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Intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices and their application in decision-making

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Abstract
This study aims to propose the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) and to present several of its basic properties. Therefore, it would be possible to improve the problem-modelling capabilities of the available intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets in the occurrence of a large number of data. Moreover, by using *ifpifs*-matrices, we suggest a new soft decision-making method, denoted by EA20, and apply it to a multi-criteria group decision-making (MCGDM) problem. We then compare the ranking performance of EA20 for five noise-removal filters with those of ten state-of-the-art soft decision-making methods. The results show that EA20 successfully models performance-based value assignment problems. Finally, we discuss *ifpifs*-matrices and EA20 for further research.

Keywords Intuitionistic fuzzy sets · Soft sets · Soft matrices · *ifpifs*-matrices · Multi-criteria group decision-making

Mathematics Subject Classification 03F55 · 90B50

1 Introduction

Recently, many mathematical tools have been developed to overcome problems involving uncertainties. Fuzzy sets Zadeh (1965) and soft sets Molodtsov (1999) are among the known mathematical tools, and so far many theoretical and applied studies have been conducted on these concepts Atmaca (2017, 2019); Bera et al. (2017); Çağman et al. (2010, 2011b); Çağman and Deli (2010b, 2012b); Çağman et al. (2011b); Çıtak and Çağman (2017); El-Shafei and Al-Shami (2020); Enginoğlu et al. (2019, 2015); Enginoğlu and Memiş (2018a); Karaaslan (2019); Maji et al. (2001a, 2003); Petchimuthu et al. (2020); Riaz and Hashmi

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(2017, 2018, 2019); Riaz et al. (2018); Riaz et al. (2020a, b); Şenel (2016, 2018a, b); Sezgin et al. (2019a, b); Sulukan et al. (2019). However, fuzzy sets, soft sets, or their hybrid versions cannot simply model some problems containing uncertainty. For example, if six of the data produced by the detector x, which sends ten signals a second, are positive and four are negative, then this case is expressed with the fuzzy value $\mu(x) = 0.6$. Since intuitionistic fuzzy sets are a generalization of fuzzy sets, intuitionistic fuzzy sets can model this problem with intuitionistic fuzzy value $\mu(x) = 0.6$ and $\nu(x) = 0.4$. However, if six of the data collected from the same detector are positive, three are negative, and one is corrupt, then this case cannot be expressed with fuzzy values but with an intuitionistic fuzzy value, i.e. $\mu(x) = 0.6$ and $\nu(x) = 0.3$. These examples too show that intuitionistic fuzzy sets are more useful than fuzzy sets. Moreover, in the presence of the data received from the detectors at different locations, soft sets are needed to attend to the problem of where to build a wind turbine. Therefore, fuzzy soft sets, fuzzy parameterized soft sets, and fuzzy parameterized fuzzy soft sets (*fpfs*-sets), which are hybrid versions of fuzzy sets and soft sets, cannot overcome such a problem. To take advantage of intuitionistic fuzzy sets and soft sets and overcome the abovementioned drawbacks, some hybrid versions of intuitionistic fuzzy sets Atanassov (1986) and soft sets, such as intuitionistic fuzzy soft sets Maji et al. (2001b), intuitionistic fuzzy parameterized soft sets Deli and Çağman (2015), and intuitionistic fuzzy parameterized fuzzy soft sets El-Yagubi and Salleh (2013), have been introduced.

However, when a problem containing uncertainties involves a large number of data, the aforesaid concepts are incapable of processing them. For this reason, the matrix representations of these concepts, such as soft matrices (Çağman and Enginoğlu 2010a), fuzzy soft matrices (Çağman et al. 2011b), fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) (Enginoğlu and Çağman 2020), and intuitionistic fuzzy soft matrices (Chetia and Das 2012), have been proposed.

Since more general forms are needed for mathematical modelling of some problems containing further uncertainties, Karaaslan (2016) has propounded the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (*ifpifs*-sets). He has also proposed a soft decision-making method by using *ifpifs*-sets and applied this method to a decision-making problem. The application provided therein has demonstrated that *ifpifs*-sets can be successfully applied to some problems containing further uncertainties. As a result, the concept of *ifpifs*-sets has allowed to model situations with parameters and objects containing intuitionistic fuzzy uncertainties. Therefore, this concept is worth being studied. On the other hand, since such concepts pose some disadvantages, such as long-running time and complexity, it is of great importance to study their matrix representations.

In addition to the type of decision-making discussed in the present paper, the related literature incorporates many studies on multi-criteria group decision-making. To exemplify, Garg and Kaur (2020) have proposed an extended TOPSIS method for multi-criteria group decision-making problems in cubic intuitionistic fuzzy environment. Çalı and Balaban (2012a) have presented a multi-criteria group decision-making method based on the integration of ELECTRE and VIKOR in the intuitionistic fuzzy environment. Enginoğlu and Arslan (2018a) have suggested a decision-making method exploiting *ifpifs*-sets and have applied it to a recruitment scenario of a company. Büyüközkan and Göçer (2017) have propounded a new multi-criteria group decision approach using the intuitionistic fuzzy analytic hierarchy process method and the intuitionistic fuzzy axiomatic design principles. Nan et al. (2016) have proposed a new extended TOPSIS by using an intuitionistic fuzzy distance measure.

In Sect. 2 of the present study, we present some of the basic definitions required in the following sections of the paper. In Sect. 3, we define the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) and investigate a number of its basic properties. This concept is first mentioned in the second author's master's thesis (Arslan 2019). In Section 4, by using *ifpifs*-matrices, we propose a new and efficient soft decision-making method, denoted by EA20 herein, which avails of some steps of two soft decision-making methods, namely EMO18a and EMO18o, provided in (Enginoğlu et al. 2018b). This method allows for selecting the optimal elements from the alternatives. In Sect. 5, to demonstrate that this method can be successfully applied to some problems involving uncertainty in the real world, we apply it to a multi-criteria group decision-making (MCGDM) problem with the determination of eligible candidates for two vacant positions advertised online. In Sect. 6, we then compare the ranking performance of EA20 with those of the ten methods avaliable in (Enginoğlu and Memiş 2018b; Enginoğlu et al. 2018b; Enginoğlu and Öngel 2020). Finally, we discuss *ifpifs*-matrices and EA20 for further research.

2 Preliminaries

In this section, we present the concepts of intuitionistic fuzzy sets (Atanassov 1986) and intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets (Karaaslan 2016) by considering the notations used across this study.

Throughout this paper, let U be a universal set, E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu, \nu \in F(E)$. Here, a fuzzy set is denoted by $\{^{\mu(x)}x : x \in E\}$ instead of $\{(x, \mu(x)) : x \in E\}$.

Definition 1 (Atanassov 1986) Let f be a function from E to $[0, 1] \times [0, 1]$. Then, the set $\left\{{}^{\mu(x)}_{\nu(x)}x : x \in E\right\}$ being the graphic of f is called an intuitionistic fuzzy set (*if*-set) over E.

Here, for all $x \in E$, $\mu(x) + \nu(x) \leq 1$. Moreover, μ and ν are called the membership function and non-membership function, respectively, and $\pi(x) = 1 - \mu(x) - \nu(x)$ is called the degree of indeterminacy of the element $x \in E$. Obviously, each ordinary fuzzy set can be written as $\left\{{}^{\mu(x)}_{1-\mu(x)}x : x \in E\right\}$.

In the present paper, the set of all *if*-sets over E is denoted by $IF(E)$ and $f \in IF(E)$. In $IF(E)$, since the $graph(f)$ and f generate each other uniquely, the notations are interchangeable. Therefore, as long as it does not cause any confusion, we denote an *if*-set $graph(f)$ by f.

Note 1 For convenience, we do not display the elements ${}^{0}_{1}x$ in an *if*-set.

Definition 2 (Karaaslan 2016) Let α be a function from f to $IF(U)$. Then, the set $\left\{\left({}^{\mu(x)}_{\nu(x)}x, \alpha\left({}^{\mu(x)}_{\nu(x)}x\right)\right) : x \in E\right\}$ being the graphic of α is called an intuitionistic fuzzy parameterized intuitionistic fuzzy soft set (*ifpifs*-set) parameterized via E over U (or briefly over U).

Throughout this paper, the set of all *ifpifs*-sets over U is denoted by $IFPIFS_E(U)$. In $IFPIFS_E(U)$, since the $graph(\alpha)$ and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it does not cause any confusion, we denote an *ifpifs*-set $graph(\alpha)$ by α.

Note 2 For convenience, we do not display the elements $({}^{0}_{1}x, 0_U)$ in an *ifpifs*-set. Here, 0_U is the empty *if*-set over U.

Example 1 Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4, u_5\}$. Then,

$$\alpha = \left\{ \left({}^{0.8}_{0.2}x_1, \left\{{}^{0}_{0.5}u_1, {}^{0.5}_{0.5}u_4\right\}\right), \left({}^{1}_{0}x_2, \left\{{}^{0.3}_{0.6}u_2, {}^{0.5}_{0.1}u_3\right\}\right), \left({}^{0.1}_{0.7}x_3, \left\{{}^{0.7}_{0}u_1, {}^{0.2}_{0.5}u_3, {}^{0.6}_{0.1}u_4\right\}\right), \left({}^{0}_{1}x_4, \left\{{}^{0.2}_{0.3}u_1\right\}\right) \right\}$$

is an *ifpifs*-set over U.

3 Intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices

In this section, we define the concept of intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices and introduce some of its basic properties. Both the concept and its properties are first presented in the second author's master's thesis (Arslan 2019).

