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A computational method for solving a problem with parameter for linear systems of integro-differential equations

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Abstract
This article presents a computational method for solving a problem with parameter for a system of Fredholm integro-differential equations. Some additional parameters are introduced and the problem under consideration is reduced to solving a system of linear algebraic equations. The coefficients and right-hand side of the system are calculated by solving the Cauchy problems for ordinary differential equations. We establish a criterion for the unique solvability of the problem under consideration. A numerical algorithm is offered for solving the problem with parameter. The results are illustrated by numerical examples.

Keywords Problem with parameter · System of integro-differential equations · Solvability criteria · Algorithm · Computational method

Mathematics Subject Classification 34B08 · 34H05 · 45J05 · 34K28

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1 Introduction

Control problems, also referred to as boundary value problems with parameters or as parameter identification problems, for ordinary differential and integro-differential equations have been extensively studied by many authors Akhmetov et al. (2002); Alimhan et al. (2015); Dauylbayev and Atakhan (2015); Dauylbaev and Mirzakulova (2017); Kiguradze (1987); Luchka and Nesterenko (2008); Nesterenko (2014); Ronto and Samoilenko (2000). Various methods have been applied to study these problems, such as methods of qualitative theory of differential equations, the calculus of variations and optimization theory, the method of upper and lower solutions, etc. However, there still remain open problems in obtaining effective criteria for the unique solvability of such problems and in developing numerical algorithms to find their optimal solutions.

Consider the following problem with a parameter for a system of Fredholm integro-differential equations with degenerate kernels:

$$\frac{dx}{dt}=A(t)x+\sum_{k=1}^{m}\int_0^T \varphi_k(t)\psi_k(s)x(s)ds+A_0(t)\mu+f(t), \quad x\in R^n,\ \mu\in R^l,\ t\in(0,T), \tag{1}$$

$$B_0\mu+Bx(0)+Cx(T)=d, \quad d\in R^{n+l}. \tag{2}$$

Here the $(n\times n)$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$, $k=\overline{1,m}$, the $(n\times l)$ matrix $A_0(t)$ and the n vector $f(t)$ are continuous on $[0,T]$; the $((n+l)\times l)$ matrix B_0 and the $((n+l)\times n)$ matrices B and C are constant; $\|x\|=\max\limits_{i=\overline{1,n}}|x_i|$.

By a solution to problem (1), (2) we mean a pair $(x^*(t),\mu^*)$, where $\mu^*\in R^n$ and $x^*(t)$ is a continuous on $[0,T]$ and continuously differentiable on $(0,T)$ vector function satisfying the system of integro-differential Eq. (1) and boundary condition (2) for $\mu=\mu^*$.

The aim of this paper was to establish a criterion for the unique solvability of problem with parameters (1), (2) and propose an algorithm for finding its solutions including its numerical implementation.

For this purpose, we use the parametrization method proposed by Dzhumabayev (1989). This is a constructive method originally developed to investigate and solve boundary value problems for ordinary differential equations. In Dzhumabayev (1989), coefficient criteria were established for the unique solvability of linear boundary value problems. An algorithm for finding their approximate solutions was developed. The method was later extended to boundary value problems, both linear and nonlinear, for various classes of equations. In particular, the parametrization method has been applied to problems for Fredholm integro-differential equations Dzhumabaev (2010, 2013); Dzhumabaev and Bakirova (2013); Dzhumabaev (2015, 2016) and linear boundary value problems with a parameter for ordinary differential equations Minglibayeva (2003); Minglibayeva and Dzhumabaev (2004).

The rest of this paper is organized as follows: Sect. 2 is devoted to the study of the unique solvability of problem (1), (2). We make a partition Δ_N of the interval $[0,T]$ into N parts and take the values of a solution at the left-end points of the subintervals as additional parameters. We then obtain a special Cauchy problem for a system of integro-differential equations with parameters on the subintervals. The unique solvability of this problem is equivalent to the invertibility of a matrix $I-G(\Delta_N)$ composed of the fundamental matrix of the differential part and the kernel matrices of the integral term. We call a partition Δ_N regular if the matrix $I-G(\Delta_N)$ has an inverse. Then, assuming Δ_N to be regular, we construct a system of linear algebraic equations in parameters by using $[I - G(\Delta_N)]^{-1}$, boundary condition (2), and the continuity conditions of a solution at the interior partition points. It is established that the unique solvability of problem (1), (2) is equivalent to the invertibility of the matrix of the constructed system.

In Sect. 3, we develop an algorithm for finding a solution to problem (1), (2). For a chosen partition Δ_N, the matrix $G(\Delta_N)$ is computed. If the matrix $I - G(\Delta_N)$ has an inverse, then we construct the above-mentioned system of linear algebraic equations. The components of $G(\Delta_N)$, the coefficients and right-hand sides of the system are determined by solving the Cauchy problems for ordinary differential equations and by calculating definite integrals of some known functions over the partition subintervals. Solving the system, we find the values of solution at the left-end points of subintervals. Using them and the initial data, we construct a function $F^*(t)$. Solving the Cauchy problem for the ordinary differential equations with the right-hand side $F^*(t)$, we find the values of the desired solution at the remaining points of $[0, T]$. Section 3 also provides the numerical implementation of the algorithm. Note that the elements of matrix $G(\Delta_N)$, the coefficients and right-hand side of the system of algebraic equations in parameters can be evaluated by the parallel computing on the partition subintervals. Section 4 presents numerical examples to demonstrate the effectiveness of the proposed method.

2 The unique solvability of the problem with parameter

Let us take a partition Δ_N of the interval $[0, T]$ by points $t_0 = 0 < t_1 < \ldots < t_N = T$. We introduce the following notation:

$C([0, T], R^n)$ is the space of continuous functions $x : [0, T] \to R^n$ with the norm $||x||_1 = \max\limits_{t \in [0,T]} ||x(t)||$;

$C([0, T], \Delta_N, R^{nN})$ is the space of function systems $x[t] = (x_1(t), x_2(t), \ldots, x_N(t))$, where $x_r : [t_{r-1}, t_r) \to R^n$, $r = \overline{1, N}$, are continuous functions having finite left-sided limits $\lim\limits_{t \to t_r - 0} x_r(t)$, with the norm $||x[\cdot]||_2 = \max\limits_{r=\overline{1,N}} \sup\limits_{t \in [t_{r-1}, t_r)} ||x_r(t)||$.

Suppose that $x(t)$ is a solution to problem (1), (2) and denote by $x_r(t)$ the restriction of $x(t)$ to the rth subinterval of the partition, i.e. $x_r(t) = x(t)$ for $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$. We introduce additional parameters λ_r, $r = \overline{1, N}$, as the values of a solution $x(t)$ at the left endpoints of the partition subintervals: $\lambda_r = x_r(t_{r-1})$. We then compose the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_{N+1})$, whose last component is the parameter μ included in problem (1), (2), i.e. $\lambda_{N+1} = \mu$.

On each partition subinterval, we make the substitution $u_r(t) = x_r(t) - \lambda_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$. The problem (1), (2) is then transformed into the multipoint boundary value problem with parameters

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{t_{j-1}}^{t_j} \varphi_k(t)\psi_k(s)[u_j(s)$$

$$+\lambda_j]ds + A_0(t)\lambda_{N+1} + f(t), \quad t \in [t_{r-1}, t_r), \tag{3}$$

$$u_r(t_{r-1}) = 0, \qquad r = \overline{1, N}, \tag{4}$$

$$B_0\lambda_{N+1} + B\lambda_1 + C\lambda_N + C \lim_{t \to T-0} u_N(t) = d, \tag{5}$$

$$\lambda_p + \lim_{t\to t_p-0} u_p(t) - \lambda_{p+1} = 0, \qquad p = \overline{1, N-1}, \tag{6}$$

where conditions (6) are imposed to ensure the continuity of a solution to (1), (2) at the interior points of the partition Δ_N. Note that conditions (6) in conjunction with integro-differential equations (3) also ensure the continuity of the derivative of a solution at these points.

A solution to problem (3)–(6) is a pair $(u^*[t], \lambda^*)$, where $u^*[t] = \big(u_1^*(t), u_2^*(t), \ldots, u_N^*(t)\big) \in C([0,T], \Delta_N, R^{nN})$ with continuously differentiable on $[t_{r-1}, t_r)$ components $u_r^*(t)$ and $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*, \lambda_{N+1}^*) \in R^{nN+l}$, satisfying the system of integro-differential equations (3), initial conditions (4), and relations (5), (6).

The problems (1), (2) and (3)–(6) are equivalent. Indeed, if a pair $(x^*(t), \mu^*)$ is a solution to problem (1), (2), then the pair $(u^*[t], \lambda^*)$ composed of the components $u_r^*(t) = x^*(t) - x^*(t_{r-1})$, $t \in [t_{r-1}, t_r)$, $\lambda_r^* = x^*(t_{r-1})$, $r = \overline{1, N}$, $\lambda_{N+1}^* = \mu^*$, is a solution to problem (3)–(6). Conversely, if a pair $(\widetilde{u}[t], \widetilde{\lambda})$, with elements $\widetilde{u}[t] \in C([0,T], \Delta_N, R^{nN})$ and $\widetilde{\lambda} \in R^{nN+l}$, is a solution to problem (3)–(6), then the pair $(\widetilde{x}(t), \widetilde{\mu})$ defined by the equalities $\widetilde{x}(t) = \widetilde{u}_r(t) + \widetilde{\lambda}_r$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, $\widetilde{x}(T) = \lim_{t\to T-0} \widetilde{u}_N(t) + \widetilde{\lambda}_N$, and $\widetilde{\mu} = \widetilde{\lambda}_{N+1}$, is a solution to the original problem (1), (2).

