Computational and Applied Mathematics (2020) 39:237
https://doi.org/10.1007/s40314-020-01280-x

A new conservative finite difference scheme for the generalized Rosenau–KdV–RLW equation

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Received: 27 June 2019 / Revised: 13 May 2020 / Accepted: 29 July 2020 / Published online: 9 August 2020

Abstract
In this paper, a new conservative fourth-order finite difference scheme is proposed for solving the generalized Rosenau–KdV–RLW equation. The solvability, convergence, and conservation of the numerical solution are discussed by the discrete energy method. The scheme is convergent of $O(\tau^2 + h^4)$ and unconditionally stable. Several numerical experiment results show that the proposed scheme is efficient and reliable.

Keywords Rosenau–KdV–RLW equation · Conservative scheme · Discrete energy method · Unconditionally stable · Unique solvability

Mathematics Subject Classification 65M12 · 65N06

1 Introduction

Nonlinear wave phenomena play an important role in engineering and sciences. In the past, many scientists have studied about different mathematical models to explain the wave behavior, such as the KdV equation (Korteweg and Vries 1895; Ozer and Kutluay 2005; Skogestad and Kalisch 2009; Kim et al. 2012; Yan et al. 2016), the Rosenau equation (Rosenau 1986, 1988; Park 1992), the Rosenau–KdV equation (Zuo 2009; Esfahani 2011; Triki and Biswas 2013; Zheng and Zhou 2014), the Rosenau–RLW equation (Pan and Zhang 2012; Wongsaijai et al. 2014, 2019), and many others (Lu and Chen 2015; Coclite and Ruvob 2017; Mohanty and Kaur 2019; Kaur and Mohanty 2019).

Communicated by Corina Giurgea.

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In this paper, we will consider the following initial-boundary value problem of the generalized Rosenau–KdV–RLW equation (Razborova et al. 2015):

$$u_t + au_x + b(u^p)_x - cu_{xxt} + du_{xxx} + u_{xxxxt} = 0, \quad x \in [\alpha, \beta], \quad t \in [0, T], \tag{1.1}$$

with an initial condition

$$u(x, 0) = \phi(x), \quad x \in [\alpha, \beta], \tag{1.2}$$

and boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0, \quad u_x(\alpha, t) = u_x(\beta, t) = 0, \quad t \in [0, T], \tag{1.3}$$

where a, b, c and d are non-negative real constants, $p \geq 2$ is a positive integer, $\phi(x)$ is a given smooth function, $u(x, t)$ is a real-valued function.

For Eq. (1.1), shock waves, solitary waves, and the asymptotic behavior with power law nonlinearity have been theoretically studied in Razborova et al. (2014) and Sanchez et al. (2015). Besides the theoretical analysis, Wongsaijai and Poochinapan (2014) proposed a three-level average implicit finite difference scheme, and Wang and Dai (2018) developed a linearly implicit finite difference scheme. However, both the schemes in Wongsaijai and Poochinapan (2014) and Wang and Dai (2018) are only second-order accurate. As pointed out in Ghiloufi and Omrani (2017), the conservative approximation properties of the scheme have possibly even more impacts on numerical results. Thus, the motivation of this research is to establish a fourth-order conservative finite difference scheme for Eqs. (1.1)–(1.3).

Theorem 1.1 *Suppose $\phi(x) \in H_0^2[\alpha, \beta]$, then the problem in Eqs. (1.1)–(1.3) satisfies the following energy conservative property:*

$$\begin{aligned} E(t) &= \int_\alpha^\beta \Big[u^2(x,t) + cu_x^2(x,t) + u_{xx}^2(x,t)\Big]\mathrm{d}x = \|u\|_{L_2}^2 + c\|u_x\|_{L_2}^2 + \|u_{xx}\|_{L_2}^2 \\ &= \int_\alpha^\beta \Big[u^2(x,0) + cu_x^2(x,0) + u_{xx}^2(x,0)\Big]\mathrm{d}x \\ &= \int_\alpha^\beta \Big[(\phi(x))^2 + c(\phi(x))_x^2 + (\phi(x))_{xx}^2\Big]\mathrm{d}x = E(0), \quad c \geq 0, \quad t \in [0, T]. \end{aligned} \tag{1.4}$$

Proof From Eq. (1.1), we have

$$u_t - cu_{xxt} + u_{xxxxt} = -au_x - b(u^p)_x - du_{xxx}. \tag{1.5}$$

Multiplying Eq. (1.5) by $2u$ and integrating on the interval $[\alpha, \beta]$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta u^2\mathrm{d}x - 2c\int_\alpha^\beta uu_{xxt}\mathrm{d}x + 2\int_\alpha^\beta uu_{xxxxt}\mathrm{d}x \\ &= -2a\int_\alpha^\beta uu_x\mathrm{d}x - 2b\int_\alpha^\beta u(u^p)_x\mathrm{d}x - 2d\int_\alpha^\beta uu_{xxx}\mathrm{d}x. \end{aligned} \tag{1.6}$$

Using the integration by parts and considering the boundary conditions in Eq. (1.3), we have

$$\int_\alpha^\beta uu_x\mathrm{d}x = \frac{1}{2}u^2\Big|_\alpha^\beta = 0, \tag{1.7}$$

$$\int_\alpha^\beta uu_{xxx}\mathrm{d}x = \Big(uu_{xx} - \frac{1}{2}u_x^2\Big)\Big|_\alpha^\beta = 0, \tag{1.8}$$

$$\int_\alpha^\beta u(u^p)_x \mathrm{d}x = \left(uu^p - \frac{1}{p+1}u^{p+1}\right)\Big|_\alpha^\beta = 0, \tag{1.9}$$

$$\int_\alpha^\beta uu_{xxt}\mathrm{d}x = \int_\alpha^\beta ud(u_{xt}) = \left(uu_{xt}\right)\Big|_\alpha^\beta - \int_\alpha^\beta u_{xt}\mathrm{d}u = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta u_x^2\mathrm{d}x, \tag{1.10}$$

$$\int_\alpha^\beta uu_{xxxxt}\mathrm{d}x = \left(uu_{xxxt}\right)\Big|_\alpha^\beta - \int_\alpha^\beta u_{xxxt}\mathrm{d}u = -\left(u_xu_{xxt}\right)\Big|_\alpha^\beta + \int_\alpha^\beta u_{xxt}u_{xx}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta u_{xx}^2\mathrm{d}x. \tag{1.11}$$

Substituting Eqs. (1.7)–(1.11) into Eq. (1.6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\alpha^\beta \left[u^2(x,t) + cu_x^2(x,t) + u_{xx}^2(x,t)\right]\mathrm{d}x = 0.$$

Therefore, we obtain $E(t) = E(0)$, $t \in [0, T]$. □

Lemma 1.2 (Wang and Dai 2018) *Suppose $\phi(x) \in H_0^2[\alpha, \beta]$, then the solution of Eqs. (1.1)–(1.3) satisfies $\|u\|_{L_2} \le C$, $\|u_x\|_{L_2} \le C$, $\|u_{xx}\|_{L_2} \le C$, and hence $\|u\|_{L_\infty} \le C$, $\|u_x\|_{L_\infty} \le C$.*

Theorem 1.3 *Suppose $\phi(x) \in H_0^2[\alpha, \beta]$, then the problem in Eqs. (1.1)–(1.3) is well posed.*