Definition 3 Let $\alpha \in IFPIFS_E(U)$. Then, $[a_{ij}]$ is called the matrix representation of α (or briefly *ifpifs*-matrix of α) and is defined by

$$[a_{ij}] := \begin{bmatrix} a_{01} & a_{02} & a_{03} & \dots & a_{0n} & \dots \\ a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{0, 1, 2, \cdots\}$ and $j \in \{1, 2, \cdots\}$,

$$a_{ij} := \begin{cases} {}^{\mu(x_j)}_{\nu(x_j)}, & i = 0 \\ \alpha\left({}^{\mu(x_j)}_{\nu(x_j)}x_j\right)(u_i), & i \neq 0 \end{cases}$$

or briefly $a_{ij} := {}^{\mu_{ij}}_{\nu_{ij}}$. Here, if $|U| = m - 1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

In this paper, as long as it does not cause any confusion, the membership and non-membership functions of $[a_{ij}]$, i.e. μ_{ij} and ν_{ij}, will be represented by μ^a_{ij} and ν^a_{ij}, respectively. Moreover, the set of all *ifpifs*-matrices parameterized via E over U is denoted by $IFPIFS_E[U]$ and $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$.

Example 2 The matrix representation of α provided in Example 1 is as follows:

$$[a_{ij}] = \begin{bmatrix} {}^{0.8}_{0.2} & {}^{1}_{0} & {}^{0.1}_{0.7} & {}^{0}_{1} \\ {}^{0}_{0.5} & {}^{0}_{1} & {}^{0.7}_{0} & {}^{0.2}_{0.3} \\ {}^{0}_{1} & {}^{0.3}_{0.6} & {}^{0}_{1} & {}^{0}_{1} \\ {}^{0}_{1} & {}^{0.5}_{0.1} & {}^{0.2}_{0.5} & {}^{0}_{1} \\ {}^{0.5}_{0.5} & {}^{0}_{1} & {}^{0.6}_{0.1} & {}^{0}_{1} \\ {}^{0}_{1} & {}^{0}_{1} & {}^{0}_{1} & {}^{0}_{1} \end{bmatrix}$$

Definition 4 Let $[a_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu_{ij} = \lambda$ and $\nu_{ij} = \varepsilon$, then $[a_{ij}]$ is called (λ, ε)-*ifpifs*-matrix and is denoted by $\left[{}^{\lambda}_{\varepsilon}\right]$. Here, $\left[{}^{0}_{1}\right]$ is called empty *ifpifs*-matrix and $\left[{}^{1}_{0}\right]$ is called universal *ifpifs*-matrix.

Definition 5 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$, $I_E := \{j : x_j \in E\}$, and $R \subseteq I_E$. For all i and j, if

$$\mu^c_{ij} = \begin{cases} \mu^a_{ij}, & j \in R \\ \mu^b_{ij}, & j \in I_E \setminus R \end{cases} \text{ and } \nu^c_{ij} = \begin{cases} \nu^a_{ij}, & j \in R \\ \nu^b_{ij}, & j \in I_E \setminus R \end{cases}$$

then $[c_{ij}]$ is called Rb-restriction of $[a_{ij}]$ and is denoted by $[(a_{Rb})_{ij}]$. Briefly, if $[b_{ij}] = \genfrac{[}{]}{0pt}{}{0}{1}$, then $[(a_R)_{ij}]$ can be used instead of $[(a_{R^0_1})_{ij}]$. It is clear that

$$(a_R)_{ij} = \begin{cases} \genfrac{}{}{0pt}{}{\mu^a_{ij}}{\nu^a_{ij}}, & j \in R \\ \genfrac{}{}{0pt}{}{0}{1}, & j \in I_E \setminus R \end{cases}$$

Example 3 For $R = \{1, 3, 4\}$ and $S = \{2, 4\}$, R^1_0-restriction and S-restriction of $[a_{ij}]$ provided in Example 2 are as follows:

$$[(a_{R^1_0})_{ij}] = \begin{bmatrix} 0.8 & 1 & 0.1 & 0 \\ 0.2 & 0 & 0.7 & 1 \\ 0 & 1 & 0.7 & 0.2 \\ 0.5 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0.2 & 0 \\ 1 & 0 & 0.5 & 1 \\ 0.5 & 1 & 0.6 & 0 \\ 0.5 & 0 & 0.1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad [(a_S)_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0.2 \\ 1 & 1 & 1 & 0.3 \\ 0 & 0.3 & 0 & 0 \\ 1 & 0.6 & 1 & 1 \\ 0 & 0.5 & 0 & 0 \\ 1 & 0.1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Definition 6 Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^a_{ij} \leq \mu^b_{ij}$ and $\nu^b_{ij} \leq \nu^a_{ij}$, then $[a_{ij}]$ is called a submatrix of $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\subseteq}[b_{ij}]$.

Definition 7 Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^a_{ij} = \mu^b_{ij}$ and $\nu^a_{ij} = \nu^b_{ij}$, then $[a_{ij}]$ and $[b_{ij}]$ are called equal *ifpifs*-matrices and is denoted by $[a_{ij}] = [b_{ij}]$.

Proposition 1 *Let* $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. *Then,*

i. $[a_{ij}]\tilde{\subseteq}\genfrac{[}{]}{0pt}{}{1}{0}$
ii. $\genfrac{[}{]}{0pt}{}{0}{1}\tilde{\subseteq}[a_{ij}]$
iii. $[a_{ij}]\tilde{\subseteq}[a_{ij}]$
iv. $\big([a_{ij}]\tilde{\subseteq}[b_{ij}] \wedge [b_{ij}]\tilde{\subseteq}[a_{ij}]\big) \Leftrightarrow [a_{ij}] = [b_{ij}]$
v. $\big([a_{ij}]\tilde{\subseteq}[b_{ij}] \wedge [b_{ij}]\tilde{\subseteq}[c_{ij}]\big) \Rightarrow [a_{ij}]\tilde{\subseteq}[c_{ij}]$
vi. $\big([a_{ij}] = [b_{ij}] \wedge [b_{ij}] = [c_{ij}]\big) \Rightarrow [a_{ij}] = [c_{ij}]$

Remark 1 From Proposition 1, it can be understood that the inclusion relation is a partial ordering relation in $IFPIFS_E[U]$.

Definition 8 Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. If $[a_{ij}]\tilde{\subseteq}[b_{ij}]$ and $[a_{ij}] \neq [b_{ij}]$, then $[a_{ij}]$ is called a proper submatrix of $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\subsetneq}[b_{ij}]$.

Definition 9 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^c_{ij} = \max\{\mu^a_{ij}, \mu^b_{ij}\}$ and $\nu^c_{ij} = \min\{\nu^a_{ij}, \nu^b_{ij}\}$, then $[c_{ij}]$ is called union of $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\cup}[b_{ij}]$.

Definition 10 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^c_{ij} = \min\{\mu^a_{ij}, \mu^b_{ij}\}$ and $\nu^c_{ij} = \max\{\nu^a_{ij}, \nu^b_{ij}\}$, then $[c_{ij}]$ is called intersection of $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\cap}[b_{ij}]$.

Example 4 Assume that two *ifpifs*-matrices $[a_{ij}]$ and $[b_{ij}]$ are as follows:

$$[a_{ij}] = \begin{bmatrix} 0.4 & 1 & 0.6 & 0.7 \\ 0.3 & 0 & 0.1 & 0 \\ 0 & 0.3 & 0.1 & 0.2 \\ 0.5 & 0 & 0.8 & 0.3 \\ 1 & 0.5 & 0.2 & 1 \\ 0 & 0.5 & 0.6 & 0 \\ 0 & 1 & 0.2 & 0 \\ 1 & 0 & 0.5 & 0.7 \end{bmatrix} \quad \text{and} \quad [b_{ij}] = \begin{bmatrix} 0.7 & 0.9 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \\ 0.6 & 0 & 0.3 & 0.2 \\ 0.3 & 1 & 0.5 & 0.1 \\ 0.1 & 0.3 & 0.7 & 0.6 \\ 0.5 & 0 & 0.1 & 0.2 \\ 0.4 & 0.5 & 0.4 & 1 \\ 0.2 & 0.5 & 0.5 & 0 \end{bmatrix}$$

Then,

$$[a_{ij}]\tilde{\cup}[b_{ij}] = \begin{bmatrix} 0.7 & 1 & 0.8 & 0.7 \\ 0.1 & 0 & 0.1 & 0 \\ 0.6 & 0.3 & 0.3 & 0.2 \\ 0.3 & 0 & 0.5 & 0.1 \\ 1 & 0.5 & 0.7 & 1 \\ 0 & 0 & 0.1 & 0 \\ 0.4 & 1 & 0.4 & 1 \\ 0.2 & 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad [a_{ij}]\tilde{\cap}[b_{ij}] = \begin{bmatrix} 0.4 & 0.9 & 0.6 & 0 \\ 0.3 & 0.1 & 0.2 & 1 \\ 0 & 0 & 0.1 & 0.2 \\ 0.5 & 1 & 0.8 & 0.3 \\ 0.1 & 0.3 & 0.2 & 0.6 \\ 0.5 & 0.5 & 0.6 & 0.2 \\ 0 & 0.5 & 0.2 & 0 \\ 1 & 0.5 & 0.5 & 0.7 \end{bmatrix}$$

