,j+1}\right) > 0, \\
G^*(x_i, y) - F(x_i, y) - \frac{1}{2}b_0(s)\left(G^*_{i,j} - F_{i,j}\right) - \frac{1}{2}b_1(s)\left(G^*_{i,j+1} - F_{i,j+1}\right) > 0, \\
G^*(x_{i+1}, y) - F(x_{i+1}, y) - \frac{1}{2}b_0(s)\left(G^*_{i+1,j} - F_{i+1,j}\right) - \frac{1}{2}b_1(s)\left(G^*_{i+1,j+1} - F_{i+1,j+1}\right) > 0.
\end{cases}
\tag{18}
$$

For $G^*(x, y_j) - F(x, y_j) - \frac{1}{2}b_0(t)\left(G^*_{i,j} - F_{i,j}\right) - \frac{1}{2}b_1(t)\left(G^*_{i+1,j} - F_{i+1,j}\right)$, we have

$$
\begin{aligned}
&G^*(x, y_j) - F(x, y_j) - \frac{1}{2}b_0(t)\left(G^*_{i,j} - F_{i,j}\right) - \frac{1}{2}b_1(t)\left(G^*_{i+1,j} - F_{i+1,j}\right) \\
&\quad = \frac{\sum_{k=0}^{5} B^*_{ik}(1-t)^{5-k}t^k}{u^x_{i,j}(1-t)^2 + \tau^x_{i,j}(1-t)t + v^x_{i,j}t^2},
\end{aligned}
$$

where

$$
\begin{aligned}
B^*_{i0} &= \frac{1}{2}u^x_{i,j}\left(G^*_{i,j} - F_{i,j}\right), \\
B^*_{i1} &= \frac{1}{2}\left[\tau^x_{i,j}\left(G^*_{i,j} - F_{i,j}\right) + 2u^x_{i,j}\left(G^*_{i+1,j} - F_{i,j} - h^x_i D^x_{i,j}\right)\right] + \frac{1}{2}u^x_{i,j}\left(G^*_{i,j} - F_{i,j}\right),
\end{aligned}
$$

$$
\begin{aligned}
B_{i2}^{*} =& \frac{1}{2}\left[\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+4u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)-v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)\right] \\
&+\left[\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)\right. \\
&\left.-\frac{3}{2}u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)\right] +u_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right), \\
B_{i3}^{*} =& \left[\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)-\frac{3}{2}v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)\right] \\
&+\frac{1}{2}\left[\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+4v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)\right. \\
&\left.-u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)\right] +v_{i,j}^{x}\left(F_{i+1,j}-G_{i+1,j}\right), \\
B_{i4}^{*} =& \frac{1}{2}\left[\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+2v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)\right] \\
&+\frac{1}{2}v_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right), \\
B_{i5}^{*} =& \frac{1}{2}v_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right).
\end{aligned}
$$

Since $G_{ij}^{*}-F_{ij}>0$, $\forall i, j$, from the above expressions, we can see that the conditions $\tau_{i,j}^{x}>0, u_{i,j}^{x}>0, v_{i,j}^{x}>0$ together with $B_{ir}^{*}\geq 0, r=1,2,3,4$ are sufficient to ensure $G^{*}(x,y_{j})-F(x,y_{j})-\frac{1}{2}b_{0}(t)\left(G^{*}{}_{i,j}-F_{i,j}\right)-\frac{1}{2}b_{1}(t)\left(G^{*}{}_{i+1,j}-F_{i+1,j}\right)>0$.

It is clear that $\tau_{i,j}^{x}>0, u_{i,j}^{x}>0, v_{i,j}^{x}>0$ together with the following conditions are sufficient to ensure $B_{ir}^{*}\geq 0, r=1,2,3,4$,

$$
\begin{cases}
\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+2u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+4u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)-v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)-\frac{3}{2}v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)-\frac{3}{2}u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+4v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)-u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+2v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)\geq 0.
\end{cases}
\tag{19}
$$