Proof Assume that u_1 and u_2 are two solutions of Eqs. (1.1)–(1.3) satisfying the initial conditions $\phi^{(1)}$ and $\phi^{(2)}$, respectively. Let $\theta = u_1 - u_2$, then θ satisfies

$$\theta_t + a\theta_x + b[(u_1)^p]_x - b[(u_2)^p]_x - c\theta_{xxt} + d\theta_{xxx} + \theta_{xxxxt} = 0,$$

and the initial-boundary conditions:

$$\theta(x,0) = \phi^{(1)} - \phi^{(2)}, \quad x \in [\alpha, \beta],$$
$$\theta(\alpha,t) = \theta(\beta,t) = 0,\ \theta_x(\alpha,t) = \theta_x(\beta,t) = 0,\ t \in [0,T].$$

Letting

$$E(t) = \int_\alpha^\beta (\theta^2 + c\theta_x^2 + \theta_{xx}^2)\mathrm{d}x,\ c \ge 0,$$

we use a similar derivation as that in the proof of Theorem 1.1 and obtain

$$\begin{aligned}\frac{\mathrm{d}E(t)}{dt} &= 2\int_\alpha^\beta (\theta\theta_t + c\theta_x\theta_{xt} + \theta_{xx}\theta_{xxt})\mathrm{d}x \\ &= -2\int_\alpha^\beta \theta\left[a\theta_x + b[(u_1)^p]_x - b[(u_2)^p]_x + d\theta_{xxx}\right]\mathrm{d}x \\ &= 2b\int_\alpha^\beta \theta\left[[(u_2)^p]_x - [(u_1)^p]_x\right]\mathrm{d}x \\ &- \left[a\theta^2 + 2d\theta\theta_{xx} - d(\theta_x)^2\right]\Big|_\alpha^\beta \\ &= 2bp\int_\alpha^\beta \theta\left[(u_2)^{p-1}(u_2)_x - (u_1)^{p-1}(u_1)_x\right]\mathrm{d}x\end{aligned}$$

$$= -2bp\int_\alpha^\beta \theta\theta_x(u_2)^{p-1}\mathrm{d}x - 2bp\int_\alpha^\beta \theta\sum_{k=0}^{p-2}(u_2)^{p-2-k}(u_1)^k(u_1)_x\theta\mathrm{d}x. \quad (1.12)$$

By Lemma 1.2, we obtain

$$\Big|\int_\alpha^\beta \theta\theta_x(u_2)^{p-1}\mathrm{d}x\Big| \le C\int_\alpha^\beta |\theta|\cdot|\theta_x|\mathrm{d}x \le C\Big(\int_\alpha^\beta \theta^2\mathrm{d}x + \int_\alpha^\beta(\theta_x)^2\mathrm{d}x\Big),$$

$$\Big|\int_\alpha^\beta \theta\sum_{k=0}^{p-2}(u_2)^{p-2-k}(u_1)^k(u_1)_x\theta\mathrm{d}x\Big| \le C\int_\alpha^\beta \theta^2\mathrm{d}x,$$

where C is a constant. Substituting the above two inequalities into Eq. (1.12), we obtain $\frac{\mathrm{d}E(t)}{dt} \le CE(t), t\in[0,T]$. This leads to $E(t)\le e^{CT}E(0), 0\le t\le T$. Thus, if $\phi^{(1)} = \phi^{(2)}$, we have $\theta(x,0)=0$ and hence $E(0)=0$, implying that $E(t)=0$. By the Sobolev inequality, we obtain $\|\theta\|_{L_\infty}=0$ and $u_1=u_2$. Furthermore, if $\theta(x,0)<\varepsilon, \theta_x(x,0)<\varepsilon, \theta_{xx}(x,0)<\varepsilon$, we obtain $E(0)<\varepsilon$ and hence $E(t)\le e^{CT}E(0)\le \varepsilon e^{CT}$, $0\le t\le T$, implying that the solution is continuously dependent on the initial condition. We conclude that Eqs. (1.1)–(1.3) are well posed. □

The rest of this paper is arranged as follows: Sect. 2 gives the detailed description of the fourth-order finite difference scheme and its discrete conservative property for Eqs. (1.1)–(1.3). Section 3 provides complete proofs on the solvability, convergence and stability of the proposed scheme with the convergence order $O(\tau^2+h^4)$. Section 4 presents some numerical simulations to verify the theoretical analysis. Finally, concluding remarks are given in Sect. 5.

2 Difference scheme and its discrete conservative law

The solution domain $\{(x,t)|\alpha\le x\le\beta, 0\le t\le T\}$ is covered by a uniform grid $\{(x_j,t_n)|x_j=\alpha+jh,\ t_n=n\tau,\ j=0,\dots,J,\ n=0,\dots,N\}$, with spacing $h=(\beta-\alpha)/J$, $\tau=T/N$. Denote $U_j^n\approx u(x_j,t_n)$, and let

$$Z_h^0 = \{U=(U_j)|U_{-1}=U_0=U_1=U_{J-1}=U_J=U_{J+1}=0\}, \quad (2.1)$$

where $j=-1,0,1,\dots,J-1,J,J+1$. For convenience, the following notations will be introduced:

$$(U_j^n)_{\tilde{x}} = \frac{1}{h}(U_{j+1}^n - U_j^n),\quad (U_j^n)_{\bar{x}} = \frac{1}{h}(U_j^n - U_{j-1}^n),\quad (U_j^n)_{\hat{x}} = \frac{1}{2h}(U_{j+1}^n - U_{j-1}^n),$$

$$\bar{U}_j^n = \frac{1}{2}(U_j^{n+1}+U_j^{n-1}),\quad (U_j^n)_{\hat{t}} = \frac{1}{2\tau}(U_j^{n+1}-U_j^{n-1}),\quad (U_j^n)_{\tilde{t}} = \frac{1}{\tau}(U_j^{n+1}-U_j^n),$$

$$\langle U^n, V^n\rangle = h\sum_{j=1}^{J-1}U_j^nV_j^n,\quad \|U^n\|^2 = \langle U^n,U^n\rangle,\quad \|U^n\|_\infty = \max_{1\le j\le J-1}|U_j^n|.$$

By setting

$$w = -au_x - b(u^p)_x + cu_{xxt} - \mathrm{d}u_{xxx} - u_{xxxxt}, \quad (2.2)$$

Eq. (1.1) can be written as $w = u_t$. Using the Taylor expansion in the variable x, we obtain

$$w_j^n = -a\Big[(U_j^n)_{\hat{x}} - \frac{h^2}{6}(\partial_x^3 u)_j^n\Big] - b\Big[[(U_j^n)^p]_{\hat{x}} - \frac{h^2}{6}(\partial_x^3 u^p)_j^n\Big] + c\Big[(U_j^n)_{\bar{x}x\hat{t}} - \frac{h^2}{12}(\partial_x^4 \partial_t u)_j^n\Big]$$
$$- d\Big[(U_j^n)_{\dot{x}} - \frac{h^2}{6}(\partial_x^5 u)_j^n\Big] - \Big[(U_j^n)_{\bar{x}x\bar{x}x\hat{t}} - \frac{h^2}{6}(\partial_x^6 \partial_t u)_j^n\Big] + O(h^4), \tag{2.3}$$

where the fourth-order operator $(U_j^n)_{\dot{x}}$ is defined as follows (Wang and Dai 2018):

$$(U_j^n)_{\dot{x}} = -\frac{1}{24h^3}(U_{j+3}^n - U_{j-3}^n) + \frac{2}{3h^3}(U_{j+2}^n - U_{j-2}^n) - \frac{29}{24h^3}(U_{j+1}^n - U_{j-1}^n)$$
$$= u_j^{(3)} + \frac{h^2}{6}u_j^{(5)} + O(h^4). \tag{2.4}$$