Proposition 2 *Let* $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. *Then,*

i. $[a_{ij}]\tilde{\cup}[a_{ij}] = [a_{ij}]$ *and* $[a_{ij}]\tilde{\cap}[a_{ij}] = [a_{ij}]$
ii. $[a_{ij}]\tilde{\cup}\begin{bmatrix}0\\1\end{bmatrix} = [a_{ij}]$ *and* $[a_{ij}]\tilde{\cap}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$
iii. $[a_{ij}]\tilde{\cup}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ *and* $[a_{ij}]\tilde{\cap}\begin{bmatrix}1\\0\end{bmatrix} = [a_{ij}]$
iv. $[a_{ij}]\tilde{\cup}[b_{ij}] = [b_{ij}]\tilde{\cup}[a_{ij}]$ *and* $[a_{ij}]\tilde{\cap}[b_{ij}] = [b_{ij}]\tilde{\cap}[a_{ij}]$
v. $([a_{ij}]\tilde{\cup}[b_{ij}])\tilde{\cup}[c_{ij}] = [a_{ij}]\tilde{\cup}([b_{ij}]\tilde{\cup}[c_{ij}])$ *and* $([a_{ij}]\tilde{\cap}[b_{ij}])\tilde{\cap}[c_{ij}] = [a_{ij}]\tilde{\cap}([b_{ij}]\tilde{\cap}[c_{ij}])$
vi. $[a_{ij}]\tilde{\cup}([b_{ij}]\tilde{\cap}[c_{ij}]) = ([a_{ij}]\tilde{\cup}[b_{ij}])\tilde{\cap}([a_{ij}]\tilde{\cup}[c_{ij}])$ *and*
$[a_{ij}]\tilde{\cap}([b_{ij}]\tilde{\cup}[c_{ij}]) = ([a_{ij}]\tilde{\cap}[b_{ij}])\tilde{\cup}([a_{ij}]\tilde{\cap}[c_{ij}])$

Proof *vi.* Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. Then,

$$\begin{aligned}
[a_{ij}]\tilde{\cup}([b_{ij}]\tilde{\cap}[c_{ij}]) &= [a_{ij}]\tilde{\cup}\begin{bmatrix} \min\left\{\mu^b_{ij},\mu^c_{ij}\right\} \\ \max\left\{\nu^b_{ij},\nu^c_{ij}\right\} \end{bmatrix} \\
&= \begin{bmatrix} \max\left\{\mu^a_{ij},\min\left\{\mu^b_{ij},\mu^c_{ij}\right\}\right\} \\ \min\left\{\nu^a_{ij},\max\left\{\nu^b_{ij},\nu^c_{ij}\right\}\right\} \end{bmatrix} \\
&= \begin{bmatrix} \min\left\{\max\left\{\mu^a_{ij},\mu^b_{ij}\right\},\max\left\{\mu^a_{ij},\mu^c_{ij}\right\}\right\} \\ \max\left\{\min\left\{\nu^a_{ij},\nu^b_{ij}\right\},\min\left\{\nu^a_{ij},\nu^c_{ij}\right\}\right\} \end{bmatrix} \\
&= \begin{bmatrix} \max\left\{\mu^a_{ij},\mu^b_{ij}\right\} \\ \min\left\{\nu^a_{ij},\nu^b_{ij}\right\} \end{bmatrix} \tilde{\cap} \begin{bmatrix} \max\left\{\mu^a_{ij},\mu^c_{ij}\right\} \\ \min\left\{\nu^a_{ij},\nu^c_{ij}\right\} \end{bmatrix} \\
&= ([a_{ij}]\tilde{\cup}[b_{ij}])\tilde{\cap}([a_{ij}]\tilde{\cup}[c_{ij}])
\end{aligned}$$

Definition 11 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^c_{ij} = \min\{\mu^a_{ij}, \nu^b_{ij}\}$ and $\nu^c_{ij} = \max\{\nu^a_{ij}, \mu^b_{ij}\}$, then $[c_{ij}]$ is called difference between $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\setminus}[b_{ij}]$.

Proposition 3 *Let* $[a_{ij}] \in IFPIFS_E[U]$. *Then,* $[a_{ij}]\tilde{\setminus}\begin{bmatrix}0\\1\end{bmatrix} = [a_{ij}]$ *and* $[a_{ij}]\tilde{\setminus}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

Remark 2 It must be noted that the difference is non-commutative and non-associative.

Definition 12 Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^b_{ij} = \nu^a_{ij}$ and $\nu^b_{ij} = \mu^a_{ij}$, then $[b_{ij}]$ is complement of $[a_{ij}]$ and is denoted by $[a_{ij}]^{\tilde{c}}$ or $[a^{\tilde{c}}_{ij}]$. It is clear that $[a_{ij}]^{\tilde{c}} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \tilde{\setminus} [a_{ij}]$.

Proposition 4 *Let* $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. *Then,*

i. $\left([a_{ij}]^{\tilde{c}}\right)^{\tilde{c}} = [a_{ij}]$
ii. $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]^{\tilde{c}} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$
iii. $[a_{ij}]\tilde{\setminus}[b_{ij}] = [a_{ij}]\tilde{\cap}[b_{ij}]^{\tilde{c}}$
iv. $[a_{ij}]\tilde{\subseteq}[b_{ij}] \Rightarrow [b_{ij}]^{\tilde{c}}\tilde{\subseteq}[a_{ij}]^{\tilde{c}}$

Proposition 5 *Let* $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. *Then, the following De Morgan's laws are valid:*

i. $\left([a_{ij}]\tilde{\cup}[b_{ij}]\right)^{\tilde{c}} = [a_{ij}]^{\tilde{c}}\tilde{\cap}[b_{ij}]^{\tilde{c}}$
ii. $\left([a_{ij}]\tilde{\cap}[b_{ij}]\right)^{\tilde{c}} = [a_{ij}]^{\tilde{c}}\tilde{\cup}[b_{ij}]^{\tilde{c}}$

Proof *i.* Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. Then,

$$([a_{ij}]\tilde{\cup}[b_{ij}])^{\tilde{c}} = \begin{bmatrix} \max\{\mu^a_{ij},\mu^b_{ij}\} \\ \min\{\nu^a_{ij},\nu^b_{ij}\} \end{bmatrix}^{\tilde{c}} = \begin{bmatrix} \min\{\nu^a_{ij},\nu^b_{ij}\} \\ \max\{\mu^a_{ij},\mu^b_{ij}\} \end{bmatrix} = \begin{bmatrix} \nu^a_{ij} \\ \mu^a_{ij} \end{bmatrix} \tilde{\cap} \begin{bmatrix} \nu^b_{ij} \\ \mu^b_{ij} \end{bmatrix} = \left[a_{ij}\right]^{\tilde{c}} \tilde{\cap} \left[b_{ij}\right]^{\tilde{c}}$$

Definition 13 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$. For all i and j, if $\mu^c_{ij} = \max\left\{\min\{\mu^a_{ij}, \nu^b_{ij}\}, \min\{\nu^a_{ij}, \mu^b_{ij}\}\right\}$ and $\nu^c_{ij} = \min\left\{\max\{\mu^a_{ij}, \nu^b_{ij}\}, \max\{\nu^a_{ij}, \mu^b_{ij}\}\right\}$, then $[c_{ij}]$ is called symmetric difference between $[a_{ij}]$ and $[b_{ij}]$ and is denoted $[a_{ij}]\tilde{\triangle}[b_{ij}]$.

Proposition 6 *Let* $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. *Then,*

i. $[a_{ij}]\tilde{\triangle}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = [a_{ij}]$
ii. $[a_{ij}]\tilde{\triangle}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = [a_{ij}]^{\tilde{c}}$
iii. $[a_{ij}]\tilde{\triangle}[b_{ij}] = [b_{ij}]\tilde{\triangle}[a_{ij}]$
iv. $[a_{ij}]\tilde{\triangle}[b_{ij}] = ([a_{ij}]\tilde{\setminus}[b_{ij}])\tilde{\cup}([b_{ij}]\tilde{\setminus}[a_{ij}])$

Remark 3 It must be noted that the symmetric difference is non-associative.

Example 5 $[a_{ij}]\tilde{\setminus}[b_{ij}]$ and $[a_{ij}]\tilde{\triangle}[b_{ij}]$ for $[a_{ij}]$ and $[b_{ij}]$ provided in Example 4 are as follows:

$$[a_{ij}]\tilde{\setminus}[b_{ij}] = \begin{bmatrix} \begin{smallmatrix}0.1\\0.7\end{smallmatrix} & \begin{smallmatrix}0.1\\0.9\end{smallmatrix} & \begin{smallmatrix}0.2\\0.8\end{smallmatrix} & \begin{smallmatrix}0.7\\0\end{smallmatrix} \\ \begin{smallmatrix}0\\0.6\end{smallmatrix} & \begin{smallmatrix}0.3\\0\end{smallmatrix} & \begin{smallmatrix}0.1\\0.8\end{smallmatrix} & \begin{smallmatrix}0.1\\0.3\end{smallmatrix} \\ \begin{smallmatrix}0.5\\0.1\end{smallmatrix} & \begin{smallmatrix}0\\0.5\end{smallmatrix} & \begin{smallmatrix}0.1\\0.7\end{smallmatrix} & \begin{smallmatrix}0.2\\0.6\end{smallmatrix} \\ \begin{smallmatrix}0\\1\end{smallmatrix} & \begin{smallmatrix}0.5\\0.5\end{smallmatrix} & \begin{smallmatrix}0.2\\0.5\end{smallmatrix} & \begin{smallmatrix}0\\1\end{smallmatrix} \end{bmatrix} \quad \text{and} \quad [a_{ij}]\tilde{\triangle}[b_{ij}] = \begin{bmatrix} \begin{smallmatrix}0.3\\0.4\end{smallmatrix} & \begin{smallmatrix}0.1\\0.9\end{smallmatrix} & \begin{smallmatrix}0.2\\0.6\end{smallmatrix} & \begin{smallmatrix}0.7\\0\end{smallmatrix} \\ \begin{smallmatrix}0.5\\0.3\end{smallmatrix} & \begin{smallmatrix}0.3\\0\end{smallmatrix} & \begin{smallmatrix}0.3\\0.5\end{smallmatrix} & \begin{smallmatrix}0.2\\0.2\end{smallmatrix} \\ \begin{smallmatrix}0.5\\0.1\end{smallmatrix} & \begin{smallmatrix}0.3\\0.5\end{smallmatrix} & \begin{smallmatrix}0.6\\0.2\end{smallmatrix} & \begin{smallmatrix}0.2\\0.6\end{smallmatrix} \\ \begin{smallmatrix}0.4\\0.2\end{smallmatrix} & \begin{smallmatrix}0.5\\0.5\end{smallmatrix} & \begin{smallmatrix}0.4\\0.5\end{smallmatrix} & \begin{smallmatrix}0.7\\0\end{smallmatrix} \end{bmatrix}$$

Definition 14 Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. If $[a_{ij}]\tilde{\cap}[b_{ij}] = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, then $[a_{ij}]$ and $[b_{ij}]$ are called disjoint.