For fixed λ_r, $r = \overline{1, N+1}$, Eqs. (3) and (4) form a special Cauchy problem for the system of Fredholm integro-differential equations.

Consider the system of differential equations:

$$\frac{du_r}{dt} = A(t)u_r + g(t), \tag{7}$$

subject to the condition

$$u_r(t_{r-1}) = u_r^0, \tag{8}$$

where $g(t)$ is a continuous on $[t_{r-1}, t_r]$ function and u_r^0 is a constant vector.

By a fundamental matrix $X_r(t)$ of

$$\frac{du_r}{dt} = A(t)u_r \tag{9}$$

or

$$\frac{dX_r}{dt} = A(t)X_r \tag{10}$$

is meant a solution of (10) such that $det X_r(t) \neq 0$.

If $X_r(t)$ is a solution of (10) and c is a constant vector, the principle of superposition states that

$$u_r(t) = X_r(t)c \tag{11}$$

is a solution of (9). Furthermore, if $X_r(t)$ is a fundamental solution of (10), then every solution of (9) subject to (8) is of the form (11) with $c = X_r^{-1}(t_{r-1})u_r(t_{r-1})$ (see Hartman (1964, p.47)), that is,

$$u_r(t) = X_r(t)X_r^{-1}(t_{r-1})u_r(t_{r-1}). \tag{12}$$

By Corollary 2.1 Hartman (1964, p.48), the solution to the initial-value problem (7), (8) can be represented by the Cauchy formula

$$u_r(t) = X_r(t)c + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)d\tau.$$

Taking into account (12), we get that the initial-value problem (7), (8) is equivalent to the system of integral equations

$$u_r(t) = X_r(t)X_r^{-1}(t_{r-1})u_r^0 + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)d\tau.$$

Setting $g(t) = A(t)\lambda_r + \sum_{j=1}^{N}\sum_{k=1}^{m}\int_{t_{j-1}}^{t_j} \varphi_k(t)\psi_k(s)[u_j(s)+\lambda_j]ds + A_0(t)\lambda_{N+1} + f(t)$ and $u_r^0 = 0$, we get that the special Cauchy problem (3), (4) is reduced to the equivalent system of integral equations

$$\begin{aligned} u_r(t) = {} & X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\lambda_r \\ & + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\sum_{j=1}^{N}\sum_{k=1}^{m}\int_{t_{j-1}}^{t_j} \varphi_k(\tau)\psi_k(s)[u_j(s)+\lambda_j]dsd\tau + \\ & + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A_0(\tau)d\tau\lambda_{N+1} \\ & + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}. \end{aligned} \tag{13}$$

Let us set $\xi_k = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \psi_k(s)u_j(s)ds$, $k = \overline{1, m}$, and rewrite system (13) in the following form:

$$\begin{aligned} u_r(t) = {} & \sum_{k=1}^{m} X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\varphi_k(\tau)d\tau\xi_k + X_r(t)\int_{t_{r-1}}^{t} X_r^{-1}(\tau)\Big[A(\tau)\lambda_r + \\ & + \sum_{k=1}^{m}\varphi_k(\tau)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\psi_k(s)ds\lambda_j + A_0(\tau)\lambda_{N+1} + f(\tau)\Big]d\tau, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}. \end{aligned} \tag{14}$$

Multiplying both sides of (14) by $\psi_p(t)$, integrating them over the interval $[t_{r-1}, t_r]$, and summing up with respect to r, we obtain the following system of linear algebraic equations in $\xi = (\xi_1, \ldots, \xi_m) \in R^{nm}$:

$$\xi_p = \sum_{k=1}^{m} G_{p,k}(\Delta_N)\xi_k + \sum_{r=1}^{N+1} V_{p,r}(\Delta_N)\lambda_r + g_p(f, \Delta_N), \quad p = \overline{1, m}, \tag{15}$$

with the $(n \times n)$ matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s)\varphi_k(s) ds d\tau, \quad k = \overline{1,m}, \tag{16}$$

$$V_{p,r}(\Delta_N) = \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s) A(s) ds d\tau + \\ + \sum_{j=1}^{N}\sum_{k=1}^{m} \int_{t_{j-1}}^{t_j} \psi_p(\tau) X_j(\tau) \int_{t_{j-1}}^{\tau} X_j^{-1}(\tau_1)\varphi_k(\tau_1) d\tau_1 d\tau \int_{t_{r-1}}^{t_r} \psi_k(s) ds, \quad r = \overline{1,N}, \tag{17}$$

the $(n \times l)$ matrices

$$V_{p,N+1}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s) A_0(s) ds d\tau, \tag{18}$$

and the vectors of dimension n

$$g_p(f, \Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(s) f(s) ds d\tau, \qquad p = \overline{1,m}. \tag{19}$$

Using the matrices $G_{p,k}(\Delta_N)$ and $V_{p,r}(\Delta_N)$, we construct the matrices $G(\Delta_N) = (G_{p,k}(\Delta_N))$, $p, k = \overline{1,m}$, and $V(\Delta_N) = (V_{p,r}(\Delta_N))$, $p = \overline{1,m}$, $r = \overline{1, N+1}$. Then the system (15) becomes

$$[I - G(\Delta_N)]\xi = V(\Delta_N)\lambda + g(f, \Delta_N), \tag{20}$$

where I is the identity matrix of order nm and $g(f, \Delta_N) = (g_1(f, \Delta_N), \dots, g_m(f, \Delta_N)) \in R^{nm}$.

Definition 2.1 A partition Δ_N is called regular if the matrix $I - G(\Delta_N)$ is invertible.

Definition 2.2 The special Cauchy problem (3), (4) is called uniquely solvable if it has a unique solution for any $\lambda \in R^{nN+l}$ and $f(t) \in C([0, T], R^n)$.

Thus, the special Cauchy problem (3), (4) is equivalent to the system of integral Eq. (13). This system, due to the kernel degeneracy, is equivalent to the system of algebraic Eq. (15) in $\xi = (\xi_1, \dots, \xi_m) \in R^{nm}$. Therefore, the special Cauchy problem is uniquely solvable if and only if the partition Δ_N, generating this problem, is regular.

Let $\sigma(m, [0, T])$ denote the set of regular partitions Δ_N of $[0, T]$ for the Eq. (1).

Since the special Cauchy problem is uniquely solvable for a partition with a sufficiently small step size $h > 0$ (see Dzhumabaev (2010), p.1152), the set $\sigma(m, [0, T])$ is not empty.

Take a partition $\Delta_N \in \sigma(m, [0, T])$ and represent the matrix $[I - G(\Delta_N)]^{-1}$ in the form

$$[I - G(\Delta_N)]^{-1} = \Big(M_{k,p}(\Delta_N)\Big), \qquad k, p = \overline{1,m},$$

where $M_{k,p}(\Delta_N)$ are square matrices of order n. Then, in view of (20), the elements of the vector $\xi \in R^{nm}$ can be determined by the equalities

$$\xi_k = \sum_{j=1}^{N+1} \Big(\sum_{p=1}^{m} M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) \Big)\lambda_j + \sum_{p=1}^{m} M_{k,p}(\Delta_N) g_p(f, \Delta_N), \quad k = \overline{1,m}. \tag{21}$$

By replacing ξ_k in (14) with the right-hand side of (21), we get the following representation of the functions: $u_r(t)$ via $\lambda_j,\ j=\overline{1,N+1}$:

$$u_r(t)=\sum_{j=1}^{N}\Big\{\sum_{k=1}^{m}X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\Big[\sum_{p=1}^{m}M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)+\int\limits_{t_{j-1}}^{t_j}\psi_k(s)ds\Big]\Big\}\lambda_j+$$
$$+X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)A(\tau)d\tau\lambda_r+$$
$$+X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}M_{k,p}(\Delta_N)V_{p,N+1}(\Delta_N)+A_0(\tau)\Big]d\tau\lambda_{N+1}+$$
$$+X_r(t)\int\limits_{t_{r-1}}^{t}X_r^{-1}(\tau)\Big[\sum_{k=1}^{m}\varphi_k(\tau)\sum_{p=1}^{m}M_{k,p}(\Delta_N)g_p(f,\Delta_N)$$
$$+f(\tau)\Big]d\tau,\quad t\in[t_{r-1},t_r),\quad r=\overline{1,N}. \tag{22}$$

We introduce the following notation:

$$D_{r,j}(\Delta_N)=\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\Big[\sum_{p=1}^{m}M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)+\int\limits_{t_{j-1}}^{t_j}\psi_k(s)ds\Big],$$
$$j\neq r,\qquad r,j=\overline{1,N}, \tag{23}$$
$$D_{r,r}(\Delta_N)=\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\Big[\sum_{p=1}^{m}M_{k,p}(\Delta_N)V_{p,r}(\Delta_N)+\int\limits_{t_{r-1}}^{t_r}\psi_k(s)ds\Big]+$$
$$+\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)A(\tau)d\tau, \tag{24}$$
$$D_{r,N+1}(\Delta_N)=\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\sum_{p=1}^{m}M_{k,p}(\Delta_N)V_{p,N+1}(\Delta_N)+$$
$$+X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)A_0(\tau)d\tau, \tag{25}$$
$$F_r(\Delta_N)=\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)\varphi_k(\tau)d\tau\sum_{p=1}^{m}M_{k,p}(\Delta_N)g_p(f,\Delta_N)+$$
$$+\sum_{k=1}^{m}X_r(t_r)\int\limits_{t_{r-1}}^{t_r}X_r^{-1}(\tau)f(\tau)d\tau,\quad r=\overline{1,N}. \tag{26}$$