From Eq. (2.2), we have

$$(\partial_x^6 \partial_t u)_j^n = -a(\partial_x^3 u)_j^n - b(\partial_x^3 u^p)_j^n + c(\partial_x^4 \partial_t u)_j^n - d(\partial_x^5 u)_j^n - (\partial_x^2 w)_j^n. \tag{2.5}$$

Substituting Eq. (2.5) into Eq. (2.3) gives

$$w_j^n = -a(U_j^n)_{\hat{x}} - b[(U_j^n)^p]_{\hat{x}} + c(U_j^n)_{\bar{x}x\hat{t}} - d(U_j^n)_{\dot{x}} - (U_j^n)_{\bar{x}x\bar{x}x\hat{t}}$$
$$+ \frac{ch^2}{12}(\partial_x^4 \partial_t u)_j^n - \frac{h^2}{6}(\partial_x^2 w)_j^n + O(h^4).$$

Using second-order accuracy for approximation, we obtain

$$U_j^n = \bar{U}_j^n + O(\tau^2), \quad w_j^n = (\partial_t u)_j^n = (U_j^n)_{\hat{t}} + O(\tau^2),$$
$$(\partial_x^2 w)_j^n = (W_j^n)_{\bar{x}x} + O(h^2), \quad (\partial_x^4 \partial_t u)_j^n = (U_j^n)_{\bar{x}x\bar{x}x\hat{t}} + O(h^2).$$

Thus, the proposed difference scheme for Eqs. (1.1)–(1.3) is written as

$$(U_j^n)_{\hat{t}} + a(\bar{U}_j^n)_{\hat{x}} + b[(\bar{U}_j^n)^p]_{\hat{x}} + d(\bar{U}_j^n)_{\dot{x}} + A(h)(U_j^n)_{\bar{x}x\bar{x}x\hat{t}} - B(h)(U_j^n)_{\bar{x}x\hat{t}} = 0, \tag{2.6}$$
$$A(h) = 1 - \frac{ch^2}{12}, \quad B(h) = c - \frac{h^2}{6}, \quad j = 2, \ldots, J-2, \quad n = 2, \ldots, N, \tag{2.7}$$
$$U_j^0 = \phi(x_j), \quad 0 \le j \le J, \tag{2.8}$$
$$U_0^n = U_J^n = 0, \; U_{-1}^n = U_1^n = 0, \; U_{J-1}^n = U_{J+1}^n = 0, \; n = 1, \ldots, N. \tag{2.9}$$

Since the scheme in Eqs. (2.6)–(2.9) is a three-level method, we need to give a two-level method to compute U^1, which is given by

$$(U_j^0)_{\tilde{t}} + a(U_j^{0.5})_{\hat{x}} + b[(U_j^{0.5})^p]_{\hat{x}} + d(U_j^{0.5})_{\dot{x}} + A(h)(U_j^0)_{\bar{x}x\bar{x}x\hat{t}} - B(h)(U_j^0)_{\bar{x}x\hat{t}} = 0, \tag{2.10}$$

where

$$U_j^{0.5} = (U_j^1 + U_j^0)/2, \quad j = 2, \ldots, J-2, \; n = 0, \ldots, N.$$

Lemma 2.1 (Hu et al. 2008; Ye et al. 2015) *For any two mesh functions $U, V \in Z_h^0$, we obtain*

$$\langle U_{\bar{x}}, V \rangle = -\langle U, V_{\bar{x}} \rangle, \langle U_{\hat{x}}, V \rangle = -\langle U, V_{\hat{x}} \rangle, \tag{2.11}$$
$$\langle U_{\bar{x}x}, V \rangle = -\langle U_{\bar{x}}, V_{\bar{x}} \rangle, \quad \langle U_{\bar{x}x}, U \rangle = -\|U_{\bar{x}}\|^2, \quad \langle U, U_{\bar{x}x\bar{x}x} \rangle = \|U_{\bar{x}x}\|^2. \tag{2.12}$$

Lemma 2.2 (Shao et al. 2013) *For any mesh function $U \in Z_h^0$, there exist two positive constants C_1 and C_2 such that*

$$\|U^n\|_\infty \le C_1\|U^n\| + C_2\|U^n_{\tilde{x}}\|. \tag{2.13}$$

Lemma 2.3 (Cai et al. 2015) *For any discrete function $U \in Z_h^0$, we have*

$$\|U^n_{\tilde{x}}\|^2 \le \frac{4}{h^2}\|U^n\|^2, \quad 0 \le n \le N. \tag{2.14}$$

Lemma 2.4 (Wongsaijai and Poochinapan 2014; Wang and Dai 2018) *For any mesh function $U \in Z_h^0$, we have*

$$\langle U_{\hat{x}}, U\rangle = 0, \quad \langle U_{\ddot{x}}, U\rangle = 0, \quad \langle U^p_{\hat{x}}, U\rangle = 0, \quad p \ge 2. \tag{2.15}$$

Theorem 2.5 *Suppose $\phi(x) \in H_0^2([\alpha, \beta])$, then the finite difference scheme in Eqs. (2.6)–(2.10) is conservative for discrete energy in sense:*

$$\begin{aligned} E^n &\equiv \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} + B(h)\Big[\frac{\|U^{n+1}_{\tilde{x}}\|^2 + \|U^n_{\tilde{x}}\|^2}{2}\Big] \\ &+ A(h)\Big[\frac{\|U^{n+1}_{\tilde{x}\bar{x}}\|^2 + \|U^n_{\tilde{x}\bar{x}}\|^2}{2}\Big] \\ &= \cdots = \|U^0\|^2 + B(h)\|U^0_{\tilde{x}}\|^2 + A(h)\|U^0_{\tilde{x}\bar{x}}\|^2 \equiv E^0, \quad n = 0, \ldots, N-1. \end{aligned} \tag{2.16}$$

Proof Taking the inner product of Eq. (2.6) with $2\bar{U}^n$, we obtain

$$\begin{aligned} &\langle U^n_{\hat{t}}, 2\bar{U}^n\rangle + A(h)\langle U^n_{\tilde{x}\bar{x}\tilde{x}\bar{x}\hat{t}}, 2\bar{U}^n\rangle - B(h)\langle U^n_{\tilde{x}\bar{x}\hat{t}}, 2\bar{U}^n\rangle \\ &+ a\langle \bar{U}^n_{\hat{x}}, 2\bar{U}^n\rangle + b\langle (\bar{U}^n)^p_{\hat{x}}, 2\bar{U}^n\rangle + d\langle \bar{U}^n_{\ddot{x}}, 2\bar{U}^n\rangle = 0. \end{aligned} \tag{2.17}$$

From Lemmas 2.1 and 2.4, we obtain

$$\begin{aligned} &\langle \bar{U}^n_{\hat{x}}, 2\bar{U}^n\rangle = 0, \quad \langle (\bar{U}^n)^p_{\hat{x}}, 2\bar{U}^n\rangle = 0, \quad \langle \bar{U}^n_{\ddot{x}}, 2\bar{U}^n\rangle = 0, \\ &\langle U^n_{\hat{t}}, 2\bar{U}^n\rangle = \|U^n\|^2_{\hat{t}}, \ \langle U^n_{\tilde{x}\bar{x}\hat{t}}, 2\bar{U}^n\rangle = -\|U^n_{\tilde{x}}\|^2_{\hat{t}}, \ \langle U^n_{\tilde{x}\bar{x}\tilde{x}\bar{x}\hat{t}}, 2\bar{U}^n\rangle = \|U^n_{\tilde{x}\bar{x}}\|^2_{\hat{t}}. \end{aligned}$$