Definition 15 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$, $I_E := \{j : x_j \in E\}$, and $R \subseteq I_E$. For all i and j, if

$$\mu^c_{ij} = \begin{cases} \max\left\{\mu^a_{ij}, \max_{k\in R}\{\nu^b_{ik}\}\right\}, & j \in R \\ \mu^a_{ij}, & j \in I_E \setminus R \end{cases} \quad \text{and } \nu^c_{ij} = \begin{cases} \min\left\{\nu^a_{ij}, \min_{k\in R}\{\mu^b_{ik}\}\right\}, & j \in R \\ \nu^a_{ij}, & j \in I_E \setminus R \end{cases}$$

then $[c_{ij}]$ is called R-relative union of $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\cup}^r_R[b_{ij}]$. Here, for brevity, "relative union" can be used instead of "I_E-relative union" and denoted by $[a_{ij}]\tilde{\cup}^r[b_{ij}]$.

Definition 16 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$, $I_E := \{j : x_j \in E\}$, and $R \subseteq I_E$. For all i and j, if

$$\mu^c_{ij} = \begin{cases} \min\left\{\mu^a_{ij}, \max_{k\in R}\{\mu^b_{ik}\}\right\}, & j \in R \\ \mu^a_{ij}, & j \in I_E \setminus R \end{cases} \quad \text{and } \nu^c_{ij} = \begin{cases} \max\left\{\nu^a_{ij}, \min_{k\in R}\{\nu^b_{ik}\}\right\}, & j \in R \\ \nu^a_{ij}, & j \in I_E \setminus R \end{cases}$$

then $[c_{ij}]$ is called R-relative intersection of $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\cap}^r_R[b_{ij}]$. Here, for brevity, "relative intersection" can be used instead of "I_E-relative intersection" and denoted by $[a_{ij}]\tilde{\cap}^r[b_{ij}]$.

Definition 17 Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$, $I_E := \{j : x_j \in E\}$, and $R \subseteq I_E$. For all i and j, if

$$\mu^c_{ij} = \begin{cases} \min\left\{\mu^a_{ij}, \max_{k\in R}\{\nu^b_{ik}\}\right\}, & j \in R \\ \mu^a_{ij}, & j \in I_E \setminus R \end{cases} \quad \text{and } \nu^c_{ij} = \begin{cases} \max\left\{\nu^a_{ij}, \min_{k\in R}\{\mu^b_{ik}\}\right\}, & j \in R \\ \nu^a_{ij}, & j \in I_E \setminus R \end{cases}$$

then $[c_{ij}]$ is called R-relative difference between $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}]\tilde{\setminus}^r_R[b_{ij}]$. Here, for brevity, "relative difference" can be used instead of "I_E-relative difference" and denoted by $[a_{ij}]\tilde{\setminus}^r[b_{ij}]$.

Proposition 7 *Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$ and R be a finite subset of I_E. Then,*

i. $[a_{ij}]\tilde{\cup}^r_R[a_{ij}] = [a_{ij}]$ *and* $[a_{ij}]\tilde{\cap}^r_R[a_{ij}] = [a_{ij}]$
ii. $[a_{ij}]\tilde{\cup}^r_R \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] = [a_{ij}]$ *and* $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\tilde{\cap}^r_R[a_{ij}] = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$
iii. $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\tilde{\cup}^r_R[a_{ij}] = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ *and* $[a_{ij}]\tilde{\cap}^r_R \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = [a_{ij}]$
iv. $\left([a_{ij}]\tilde{\cup}^r_R[b_{ij}]\right)\tilde{\cup}^r_R[c_{ij}] = [a_{ij}]\tilde{\cup}^r_R\left([b_{ij}]\tilde{\cup}^r_R[c_{ij}]\right)$ *and* $\left([a_{ij}]\tilde{\cap}^r_R[b_{ij}]\right)\tilde{\cap}^r_R[c_{ij}] = [a_{ij}]\tilde{\cap}^r_R\left([b_{ij}]\tilde{\cap}^r_R[c_{ij}]\right)$

Proof *iv.* Let $[a_{ij}], [b_{ij}], [c_{ij}] \in IFPIFS_E[U]$, $R = \{n_1, n_2, \ldots, n_s\}$ and ℓ be denote $[a_{ij}]\tilde{\cup}^r_R([b_{ij}]\tilde{\cup}^r_R[c_{ij}])$. Then,

$$\ell=[a_{ij}]\tilde{\cup}^r_R\begin{bmatrix}\begin{cases}\max\left\{\mu^b_{ij},\min_{t\in R}\{\mu^c_{it}\}\right\}, & j\in R\\ \mu^b_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\nu^b_{ij},\max_{t\in R}\{\nu^c_{it}\}\right\}, & j\in R\\ \nu^b_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{cases}\max\left\{\mu^a_{ij},\min_{k\in R}\left\{\max\left\{\mu^b_{ik},\min_{t\in R}\{\mu^c_{it}\}\right\}\right\}\right\}, & j\in R\\ \mu^a_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\nu^a_{ij},\max_{k\in R}\left\{\min\left\{\nu^b_{ik},\max_{t\in R}\{\nu^c_{it}\}\right\}\right\}\right\}, & j\in R\\ \nu^a_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{cases}\max\left\{\mu^a_{ij},\min\left\{\max\left\{\mu^b_{in_1},\min_{t\in R}\{\mu^c_{it}\}\right\},\dots,\max\left\{\mu^b_{in_s},\min_{t\in R}\{\mu^c_{it}\}\right\}\right\}\right\}, & j\in R\\ \mu^a_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\nu^a_{ij},\max\left\{\min\left\{\nu^b_{in_1},\max_{t\in R}\{\nu^c_{it}\}\right\},\dots,\min\left\{\nu^b_{in_s},\max_{t\in R}\{\nu^c_{it}\}\right\}\right\}\right\}, & j\in R\\ \nu^a_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{cases}\max\left\{\mu^a_{ij},\max\left\{\min_{k\in R}\left\{\mu^b_{ik}\right\},\min_{t\in R}\{\mu^c_{it}\}\right\}\right\}, & j\in R\\ \mu^a_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\nu^a_{ij},\min\left\{\max_{k\in R}\left\{\nu^b_{ik}\right\},\max_{t\in R}\{\nu^c_{it}\}\right\}\right\}, & j\in R\\ \nu^a_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{cases}\max\left\{\max\left\{\mu^a_{ij},\min_{k\in R}\left\{\mu^b_{ik}\right\}\right\},\min_{t\in R}\{\mu^c_{it}\}\right\}, & j\in R\\ \mu^a_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\min\left\{\nu^a_{ij},\max_{k\in R}\left\{\nu^b_{ik}\right\}\right\},\max_{t\in R}\{\nu^c_{it}\}\right\}, & j\in R\\ \nu^a_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}$$

$$=\begin{bmatrix}\begin{cases}\max\left\{\mu^a_{ij},\min_{k\in R}\{\mu^b_{ik}\}\right\}, & j\in R\\ \mu^a_{ij}, & j\in I_E\setminus R\end{cases}\\ \begin{cases}\min\left\{\nu^a_{ij},\max_{k\in R}\{\nu^b_{ik}\}\right\}, & j\in R\\ \nu^a_{ij}, & j\in I_E\setminus R\end{cases}\end{bmatrix}\tilde{\cup}^r_R[c_{ij}]$$

$$=\left([a_{ij}]\tilde{\cup}^r_R[b_{ij}]\right)\tilde{\cup}^r_R[c_{ij}]$$

Remark 4 It must be noted that the relative union and relative intersection of *ifpifs*-matrices are non-commutative and non-distributive.

Example 6 Relative union and R-relative intersection, for $R=\{1,3\}$, of $[a_{ij}]$ and $[b_{ij}]$ provided in Example 4 are as follows:

$$[a_{ij}]\tilde{\cup}^r[b_{ij}]=\begin{bmatrix}0.4&1&0.6&0.7\\0.3&0&0.1&0\\0&0.3&0.1&0.2\\0.5&0&0.8&0.3\\1&0.5&0.2&1\\0&0.5&0.5&0\\0.4&1&0.4&0.4\\0.5&0&0.5&0.5\end{bmatrix}\quad\text{and}\quad[a_{ij}]\tilde{\cap}^r_R[b_{ij}]=\begin{bmatrix}0.4&1&0.6&0.7\\0.3&0&0.1&0\\0&0.3&0.1&0.2\\0.5&0&0.8&0.3\\0.7&0.5&0.2&1\\0.1&0.5&0.6&0\\0&1&0.2&0\\1&0&0.5&0.7\end{bmatrix}$$

Proposition 8 *Let* $[a_{ij}],[b_{ij}]\in IFPIFS_E[U]$. *Then, the following De Morgan's laws are valid:*

i. $\left([a_{ij}]\tilde{\cup}^r_R[b_{ij}]\right)^{\tilde{c}}=[a_{ij}]^{\tilde{c}}\tilde{\cap}^r_R[b_{ij}]^{\tilde{c}}$

ii. $\left([a_{ij}]\tilde{\cap}^r_R[b_{ij}]\right)^{\tilde{c}}=[a_{ij}]^{\tilde{c}}\tilde{\cup}^r_R[b_{ij}]^{\tilde{c}}$