Then from (22) we get

$$\lim_{t\to t_r-0}u_r(t)=\sum_{j=1}^{N+1}D_{r,j}(\Delta_N)\lambda_j+F_r(\Delta_N). \tag{27}$$

If we substitute the right-hand side of (27) into the boundary condition (5) and the continuity condition (6), we obtain the following system of linear algebraic equations in parameters

$\lambda_r,\ r=\overline{1,N+1}$:

$$[B+CD_{N,1}(\Delta_N)]\lambda_1+\sum_{j=2}^{N-1}CD_{N,j}(\Delta_N)\lambda_j+C[I+D_{N,N}(\Delta_N)]\lambda_N+$$

$$+[B_0+CD_{N,N+1}(\Delta_N)]\lambda_{N+1}=d-CF_N(\Delta_N), \tag{28}$$

$$[I+D_{p,p}(\Delta_N)]\lambda_p-[I-D_{p,p+1}(\Delta_N)]\lambda_{p+1}$$

$$+\sum_{\substack{j=1\\ j\neq p,\ j\neq p+1}}^{N+1}D_{p,j}(\Delta_N)\lambda_j=-F_p(\Delta_N),\quad p=\overline{1,N-1}. \tag{29}$$

Let $Q_*(\Delta_N)$ denote the matrix corresponding to the left-hand side of this system. Then we can represent (28), (29) as follows:

$$Q_*(\Delta_N)\lambda=-F_*(\Delta_N),\quad \lambda\in R^{nN+l}, \tag{30}$$

where $F_*(\Delta_N)=\Big(-d+CF_N(\Delta_N),F_1(\Delta_N),\ldots,F_{N-1}(\Delta_N)\Big)\in R^{nN+l}$.

Lemma 2.1 *For $\Delta_N\in\sigma(m,[0,T])$ the following assertions hold:*
(i) the vector $\lambda^*=(\lambda_1^*,\ldots,\lambda_{N+1}^*)\in R^{nN+l}$, composed of the values of a solution $(x^*(t),\mu^*)$ to problem (1), (2) *at the partition points $\lambda_r^*=x^*(t_{r-1})$, $r=\overline{1,N}$, and $\lambda_{N+1}^*=\mu^*$, satisfies the system* (30)*;*
(ii) if $\widetilde{\lambda}=(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_{N+1})\in R^{nN+l}$ is a solution to system (30) and the function system $\widetilde{u}[t]=(\widetilde{u}_1(t),\ldots,\widetilde{u}_N(t))$ is a solution to the special Cauchy problem (3), (4) *with $\lambda_r=\widetilde{\lambda}_r$, $r=\overline{1,N+1}$, then the pair $(\widetilde{x}(t),\widetilde{\mu})$, where the function $\widetilde{x}(t)$ and the parameter $\widetilde{\mu}$ are defined by the equalities:*

$$\widetilde{x}(t)=\widetilde{\lambda}_r+\widetilde{u}_r(t),\quad t\in[t_{r-1},t_r),\quad r=\overline{1,N},\qquad \widetilde{x}(T)=\widetilde{\lambda}_N+\lim_{t\to T-0}\widetilde{u}_N(t),\qquad \widetilde{\mu}=\widetilde{\lambda}_{N+1},$$

is a solution to problem (1), (2).

The proof of Lemma 2.1 is similar to that of Lemma 1 in Dzhumabaev (2010, p. 1155).
Let us introduce the following notation:

$$\alpha=\max_{t\in[0,T]}\|A(t)\|,\quad \alpha_0=\max_{t\in[0,T]}\|A_0(t)\|,\quad \overline{\omega}=\max_{r=\overline{1,N}}(t_r-t_{r-1}),$$

$$\overline{\varphi}(m)=\max_{r=\overline{1,N}}\int_{t_{r-1}}^{t_r}\sum_{k=1}^{m}\|\varphi_k(t)\|dt,\quad \overline{\psi}(T)=\max_{p=\overline{1,m}}\int_0^T\|\psi_p(t)\|dt.$$

Theorem 2.1 *Let $\Delta_N\in\sigma(m,[0,T])$ and the matrix $Q_*(\Delta_N):R^{nN+l}\to R^{nN+l}$ be invertible. Then problem* (1), (2) *has a unique solution $(x^*(t),\mu^*)$ for any $f(t)\in C([0,T],R^n)$, $d\in R^{n+l}$, and the estimate*

$$\max(\|x^*\|_1,||\mu^*||)\le\mathcal{N}(m,\Delta_N)\max(\|d\|,\|f\|_1), \tag{31}$$

holds, where

$$\mathcal{N}(m,\Delta_N)=e^{\alpha\overline{\omega}}\Big\{\overline{\varphi}(m)\Big[\|[I-G(\Delta_N)]^{-1}\|\cdot\overline{\psi}(T)\Big(e^{\alpha\overline{\omega}}-1+e^{\alpha\overline{\omega}}\cdot\overline{\varphi}(m)\cdot\overline{\psi}(T)\Big)+$$

$$+\overline{\psi}(T)+\alpha_0\Big]+1+\alpha_0\Big\}\gamma_*(\Delta_N)(1+\|C\|)\max\Big\{1,\overline{\omega}e^{\alpha\overline{\omega}}\Big[1+e^{\alpha\overline{\omega}}\cdot\overline{\varphi}(m)\cdot$$

$$\|[I - G(\Delta_N)]^{-1}\| \cdot \overline{\psi}(T)\Big\} + e^{\alpha\overline{\omega}}\overline{\omega}\Big[\overline{\varphi}(m) \cdot \|[I - G(\Delta_N)]^{-1}\| \cdot \overline{\psi}(T) \cdot e^{\alpha\overline{\omega}} + 1\Big]. \tag{32}$$

Definition 2.3 Problem (1), (2) is said to be well-posed if it has a unique solution $(x(t), \mu)$ for any pair $(f(t), d)$, with $f(t) \in C([0, T], R^n)$ and $d \in R^{n+l}$, and the following inequality holds:

$$\max(\|x\|_1, ||\mu||) \leq K \max(\|f\|_1, \|d\|),$$

where K is a constant, independent of $f(t)$ and d.

Theorem 2.2 *Problem (1), (2) is well-posed if and only if for any $\Delta_N \in \sigma(m, [0, T])$ the matrix $Q_*(\Delta_N) : R^{nN+l} \to R^{nN+l}$ is invertible.*

The proofs of Theorems 2.1 and 2.2 repeat with minor changes in the proofs of Theorems 2.1 and 2.2 in Dzhumabaev (2016, pp. 347–349).

3 An algorithm for solving problem (1), (2) and its numerical realization

An essential part of the proposed algorithm is solving auxiliary Cauchy problems for ordinary differential equations on the partition subintervals:

$$\frac{dx}{dt} = A(t)x + P(t), \qquad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N}. \tag{33}$$

Here $P(t)$ is either a square matrix of order n or a vector of dimension n, both continuous on $[t_{r-1}, t_r]$, $r = \overline{1, N}$. Hence a solution to problem (33) is either a square matrix or a vector. Let $E_{*,r}(A(\cdot), P(\cdot), t)$ denote such a solution. We then have

$$E_{*,r}(A(\cdot), P(\cdot), t) = X_r(t) \int_{t_{r-1}}^{t} X^{-1}(\tau)P(\tau)d\tau, \quad t \in [t_{r-1}, t_r], \tag{34}$$

where $X_r(t)$ is a fundamental matrix of a homogeneous differential equation corresponding to (33) on the r-th subinterval.

An appropriate choice of a regular partition is another important part of the algorithm. We can start with Δ_1, when the interval[0; T] is not partitioned.

Let us now formulate the Algorithm for solving problem (1), (2).

I. Choose a partition Δ_N, $N = 1, 2, \ldots$.

II. Solve the $N \cdot m$ auxiliary Cauchy problems for matrix ordinary differential equations

$$\frac{dx}{dt} = A(t)x + \varphi_k(t), \qquad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \tag{35}$$

to get the matrix functions

$$E_{*,r}(A(\cdot), \varphi_k(\cdot), t), \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N}, \quad k = \overline{1, m}. \tag{36}$$

III. Multiply each $(n \times n)$ matrix (36) by the $(n \times n)$ matrix $\psi_p(t)$, $p = \overline{1, m}$, and integrate the product over $[t_{r-1}, t_r]$:

$$\widehat{\psi}_{p,r}(\varphi_k) = \int_{t_{r-1}}^{t_r} \psi_p(t)E_{*,r}(A(\cdot), \varphi_k(\cdot), t)dt. \tag{37}$$

Summing up (37) with respect to r, we obtain the $(n \times n)$ matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \widehat{\psi}_{p,r}(\varphi_k), \quad p, k = \overline{1, m},$$

which follows from (18) and (34).

Compose the $(nm \times nm)$ matrix $G(\Delta_N) = (G_{p,k}(\Delta_N))$, $p, k = \overline{1, m}$, and check whether the matrix $[I - G(\Delta_N)] : R^{nm} \to R^{nm}$ is invertible.

If so, find its inverse and represent it in the form $[I - G(\Delta_N)]^{-1} = (M_{p,k}(\Delta_N))$, where $M_{p,k}(\Delta_N))$ are square matrices of order n, $p, k = \overline{1, N}$. Then move on to the next step of Algorithm.

If there is no inverse of $[I - G(\Delta_N)]$, i.e. the partition Δ_N is not regular, then take a new partition of interval $[0, T]$, and the algorithm starts over. A simple way for selecting a new partition is to choose the partition Δ_{2N}, where each interval of the partition Δ_N is divided into two parts.