Thus, Eq. (2.17) can be rewritten as

$$\|U^n\|^2_{\hat{t}} + A(h)\|U^n_{\tilde{x}\bar{x}}\|^2_{\hat{t}} + B(h)\|U^n_{\tilde{x}}\|^2_{\hat{t}} = 0.$$

This is equivalent to

$$\|U^{n+1}\|^2 + A(h)\|U^{n+1}_{\tilde{x}\bar{x}}\|^2 + B(h)\|U^{n+1}_{\tilde{x}}\|^2 = \|U^{n-1}\|^2 + A(h)\|U^{n-1}_{\tilde{x}\bar{x}}\|^2 + B(h)\|U^{n-1}_{\tilde{x}}\|^2,$$

where $n = 1, \ldots, N-1$. This further yields

$$\begin{aligned} E^n &\equiv \frac{\|U^{n+1}\|^2 + \|U^n\|^2}{2} + B(h)\Big[\frac{\|U^{n+1}_{\tilde{x}}\|^2 + \|U^n_{\tilde{x}}\|^2}{2}\Big] + A(h)\Big[\frac{\|U^{n+1}_{\tilde{x}\bar{x}}\|^2 + \|U^n_{\tilde{x}\bar{x}}\|^2}{2}\Big] \\ &= \frac{\|U^n\|^2 + \|U^{n-1}\|^2}{2} + B(h)\Big[\frac{\|U^n_{\tilde{x}}\|^2 + \|U^{n-1}_{\tilde{x}}\|^2}{2}\Big] + A(h)\Big[\frac{\|U^n_{\tilde{x}\bar{x}}\|^2 + \|U^{n-1}_{\tilde{x}\bar{x}}\|^2}{2}\Big] \\ &= E^{n-1} = \cdots = E^0, \quad n = 1, \ldots, N-1. \end{aligned} \tag{2.18}$$

Similarly, taking the inner product of Eq. (2.10) with $2u^{0.5}$, we obtain

$$\langle U^0_{\tilde{t}}, 2u^{0.5}\rangle + a\langle (U^{0.5}_{\hat{x}}), 2u^{0.5}\rangle + b\langle (U^{0.5})^p_{\hat{x}}, 2u^{0.5}\rangle + d\langle U^{0.5}_{\ddot{x}}, 2u^{0.5}\rangle + A(h)\langle U^0_{\tilde{x}\bar{x}\tilde{x}\bar{x}\hat{t}}, 2u^{0.5}\rangle - B(h)\langle U^0_{\tilde{x}\bar{x}\hat{t}}, 2u^{0.5}\rangle = 0. \tag{2.19}$$

From Lemmas 2.1 and 2.4, we obtain

$$\langle (U^{0.5}_{\hat{x}}), 2u^{0.5}\rangle = 0, \quad \langle (U^{0.5})^p_{\hat{x}}, 2u^{0.5}\rangle = 0, \quad \langle U^{0.5}_{\ddot{x}}, 2u^{0.5}\rangle = 0, \tag{2.20}$$

$$\langle U^0_{\tilde{t}}, 2u^{0.5}\rangle = \|U^0\|^2_{\tilde{t}}, \quad \langle U^0_{\tilde{x}\bar{x}\tilde{t}}, 2U^{0.5}\rangle = -\|U^0_{\tilde{x}}\|^2_{\tilde{t}}, \quad \langle U^0_{\tilde{x}\bar{x}\tilde{x}\bar{x}\tilde{t}}, 2U^{0.5}\rangle = \|U^0_{\tilde{x}\bar{x}}\|^2_{\tilde{t}}. \tag{2.21}$$

Substituting Eqs. (2.20)–(2.21) into Eq. (2.19), we obtain

$$\|U^0\|^2_{\tilde{t}} + A(h)\|U^0_{\tilde{x}\bar{x}}\|^2_{\tilde{t}} + B(h)\|U^0_{\tilde{x}}\|^2_{\tilde{t}} = 0.$$

This is equivalent to

$$\|U^1\|^2 + A(h)\|U^1_{\tilde{x}\bar{x}}\|^2 + B(h)\|U^1_{\tilde{x}}\|^2 = \|U^0\|^2 + A(h)\|U^0_{\tilde{x}\bar{x}}\|^2 + B(h)\|U^0_{\tilde{x}}\|^2.$$

Thus, from the above equation and the definition of E^0, we have

$$E^0 = \frac{\|U^1\|^2 + \|U^0\|^2}{2} + B(h)\left[\frac{\|U^1_{\tilde{x}}\|^2 + \|U^0_{\tilde{x}}\|^2}{2}\right] + A(h)\left[\frac{\|U^1_{\tilde{x}\bar{x}}\|^2 + \|U^0_{\tilde{x}\bar{x}}\|^2}{2}\right] = \|U^0\|^2 + B(h)\|U^0_{\tilde{x}}\|^2 + A(h)\|U^0_{\tilde{x}\bar{x}}\|^2.$$

□

Theorem 2.6 *Suppose $\phi(x) \in H^2_0([\alpha, \beta])$, then the solution U^n of Eqs. (2.6)–(2.10) satisfies $\|U^n\| \le C$, $\|U^n_{\tilde{x}}\| \le C$, $\|U^n_{\tilde{x}\bar{x}}\| \le C$, which yield*

$$\|U^n\|_\infty \le C, \quad \|U^n_{\tilde{x}}\|_\infty \le C, \quad \|U^n_{\tilde{x}}\|_\infty \le C, \quad \|U^n_{\hat{x}}\|_\infty \le C, \quad 0 \le n \le N.$$

Proof We prove the theorem by the mathematical induction. From Eq. (2.8), we obtain $\|U^0\| \le C$. We assume that

$$\|U^k\| \le C, \quad \|U^k\|_\infty \le C, \quad k = 0, 1, 2, \ldots, n.$$

From Lemma 2.3, we obtain

$$\|U^{n+1}_{\tilde{x}}\|^2 \le \frac{4}{h^2}\|U^{n+1}\|^2, \quad \|U^{n+1}_{\tilde{x}\bar{x}}\|^2 \le \frac{4}{h^2}\|U^{n+1}_{\tilde{x}}\|^2,$$

which yield

$$\begin{aligned} &\|U^{n+1}\|^2 + B(h)\|U^{n+1}_{\tilde{x}}\|^2 + A(h)\|U^{n+1}_{\tilde{x}\bar{x}}\|^2 \\ &\ge \|U^{n+1}\|^2 + c\|U^{n+1}_{\tilde{x}}\|^2 + \|U^{n+1}_{\tilde{x}\bar{x}}\|^2 - \frac{h^2}{6}\frac{4}{h^2}\|U^{n+1}\|^2 \\ &\quad - \frac{ch^2}{12}\frac{4}{h^2}\|U^{n+1}_{\tilde{x}}\|^2 \\ &\ge \frac{1}{3}\|U^{n+1}\|^2 + \frac{2c}{3}\|U^{n+1}_{\tilde{x}}\|^2 + \|U^{n+1}_{\tilde{x}\bar{x}}\|^2 \ge 0, \quad c \ge 0, \end{aligned} \tag{2.22}$$

implying that $E^{n+1} \geq 0$. From Eqs. (2.16) and (2.22), we obtain $\|U^{n+1}\| \leq C$, $\|U_{\tilde{x}}^{n+1}\| \leq C$, $\|U_{\tilde{x}\bar{x}}^{n+1}\| \leq C$. By Lemma 2.2, we further obtain

$$\|U^{n+1}\|_\infty \leq C, \quad \|U_{\tilde{x}}^{n+1}\|_\infty \leq C, \quad \|U_{\bar{x}}^{n+1}\|_\infty \leq C, \quad \|U_{\hat{x}}^{n+1}\|_\infty \leq C,$$

which completes the proof. □

3 Solvability, convergence and stability

Theorem 3.1 *The finite difference scheme in Eqs.* (2.6)–(2.10) *is uniquely solvable.*