Further, we can see that if $G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}<0$ and $G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}<0$, the following conditions are sufficient to ensure the constrains (19) hold

$$
\begin{cases}
\tau_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)+4u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\right)-\frac{3}{2}v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i,j}\right)\geq 0, \\
\tau_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)+4v_{i,j}^{x}\left(G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\right)-\frac{3}{2}u_{i,j}^{x}\left(G_{i+1,j}^{*}-F_{i+1,j}\right)\geq 0.
\end{cases}
\tag{20}
$$

In addition, for $G_{i+1,j}^{*}-F_{i,j}-h_{i}^{x}D_{i,j}^{x}\geq 0$ and $G_{i,j}^{*}-F_{i+1,j}+h_{i}^{x}D_{i+1,j}^{x}\geq 0$, it can be easy to check that the following conditions are sufficient to ensure the constrains (19) hold

$$\begin{cases} \tau_{i,j}^x\left(G_{i,j}^*-F_{i,j}\right)-\frac{3}{2}v_{i,j}^x\left(G_{i,j}^*-F_{i,j}\right)\geq 0, \\ \tau_{i,j}^x\left(G_{i+1,j}^*-F_{i+1,j}\right)-\frac{3}{2}u_{i,j}^x\left(G_{i+1,j}^*-F_{i+1,j}\right)\geq 0. \end{cases} \tag{21}$$

From the above analysis, we can immediately obtain the following sufficient conditions for $G^*(x,y_j)-F(x,y_j)-\frac{1}{2}b_0(t)\left(G_{i,j}^*-F_{i,j}\right)-\frac{1}{2}b_1(t)\left(G_{i+1,j}^*-F_{i+1,j}\right)>0$

$$\begin{cases} u_{i,j}^x>0,\quad v_{i,j}^x>0, \\ \tau_{i,j}^x=\max\left\{-4u_{i,j}^x\left(\dfrac{G_{i+1,j}^*-F_{i,j}-h_i^xD_{i,j}^x}{G_{i,j}^*-F_{i,j}}\right)+\dfrac{3}{2}v_{i,j}^x,\right. \\ \qquad\left.-4v_{i,j}^x\left(\dfrac{G_{i,j}^*-F_{i+1,j}+h_i^xD_{i+1,j}^x}{G_{i+1,j}^*-F_{i+1,j}}\right)+\dfrac{3}{2}u_{i,j}^x,\ \dfrac{3}{2}u_{i,j}^x,\ \dfrac{3}{2}v_{i,j}^x\right\}+\rho_{i,j}^x, \end{cases} \tag{22}$$

where $1\leq i\leq n-1$, $1\leq j\leq m$, and $\rho_{i,j}^x,\rho_{i,j}^y\geq 0$ serve as free control parameters.

In the same way, we can derive similar sufficient conditions for $G^*(x,y_{j+1})-F(x,y_{j+1})-\frac{1}{2}b_0(t)\left(G_{i,j+1}^*-F_{i,j+1}\right)-\frac{1}{2}b_1(t)\left(G_{i+1,j+1}^*-F_{i+1,j+1}\right)>0$, $G^*(x_i,y)-F(x_i,y)-\frac{1}{2}b_0(s)\left(G_{i,j}^*-F_{i,j}\right)-\frac{1}{2}b_1(s)\left(G_{i,j+1}^*-F_{i,j+1}\right)>0$ and $G^*(x_{i+1},y)-F(x_{i+1},y)-\frac{1}{2}b_0(s)\left(G_{i+1,j}^*-F_{i+1,j}\right)-\frac{1}{2}b_1(s)\left(G_{i+1,j+1}^*-F_{i+1,j+1}\right)>0$.

In conclusion, the following conditions are sufficient to ensure $G^*(x,y)-S(x,y)>0$ $(\forall(x,y)\in\pi_{i,j})$