Proof *i.* Let $[a_{ij}], [b_{ij}] \in IFPIFS_E[U]$. Then,

$$
\begin{aligned}
([a_{ij}]\tilde{\cup}^r_R[b_{ij}])^{\tilde{c}} &= \begin{bmatrix} \begin{cases} \max\left\{\mu^a_{ij}, \min_{k\in R}\{\mu^b_{ik}\}\right\}, & j \in R \\ \mu^a_{ij}, & j \in I_E \setminus R \end{cases} \\ \begin{cases} \min\left\{\nu^a_{ij}, \max_{k\in R}\{\nu^b_{ik}\}\right\}, & j \in R \\ \nu^a_{ij}, & j \in I_E \setminus R \end{cases} \end{bmatrix}^{\tilde{c}} \\
&= \begin{bmatrix} \begin{cases} \min\left\{\nu^a_{ij}, \max_{k\in R}\{\nu^b_{ik}\}\right\}, & j \in R \\ \nu^a_{ij}, & j \in I_E \setminus R \end{cases} \\ \begin{cases} \max\left\{\mu^a_{ij}, \min_{k\in R}\{\mu^b_{ik}\}\right\}, & j \in R \\ \mu^a_{ij}, & j \in I_E \setminus R \end{cases} \end{bmatrix} \\
&= \begin{bmatrix} \nu^a_{ij} \\ \mu^a_{ij} \end{bmatrix} \tilde{\cap}^r_R \begin{bmatrix} \nu^b_{ij} \\ \mu^b_{ij} \end{bmatrix} \\
&= [a_{ij}]^{\tilde{c}} \tilde{\cap}^r_R [b_{ij}]^{\tilde{c}}
\end{aligned}
$$

Definition 18 Let $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in IFPIFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $\mu^c_{ip} = \min\left\{\mu^a_{ij}, \mu^b_{ik}\right\}$ and $\nu^c_{ip} = \max\left\{\nu^a_{ij}, \nu^b_{ik}\right\}$, then $[c_{ip}]$ is called and-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}]\wedge[b_{ik}]$.

Definition 19 Let $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in IFPIFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $\mu^c_{ip} = \max\left\{\mu^a_{ij}, \mu^b_{ik}\right\}$ and $\nu^c_{ip} = \min\left\{\nu^a_{ij}, \nu^b_{ik}\right\}$, then $[c_{ip}]$ is called or-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}]\vee[b_{ik}]$.

Definition 20 Let $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in IFPIFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $\mu^c_{ip} = \min\left\{\mu^a_{ij}, \nu^b_{ik}\right\}$ and $\nu^c_{ip} = \max\left\{\nu^a_{ij}, \mu^b_{ik}\right\}$, then $[c_{ip}]$ is called andnot-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}]\overline{\wedge}[b_{ik}]$.

Definition 21 Let $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in IFPIFS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $\mu^c_{ip} = \max\left\{\mu^a_{ij}, \nu^b_{ik}\right\}$ and $\nu^c_{ip} = \min\left\{\nu^a_{ij}, \mu^b_{ik}\right\}$, then $[c_{ip}]$ is called ornot-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}]\underline{\vee}[b_{ik}]$.

Example 7 For $[a_{ij}]$ and $[b_{ik}]$ provided in Example 4, $[a_{ij}]\vee[b_{ik}]$ is as follows:

$$
[a_{ij}]\vee[b_{ik}] = \begin{bmatrix}
0.7 & 0.9 & 0.8 & 0.4 & 1 & 1 & 1 & 1 & 0.7 & 0.9 & 0.8 & 0.6 & 0.7 & 0.9 & 0.8 & 0.7 \\
0.1 & 0.1 & 0.2 & 0.3 & 0 & 0 & 0 & 0 & 0.1 & 0.1 & 0.1 & 0.1 & 0 & 0 & 0 & 0 \\
0.6 & 0 & 0.3 & 0.2 & 0.6 & 0.3 & 0.3 & 0.3 & 0.6 & 0.1 & 0.3 & 0.2 & 0.6 & 0.2 & 0.3 & 0.2 \\
0.3 & 0.5 & 0.5 & 0.1 & 0 & 0 & 0 & 0 & 0.3 & 0.8 & 0.5 & 0.1 & 0.3 & 0.3 & 0.3 & 0.1 \\
1 & 1 & 1 & 1 & 0.5 & 0.5 & 0.7 & 0.6 & 0.2 & 0.3 & 0.7 & 0.6 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0.1 & 0.2 & 0.5 & 0 & 0.1 & 0.2 & 0 & 0 & 0 & 0 \\
0.4 & 0.5 & 0.4 & 1 & 1 & 1 & 1 & 1 & 0.4 & 0.5 & 0.4 & 1 & 0.4 & 0.5 & 0.4 & 1 \\
0.2 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.5 & 0.5 & 0 & 0.2 & 0.5 & 0.5 & 0
\end{bmatrix}
$$

Proposition 9 *Let* $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$, *and* $[c_{il}]_{m\times n_3} \in IFPIFS_{E_3}[U]$. *Then,*

i. $([a_{ij}] \wedge [b_{ik}]) \wedge [c_{il}] = [a_{ij}] \wedge ([b_{ik}] \wedge [c_{il}])$
ii. $([a_{ij}] \vee [b_{ik}]) \vee [c_{il}] = [a_{ij}] \vee ([b_{ik}] \vee [c_{il}])$

Proposition 10 *Let* $[a_{ij}]_{m\times n_1} \in IFPIFS_{E_1}[U]$ *and* $[b_{ik}]_{m\times n_2} \in IFPIFS_{E_2}[U]$. *Then, the following De Morgan's laws are valid:*

i. $([a_{ij}] \vee [b_{ik}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \wedge [b_{ik}]^{\tilde{c}}$
ii. $([a_{ij}] \wedge [b_{ik}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \vee [b_{ik}]^{\tilde{c}}$
iii. $([a_{ij}] \veebar [b_{ik}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \barwedge [b_{ik}]^{\tilde{c}}$
iv. $([a_{ij}] \barwedge [b_{ik}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \veebar [b_{ik}]^{\tilde{c}}$

Proof *i.* Let $[a_{ij}]$ and $[b_{ik}]$ be two *ifpifs*-matrices over U. Then,

$$([a_{ij}] \vee [b_{ik}])^{\tilde{c}} = \begin{bmatrix} \max\{\mu^a_{ij},\mu^b_{ik}\} \\ \min\{\nu^a_{ij},\nu^b_{ik}\} \end{bmatrix}^{\tilde{c}} = \begin{bmatrix} \min\{\nu^a_{ij},\nu^b_{ik}\} \\ \max\{\mu^a_{ij},\mu^b_{ik}\} \end{bmatrix} = \begin{bmatrix} \nu^a_{ij} \\ \mu^a_{ij} \end{bmatrix} \wedge \begin{bmatrix} \nu^b_{ik} \\ \mu^b_{ik} \end{bmatrix} = \left[a_{ij}\right]^{\tilde{c}} \wedge \left[b_{ij}\right]^{\tilde{c}}$$

Remark 5 It must be noted that the aforementioned products of *ifpifs*-matrices are non-commutative and non-distributive. Moreover, ornot-product and andnot-product are non-associative.

4 A soft decision-making method: EA20

The available literature presents studies on decision functions, such as $uni\text{-}int$ ($max\text{-}min$) and $int\text{-}uni$ ($min\text{-}max$). Çağman and Enginoğlu (2011a) are the first to define the $uni\text{-}int$ decision function in soft sets and propose the $uni\text{-}int$ soft decision-making method. Thereafter, Enginoğlu and Memiş (2018b) have configured this method via *fpfs*-matrices, which is denoted by CE10. Moreover, Enginoğlu et al. (2018b) have propounded two new soft decision-making methods, denoted by EMO18a and EMO18o, which are equivalent under certain conditions to CE10, constructed by and-product (CE10a) and or-product (CE10o), respectively. They then have shown that EMO18a and EMO18o outperform CE10a and CE10o in terms of time, respectively. In this section, we introduce a new soft decision-making method via *ifpifs*-matrices, which is denoted by EA20 and a configuration of EMO18a and EMO18o, presented in (Enginoğlu et al. 2018b).

EA20 Algorithm Steps

Step 1. Construct two *ifpifs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ik}]_{m\times n}$ by considering the set of alternatives $U = \{u_1, u_2, \ldots, u_{m-1}\}$ and the parameters set $E = \{e_1, e_2, \ldots, e_n\}$.
Step 2. Obtain the score matrix $[s_{i1}]$ defined by $s_{i1} := \mu_{i1} - \nu_{i1}$

Here,

$$\mu_{i1} := \max\left\{\max_j \min_k \left(\mu^a_{ij}, \mu^b_{ik}\right), \max_k \min_j \left(\mu^a_{ij}, \mu^b_{ik}\right)\right\}$$

and

$$\nu_{i1} := \min\left\{\max_j \min_k \left(\nu^a_{ij}, \nu^b_{ik}\right), \max_k \min_j \left(\nu^a_{ij}, \nu^b_{ik}\right)\right\}$$

such that $i \in \{1, 2, \dots, m-1\}$, $I_a := \left\{ j \mid \mu^a_{0j} \neq 0 \vee \nu^a_{0j} \neq 1 \right\}$, $I_b := \left\{ k \mid \mu^b_{0k} \neq 0 \vee \nu^b_{0k} \neq 1 \right\}$, and

$$\max{}_j \min{}_k(\mu^a_{ij}, \mu^b_{ik}) := \begin{cases} \max\left\{ \max\limits_{j\in I_a}\{\mu^a_{0j}\mu^a_{ij}\}, \min\limits_{k\in I_b}\{\mu^b_{0k}\mu^b_{ik}\} \right\}, & I_a \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\max{}_k \min{}_j(\mu^a_{ij}, \mu^b_{ik}) := \begin{cases} \max\left\{ \max\limits_{k\in I_b}\{\mu^b_{0k}\mu^b_{ik}\}, \min\limits_{j\in I_a}\{\mu^a_{0j}\mu^a_{ij}\} \right\}, & I_a \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\max{}_j \min{}_k(\nu^a_{ij}, \nu^b_{ik}) := \begin{cases} \min\left\{ \max\limits_{j\in I_a}\{\nu^a_{0j}\nu^a_{ij}\}, \min\limits_{k\in I_b}\{\nu^b_{0k}\nu^b_{ik}\} \right\}, & I_a \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

$$\max{}_k \min{}_j(\nu^a_{ij}, \nu^b_{ik}) := \begin{cases} \min\left\{ \max\limits_{k\in I_b}\{\nu^b_{0k}\nu^b_{ik}\}, \min\limits_{j\in I_a}\{\nu^a_{0j}\nu^a_{ij}\} \right\}, & I_a \neq \emptyset \text{ and } I_b \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

Step 3. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1} + |\min\limits_i s_{i1}|}{\max\limits_i s_{i1} + |\min\limits_i s_{i1}|}$.