Proof Using the mathematical induction, we can determine U^0 uniquely by Eq. (2.8) and choose Eq. (2.10) to compute U^1. Assume that $U^{1(\gamma_1)}$ and $U^{1(\gamma_2)}$ are two solutions of Eq. (2.10) and let $U^{1(\gamma)} = U^{1(\gamma_1)} - U^{1(\gamma_2)}$, then $U^{1(\gamma)}$ satisfies the following equation:

$$\frac{1}{\tau}U_j^{1(\gamma)} + \frac{a}{2}(U_j^{1(\gamma)})_{\hat{x}} + \frac{\mathrm{d}}{2}(U_j^{1(\gamma)})_{\dot{\hat{x}}} - \frac{1}{\tau}B(h)(U_j^{1(\gamma)})_{\tilde{x}\bar{x}} + \frac{1}{\tau}A(h)(U_j^{1(\gamma)})_{\tilde{x}\bar{x}\tilde{x}\bar{x}} = 0, \tag{3.1}$$

where $j = 2, \ldots, J-2$. By taking an inner product on both sides of Eq. (3.1) with $U^{1(\gamma)}$, we have

$$\|U^{1(\gamma)}\|^2 + B(h)\|U_{\tilde{x}}^{1(\gamma)}\|^2 + A(h)\|U_{\tilde{x}\bar{x}}^{1(\gamma)}\|^2 = 0. \tag{3.2}$$

From Theorem 2.6, we get

$$\begin{aligned}&\|U^{1(\gamma)}\|^2 + B(h)\|U_{\tilde{x}}^{1(\gamma)}\|^2 + A(h)\|U_{\tilde{x}\bar{x}}^{1(\gamma)}\|^2 \\ &\quad \geq \frac{1}{3}\|U^{1(\gamma)}\|^2 + \frac{2c}{3}\|U_{\tilde{x}}^{1(\gamma)}\|^2 + \|U_{\tilde{x}\bar{x}}^{1(\gamma)}\|^2 \geq 0.\end{aligned} \tag{3.3}$$

Thus, from Eqs. (3.2) and (3.3), we obtain

$$\|U^{1(\gamma)}\|^2 = 0, \quad \|U_{\tilde{x}}^{1(\gamma)}\|^2 = 0, \quad \|U_{\tilde{x}\bar{x}}^{1(\gamma)}\|^2 = 0.$$

Therefore, Eq. (3.1) has the only one solution and U^1 is uniquely solvable. Now, suppose $U^0, U^1, \ldots, U^n$ to be solved uniquely. By consider the homogeneous Eq. (2.6) for U^{n+1}, we have

$$\begin{aligned}&\frac{1}{\tau}U_j^{n+1} - \frac{1}{\tau}B(h)(U_j^{n+1})_{\tilde{x}\bar{x}} + \frac{1}{\tau}A(h)(U_j^{n+1})_{\tilde{x}\bar{x}\tilde{x}\bar{x}} \\ &\quad + a(U_j^{n+1})_{\hat{x}} + b[(U_j^{n+1})^p]_{\hat{x}} + d(U_j^{n+1})_{\dot{\hat{x}}} = 0.\end{aligned} \tag{3.4}$$

Computing an inner product of Eq. (3.4) with U^{n+1}, we obtain

$$\|U^{n+1}\|^2 + A(h)\|U_{\tilde{x}\bar{x}}^{n+1}\|^2 + B(h)\|U_{\tilde{x}}^{n+1}\|^2 = 0. \tag{3.5}$$

This together with Eq. (2.22) gives

$$\|U^{n+1}\|^2 = 0, \quad \|U_{\tilde{x}}^{n+1}\|^2 = 0, \quad \|U_{\tilde{x}\bar{x}}^{n+1}\|^2 = 0. \tag{3.6}$$

Therefore, Eq. (3.4) has the only one solution and U^{n+1} is uniquely solvable. □

Theorem 3.2 *Suppose $\phi(x) \in H_0^2([\alpha, \beta])$, then the solution U^n converges to the solution u^n in the sense of $\|\cdot\|_\infty$, and the convergence rate is $O(\tau^2 + h^4)$.*

Proof Let $e_j^n = u_j^n - U_j^n$, then the truncation error can be obtained as follows:

$$R_j^n = (e_j^n)_{\hat{t}} - B(h)(e_j^n)_{\bar{x}x\hat{t}} + a(\bar{e}_j^n)_{\hat{x}} + d(\bar{e}_j^n)_{\ddot{x}} + A(h)(e_j^n)_{\bar{x}x\bar{x}x\hat{t}} + b[(\bar{u}_j^n)^p - (\bar{U}_j^n)^p]_{\hat{x}}, \tag{3.7}$$

where $R_j^n = O(\tau^2 + h^4)$. By taking an inner product on both sides of Eq. (3.7) with $2\bar{e}_j^n$, we have

$$\begin{aligned}
&(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + B(h)(\|e_{\tilde{x}}^{n+1}\|^2 - \|e_{\tilde{x}}^{n-1}\|^2)\\
&\quad + A(h)(\|e_{\tilde{x}\bar{x}}^{n+1}\|^2 - \|e_{\tilde{x}\bar{x}}^{n-1}\|^2)\\
&= 2\tau\langle R^n, 2\bar{e}^n\rangle - 2\tau\langle a(\bar{e}^n)_{\hat{x}} + d(\bar{e}^n)_{\ddot{x}}, 2\bar{e}^n\rangle\\
&\quad - 2\tau\langle b[(\bar{u}^n)^p - (\bar{U}^n)^p]_{\hat{x}}, 2\bar{e}^n\rangle.
\end{aligned} \tag{3.8}$$

According to Lemmas 1.2, 2.1 and Theorem 2.6, we obtain

$$\begin{aligned}
\langle[(\bar{u}^n)^p - (\bar{U}^n)^p]_{\hat{x}}, 2\bar{e}^n\rangle &= -h\sum_{j=1}^{J-1}\left\{\left[(\bar{u}_j^n)^p - (\bar{U}_j^n)^p\right]\cdot 2(\bar{e}_j^n)_{\hat{x}}\right\}\\
&= -h\sum_{j=1}^{J-1}\left\{\sum_{k=1}^{p-1}\left[(\bar{u}_j^n)^{p-k}(\bar{U}_j^n)^k(\bar{e}_j^n)\right]\cdot 2(\bar{e}_j^n)_{\hat{x}}\right\}\\
&\le C(\|e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e_{\tilde{x}}^{n-1}\|^2 + \|e_{\tilde{x}}^{n+1}\|^2).
\end{aligned} \tag{3.9}$$