IV. By solving again the auxiliary Cauchy problem for ordinary differential equations

$$\begin{aligned}
\frac{dx}{dt} &= A(t)x + A(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \\
\frac{dx}{dt} &= A(t)x + A_0(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \\
\frac{dx}{dt} &= A(t)x + f(t), \quad x(t_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, N},
\end{aligned}$$

find their respective solutions $E_{*,r}(A(\cdot), A(\cdot), t)$, $E_{*,r}(A(\cdot), A_0(\cdot), t)$, and $E_{*,r}(A(\cdot), f(\cdot), t)$, $r = \overline{1, N}$.

V. Evaluate the integrals

$$\widehat{\psi}_{p,r} = \int_{t_{r-1}}^{t_r} \psi_p(t)dt, \quad \widehat{\psi}_{p,r}(A) = \int_{t_{r-1}}^{t_r} \psi_p(t)E_{*,r}(A(\cdot), A(\cdot), t)dt,$$

$$\widehat{\psi}_{p,r}(A_0) = \int_{t_{r-1}}^{t_r} \psi_p(t)E_{*,r}(A(\cdot), A_0(\cdot), t)dt, \quad \widehat{\psi}_{p,r}(f) = \int_{t_{r-1}}^{t_r} \psi_p(t)E_{*,r}(A(\cdot), f(\cdot), t)dt.$$

By equalities (19), (20), and (34), determine the $(n \times n)$ matrices

$$V_{p,r}(\Delta_N) = \widehat{\psi}_{p,r}(A) + \sum_{j=1}^{N} \sum_{k=1}^{m} \widehat{\psi}_{p,j}(\varphi_k) \cdot \widehat{\psi}_{k,r}, \quad r = \overline{1, N},$$

the $(n \times l)$ matrices

$$V_{p,N+1}(\Delta_N) = \sum_{r=1}^{N} \widehat{\psi}_{p,r}(A_0), \quad p = \overline{1, m},$$

and the n vectors

$$g_p(f, \Delta_N) = \sum_{r=1}^{N} \widehat{\psi}_{p,r}(\Delta_N), \quad p = \overline{1, m}, \quad r = \overline{1, N}.$$

VI. Form the system of linear algebraic equations in parameters

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \quad \lambda \in R^{nN+l}. \tag{38}$$

The elements of the matrix $Q_*(\Delta_N) : R^{nN+l} \to R^{nN+l}$ and the vector $F_*(\Delta_N) = (-d + CF_N(\Delta_N), F_1(\Delta_N), \ldots, F_{N-1}(\Delta_N)) \in R^{nN+l}$ are defined by the equalities (23), (24), (25), and (26), where, in view of (34), we replace

$$X_r(t_r)\int\limits_{t_{r-1}}^{t_r} X_r^{-1}(\tau)\varphi_k(\tau)d\tau \quad \text{and} \quad X_r(t_r)\int\limits_{t_{r-1}}^{t_r} X_r^{-1}(\tau)f(\tau)d\tau$$

with $E_{*,r}(A(\cdot), \varphi_k(\cdot), t_r)$ and $E_{*,r}(A(\cdot), f(\cdot), t_r)$, respectively.

As it follows from Theorem 2.2, the invertibility of matrix $Q_*(\Delta_N)$ is equivalent to the well-posedness of problem (1), (2). By solving the system (38), find $\lambda^* = (\lambda_1^*, \ldots, \lambda_{N+1}^*) \in R^{nN+l}$.

VII. Determine the components of $\xi^* = (\xi_1^*, \ldots, \xi_m^*) \in R^{nm}$ by the equalities

$$\xi_k^* = \sum_{j=1}^{N+1}\Big(\sum_{p=1}^{m} M_{k,p}(\Delta_N)V_{p,j}(\Delta_N)\Big)\lambda_j^* + \sum_{p=1}^{m} M_{k,p}(\Delta_N)g_p(f, \Delta_N) \tag{39}$$

and construct the function

$$\mathcal{F}^*(t) = \sum_{k=1}^{m}\varphi_k(t)\Big[\xi_k^* + \sum_{r=1}^{N}\int\limits_{t_{r-1}}^{t_r}\psi_k(s)ds\lambda_r^*\Big] + A_0(t)\lambda_{N+1}^* + f(t). \tag{40}$$

Recall that $\lambda_r^* = x^*(t_{r-1})$, $r = \overline{1, N}$, $\lambda_{N+1}^* = \mu^*$, where $(x^*(t), \mu^*)$ is a solution to the problem with parameter (1), (2). Therefore, the solution of system (38) provides us with the values of the function $x^*(t)$ at the left-end points of the partition subintervals and the parameter μ^*.

The values of $x^*(t)$ at the remaining points of the subinterval $[t_{r-1}, t_r)$ determine by solving the following Cauchy problem for the ordinary differential equation:

$$\frac{dx}{dt} = A(t)x + \mathcal{F}^*(t), \quad x(t_{r-1}) = \lambda_r^*, \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}.$$

Thus, the offered algorithm consists of seven interconnected steps.

If the fundamental matrices $X_r(t)$, $r = \overline{1, N}$, are known, then equalities (23), (24), (25), and (26) allow us to construct the system (38). Using the solution λ^* to (38), by (39) and (40), we construct the function $\mathcal{F}^*(t)$. Therefore, the solution to the problem with parameter (1), (2) is defined by the equalities

$$x^*(t) = X_r(t)X_r^{-1}(t_{r-1})\lambda_r^* + X_r(t)\int\limits_{t_{r-1}}^{t} X_r^{-1}(\tau)\mathcal{F}^*(\tau)d\tau, \quad t \in [t_{r-1}, t_r),\ r = \overline{1, N}, \tag{41}$$

$$x^*(T) = X_N(T)X_N^{-1}(t_{N-1})\lambda_N^* + X_N(T)\int\limits_{t_{N-1}}^{T} X_N^{-1}(\tau)\mathcal{F}^*(\tau)d\tau, \tag{42}$$

$$\mu^* = \lambda_{N+1}^*. \tag{43}$$

However, it is not always possible to construct a fundamental matrix for a system of ordinary differential equations with variable coefficients. We, therefore, offer the numerical implementation of Algorithm that involves numerical solution of auxiliary Cauchy problems and numerical integration.

The **numerical algorithm for solving problem** (1), (2) performs as follows:

I. Take a partition $\Delta_N : t_0 = 0 < t_1 < \ldots < t_{N-1} < t_N = T$. Divide each subinterval $[t_{r-1}, t_r]$, $r = \overline{1, N}$, into N_r parts with step size $h_r = (t_r - t_{r-1})/N_r$.

Let $\widehat{t}$ be a variable taking on the discrete values $\widehat{t} = t_{r-1}, t_{r-1} + h_r, \ldots, t_{r-1} + (N_r - 1)h_r, t_r$ on the subinterval $[t_{r-1}, t_r]$. We denote the set of such points by $\{t_{r-1}, t_r\}$.

II. Find the numerical solutions to Cauchy problems (33) by using one of numerical methods for solving initial value problems for ordinary differential equations. Determine the values of the $(n \times n)$ matrices $E^{h_r}_{*,r}(A(\cdot), \varphi_k(\cdot), \widehat{t})$ on the set $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$, $k = \overline{1, m}$.

III. Using the values of $(n \times n)$ matrices $\psi_k(s)$ and $E^{h_r}_{*,r}\Big(A(\cdot), \varphi(\cdot), \widehat{t}\Big)$ on $\{t_{r-1}, t_r\}$, and applying a numerical quadrature rule, calculate the $(n \times n)$ matrices

$$\widehat{\psi}^{h_r}_{p,r}(\varphi_k) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), \varphi_k(\cdot), \tau) d\tau, \quad p, k = \overline{1, m}, \quad r = \overline{1, N}.$$

Summing up the matrices $\widehat{\varphi}^{h_r}_{p,r}(\psi_k)$ with respect to r, determine the $(n \times n)$ matrices $G^{\widetilde{h}}_{p,k}(\Delta_N) = \sum\limits_{r=1}^{N} \widehat{\varphi}^{h_r}_{p,r}(\psi_k)$, where $\widetilde{h} = (h_1, h_2, \ldots, h_N) \in R^n$. Using them, compose the $nm \times nm$ matrix $G^{\widetilde{h}}(\Delta_N) = (G^{\widetilde{h}}_{p,k}(\Delta_N))$, $p, k = \overline{1, m}$.

Check whether the matrix $I - G^{\widetilde{h}}(\Delta_N)$ is invertible. If so, calculate its inverse $[I - G^{\widetilde{h}}(\Delta_N)]^{-1} = (M^{\widetilde{h}}_{p,k}(\Delta_N))$, $p, k = \overline{1, m}$, and move on to the next step.

If the matrix is not invertible, take a new partition. In particular, each subinterval can be divided into two parts. Then go back to Step I.

IV. Solve numerically the Cauchy problem (33), (35) and find the values of the $(n \times n)$ matrix $E_{*,r}(A(\cdot), A(\cdot), \widehat{t})$, the $(n \times l)$ matrix $E_{*,r}(A(\cdot), A_0(\cdot), \widehat{t})$, and the n vector $E_{*,r}(A(\cdot), f(\cdot), \widehat{t})$ on the grid $\{t_{r-1}, t_r\}$, $r = \overline{1, N}$.