Step 4. Choose the most suitable alternatives u_k with respect to $\mu(u_k)$.

5 A recruitment scenario for EA20

Recruitment poses difficulties faced by the human resources (HR) department of a company. Among the important factors in the success or failure of the company in the future is its workforce. A critical problem encountered by HR is to select the most suitable candidate to meet the requirements. The recruitment problem can be considered an MCGDM problem that generally consists of selecting the most desirable alternatives from all the eligible alternatives.

However, it is sometimes impossible to state the criteria with crisp values (0 or 1). To cope with this issue, intuitionistic fuzzy sets which are a generalization of fuzzy sets and characterise linguistic variables by intuitionistic fuzzy values can be used. Some studies on intuitionistic fuzzy MCGDM or recruitment process can be listed as (Boran et al. 2011; Das and Kar 2014; Karaaslan 2016; Mondal and Roy 2014; Pramanik and Mukhopadhyaya 2011).

In this section, a hypothetical recruitment scenario of a company is modelled. Assume that ten candidates, denoted by $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10}\}$, have applied for two vacant positions. Let the parameter set determined by HR of the company and a member of the board of directors appointed for this recruitment be $E = \{x_1, x_2, x_3, x_4, x_5\}$ such that $x_1 =$ "*experience*", $x_2 =$ "*rhetoric*", $x_3 =$ "*technological competence*", $x_4 =$ "*age*", and $x_5 =$ "*work ethic*". Besides, let the intuitionistic fuzzy sets f_1 and f_2 over E determined by these two decision-makers be

$$f_1 := \left\{ {}^{0.45}_{0.23}x_1, {}^{0.36}_{0.17}x_2, {}^{0.62}_{0.12}x_3, {}^{0.27}_{0.34}x_4, {}^{0.32}_{0.29}x_5 \right\} \text{ and } f_2 := \left\{ {}^{0.26}_{0.41}x_1, {}^{0.17}_{0.33}x_2, {}^{0.35}_{0.20}x_3, {}^{0.49}_{0.23}x_4, {}^{0.28}_{0.25}x_5 \right\}$$

Here, the intuitionistic fuzzy value of the parameter "age", determined as ${}^{0.27}_{0.34}$, by HR, is produced by defaulting to Table 1. HR, first, sets the amount of minimum sales needed for the first three quarters of the year and the annual amount of target sales. In this study, these values are €80000 and €100000, respectively. Then, HR evaluates the percentages of each age range specified as "less than 30 years", "30-40 years", and "more than 40 years". Finally,

Table 1 By-percentage distribution of the employees in the three age ranges by the given sales amount for the first three quarters

	less than €80000	€80000-€100000	more than €100000
less than 30 years	7	23	70
30-40 years	10	85	5
more than 40 years	85	9	6

HR obtains the values of contribution of the parameter "age" to high success by considering these percentages.

In the last column of Table 1, it can be observed that 70%, 5%, and 6% of the employees within three age ranges have achieved more sales than the annual targeted amount. Since $\frac{70+5+6}{300} = 0.27$, the degree of the precise positive effect of the parameter "age" on success is 0.27. Moreover, according to the first column, 7%, %10, and 85% of employees for three age ranges have not been able to exceed the minimum sales amount in the first three quarters. Hence, the degree of precise negative effect of employees on success is 0.34 because $\frac{7+10+85}{300} = 0.34$. Besides, since it is unclear whether employees that have achieved a total sale of €80000-€100000 will exceed the target amount, the degree of the effect of these employees on success is indeterminate. Therefore, the degree of indeterminacy of the age parameter is 0.39. Consequently, the intuitionistic fuzzy value of the parameter is ${}^{0.27}_{0.34}x_4$. The intuitionistic fuzzy values of the other parameters have been obtained similarly. The application of EA20 is as follows:

Step 1. Let two *ifpifs*-matrices $[a_{ij}]$ and $[b_{ik}]$, showing how suitable the candidates are for the parameters, be as follows:

$$[a_{ij}] = \begin{bmatrix} \substack{0.45\\0.23} & \substack{0.36\\0.17} & \substack{0.62\\0.12} & \substack{0.27\\0.34} & \substack{0.32\\0.29} \\ \substack{0\\0.6} & \substack{0.2\\0.1} & \substack{0\\0.4} & \substack{0.4\\0.2} & \substack{0.3\\0.4} \\ \substack{0.8\\0} & \substack{0.2\\0.2} & \substack{0.3\\0.1} & \substack{0.7\\0.2} & \substack{0.7\\0.1} \\ \substack{0.5\\0.4} & \substack{0.1\\0} & \substack{0.1\\0.6} & \substack{0.1\\0} & \substack{0\\0.3} \\ \substack{0.1\\0.5} & \substack{0.6\\0.2} & \substack{0.3\\0.5} & \substack{0\\0.5} & \substack{0.2\\0.4} \\ \substack{0.4\\0.1} & \substack{0.5\\0.1} & \substack{0.3\\0.1} & \substack{0.1\\0.4} & \substack{0.3\\0.2} \\ \substack{0.7\\0.1} & \substack{0\\0.2} & \substack{0.1\\0.5} & \substack{0\\0.6} & \substack{0.1\\0.5} \\ \substack{0.7\\0.2} & \substack{0.8\\0} & \substack{0.7\\0.1} & \substack{0.5\\0.3} & \substack{0.2\\0.2} \\ \substack{0.6\\0.2} & \substack{0.5\\0.1} & \substack{0\\0.8} & \substack{0.2\\0.6} & \substack{0.3\\0.6} \\ \substack{0.7\\0.1} & \substack{0.6\\0.3} & \substack{0.5\\0.2} & \substack{0.1\\0.4} & \substack{0.8\\0} \\ \substack{0\\0.1} & \substack{0.1\\0.5} & \substack{0\\0.6} & \substack{0.6\\0.2} & \substack{0.3\\0.2} \end{bmatrix} \text{ and } [b_{ik}] = \begin{bmatrix} \substack{0.26\\0.41} & \substack{0.17\\0.33} & \substack{0.35\\0.2} & \substack{0.49\\0.23} & \substack{0.28\\0.25} \\ \substack{0\\0.8} & \substack{0.2\\0.2} & \substack{0.6\\0} & \substack{0.1\\0} & \substack{0.6\\0.2} \\ \substack{0.4\\0.5} & \substack{0\\0.9} & \substack{0.8\\0} & \substack{0.8\\0.1} & \substack{0.3\\0} \\ \substack{0.4\\0.4} & \substack{0.2\\0.3} & \substack{0.3\\0.5} & \substack{0\\0.8} & \substack{0.2\\0.4} \\ \substack{0.2\\0} & \substack{0.3\\0.5} & \substack{0\\0.6} & \substack{0.2\\0.5} & \substack{0.2\\0} \\ \substack{0\\0.9} & \substack{0\\0.1} & \substack{0.2\\0.6} & \substack{0.6\\0.2} & \substack{0\\0.7} \\ \substack{0.7\\0.1} & \substack{0.4\\0.1} & \substack{0\\0} & \substack{0.9\\0} & \substack{0.3\\0.4} \\ \substack{0.3\\0.3} & \substack{0.3\\0.2} & \substack{0.3\\0.1} & \substack{0.8\\0.1} & \substack{0.2\\0} \\ \substack{0.1\\0.1} & \substack{0\\0.3} & \substack{0.3\\0} & \substack{0.5\\0.2} & \substack{0.2\\0.4} \\ \substack{0.3\\0.4} & \substack{0\\0.6} & \substack{0.3\\0.1} & \substack{0.5\\0.4} & \substack{0.3\\0.1} \\ \substack{0.2\\0.6} & \substack{0.3\\0.1} & \substack{0.7\\0.1} & \substack{0.5\\0} & \substack{0.5\\0.3} \end{bmatrix}$$

Here, the candidate u_4 has declared that he/she knows how to use three of the ten computer programs determined by HR and presents their valid certificates, he/she recognises two of the others but holds no valid certificates, and he/she does not know the other five. Therefore, $a_{43} = {}^{\mu^a_{43}}_{\nu^a_{43}} = {}^{0.3}_{0.5}$. Similarly, since the candidate u_3 has positive referrals from three of his/her five previous companies, negative referrals from one, and no referral from the other, HR has assigned ${}^{0.6}_{0.2}$ as the intuitionistic fuzzy membership value of the candidate. Thus,

$a_{81} = \frac{\mu^a_{81}}{\nu^a_{81}} = \frac{0.6}{0.2}$. The other intuitionistic fuzzy values in the matrices are similarly obtained by the decision-makers.