From Lemmas 2.1, 2.3 and 2.4, we obtain

$$\langle\bar{e}_{\hat{x}}^n, 2\bar{e}^n\rangle = 0, \quad \langle\bar{e}_{\ddot{x}}^n, 2\bar{e}^n\rangle = 0, \tag{3.10}$$

$$\|e_{\hat{x}}^n\|^2 \le \|e_{\tilde{x}}^n\|^2 \le \frac{1}{2}\|e^n\|^2 + \frac{1}{2}\|e_{\tilde{x}\bar{x}}^n\|^2, \tag{3.11}$$

$$\langle R^n, 2\bar{e}^n\rangle \le \|R^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2). \tag{3.12}$$

Substituting Eqs. (3.9)–(3.12) into Eq. (3.8) gives

$$\begin{aligned}
&(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + B(h)(\|e_{\tilde{x}}^{n+1}\|^2 - \|e_{\tilde{x}}^{n-1}\|^2) + A(h)(\|e_{\tilde{x}\bar{x}}^{n+1}\|^2 - \|e_{\tilde{x}\bar{x}}^{n-1}\|^2)\\
&\le 2\tau\|R^n\|^2 + C\tau(\|e_{\tilde{x}\bar{x}}^{n+1}\|^2 + \|e_{\tilde{x}\bar{x}}^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2).
\end{aligned} \tag{3.13}$$

Setting

$$\Lambda^n \equiv \|e^n\|^2 + \|e^{n-1}\|^2 + B(h)(\|e_{\tilde{x}}^n\|^2 + \|e_{\tilde{x}}^{n-1}\|^2) + A(h)(\|e_{\tilde{x}\bar{x}}^n\|^2 + \|e_{\tilde{x}\bar{x}}^{n-1}\|^2),$$

then we have from Eq. (2.22) that

$$\Lambda^{n+1} \ge \frac{1}{3}(\|e^{n+1}\|^2 + \|e^n\|^2) + \frac{2c}{3}(\|e_{\tilde{x}}^{n+1}\|^2 + \|e_{\tilde{x}}^n\|^2) + (\|e_{\tilde{x}\bar{x}}^{n+1}\|^2 + \|e_{\tilde{x}\bar{x}}^n\|^2) \ge 0.$$

And, Eq. (3.13) can be rewritten as

$$\Lambda^{n+1} - \Lambda^n \le 2\tau\|R^n\|^2 + C\tau(\Lambda^{n+1} + \Lambda^n).$$

Hence, we obtain

$$(1 - C\tau)(\Lambda^{n+1} - \Lambda^n) \le 2\tau\|R^n\|^2 + 2C\tau\Lambda^n.$$

Table 1 Comparison of errors with $\tau = h$ at $T = 20$

Norm	Scheme	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.025$
L_2	Hu et al. (2013)	–	3.045414E−03	7.631169E−04	1.905450E−04
	Present	7.829724E−03	1.946370E−03	4.860569E−04	1.215484E−04
L_∞	Hu et al. (2013)	–	1.131442E−03	2.835874E−04	7.097948E−05
	Present	3.034254E−03	7.533525E−04	1.880891E−04	4.701348E−05

If τ is sufficiently small, which satisfies $1 - C\tau > 0$, then we obtain

$$\Lambda^{n+1} - \Lambda^n \leq C\tau \|R^n\|^2 + C\tau \Lambda^n. \tag{3.14}$$

Summarizing Eq. (3.14) from 1 to n, we get

$$\Lambda^{n+1} \leq \Lambda^1 + C\tau \sum_{k=1}^{n} \|R^k\|^2 + C\tau \sum_{k=1}^{n} \Lambda^k.$$

Since $e^0 = 0$ and Eq. (2.10) is used to compute U^1, we obtain

$$e^0 = 0, \quad \Lambda^1 = O(\tau^2 + h^4)^2,$$

and notice that

$$\tau \sum_{k=1}^{n} \|R^k\|^2 \leq n\tau \max_{1\leq k\leq n} \|R^k\|^2 \leq T \cdot O(\tau^2 + h^4)^2,$$

we have

$$\Lambda^{n+1} \leq O(\tau^2 + h^4)^2 + C\tau \sum_{k=1}^{n} \Lambda^k.$$

From discrete Gronwall's inequality Wang et al. (2019), we obtain $\Lambda^n \leq O(\tau^2 + h^4)^2$, implying

$$\|e^{n+1}\|^2 \leq O(\tau^2 + h^4)^2, \quad \|e^{n+1}_{\tilde{x}\bar{x}}\|^2 \leq O(\tau^2 + h^4)^2. \tag{3.15}$$

Furthermore, it follows from Eqs. (3.11), (3.15) and Lemma 2.2 that

$$\|e^{n+1}_{\tilde{x}}\| \leq O(\tau^2 + h^4), \quad \|e^{n+1}\|_\infty \leq O(\tau^2 + h^4). \tag{3.16}$$

This completes the proof. □

Theorem 3.3 *Under the conditions of Theorem 3.2, the solution U^n of Eqs.* (2.6)–(2.10) *is unconditionally stable in norm* $\|\cdot\|_\infty$.

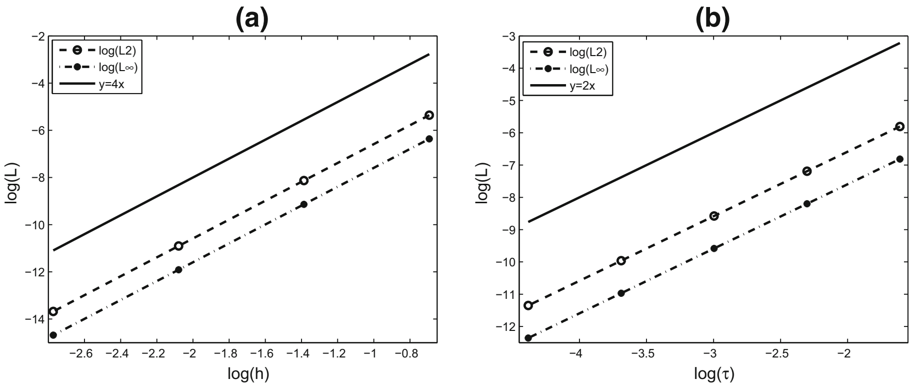

Fig. 1 The spatial and temporal convergence orders

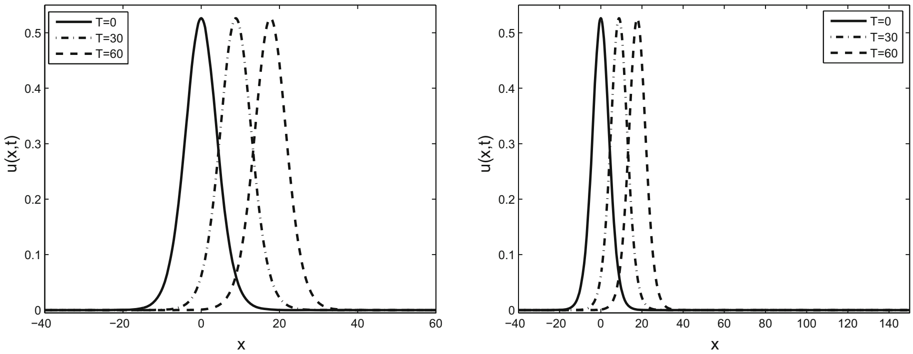

Fig. 2 Numerical solutions of the Rosenau–KdV equation with $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 60$ (left) and $\alpha = -40$, β=150 (right)