V. On the set $\{t_{r-1}, t_r\}$, evaluate the definite integrals

$$\widehat{\psi}^{h_r}_{p,r} = \int\limits_{t_{r-1}}^{t_r} \psi_p(s) ds, \quad \widehat{\psi}^{h_r}_{p,r}(A) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), A(\cdot), \tau) d\tau,$$

$$\widehat{\psi}^{h_r}_{p,r}(A_0) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), A_0(\cdot), \tau) d\tau,$$

$$\widehat{\psi}^{h_r}_{p,r}(f) = \int\limits_{t_{r-1}}^{t_r} \psi_p(\tau) E^{h_r}_{*,r}(A(\cdot), f(\cdot), \tau) d\tau, \quad r = \overline{1, N}, \quad p = \overline{1, m}.$$

Determine the $(n \times n)$ matrices $V^{\widetilde{h}}_{p,r}(\Delta_N)$, the $(n \times l)$ matrices $V^{\widetilde{h}}_{p,N+1}(\Delta_N)$, and the n vectors $g^{\widetilde{h}}_p(f, \Delta_N)$ by the respective equalities

$$V^{\widetilde{h}}_{p,r}(\Delta_N) = \widehat{\psi}^{h_r}_{p,r}(A) + \sum_{j=1}^{N} \sum_{k=1}^{m} \widehat{\psi}^{h_j}_{p,j}(\varphi_k) \cdot \widehat{\psi}^{h_r}_{k,r}, \qquad r = \overline{1, N},$$

$$V_{p,N+1}^{\widetilde{h}}(\Delta_N)=\sum_{r=1}^{N}\widehat{\psi}_{p,r}^{h_r}(A_0),\qquad g_p^{\widetilde{h}}(f,\Delta_N)=\sum_{r=1}^{N}\widehat{\psi}_{p,r}^{h_r}(f),\qquad p=\overline{1,m}.$$

VI. Construct the system of linear algebraic equations in parameters

$$Q_*^{\widetilde{h}}(\Delta_N)\lambda=-F_*^{\widetilde{h}}(\Delta_N),\quad \lambda\in R^{nN+l}, \tag{44}$$

where the elements of the matrix $Q_*^{\widetilde{h}}(\Delta_N)$ and the vector

$$F_*^{\widetilde{h}}(\Delta_N)=(-d+CF_N^{\widetilde{h}}(\Delta_N),F_1^{\widetilde{h}}(\Delta_N),\ldots,F_{N-1}^{\widetilde{h}}(\Delta_N))$$

are defined by the equalities

$$D_{r,j}^{\widetilde{h}}(\Delta_N)=\sum_{k=1}^{m}E_{*,r}^{h_r}(A(\cdot),\varphi_k(\cdot),t_r)\Big[\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)V_{p,j}^{\widetilde{h}}(\Delta_N)$$
$$+\widehat{\psi}_{k,j}^{h_j}\Big],\quad j\neq r,\quad r,j=\overline{1,N},$$
$$D_{r,r}^{\widetilde{h}}(\Delta_N)=\sum_{k=1}^{m}E_{*,r}^{h_r}(A(\cdot),\varphi_k(\cdot),t_r)\Big[\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)V_{p,r}^{\widetilde{h}}(\Delta_N)+\widehat{\psi}_{k,r}^{h_r}\Big]$$
$$+E_{*,r}^{h_r}(A(\cdot),A(\cdot),t_r),\quad r=\overline{1,N},$$
$$D_{r,N+1}^{\widetilde{h}}(\Delta_N)=\sum_{k=1}^{m}E_{*,r}^{h_r}(A(\cdot),\varphi_k(\cdot),t_r)\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)V_{p,N+1}^{\widetilde{h}}(\Delta_N)$$
$$+E_{*,r}^{h_r}(A(\cdot),A_0(\cdot),t_r),\quad r=\overline{1,N},$$
$$F_r^{\widetilde{h}}(\Delta_N)=\sum_{k=1}^{m}E_{*,r}^{h_r}(A(\cdot),\varphi_k(\cdot),t_r)\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)g_p^{\widetilde{h}}(\Delta_N)$$
$$+E_{*,r}^{h_r}(A(\cdot),f(\cdot),t_r),\quad r=\overline{1,N}.$$

Using the constructed matrix $Q_*^{\widetilde{h}}(\Delta_N)$, we can establish the well-posedness of problem (1), (2). Suppose that the matrix $Q_*^{\widetilde{h}}(\Delta_N)$ is invertible and the estimate $||Q_*(\Delta_N)-Q_*^{\widetilde{h}}(\Delta_N)||\leq\varepsilon(\widetilde{h})$ holds. By Theorem 4 Dzhumabaev (2015, p.212), if the inequality $||[Q_*^{\widetilde{h}}(\Delta_N)]^{-1}||\cdot\varepsilon(\widetilde{h})<1$ holds, then $Q_*(\Delta_N)$ is invertible. Thus it follows from Theorem 2.2 that the problem (1), (2) is well-posed.

By solving system (44) find $\lambda^{\widetilde{h}}\in R^{nN+l}$. As noted above, the elements of $\lambda^{\widetilde{h}}=(\lambda_1^{\widetilde{h}},\ldots,\lambda_{N+1}^{\widetilde{h}})$ are the values of approximate solution to problem (1), (2), i.e. the approximate values of $x(t)$ at the left endpoints of the subintervals: $x^{\widetilde{h}_r}(t_{r-1})=\lambda_r^{\widetilde{h}}$, $r=\overline{1,N}$, and the approximate value of the parameter μ: $\mu^{\widetilde{h}_r}=\lambda_{N+1}^{\widetilde{h}}$.

VII. To define the values of an approximate solution at the remaining points of the set $\{t_{r-1},t_r\}$, we first find

$$\xi_k^{\widetilde{h}}=\sum_{j=1}^{N+1}\Big(\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)V_{p,j}^{\widetilde{h}}(\Delta_N)\Big)\lambda_j^{h}+\sum_{p=1}^{m}M_{k,p}^{\widetilde{h}}(\Delta_N)g_p^{\widetilde{h}}(f,\Delta_N),\quad k=\overline{1,m},$$

and then numerically solve the Cauchy problems

$$\frac{dx}{dt}=A(t)x+\mathcal{F}^{\widetilde{h}}(t),\quad x(t_{r-1})=\lambda_r^{\widetilde{h}},\quad t\in[t_{r-1},t_r],\quad r=\overline{1,N}.$$

Here

$$\mathcal{F}^{\widetilde{h}}(t) = \sum_{k=1}^{m} \varphi_k(t)\Big(\xi_k^{\widetilde{h}} + \sum_{j=1}^{N} \widehat{\psi}_{k,j}^{h_j} \lambda_j^h\Big) + A_0(t)\lambda_{N+1}^h + f(t).$$

Thus the algorithm offered provides us with the numerical solution to the problem (1), (2).

4 Numerical examples

Example 1 Consider the following problem with parameter for the system of integro-differential equations:

$$\frac{dx}{dt} = A(t)x + \varphi(t)\int_0^T \psi(\tau)x(\tau)d\tau + A_0(t)\mu + f(t), \qquad x \in R^2, \quad \mu \in R^1, \quad (45)$$

$$B_0\mu + Bx(0) + Cx(T) = d, \qquad d \in R^3, \quad (46)$$

where

$$T = 1, \quad A(t) = \begin{pmatrix} t & t^2 \\ 0 & t-4 \end{pmatrix}, \quad A_0(t) = \begin{pmatrix} t+5 \\ t^3-2 \end{pmatrix},$$

$$\varphi(t) = \begin{pmatrix} 3t & t^3 \\ 4 & t-2 \end{pmatrix}, \quad \psi(t) = \begin{pmatrix} t & t^2-2 \\ t+4 & 3 \end{pmatrix}, \quad f(t) = \begin{pmatrix} 5t^4 - t^5 - t^6 + \frac{13t^3}{7} - \frac{318t}{7} - 45 \\ 10t^2 - 5t^3 - t^4 - \frac{267t}{7} + \frac{85}{21} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -5 \\ 0 & 6 \\ -4 & 11 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 0 \\ -12 & 5 \\ 9 & 17 \end{pmatrix}, \quad d = \begin{pmatrix} 36 \\ 9 \\ -153 \end{pmatrix}.$$

To implement the numerical algorithm for solving problem (45),(46), we use Simpson's rule for estimation of definite integrals and the fourth-order Runge-Kutta method. To do this, we divide each interval [0, 0.5] and [0.5, 1] into $N = 10$ subintervals with the step $h = 0.05$.

We compute the matrix $I - G^{\widetilde{h}}(\Delta_2) = \begin{pmatrix} 1.6665474 & -0.4153697 \\ -3.6533113 & 1.5407901 \end{pmatrix}$, where I is the second-order identity matrix. The invertibility of this matrix implies the regularity of Δ_2.