$$\begin{cases} u_{i,k}^x>0,\quad v_{i,k}^x>0, \\ \tau_{i,k}^x=\max\left\{-4u_{i,k}^x\left(\dfrac{G_{i+1,k}^*-F_{i,k}-h_i^xD_{i,k}^x}{G_{i,k}^*-F_{i,k}}\right)+\dfrac{3}{2}v_{i,k}^x,\right. \\ \qquad\left.-4v_{i,k}^x\left(\dfrac{G_{i,k}^*-F_{i+1,k}+h_i^xD_{i+1,k}^x}{G_{i+1,k}^*-F_{i+1,k}}\right)+\dfrac{3}{2}u_{i,k}^x,\ \dfrac{3}{2}u_{i,k}^x,\ \dfrac{3}{2}v_{i,k}^x\right\}+\rho_{i,k}^x, \\ u_{l,j}^y>0,\quad v_{l,j}^y>0, \\ \tau_{l,j}^y=\max\left\{-4u_{l,j}^y\left(\dfrac{G_{l,j+1}^*-F_{l,j}-h_j^yD_{l,j}^y}{G_{l,j}^*-F_{l,j}}\right)+\dfrac{3}{2}v_{l,j}^y,\right. \\ \qquad\left.-4v_{l,j}^y\left(\dfrac{G_{l,j}^*-F_{l,j+1}+h_j^yD_{l,j+1}^y}{G_{l,j+1}^*-F_{l,j+1}}\right)+\dfrac{3}{2}u_{l,j}^y,\ \dfrac{3}{2}u_{l,j}^y,\ \dfrac{3}{2}v_{l,j}^y\right\}+\rho_{l,j}^y, \end{cases} \tag{23}$$

where $k=j,j+1$, $l=i,i+1$, and $\rho_{i,k}^x,\rho_{l,j}^y\geq 0$ serve as free control parameters.

Summarizing the above discussion, combining (17) with (23), we can see that the following explicit conditions are sufficient to ensure $G(x,y)<S(x,y)<G^*(x,y)$ $(\forall(x,y)\in\pi_{i,j}$, $i=1,2,\ldots,n-1$, $j=1,2,\ldots,m-1)$

$$
\begin{cases}
u_{i,k}^x > 0, \quad v_{i,k}^x > 0, \\
\tau_{i,k}^x = \max\left\{-4u_{i,k}^x\left(\dfrac{F_{i,k}-G_{i+1,k}+h_i^x D_{i,k}^x}{F_{i,k}-G_{i,k}}\right)+\dfrac{3}{2}v_{i,k}^x,\right. \\
\qquad\qquad -4v_{i,k}^x\left(\dfrac{F_{i+1,k}-G_{i,k}-h_i^x D_{i+1,k}^x}{F_{i+1,k}-G_{i+1,k}}\right)+\dfrac{3}{2}u_{i,k}^x, \\
\qquad\qquad -4u_{i,k}^x\left(\dfrac{G_{i+1,k}^*-F_{i,k}-h_i^x D_{i,k}^x}{G_{i,k}^*-F_{i,k}}\right)+\dfrac{3}{2}v_{i,k}^x, \\
\qquad\qquad \left. -4v_{i,k}^x\left(\dfrac{G_{i,k}^*-F_{i+1,k}+h_i^x D_{i+1,k}^x}{G_{i+1,k}^*-F_{i+1,k}}\right)+\dfrac{3}{2}u_{i,k}^x, \dfrac{3}{2}u_{i,k}^x, \dfrac{3}{2}v_{i,k}^x\right\}+\rho_{i,k}^x, \\
u_{l,j}^y > 0, \quad v_{l,j}^y > 0, \\
\tau_{l,j}^y = \max\left\{-4u_{l,j}^y\left(\dfrac{F_{l,j}-G_{l,j+1}+h_j^y D_{l,j}^y}{F_{l,j}-G_{l,j}}\right)+\dfrac{3}{2}v_{l,j}^y,\right. \\
\qquad\qquad -4v_{l,j}^y\left(\dfrac{F_{l,j+1}-G_{l,j}-h_j^y D_{l,j+1}^y}{F_{l,j+1}-G_{l,j+1}}\right)+\dfrac{3}{2}u_{l,j}^y, \\
\qquad\qquad -4u_{l,j}^y\left(\dfrac{G_{l,j+1}^*-F_{l,j}-h_j^y D_{l,j}^y}{G_{l,j}^*-F_{l,j}}\right)+\dfrac{3}{2}v_{l,j}^y, \\
\qquad\qquad \left. -4v_{l,j}^y\left(\dfrac{G_{l,j}^*-F_{l,j+1}+h_j^y D_{l,j+1}^y}{G_{l,j+1}^*-F_{l,j+1}}\right)+\dfrac{3}{2}u_{l,j}^y, \dfrac{3}{2}u_{l,j}^y, \dfrac{3}{2}v_{l,j}^y\right\}+\rho_{l,j}^y,
\end{cases}
\tag{24}
$$

where $k = 1, 2, \ldots, m$, $l = 1, 2, \ldots, n$, and $\rho_{i,k}^x, \rho_{l,j}^y \geq 0$ serve as free control parameters.