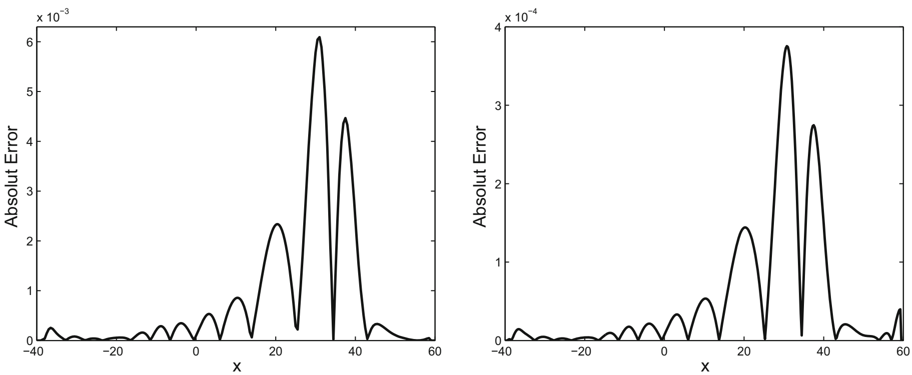

Fig. 3 Absolute error distribution at $T = 30$ with $\tau = h/2$, $\tau = h^2$, $h = 0.5$ (left) and $h = 0.25$ (right)

Table 2 Comparison of errors at $T = 40$ with $\tau = 0.1$, $h = 0.25$

Norm	Scheme	$p = 2$	$p = 4$	$p = 8$	$p = 16$
L_2	Pan and Zhang (2012)	7.87770E−03	1.73066E−02	1.80583E−02	1.37857E−02
	Wongsaijai and Poochinapan (2014)	2.36080E−03	4.72540E−03	4.67130E−03	3.84380E−02
	Present	1.99758E−03	3.92115E−03	3.54784E−03	4.20535E−03
L_∞	Pan and Zhang (2012)	2.88972E−03	6.47969E−03	6.66740E−03	5.05919E−03
	Wongsaijai and Poochinapan (2014)	8.86700E−04	1.81252E−03	1.75739E−03	1.30630E−03
	Present	7.61239E−04	1.53212E−03	1.37125E−03	9.37760E−04

4 Numerical experiments

In this section, we choose numerical experiments to verify the correctness of our theoretical analysis results. The L_∞ and L_2 error norms of the solution obtained from Eqs. (2.6)–(2.10) are defined as

$$L_2 = \|e^n\|_2 = \left[h \sum_{j=1}^{J-1} |e_j^n|^2\right]^{\frac{1}{2}}, \quad L_\infty = \|e^n\|_\infty = \max_{1 \le j \le J-1} |e_j^n|.$$

Example 1 Consider the Rosenau–KdV equation in the case of $a = 1, b = 0.5, c = 0, d = 1$ and $p = 2$ as follows (Wongsaijai and Poochinapan 2014):

$$u_t + u_x + 0.5(u^2)_x + u_{xxx} + u_{xxxxt} = 0, \quad \alpha \le x \le \beta, \quad t \in [0, T], \tag{4.1}$$

subject to the initial condition

$$u(x, 0) = \phi(x) = 35\Big(\frac{\sqrt{313}}{312} - \frac{1}{24}\Big) \operatorname{sech}^4 \left[\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}x\right], \quad \alpha \le x \le \beta,$$

and the boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0, \quad u_x(\alpha, t) = u_x(\beta, t) = 0.$$

The analytical solution is

$$u(x, t) = 35\Big(\frac{\sqrt{313}}{312} - \frac{1}{24}\Big) \operatorname{sech}^4 \left\{\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}\Big[x - \frac{1}{2}(1 + \frac{\sqrt{313}}{13})t\Big]\right\}. \tag{4.2}$$

First, using different h, $x \in [-70, 100]$, $T = 20$ and $\tau = h$, the comparison of error results was listed in Table 1. As seen, the results from the present scheme are more accurate than that obtained by the scheme in Hu et al. (2013). Then the spatial and temporal convergence orders for U^n at $T = 10$ with different space and time steps were drawn in Fig. 1, where $h = 0.5, 0.25, 0.125, 0.0625$, $\tau = h^2$ in Fig. 1a, and $\tau = 0.2, 0.1, 0.05, 0.025, 0.0125$, $h = \sqrt{\tau}$ in Fig. 1b. From Fig. 1, the convergence rate $O(\tau^2 + h^4)$ is verified. Furthermore, the solution profiles were plotted in Fig. 2 using $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 60$, 150. The solitons at $t = 30$ and 60 are in excellent agreement with the solitons at $t = 0$. Finally, Fig. 3 shows absolute errors at $T = 30$ with $h = 1/2$, 1/4 and $\tau = h^2$. From Fig. 3, we can see that the maximum errors are around orders of 10^{-3} and 10^{-4}, respectively. The above results indicate that our scheme in Eqs. (2.6)–(2.10) can be applied to simulate solitary propagations.

Table 3 Comparison of rates of convergence and CPU time at $T = 40$ with $p = 4$, $h = 0.5$ and $\tau = h^2$

Norm	Scheme	τ, h	$\tau/4, h/2$	$\tau/16, h/4$
L_2	Pan and Zhang (2012)	6.41825E−02	1.85385E−02	4.79643E−03
	Rate	–	1.79165	1.95050
L_∞	Pan and Zhang (2012)	2.38960E−02	6.96030E−03	1.80409E−03
	Rate	–	1.77955	1.94788
	CPU time	1.252	13.534	157.561
L_2	Present	3.20547E−02	1.97080E−03	1.23081E−04
	Rate	–	4.02369	4.00110
L_∞	Present	1.22483E−02	7.52289E−04	4.69771E−05
	Rate	–	4.02515	4.00126
	CPU time	0.125	2.277	51.854

Table 4 Comparison of rates of convergence and CPU time at $T = 40$ with $p = 8$, $h = 0.5$ and $\tau = h^2$

Norm	Scheme	τ, h	$\tau/4, h/2$	$\tau/16, h/4$
L_2	Pan and Zhang (2012)	6.44908E−02	1.99919E−02	5.25426E−03
	Rate	–	1.68968	1.92785
L_∞	Pan and Zhang (2012)	2.35870E−02	7.39615E−03	1.94938E−03
	Rate	–	1.67314	1.92376
	CPU time	1.371416	14.862871	175.068007
L_2	Present	3.18079E−02	1.94289E−03	1.22300E−04
	Rate	–	4.03311	3.98971
L_∞	Present	1.19512E−02	7.27869E−04	4.54605E−05
	Rate	–	4.03734	4.00099
	CPU time	0.156	2.621	54.522

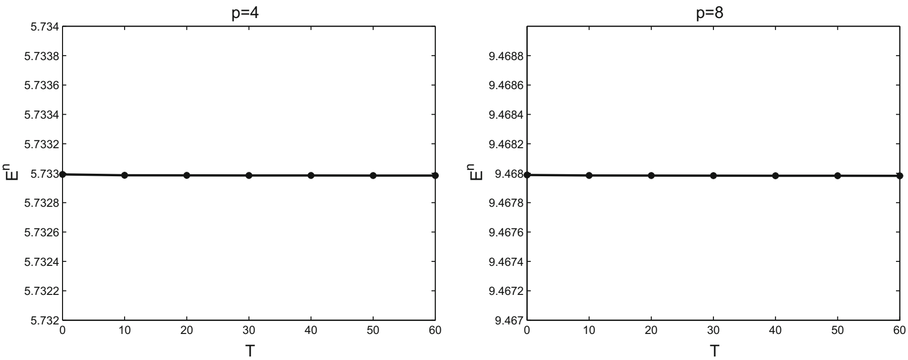

Fig. 4 Discrete energy by the present scheme with $h = 0.25$, $\tau = h^2$, $p = 4$ (left) and $p = 8$ (right)