We then construct the system of linear algebraic equations with respect to parameters

$$Q_*^{\widetilde{h}}(\Delta_2)\lambda = -F_*^{\widetilde{h}}(\Delta_2), \quad \lambda \in R^5, \quad (47)$$

where

$$Q_*^{\widetilde{h}}(\Delta_2) = \begin{pmatrix} 9.5083951 & -7.0229286 & 16.5414502 & -0.0873712 & 41.3179432 \\ -30.5783061 & 12.889927 & -66.2339239 & 0.2548226 & -151.3556546 \\ 16.6731188 & 0.8451376 & 49.3927316 & -0.5839651 & 109.2686315 \\ 1.6033622 & -0.1482411 & -0.2174947 & -0.0605072 & 4.7172016 \\ -0.5501459 & -0.0171558 & -0.6994877 & -1.2253692 & -1.908402 \end{pmatrix},$$

$$F_*^{\widetilde{h}}(\Delta_2) = \Big(-508.2954741, 1863.211432, -1321.586001, -51.2392, 19.6852962 \Big)'.$$

By solving (47) we find $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, \lambda_3^{\widetilde{h}}) \in R^5$ with

$$\lambda_1^{\widetilde{h}} = \begin{pmatrix} 6.0000271 \\ 0.0000055 \end{pmatrix}, \quad \lambda_2^{\widetilde{h}} = \begin{pmatrix} 4.7812568 \\ -3.3750023 \end{pmatrix}, \quad \lambda_3^{\widetilde{h}} = 8.9999902.$$

Table 1 Comparison of exact and numerical solutions to problem (45), (46)

t	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$x_1^*(t)$	$x_2^*(t)$
0	6.0000271	0.0000055	6	0
0.05	5.987525	−0.349872	5.9875003	−0.349875
0.1	5.9500324	−0.6989988	5.95001	−0.699
0.15	5.8875962	−1.0466252	5.8875759	−1.046625
0.2	5.8003381	−1.3920012	5.80032	−1.392
0.25	5.6884926	−1.7343768	5.6884766	−1.734375
0.3	5.5524441	−2.0730023	5.55243	−2.073
0.35	5.3927644	−2.4071275	5.3927522	−2.407125
0.4	5.2102503	−2.7360025	5.21024	−2.736
0.45	5.0059613	−3.0588775	5.0059528	−3.058875
0.5	4.7812568	−3.3750023	4.78125	−3.375
0.55	4.5378336	−3.6836271	4.5378284	−3.683625
0.6	4.2777636	−3.9840018	4.27776	−3.984
0.65	4.0035313	−4.2753765	4.0035291	−4.275375
0.7	3.7180709	−4.5570011	3.71807	−4.557
0.75	3.4248045	−4.8281257	3.4248047	−4.828125
0.8	3.1276788	−5.0880003	3.12768	−5.088
0.85	2.8312034	−5.3358749	2.8312053	−5.335875
0.9	2.5404876	−5.5709995	2.54049	−5.571
0.95	2.2612783	−5.7926241	2.2612809	−5.792625
1	1.9999975	−5.9999987	2	−6
	$\widetilde{\mu} = 8.9999902$		$\mu^* = 9$	

To define the values of an approximate solution at the remaining points of set $\{t_{r-1}, t_r\}$, $r = \overline{1,2}$, we first find $\xi^{\widetilde{h}} = \begin{pmatrix} 1.8759368 \\ -8.9001442 \end{pmatrix}$, and then solve the Cauchy problems:

$$\frac{d\widetilde{x}}{dt} = A(t)\widetilde{x} + \varphi(t)\xi^{\widetilde{h}} + \varphi(t)\int_0^{0.5} \psi(\tau)d\tau\lambda_1^{\widetilde{h}} + \varphi(t)\int_{0.5}^{1} \psi(\tau)d\tau\lambda_2^{\widetilde{h}} + A_0(t)\lambda_3^{\widetilde{h}} + f(t),$$

$$\widetilde{x}(t_{r-1}) = \lambda_r^{\widetilde{h}}, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,2}.$$

The exact solution to problem with parameter (45), (46) is the pair $(x^*(t), \mu^*)$ with

$$x^*(t) = \begin{pmatrix} t^5 - 5t^2 + 6 \\ t^3 - 7t \end{pmatrix}, \mu^* = 9.$$

Table 1 provides the values of the exact solution and the numerical solution $(\widetilde{x}(t), \widetilde{\mu})$. The calculations were carried out in the MathCad software package.

The error estimates obtained by using the Runge–Kutta method are as follows:

$$\|\mu^* - \widetilde{\mu}\| < 0.00001, \qquad \max_{j=\overline{0,20}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.00003.$$

Example 2 Consider the problem with parameter for system of integro-differential equations

$$\frac{dx}{dt} = A(t)x + \varphi(t)\int_0^1 \psi(\tau)x(\tau)d\tau + A_0(t)\mu + f(t), \qquad x \in R^2, \quad \mu \in R^2, \quad (48)$$

$$B_0\mu + Bx(0) + Cx(T) = d, \quad d \in R^4, \quad (49)$$

where $A(t) = \begin{pmatrix} e^t & 1 \\ t^3 & cos(t) \end{pmatrix}$, $A_0(t) = \begin{pmatrix} 4 & t^2 \\ sin(t) & 0 \end{pmatrix}$, $\varphi(t) = \begin{pmatrix} t & t^2 \\ 0 & t+3 \end{pmatrix}$, $\psi(t) = \begin{pmatrix} t^3 & t-2 \\ 0 & e^t \end{pmatrix}$,

$$f(t) = \begin{pmatrix} \frac{43t}{56} + t^2 + 4t^3 - 28 - t^2e + e^t(4 - 6t^3 - t^4) \\ 5t - 3e - 7sin(t) - t^2cos(t) - te + 4t^3 - 6t^6 - t^7 + tcos(t) + 8 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 4 & 5 \\ 2 & 3 \\ 0 & -3 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 7 \\ 3 & 0 \\ -5 & 2 \\ 9 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 9 \\ 0 & -8 \\ 19 & 4 \\ 9 & 7 \end{pmatrix}, \quad d = \begin{pmatrix} 122 \\ 59 \\ 20 \\ 36 \end{pmatrix}.$$

We use the numerical implementation of algorithm. The accuracy of the solution depends on that of solving the Cauchy problems on the subintervals. We provide the results of the numerical implementation of algorithm based on the Bulirsch–Stoer method Atkinson et al. (2009), Butcher (2000), Stoer and Bulirsch (2002) by partitioning the subintervals [0, 0.5], [0.5, 1] with step size h = 0.05.

The exact solution to problem with parameter (48), (49) is the pair $(x^*(t), \mu^*)$ with

$$x^*(t) = \begin{pmatrix} t^4 + 6t^3 - 4 \\ t^2 - t \end{pmatrix}, \mu^* = \begin{pmatrix} 7 \\ 19 \end{pmatrix}.$$

In Table 2, the values of the exact solution and numerical solution $(x^*(t_k), \mu^*)$ and $(\widetilde{x}(t_k), \widetilde{\mu})$, $k = \overline{0, 20}$, are shown.

Example 3 Consider the following problem with parameter for the system of integro-differential equations:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{2}\int_0^1 \varphi_k(t)\psi_k(\tau)x(\tau)d\tau + A_0(t)\mu + f(t), \qquad x \in R^2, \quad \mu \in R^3, \quad (50)$$

$$B_0\mu + Bx(0) + Cx(T) = d, \qquad d \in R^5, \quad (51)$$

where

$$A(t) = \begin{pmatrix} sin(t) & 1 \\ t^2 & 0 \end{pmatrix}, \qquad A_0(t) = \begin{pmatrix} t & 1 & t+2 \\ t^2 - 7 & 2t & 8 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix},$$

$$\varphi_1(t) = \begin{pmatrix} t & 0 \\ 2t^3 & t-3 \end{pmatrix}, \quad \varphi_2(t) = \begin{pmatrix} 1 & t \\ t^3 & t+5 \end{pmatrix}, \quad \psi_1(t) = \begin{pmatrix} t & e^t \\ t^2 & 4t \end{pmatrix}, \quad \psi_2(t) = \begin{pmatrix} 2 & t^2 \\ t & e^t \end{pmatrix},$$

$$f_1(t) = -\frac{1}{6\pi}(163\pi - 6\pi^2\cos(\pi t) + 6\pi\sin(t)t + 6\pi\sin(t)\sin(\pi t) + 6\pi t^3 + 30\pi t^2 - 56\pi t + 12t + 144\pi e t + 24),$$

Table 2 Comparison of exact and numerical solutions to problem (48), (49)

t	$\widetilde{x}_1(t)$	$x_1^*(t)$	$\lvert x_1^*(t)-\widetilde{x}_1(t)\rvert$	$\widetilde{x}_2(t)$	$x_2^*(t)$	$\lvert x_2^*(t)-\widetilde{x}_2(t)\rvert$
0	−4.00000058	−4	0.00000058	−0.00000066	0	0.00000066
0.05	−3.99924493	−3.99924375	0.00000118	−0.04750057	−0.0475	0.00000057
0.1	−3.99390172	−3.9939	0.00000172	−0.09000049	−0.09	0.00000049
0.15	−3.97924596	−3.97924375	0.00000221	−0.12750041	−0.1275	0.00000041
0.2	−3.95040265	−3.9504	0.00000265	−0.16000032	−0.16	0.00000032
0.25	−3.90234679	−3.90234375	0.00000304	−0.18750024	−0.1875	0.00000024
0.3	−3.82990337	−3.8299	0.00000337	−0.21000017	−0.21	0.00000017
0.35	−3.72774739	−3.72774375	0.00000364	−0.22750009	−0.2275	0.00000009
0.4	−3.59040386	−3.5904	0.00000386	−0.24000002	−0.24	0.00000002
0.45	−3.41224776	−3.41224375	0.00000401	−0.24749996	−0.2475	0.00000004
0.5	−3.18750409	−3.1875	0.00000409	−0.2499999	−0.25	0.00000010
0.55	−2.91024786	−2.91024375	0.00000411	−0.24749986	−0.2475	0.00000014
0.6	−2.57440405	−2.5744	0.00000405	−0.23999983	−0.24	0.00000017
0.65	−2.17374767	−2.17374375	0.00000392	−0.22749981	−0.2275	0.00000019
0.7	−1.70190369	−1.7019	0.00000369	−0.2099998	−0.21	0.00000020
0.75	−1.15234711	−1.15234375	0.00000336	−0.1874998	−0.1875	0.00000020
0.8	−0.51840292	−0.5184	0.00000292	−0.15999982	−0.16	0.00000018
0.85	0.2067539	0.20675625	0.00000235	−0.12749984	−0.1275	0.00000016
0.9	1.03009837	1.0301	0.00000163	−0.08999986	−0.09	0.00000014
0.95	1.95875552	1.95875625	0.00000073	−0.04749986	−0.0475	0.00000014
1	3.00000038	3	0.00000038	0.00000017	0	0.00000017
	$\widetilde{\mu}_1$	μ_1^*	$\lvert\mu_1^*-\widetilde{\mu}_1\rvert$	$\widetilde{\mu}_2$	μ_2^*	$\lvert\mu_2^*-\widetilde{\mu}_2\rvert$
	6.99999721	7	0.00000279	19.0000029	19	0.0000029