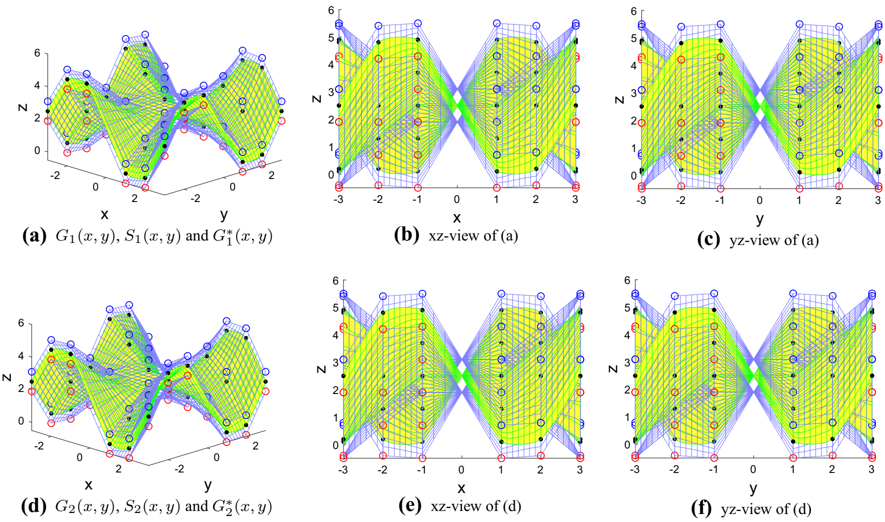
We shall give some graphic examples to show that the proposed C^1 interpolation surface $S(x, y)$ given in (3) can be constrained strictly between two piecewise bi-cubic blending linear interpolation surfaces. The corresponding first partial derivatives $D_{i,j}^x, D_{i,j}^y$ are computed by (4), and (5). In the following figures, for all possible i, j, the data points $(x_i, y_j, F_{i,j})$ have been marked with solid black dots, $(x_i, y_j, G_{i,j}^*)$ with hollow blue dots, and $(x_i, y_j, G_{i,j})$ with red blue dots. And the resulting C^1 constrained interpolation surfaces $S_N(x, y)$, $N = 1, 2, \ldots, 6$ have been marked with yellow color.

Example 1 In this example, the data set $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 6\}$ is given in Table 4. The data sets $\{(x_i, y_i, G_{i,j}), i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 6\}$ and $\{(x_i, y_i, G_{i,j}^*), i = 1, 2, \ldots, 6, j = 1, 2, \ldots, 6\}$ are given by $G_{i,j} = F_{i,j} - 0.6$, $G_{i,j}^* = F_{i,j} + 0.6$, $\forall i, j$. Figure 4a shows the interpolation surface $S_1(x, y)$ generated using the sufficient conditions (24) with all $u_{i,k}^x = v_{i,k}^x = u_{l,j}^y = v_{l,j}^y = 1$ and $\rho_{i,k}^x = \rho_{l,j}^y = 0$. Figure 4d shows the interpolation surface $S_2(x, y)$ generated by changing the free control parameters $\rho_{i,j}^x$ and $\rho_{i,j}^y$, $i = 3, 4$, $j = 3, 4$ from 0 to 1.2.