Step 2. If we apply EA20 to the $[a_{ij}]$ and $[b_{ik}]$, then the score matrix $[s_{i1}]$ is as follows:

$$[s_{i1}] = [0.2100\ 0.3920\ 0.2250\ 0.2160\ 0.2820\ 0.4410\ 0.4340\ 0.2700\ 0.3150\ 0.2450]^T$$

Here, since $I_a = \{1, 2, 3, 4, 5\}$ and $I_b = \{1, 2, 3, 4, 5\}$,

$$\mu_{51} = \max\left\{\max\left\{\max_{j\in I_a}\left\{\mu^a_{0j}\mu^a_{5j}\right\}, \min_{k\in I_b}\left\{\mu^b_{0k}\mu^b_{5k}\right\}\right\}, \max\left\{\max_{k\in I_b}\left\{\mu^b_{0k}\mu^b_{5k}\right\}, \min_{j\in I_a}\left\{\mu^a_{0j}\mu^a_{5j}\right\}\right\}\right\}$$

$$\nu_{51} = \min\left\{\min\left\{\max_{j\in I_a}\left\{\nu^a_{0j}\nu^a_{5j}\right\}, \min_{k\in I_b}\left\{\nu^b_{0k}\nu^b_{5k}\right\}\right\}, \min\left\{\max_{k\in I_b}\left\{\nu^b_{0k}\nu^b_{5k}\right\}, \min_{j\in I_a}\left\{\nu^a_{0j}\nu^a_{5j}\right\}\right\}\right\}$$

and $s_{51} = \mu_{51} - \nu_{51} = 0.2940 - 0.0120 = 0.2820$. The other score values may be found by a similar way.

Step 3. The decision set is as follows:

$$\left\{{}^{0.6452}u_1, {}^{0.9247}u_2, {}^{0.6682}u_3, {}^{0.6544}u_4, {}^{0.7558}u_5, {}^{1}u_6, {}^{0.9892}u_7, {}^{0.7373}u_8, {}^{0.8065}u_9, {}^{0.6989}u_{10}\right\}$$

The optimal ranking order of the ten candidates is $u_1 \prec u_4 \prec u_3 \prec u_{10} \prec u_8 \prec u_5 \prec u_9 \prec u_2 \prec u_7 \prec u_6$. The scores show that u_6 and u_7 are more eligible for the vacant positions than the others. Thus, the candidates u_6 and u_7 are selected for the positions at stake.

6 Comparison results

In this section, we first provide the definitions of fuzzy parameterized fuzzy soft sets and fuzzy parameterized fuzzy soft matrices by taking into account the notations used throughout this paper.

Definition 22 (Çağman et al. 2012b; Enginoğlu 2012) Let U be a universal set, E be a parameter set, $\mu \in F(E)$, and α be a function from μ to $F(U)$. Then, the set $\left\{\left({}^{\mu(x)}x, \alpha({}^{\mu(x)}x)\right) : x \in E\right\}$ being the graphic of α is called an fuzzy parameterized fuzzy soft set (*fpfs*-set) parameterized via E over U (or briefly over U).

In the present paper, the set of all *fpfs*-sets over U is denoted by $FPFS_E(U)$. In $FPFS_E(U)$, since the $graph(\alpha)$ and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it does not cause any confusion, we denote an *fpfs*-set $graph(\alpha)$ by α.

Definition 23 (Enginoğlu and Çağman 2020) Let $\alpha \in FPFS_E(U)$. Then, $[a_{ij}]$ is called the matrix representation of α (or briefly *fpfs*-matrix of α) and is defined by

$$[a_{ij}] := \begin{bmatrix} a_{01} & a_{02} & a_{03} & \dots & a_{0n} & \dots \\ a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

Table 2 SSIM results for the Cameraman image

Algorithm	10%	20%	30%	40%	50%	60%	70%	80%	90%
PSMF	0.9722	0.9454	0.9044	0.8036	0.6215	0.1178	0.0576	0.0290	0.0129
DBA	0.9883	0.9664	0.9324	0.8795	0.8167	0.7413	0.6650	0.5841	0.4858
MDBUTMF	0.9501	0.8388	0.7740	0.8249	0.9014	0.9178	0.8954	0.7864	0.4062
NAFSM	0.9797	0.9642	0.9494	0.9340	0.9198	0.8975	0.8745	0.8344	0.7246
DAMF	0.9963	0.9911	0.9844	0.9760	0.9659	0.9511	0.9323	0.9008	0.8373

Table 3 SSIM results for the Lena image

Algorithm	10%	20%	30%	40%	50%	60%	70%	80%	90%
PSMF	0.9840	0.9631	0.9163	0.7854	0.5640	0.1115	0.0542	0.0263	0.0123
DBA	0.9758	0.9422	0.8952	0.8308	0.7549	0.6651	0.5673	0.4442	0.3458
MDBUTMF	0.9542	0.8686	0.8137	0.8449	0.8841	0.8835	0.8521	0.7392	0.3395
NAFSM	0.9838	0.9667	0.9481	0.9293	0.9055	0.8809	0.8495	0.8043	0.6868
DAMF	0.9902	0.9792	0.9652	0.9503	0.9303	0.9090	0.8788	0.8382	0.7697

such that for $i \in \{0, 1, 2, \cdots\}$ and $j \in \{1, 2, \cdots\}$,

$$a_{ij} := \begin{cases} \mu(x_j), & i = 0 \\ \alpha(^{\mu(x_j)}x_j)(u_i), & i \neq 0 \end{cases}$$

Here, if $|U| = m - 1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

Hereinafter, the set of all *fpfs*-matrices parameterized via E over U is denoted by $FPFS_E[U]$.

Since EA20 is the first method to be proposed in $IFPIFS_E[U]$, it is impossible to compare this method with another. On the other hand, an *fpfs*-matrix can be regarded as an *ifpifs*-matrix, in which the sum of the membership degree and the non-membership degree of each entry is 1. Thus, EA20 can be compared with the state-of-the-art methods constructed in $FPFS_E[U]$. Even so, the comparisons will be relative because there is no known criterion except intuition for the validity of the results obtained by the methods used in $FPFS_E[U]$.

Second, in order to make this comparison, we apply EA20 to a real-life problem in image processing in addition to the hypothetical recruitment process problem given in Sect. 5. In other words, we compare EA20 with the state-of-the-art soft decision-making methods in $FPFS_E[U]$, i.e. E15, ZZ16, ZZ16-2 (Enginoğlu et al. 2018b), YJ11, YJ11/2, BNS12, MR13 (NB14), MR13/2, Z14 (Enginoğlu and Öngel 2020), and EMO18o (Enginoğlu et al. 2018b), in the problem of performance ranking of the filters used in image denoising, namely Progressive Switching Median Filter (PSMF) (Wang and Zhang 1999), Decision-Based Algorithm (DBA) (Pattnaik et al. 2012), Modified Decision-Based Unsymmetrical Trimmed Median Filter (MDBUTMF) (Esakkirajan et al. 2011), Noise Adaptive Fuzzy Switching Median Filter (NAFSMF) (Toh and Isa 2010), and Different Applied Median Filter (DAMF) (Erkan et al. 2018), by using the Structural Similarity (SSIM) Wang et al. (2004) results of these filters (Erkan et al. 2018) available in Tables 2 and 3 for two traditional images "Cameraman" and "Lena". Here, the notations, such as E15 and ZZ16, denote the first letter/letters of the authors' surnames and the last two digits of the publication years of the papers (see (Enginoğlu et al. 2018b)).

Suppose that the success at high noise densities is more important than in the presence of other densities (see (Enginoğlu et al. 2018a, b; Erkan et al. 2018)). In this case, the values in Tables 2 and 3 can be represented with *fpfs*-matrices $[a_{ij}]$ and $[b_{ik}]$ as follows:

$$[a_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9722 & 0.9454 & 0.9044 & 0.8036 & 0.6215 & 0.1178 & 0.0576 & 0.0290 & 0.0129 \\ 0.9883 & 0.9664 & 0.9324 & 0.8795 & 0.8167 & 0.7413 & 0.6650 & 0.5841 & 0.4858 \\ 0.9501 & 0.8388 & 0.7740 & 0.8249 & 0.9014 & 0.9178 & 0.8954 & 0.7864 & 0.4062 \\ 0.9797 & 0.9642 & 0.9494 & 0.9340 & 0.9198 & 0.8975 & 0.8745 & 0.8344 & 0.7246 \\ 0.9963 & 0.9911 & 0.9844 & 0.9760 & 0.9659 & 0.9511 & 0.9323 & 0.9008 & 0.8373 \end{bmatrix}$$

and

$$[b_{ik}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9840 & 0.9631 & 0.9163 & 0.7854 & 0.5640 & 0.1115 & 0.0542 & 0.0263 & 0.0123 \\ 0.9758 & 0.9422 & 0.8952 & 0.8308 & 0.7549 & 0.6651 & 0.5673 & 0.4442 & 0.3458 \\ 0.9542 & 0.8686 & 0.8137 & 0.8449 & 0.8841 & 0.8835 & 0.8521 & 0.7392 & 0.3395 \\ 0.9838 & 0.9667 & 0.9481 & 0.9293 & 0.9055 & 0.8809 & 0.8495 & 0.8043 & 0.6868 \\ 0.9902 & 0.9792 & 0.9652 & 0.9503 & 0.9303 & 0.9090 & 0.8788 & 0.8382 & 0.7697 \end{bmatrix}$$