$$
\begin{aligned}
f_2(t) = &-\frac{1}{60\pi^3}(240\pi^3t^2 - 397\pi^3t - 1150\pi^3t^3 + 60\pi^3t^2\sin(\pi t)\\
&+360\pi^2t^3 + 1440\pi^3et^3 +\\
&+120\pi^2t - 240t - 6749\pi^3 + 120\pi^2 + 720 + 720\pi^3te + 3600\pi^3e),
\end{aligned}
$$

$$
B_0 = \begin{pmatrix} 1 & 2 & 5\\ 3 & 0 & 1\\ 4 & 2 & 1\\ 12 & 1 & 4\\ 2 & 0 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 5\\ 1 & 2\\ 5 & 6\\ 2 & -5\\ 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} -8 & 9\\ 5 & 2\\ 6 & 5\\ 15 & 6\\ 1 & 3 \end{pmatrix}, \quad d = \begin{pmatrix} 216\\ 83\\ 164\\ 176\\ 60 \end{pmatrix}.
$$

The exact solution to problem with parameter (50), (51) is the pair $(x^*(t), \mu^*)$ with $x^*(t) = \begin{pmatrix} t + sin(\pi t)\\ t^3 + 5t^2 + 9 \end{pmatrix}$, $\mu^* = \begin{pmatrix} 7\\ -4\\ 9 \end{pmatrix}$.

Case 1. Let $N = 1$. We introduce the additional parameters $\lambda_1, \lambda_2 \in R^2$ setting $\lambda_1 = x(0)$ and $\lambda_2 = \mu$. Let $\xi_k = \int\limits_0^1 \psi_k(s)u(s)ds$, $k = 1, 2$, where $u(s) = x(s) - \lambda_1$. Then for the function $u(t)$ we have the equality

Table 3 Comparison of numerical solutions to problem (50, 51). Case 1

t	Adams method		Runge-Kutta method		Bulirsch-Stoer method	
	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
0	0.0482514	8.980579	0.025551	8.9897177	0.02405	8.9903231
0.1	0.4499408	9.0389214	0.4306937	9.0446196	0.4294176	9.0449953
0.2	0.8224944	9.2021676	0.8061773	9.2049413	0.8050918	9.2051211
0.3	1.1383968	9.4763998	1.1245971	9.4767265	1.1236752	9.4767416
0.4	1.3758101	9.8677329	1.364198	9.8660358	1.3634185	9.8659143
0.5	1.5206706	10.38232	1.5109927	10.3789505	1.5103394	10.3787158
0.6	1.5680604	11.0263565	1.5601231	11.0215746	1.5595835	11.021244
0.7	1.5226728	11.8060815	1.5163294	11.8000353	1.5158938	11.7996189
0.8	1.3983499	12.7277792	1.3934832	12.7204846	1.3931437	12.719983
0.9	1.2167217	13.7977804	1.2132287	13.7890996	1.2129778	13.7885031
1	1.0051	15.0224688	1.0028668	15.012085	1.0026965	15.0113717
	$\widetilde{\mu}_1 = 6.9771984$		$\widetilde{\mu}_1 = 6.9879316$		$\widetilde{\mu}_1 = 6.9886448$	
	$\widetilde{\mu}_2 = -4.0832155$		$\widetilde{\mu}_2 = -4.0440641$		$\widetilde{\mu}_2 = -4.0414742$	
	$\widetilde{\mu}_3 = 8.9862406$		$\widetilde{\mu}_3 = 8.9927147$		$\widetilde{\mu}_3 = 8.9931433$	

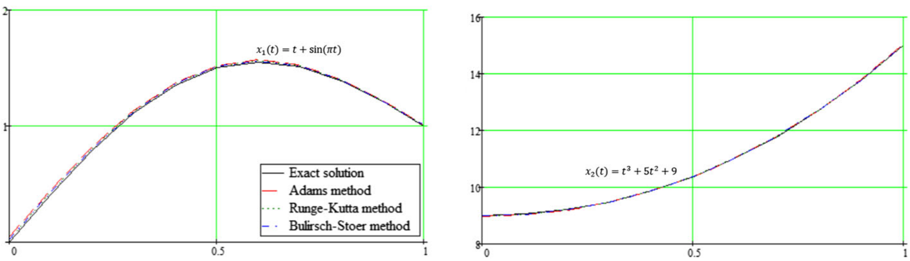

Fig. 1 Comparison of the exact solutions and numerical solutions to problem (50, 51) ($N = 2$)

$$
\begin{aligned}
u(t) = {} & E_{*,1}^{h_1}(A(\cdot), A(\cdot), \widehat{t})\lambda_1 \\
& + E_{*,1}^{h_1}(A(\cdot), A_0(\cdot), \widehat{t})\lambda_2 + E_{*,1}^{h_1}(A(\cdot), f(\cdot), \widehat{t}) + E_{*,1}^{h_1}(A(\cdot), \varphi_1(\cdot), \widehat{t})\xi_1 + \\
& + E_{*,1}^{h_1}(A(\cdot), \varphi_2(\cdot), \widehat{t})\xi_2 + E_{*,1}^{h_1}(A(\cdot), \varphi_1(\cdot), \widehat{t}) \int_0^1 \psi_1(s) ds \lambda_1 \\
& + E_{*,1}^{h_1}(A(\cdot), \varphi_2(\cdot), \widehat{t}) \int_0^1 \psi_2(s) ds \lambda_1 .
\end{aligned} \tag{52}
$$

Multiplying both sides of (52) by $\psi_p(t)$, $p = 1, 2$, and then integrating them over the interval [0, 1], we get the system of linear algebraic equations in $\xi = (\xi_1, \xi_2) \in R^4$. By solving this system we determine ξ and substitute the corresponding expression into the right-hand side of (52). We then obtain the representation of function $u(t)$ through λ_1 and λ_2.

To implement the numerical algorithm for solving problem (50),(51), we use Simpson's rule for estimation of definite integrals and the fourth-order Runge–Kutta method, the Adams method, and the Bulirsch–Stoer method to solve auxiliary Cauchy problems for ordinary differential equations. The calculations were carried out in the MathCad software package.

For the chosen step size $h = 0.1$, the algorithm was performed three times, by separately using the Adams method, the fourth-order Runge–Kutta method, and the Bulirsch–Stoer

Table 4 Comparison of numerical solutions to problem (50),(51). Case 2

t	Adams method		Runge-Kutta method		Bulirsch-Stoer method	
	$\lvert x_1^*(t) - \widetilde{x}_1(t)\rvert$	$\lvert x_2^*(t) - \widetilde{x}_2(t)\rvert$	$\lvert x_1^*(t) - \widetilde{x}_1(t)\rvert$	$\lvert x_2^*(t) - \widetilde{x}_2(t)\rvert$	$\lvert x_1^*(t) - \widetilde{x}_1(t)\rvert$	$\lvert x_2^*(t) - \widetilde{x}_2(t)\rvert$
0	0.0269675	0.0108602	0.0015882	0.0006391	0.0014922	0.0006004
0.05	0.0248291	0.0087321	0.0014627	0.0005132	0.0013742	0.0004821
0.1	0.0228632	0.0067616	0.0013474	0.0003966	0.0012658	0.0003726
0.15	0.0210526	0.0049444	0.0012412	0.0002890	0.0011660	0.0002715
0.2	0.0193803	0.0032753	0.0011432	0.0001901	0.0010738	0.0001786
0.25	0.0178316	0.0017481	0.0010525	0.0000996	0.0009886	0.0000936
0.3	0.0163908	0.0003555	0.0009684	0.0000170	0.0009095	0.0000160
0.35	0.0150488	0.0009111	0.0008901	0.0000582	0.0008359	0.0000546
0.4	0.0137924	0.0020616	0.0008169	0.0001265	0.0007670	0.0001188
0.45	0.0126113	0.0031073	0.0007482	0.0001887	0.0007025	0.0001772
0.5	0.0114955	0.0040611	0.0006833	0.0002456	0.0006415	0.0002306
0.55	0.0106086	0.0051473	0.0006220	0.0002978	0.0005839	0.0002797
0.6	0.0096166	0.0059645	0.0005636	0.0003465	0.0005291	0.0003254
0.65	0.0086672	0.0067379	0.0005079	0.0003926	0.0004768	0.0003687
0.7	0.0077588	0.0074873	0.0004545	0.0004373	0.0004267	0.0004107
0.75	0.0068860	0.0082326	0.0004033	0.0004818	0.0003786	0.0004525
0.8	0.0060468	0.0089966	0.0003541	0.0005274	0.0003324	0.0004953
0.85	0.0052409	0.0098031	0.0003070	0.0005755	0.0002882	0.0005406
0.9	0.0044676	0.0106778	0.0002618	0.0006278	0.0002458	0.0005896
0.95	0.0037303	0.0116487	0.0002188	0.0006857	0.0002054	0.0006441
1	0.0030334	0.0127456	0.0001782	0.0007512	0.0001673	0.0007056
	$\lvert\mu_1^* - \widetilde{\mu}_1\rvert = 0.0228016$		$\lvert\mu_1^* - \widetilde{\mu}_1\rvert = 0.0120684$		$\lvert\mu_1^* - \widetilde{\mu}_1\rvert = 0.0113552$	
	$\lvert\mu_2^* - \widetilde{\mu}_2\rvert = 0.0832155$		$\lvert\mu_2^* - \widetilde{\mu}_2\rvert = 0.0440641$		$\lvert\mu_2^* - \widetilde{\mu}_2\rvert = 0.0414742$	
	$\lvert\mu_3^* - \widetilde{\mu}_3\rvert = 0.0137594$		$\lvert\mu_3^* - \widetilde{\mu}_3\rvert = 0.0072853$		$\lvert\mu_3^* - \widetilde{\mu}_3\rvert = 0.0068567$	

method. Table 3 and Fig. 1 provide the comparative results obtained for the numerical solution $(\widetilde{x}(t), \widetilde{\mu})$ to problem (50),(51).