Example 2 In this example, the data set $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$ is given in Table 5. The data sets $\{(x_i, y_i, G_{i,j}), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$and $\{(x_i, y_i, G_{i,j}^*), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$ are given by $G_{i,j} = F_{i,j} - 0.2$, $G_{i,j}^* = F_{i,j} + 0.2$, $\forall i, j$. Figure 5a shows the interpolation surface $S_3(x, y)$ generated using the sufficient conditions (24) with all $u_{i,k}^x = v_{i,k}^x = u_{l,j}^y = v_{l,j}^y = 1$ and $\rho_{i,k}^x = \rho_{l,j}^y = 0$. Figure 5d shows the interpolation surface $S_4(x, y)$ generated by changing the free control parameters $\rho_{i,j}^x$ and $\rho_{i,j}^y$, $i = 3, 4, 5$, $j = 3, 4, 5$, from 0 to 1.

Table 4 The 3D data set given in Hussain et al. (2014)

y/x	-3	-2	-1	1	2	3
-3	2.5	4.8077	4.9	0.1	0.1923	2.5
-2	0.1923	2.5	3.7	1.3	2.5	4.8077
-1	0.1	1.3	2.5	2.5	3.7	4.9
1	4.9	3.7	2.5	2.5	1.3	0.1
2	4.8077	2.5	1.3	3.7	2.5	0.1923
3	2.5	0.1923	0.1	4.9	4.8077	2.5



(a) $G_1(x, y)$, $S_1(x, y)$ and $G_1^*(x, y)$ **(b)** xz-view of (a) **(c)** yz-view of (a)

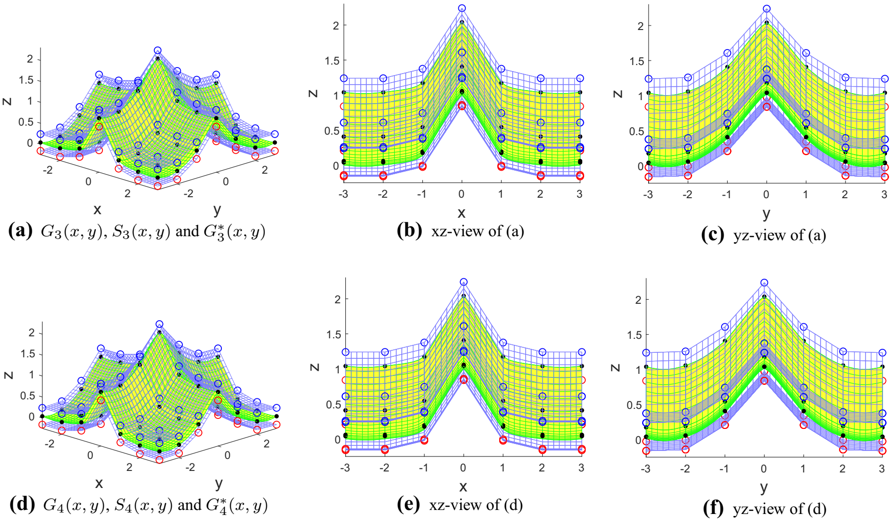
(d) $G_2(x, y)$, $S_2(x, y)$ and $G_2^*(x, y)$ **(e)** xz-view of (d) **(f)** yz-view of (d)

Fig. 4 Interpolation surfaces for examples 1

Table 5 The 3D data set given in Abbas et al. (2014)

y/x	-3	-2	-1	0	1	2	3
-3	0.0401	0.0404	0.1755	1.0401	0.1755	0.0404	0.0401
-2	0.0583	0.0586	0.1936	1.0583	0.1936	0.0586	0.0583
-1	0.4078	0.4082	0.5432	1.4079	0.5432	0.4082	0.4078
0	1.0400	1.0403	1.1753	2.0400	1.1753	1.0403	1.0400
1	0.4078	0.4082	0.5432	1.4079	0.5432	0.4082	0.4078
2	0.0583	0.0586	0.1936	1.0583	0.1936	0.0586	0.0583
3	0.0401	0.0404	0.1755	1.0401	0.1755	0.0404	0.0401

Example 3 In this example, the data set $\{(x_i, y_i, F_{i,j}), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$ is given in Table 6. The data sets $\{(x_i, y_i, G_{i,j}), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$ and $\{(x_i, y_i, G^*_{i,j}), i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\}$ are given by $G_{i,j} = F_{i,j} - 10$, $G^*_{i,j} = F_{i,j} + 10$, $\forall i, j$. Figure 6a shows the interpolation surface $S_3(x, y)$ generated using the sufficient conditions (24) with all $u^x_{i,k} = v^x_{i,k} = u^y_{l,j} = v^y_{l,j} = 1$ and $\rho^x_{i,k} = \rho^y_{l,j} = 0$. Figure 6d shows the interpolation surface $S_4(x, y)$ generated by changing the free control parameters $\rho^x_{i,j}$ and $\rho^y_{i,j}$, $i = 3, 4, 5$, $j = 3, 4, 5$ from 0 to 0.8.