Moreover, the *fpfs*-matrices $[a_{ij}]$ and $[b_{ik}]$ can be written as the *ifpifs*-matrices $[c_{ij}]$ and $[d_{ik}]$, respectively, as follows:

$$[c_{ij}] := \begin{bmatrix} \substack{0.1\\0.9} & \substack{0.2\\0.8} & \substack{0.3\\0.7} & \substack{0.4\\0.6} & \substack{0.5\\0.5} & \substack{0.6\\0.4} & \substack{0.7\\0.3} & \substack{0.8\\0.2} & \substack{0.9\\0.1} \\ \substack{0.9722\\0.0278} & \substack{0.9454\\0.0546} & \substack{0.9044\\0.0956} & \substack{0.8036\\0.1964} & \substack{0.6215\\0.3785} & \substack{0.1178\\0.8822} & \substack{0.0576\\0.9424} & \substack{0.0290\\0.9710} & \substack{0.0129\\0.9871} \\ \substack{0.9883\\0.0117} & \substack{0.9664\\0.0336} & \substack{0.9324\\0.0676} & \substack{0.8795\\0.1205} & \substack{0.8167\\0.1833} & \substack{0.7413\\0.2587} & \substack{0.6650\\0.3350} & \substack{0.5841\\0.4159} & \substack{0.4858\\0.5142} \\ \substack{0.9501\\0.0499} & \substack{0.8388\\0.1612} & \substack{0.7740\\0.2260} & \substack{0.8249\\0.1751} & \substack{0.9014\\0.0986} & \substack{0.9178\\0.0822} & \substack{0.8954\\0.1046} & \substack{0.7864\\0.2136} & \substack{0.4062\\0.5938} \\ \substack{0.9797\\0.0203} & \substack{0.9642\\0.0358} & \substack{0.9494\\0.0506} & \substack{0.9340\\0.0660} & \substack{0.9198\\0.0802} & \substack{0.8975\\0.1025} & \substack{0.8745\\0.1255} & \substack{0.8344\\0.1656} & \substack{0.7246\\0.2754} \\ \substack{0.9963\\0.0037} & \substack{0.9911\\0.0089} & \substack{0.9844\\0.0156} & \substack{0.9760\\0.0240} & \substack{0.9659\\0.0341} & \substack{0.9511\\0.0489} & \substack{0.9323\\0.0677} & \substack{0.9008\\0.0992} & \substack{0.8373\\0.1627} \end{bmatrix}$$

and

$$[d_{ik}] := \begin{bmatrix} \substack{0.1\\0.9} & \substack{0.2\\0.8} & \substack{0.3\\0.7} & \substack{0.4\\0.6} & \substack{0.5\\0.5} & \substack{0.6\\0.4} & \substack{0.7\\0.3} & \substack{0.8\\0.2} & \substack{0.9\\0.1} \\ \substack{0.9840\\0.0160} & \substack{0.9631\\0.0369} & \substack{0.9163\\0.0837} & \substack{0.7854\\0.2146} & \substack{0.5640\\0.4360} & \substack{0.1115\\0.8885} & \substack{0.0542\\0.9458} & \substack{0.0263\\0.9737} & \substack{0.0123\\0.9877} \\ \substack{0.9758\\0.0242} & \substack{0.9422\\0.0578} & \substack{0.8952\\0.1048} & \substack{0.8308\\0.1692} & \substack{0.7549\\0.2451} & \substack{0.6651\\0.3349} & \substack{0.5673\\0.4327} & \substack{0.4442\\0.5558} & \substack{0.3458\\0.6542} \\ \substack{0.9542\\0.0458} & \substack{0.8686\\0.1314} & \substack{0.8137\\0.1863} & \substack{0.8449\\0.1551} & \substack{0.8841\\0.1159} & \substack{0.8835\\0.1165} & \substack{0.8521\\0.1479} & \substack{0.7392\\0.2608} & \substack{0.3395\\0.6605} \\ \substack{0.9838\\0.0162} & \substack{0.9667\\0.0333} & \substack{0.9481\\0.0519} & \substack{0.9293\\0.0707} & \substack{0.9055\\0.0945} & \substack{0.8809\\0.1191} & \substack{0.8495\\0.1505} & \substack{0.8043\\0.1957} & \substack{0.6868\\0.3132} \\ \substack{0.9902\\0.0098} & \substack{0.9792\\0.0208} & \substack{0.9652\\0.0348} & \substack{0.9503\\0.0497} & \substack{0.9303\\0.0697} & \substack{0.9090\\0.0910} & \substack{0.8788\\0.1212} & \substack{0.8382\\0.1618} & \substack{0.7697\\0.2303} \end{bmatrix}$$

If we apply EA20 to $[c_{ij}]$ and $[d_{ik}]$ and the ten state-of-art soft decision-making methods to $[a_{ij}]$ and $[b_{ik}]$, then the values of score matrices are as in Table 4:

Similarly, if we apply EA20 to $[c_{ij}]$ and $[d_{ik}]$ and the ten state-of-art soft decision-making methods to $[a_{ij}]$ and $[b_{ik}]$, then the values of decision sets are as in Table 5:

Finally, the performance-based ranking orders of the filters for the 11 soft decision-making algorithms are as in Table 6:

The scores show that DAMF outperforms the other methods and PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF sorting is valid (Enginoğlu and Çağman 2020; Enginoğlu and Memiş 2018a; Enginoğlu et al. 2018a, b; Erkan et al. (2018). The results in Table 6 show that the ranking results obtained by EA20 confirm the results of the filters for the ten soft decision-making methods. In other words, EA20 produces a valid ranking of the alternatives.

Table 4 Values in the score matrices of filters for the 11 soft decision-making methods

Algorithms \ Filters	PSMF	DBA	MDBUTMF	NAFSM	DAMF
E15	2.6346	5.8739	6.7645	7.6648	8.1175
ZZ16	0.0415	0.1493	0.1923	0.2308	0.2440
ZZ16-2	0.2578	0.5192	0.5663	0.6242	0.6659
YJ11	0.1434	0.3030	0.3656	0.4205	0.4393
YJ11/2	0.1493	0.3496	0.3860	0.4312	0.4627
BNS12	0.3161	1.0567	1.5551	1.9563	2.2252
MR13 (NB14)	0.1434	0.3030	0.3656	0.4205	0.4393
MR13/2	0.0351	0.1174	0.1728	0.2174	0.2472
Z14	1.3439	3.1468	3.4741	3.8804	4.1640
EMO18o	0.3214	0.4673	0.6291	0.6675	0.7536
EA20	0.3070	0.4568	0.5977	0.6529	0.7502

Table 5 Values in the decision sets of filters for the 11 soft decision-making methods

Algorithms \ Filters	PSMF	DBA	MDBUTMF	NAFSM	DAMF
E15	0.3246	0.7236	0.8333	0.9442	1
ZZ16	0.1700	0.6118	0.7882	0.9462	1
ZZ16-2	0.3872	0.7797	0.8505	0.9374	1
YJ11	0.3265	0.6898	0.8323	0.9572	1
YJ11/2	0.3227	0.7557	0.8343	0.9319	1
BNS12	0.1420	0.4749	0.6989	0.8791	1
MR13 (NB14)	0.3265	0.6898	0.8323	0.9572	1
MR13/2	0.1420	0.4749	0.6989	0.8791	1
Z14	0.3227	0.7557	0.8343	0.9319	1
EMO18o	0.4266	0.6201	0.8349	0.8858	1
EA20	0.5808	0.7224	0.8558	0.9080	1

Table 6 Ranking orders of filters for the eleven soft decision-making methods

Algorithms	Ranking orders
E15	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
ZZ16	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
ZZ16-2	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
YJ11	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
YJ11/2	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
BNS12	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
MR13 (NB14)	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
MR13/2	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
Z14	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
EMO18o	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF
EA20	PSMF $\prec$ DBA $\prec$ MDBUTMF $\prec$ NAFSMF $\prec$ DAMF

Table 7 Running time of EA20 for the objects and parameters ranging from 100 to 1000 (in second)

(u, x)	100	200	300	400	500	600	700	800	900	1000
100	0.0526	0.0099	0.0090	0.0101	0.0121	0.0125	0.0139	0.0156	0.0168	0.0360
200	0.0093	0.0123	0.0152	0.0179	0.0207	0.0241	0.0277	0.0312	0.0337	0.0377
300	0.0138	0.0178	0.0228	0.0310	0.0429	0.0487	0.0571	0.0614	0.0706	0.0714
400	0.0194	0.0389	0.0332	0.0538	0.0560	0.0579	0.0739	0.0912	0.0953	0.1062
500	0.0296	0.0509	0.0564	0.0558	0.0641	0.0729	0.0850	0.0966	0.1104	0.1292
600	0.0281	0.0369	0.0505	0.0638	0.0778	0.0909	0.1051	0.1219	0.1624	0.1698
700	0.0326	0.0451	0.0589	0.0741	0.0904	0.1072	0.1267	0.1514	0.1760	0.1988
800	0.0375	0.0497	0.0680	0.0874	0.1063	0.1262	0.1463	0.1708	0.2003	0.2342
900	0.0414	0.0570	0.0782	0.1015	0.1294	0.1451	0.1710	0.1968	0.2296	0.2664
1000	0.0473	0.0641	0.0891	0.1148	0.1388	0.1702	0.1929	0.2207	0.2566	0.3030

7 Conclusion

In this paper, we defined the concept of *ifpifs*-matrices. We then suggested a new soft decision-making method, denoted by EA20, and applied it to a recruitment scenario of a company. This application showed that *ifpifs*-matrices can be successfully applied to the problems that contain greater uncertainties. Moreover, we provided an application that assigned performance-based values to noise-removal filters. Since EA20 is the first method in $IFPIFS_E[U]$, we present the *fpfs*-matrices therein in the form of *ifpifs*-matrices.

Furthermore, for *ifpifs*-matrices $[a_{ij}]$ and $[b_{ik}]$ in Sect. 5, the average running time of EA20 is 0.0004 sec. for 1000 tests. Moreover, in Table 7, we present the running time values of EA20 by using MATLAB R2019b and a laptop with 2.4 GHz i3 Dual Core CPU and 4GB RAM for the parameters and objects ranging from 100 to 1000. The results show that the method can be translated into a technological product.

In the future, different soft decision-making methods can be developed by using operations of *ifpifs*-matrices, such as R-relative union/intersection/difference and and/or/andnot/ornot-products. Additionally, to model more severe uncertainties than the aforesaid, *ifpifs*-matrices can be expanded to interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft matrices through the closed subintervals of [0, 1]. Besides, defining the distance and similarity measurements of *ifpifs*-matrices can prove beneficial in such areas as medical diagnosis and pattern recognition.

Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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