The error estimates obtained by using the three methods are as follows:

Adams method: $\|\mu^* - \widetilde{\mu}\| < 0.0832155$, $\max\limits_{j=\overline{0,10}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.0482514$;

Runge–Kutta method: $\|\mu^* - \widetilde{\mu}\| < 0.0440641$, $\max\limits_{j=\overline{0,10}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.0255510$;

Bulirsch–Stoer method: $\|\mu^* - \widetilde{\mu}\| < 0.0414742$, $\max\limits_{j=\overline{0,10}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.0240500$.

Case 2. Let $N = 2$. We perform the numerical algorithm by the partitioning the interval $[0, 1]$ with step size $h = 0.5$ and the subintervals $[0, 0.5]$, $[0.5, 1]$ with step size $h_1 = 0.05$. The results are presented in Table 4.

Case 3. Let us take $N = 4$ and perform the numerical algorithm for solving problem (50),(51). The four partition subintervals $[0, 0.25]$, $[0.25, 0.5]$, $[0.5, 0.75]$, $[0.75, 1]$ are in turn divided with step size $h_1 = 0.025$.

Again, we implement the algorithm three times using different methods for solving auxiliary Cauchy problems.

The results are presented in Table 5 and Fig. 2.

To compare the results, we obtain the following error estimates:

Adams method: $\|\mu^* - \widetilde{\mu}\| < 0.0309316$, $\max\limits_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.01793120$;

Runge–Kutta method: $\|\mu^* - \widetilde{\mu}\| < 0.0001708$, $\max\limits_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.000099$;

Bulirsch–Stoer method: $\|\mu^* - \widetilde{\mu}\| < 0.0001604$, $\max\limits_{j=\overline{0,40}} \|x^*(t_j) - \widetilde{x}(t_j)\| < 0.000093$.

Conclusion

The proposed computational method for solving problems with parameters for integro-differential equations is based on the parametrization method with the choice of a regular partition. The algorithm includes two auxiliary problems: the Cauchy problems for ordinary differential equations and the evaluation of definite integrals. The numerical solutions to the Cauchy problems were obtained by the Adams method, the fourth-order Runge–Kutta method, and the Bulirsch–Stoer method; the integrals were evaluated by Simpson's rule. If we use other numerical or approximate methods, we obtain a new numerical or approximate implementation of the algorithm. By choosing various regular partitions, we obtain a family of algorithms.

The proposed method can be extended to problems for impulsive integro-differential equations, integro-differential equations of mixed type, and fractional integro-differential equations. One of the possible options for the further development of the proposed computational method is its combination with computational methods for fractional dynamical systems Burgos et al. (2019), Harrat et al. (2018), Kim et al. (2020), Liu et al. (2018), Manimaran et al. (2019).

Table 5 Comparison of numerical solutions to problem (50),(51). Case 3

t	Adams method		Runge–Kutta method		Bulirsch–Stoer method	
	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
0	0.0179312	8.9927756	0.000099	8.9999601	0.000093	8.9999626
0.025	0.1206645	8.9966377	0.1035541	9.0031048	0.1035484	9.0031069
0.05	0.2229442	9.006817	0.2065257	9.012593	0.2065201	9.0125949
0.075	0.3242882	9.0234076	0.3085329	9.0285186	0.3085276	9.0285203
0.1	0.4242201	9.0465036	0.409101	9.0509753	0.4090959	9.0509768
0.125	0.5222723	9.076199	0.5077641	9.0800568	0.5077592	9.0800581
0.15	0.6179893	9.1125881	0.6040679	9.115857	0.6040632	9.1158581
0.175	0.7109304	9.1557651	0.6975728	9.1584695	0.6975683	9.1584704
0.2	0.8006721	9.2058242	0.7878565	9.2079881	0.7878522	9.2079889
0.225	0.8868103	9.2628597	0.8745165	9.2645067	0.8745123	9.2645072
0.25	0.9689639	9.3269659	0.9571724	9.3281188	0.9571684	9.3281192
0.275	1.0468294	9.3983148	1.0354689	9.3989183	1.0354651	9.3989185
0.3	1.1199735	9.4768458	1.1090774	9.4769989	1.1090737	9.476999
0.325	1.1881462	9.5627308	1.1776981	9.5624545	1.1776945	9.5624544
0.35	1.2510769	9.6560644	1.241062	9.6553786	1.2410586	9.6553784
0.375	1.3085282	9.7569412	1.2989327	9.7558652	1.2989295	9.7558648
0.4	1.3602965	9.8654559	1.3511075	9.8640079	1.3511043	9.8640074
0.425	1.4062146	9.9817032	1.3974187	9.9799005	1.3974157	9.9798999
0.45	1.4461494	10.105778	1.437735	10.1036368	1.4377321	10.103636
0.475	1.4800058	10.2377749	1.4719619	10.2353105	1.4719592	10.2353096
0.5	1.5077261	10.377789	1.5000426	10.3750153	1.50004	10.3750144
0.525	1.5293173	10.5259487	1.521958	10.5228451	1.5219555	10.5228441
0.55	1.5447462	10.6822821	1.5377271	10.6788936	1.5377247	10.6788924
0.575	1.5540941	10.846918	1.5474069	10.8432545	1.5474046	10.8432533
0.6	1.5574546	11.0199514	1.5510917	11.0160216	1.5510895	11.0160203
0.625	1.5549583	11.2014775	1.5489129	11.1972887	1.5489109	11.1972873
0.65	1.5467723	11.3915916	1.5410382	11.3871495	1.5410362	11.387148
0.675	1.5331006	11.5903893	1.5276702	11.5856978	1.5276683	11.5856962
0.7	1.5141784	11.7979659	1.5090453	11.7930273	1.5090436	11.7930256
0.725	1.4902741	12.0144167	1.4854327	12.0092318	1.4854311	12.00923
0.75	1.4616872	12.2398374	1.4571319	12.234405	1.4571304	12.2344032
0.775	1.4287557	12.4743399	1.4244717	12.4686408	1.4244702	12.4686389
0.8	1.3918163	12.7179873	1.3878073	12.7120329	1.387806	12.7120309
0.825	1.3512595	12.970892	1.3475192	12.964675	1.3475179	12.9646729
0.85	1.3074863	13.2331496	1.3040096	13.2266609	1.3040085	13.2266587
0.875	1.2609193	13.5048559	1.2577011	13.4980844	1.2577001	13.4980821
0.9	1.2119984	13.7861072	1.2090333	13.7790391	1.2090323	13.7790368
0.925	1.1611788	14.0769998	1.1584603	14.069619	1.1584594	14.0696165
0.95	1.1089262	14.3776294	1.1064481	14.3699178	1.1064473	14.3699152
0.975	1.0557158	14.6880923	1.0534715	14.6800291	1.0534707	14.6800264

Table 5 continued

t	Adams method		Runge–Kutta method		Bulirsch–Stoer method	
	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$	$\widetilde{x}_1(t)$	$\widetilde{x}_2(t)$
1	1.0020287	15.0084851	1.0000111	15.0000468	1.0000104	15.000044
	$\widetilde{\mu}_1 = 6.9915036$		$\widetilde{\mu}_1 = 6.9999532$		$\widetilde{\mu}_1 = 6.9999561$	
	$\widetilde{\mu}_2 = -4.0309316$		$\widetilde{\mu}_2 = -4.0001708$		$\widetilde{\mu}_2 = -4.0001604$	
	$\widetilde{\mu}_3 = 8.9948835$		$\widetilde{\mu}_3 = 8.9999718$		$\widetilde{\mu}_3 = 8.9999735$	

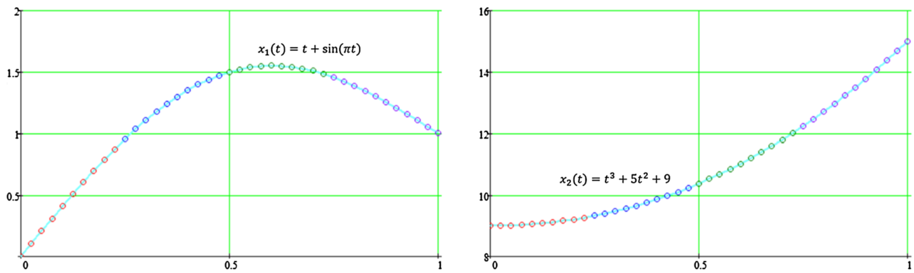

Fig. 2 The exact solution (light blue solid line) and the numerical solution values obtained by the Bulirsch–Stoer method ('o')

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