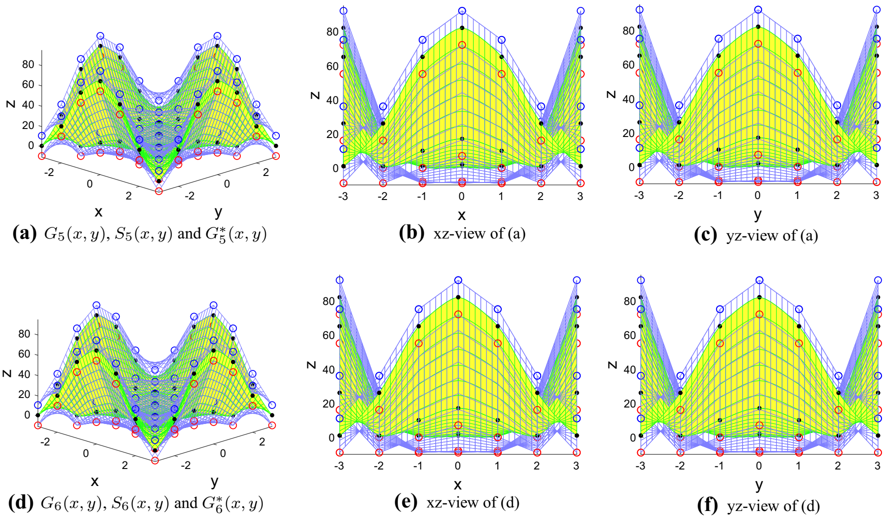
(a) $G_3(x, y)$, $S_3(x, y)$ and $G_3^*(x, y)$ **(b)** xz-view of (a) **(c)** yz-view of (a)

(d) $G_4(x, y)$, $S_4(x, y)$ and $G_4^*(x, y)$ **(e)** xz-view of (d) **(f)** yz-view of (d)

Fig. 5 Interpolation surfaces for example 2

Table 6 The 3D data set given in Sarfraz et al. (2010)

y/x	−3	−2	−1	0	1	2	3
−3	1	26	65	82	65	26	1
−2	26	1	10	17	10	1	26
−1	65	10	1	2	1	10	65
0	82	17	2	1	2	17	82
1	65	10	1	2	1	10	65
2	26	1	10	17	10	1	26
3	1	26	65	82	65	26	1



(a) $G_5(x, y)$, $S_5(x, y)$ and $G_5^*(x, y)$ **(b)** xz-view of (a) **(c)** yz-view of (a)

(d) $G_6(x, y)$, $S_6(x, y)$ and $G_6^*(x, y)$ **(e)** xz-view of (d) **(f)** yz-view of (d)

Fig. 6 Interpolation surfaces for example 3

We conclude from Figs. 4, 5 and 6 that using the sufficient conditions (24), we can generate interpolation surfaces lying strictly between two given piecewise bi-cubic blending linear interpolation surfaces and the shape of the resulting interpolation surfaces can be locally adjusted using the free control parameters.

5 Conclusion

Based on the C^1 rational interpolation splines in one and two dimensions given in Qin et al. (2016), simple and explicit sufficient conditions have been developed for generating C^1 interpolation curves and surfaces with region control. The numerical results show that our method is effective and practical. What's more, in our work, the constraint conditions are explicit so that it is easy to apply, while the conditions given in Duan et al. (1999, 2000, 2005) and Duan et al. (2006) are non-explicit. There are still some problems that deserve further study, for example, the construction of monotonicity- and/or convexity-preserving interpolation surfaces with C^1 continuity. These will be our future work.

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