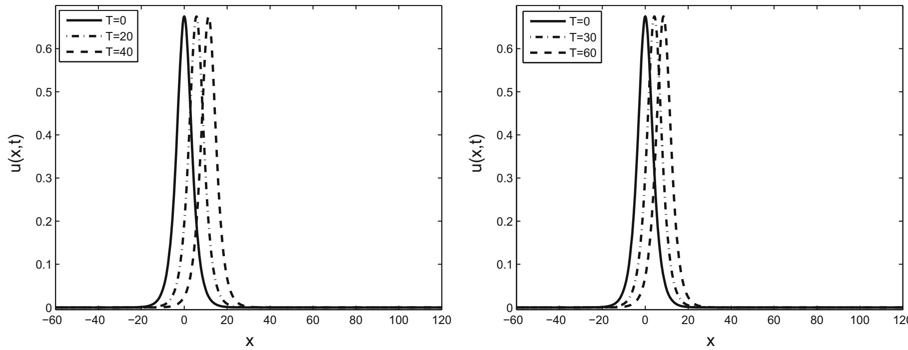

Fig. 5 Numerical solutions of the Rosenau–RLW equation with $p = 4, \tau = h^2, \alpha = -60, \beta = 120, h = 0.25$ (left) $h = 0.125$ (right)

Example 2 Consider the generalized Rosenau–RLW equation in the case of $a = 1$, $b = 1$, $c = 1$, $d = 0$ and $p \geq 2$ as follows

$$u_t + u_x + (u^p)_x - u_{xxt} + u_{xxxxt} = 0, \quad -60 \leq x \leq 120, \quad t \in [0, 40], \tag{4.3}$$

subject to the initial condition

$$u(x, 0) = \phi(x) = \exp\left[\frac{1}{p-1} \ln \frac{(p+1)(3p+1)(p+3)}{2(p^2+3)(p^2+4p+7)}\right] \operatorname{sech}^{\frac{4}{p-1}}(k_1 x),$$

and the boundary conditions

$$u(-60, t) = u(120, t) = 0, \quad u_x(-60, t) = u_x(120, t) = 0.$$

The exact solitary wave solution is

$$u(x, t) = \exp\left[\frac{1}{p-1} \ln \frac{(p+1)(3p+1)(p+3)}{2(p^2+3)(p^2+4p+7)}\right] \operatorname{sech}^{\frac{4}{p-1}}[k_1(x - k_2 t)],$$

where

$$k_1 = \frac{p-1}{\sqrt{4p^2+8p+20}}, \quad k_2 = \frac{p^4+4p^3+14p^2+20p+25}{p^4+4p^3+10p^2+12p+21}, \quad p \geq 2.$$

First, we made a comparison between our scheme and the scheme in Pan and Zhang (2012); Wongsaijai and Poochinapan (2014). The results in term of errors at $T = 40$, and different p were listed in Tables 2, 3 and 4. One may see that the computational efficiency of the present scheme is clearly better than the ones obtained by the schemes in Pan and Zhang (2012); Wongsaijai and Poochinapan (2014). Then the conservative invariant E^n at different times $t \in [0, 60]$ was listed in Table 5, where $p = 4, 8$, $h = 0.25$ and $\tau = h^2$. We also showed the conservative law of discrete energy E^n in Fig. 4. The obtained results in Table 5 and Fig. 4 testify that the present scheme is conservative for energy, which coincides with the theory. Finally, we simulated the wave graph of the numerical solution of Eq. (4.3). The wave graph comparison of numerical solutions obtained using $h = 0.25, 0.125$, $\tau = h^2$ at various times are given in Figs. 5 and 6 for $p = 4$ and $p = 8$, respectively. From Figs. 5 and 6, we can see that the heights of the wave graph at different times are almost identical, which implies that the energy computed by the scheme is conservative.

Table 5 Discrete energy E^n of the scheme with $p = 4, 8$, $h = 0.25$, $\tau = h^2$

T	$p = 4$	$p = 8$	T	$p = 4$	$p = 8$
0	5.73299163669502	9.46798776268510	40	5.73298452807903	9.46798312378886
10	5.73298584179234	9.46798468168063	50	5.73298429973518	9.46798257522380
20	5.73298526560656	9.46798415838727	60	5.73298412584239	9.46798199389881
30	5.73298483514227	9.46798364683599			

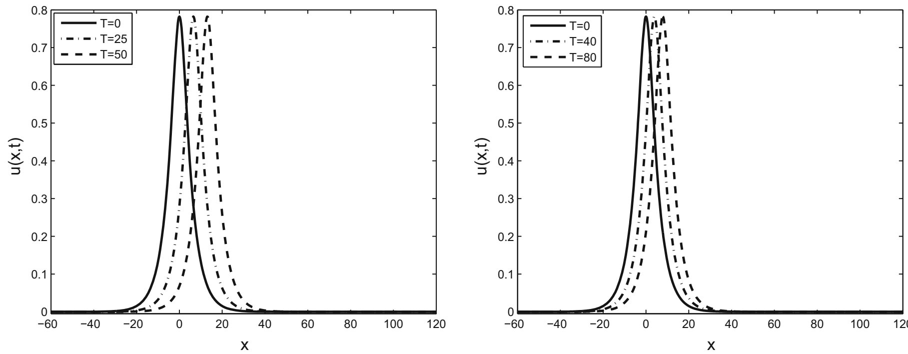

Fig. 6 Numerical solutions of the Rosenau–RLW equation with $p = 8$, $\tau = h^2$, $\alpha = -60$, $\beta = 120$, $h = 0.25$ (left) $h = 0.125$ (right)

Example 3 Consider the Rosenau–KdV–RLW equation in the case of $a = 1$, $b = 0.5$, $c = 1$, $d = 1$ and $p = 2$ as follows:

$$u_t + u_x + 0.5(u^2)_x - u_{xxt} + u_{xxx} + u_{xxxxt} = 0, \quad \alpha \le x \le \beta, \quad t \in [0, T], \tag{4.4}$$

subject to the initial condition

$$u(x, 0) = \phi(x) = -\frac{5}{456}\left(25 - 13\sqrt{457}\right)\operatorname{sech}^4(k_3 x), \quad \alpha \le x \le \beta,$$

and boundary conditions

$$u(\alpha, t) = u(\beta, t) = 0, \quad u_x(\alpha, t) = u_x(\beta, t) = 0.$$

The exact solitary wave solution is

$$u(x, t) = -\frac{5}{456}\left(25 - 13\sqrt{457}\right)\operatorname{sech}^4(k_3 x)[k_3(x - k_4 t)],$$

where

$$k_3 = \left(\frac{-13 + \sqrt{457}}{288}\right)^{1/2}, \; k_4 = \frac{241 + 13\sqrt{457}}{266}.$$

First, we compared the errors at $T = 30$ and different h, using $\alpha = -40$ and $\beta = 100$ as reported in Table 6. It is clear that the errors obtained by the present scheme are slightly smaller than the ones obtained by the method in Wongsaijai and Poochinapan (2014). Then we drew the absolute errors distributions with $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 160$ at $T = 30, 60$ in Fig. 7. We found that the maximum error obtained by the present scheme takes place around the peak amplitude of solitary waves. At last, the curves of the numerical

Table 6 Comparison of errors at $T = 30$ with $\tau = h$

h	$\|e\|$ Wongsaijai and Poochinapan (2014)	$\|e\|$ Present	$\|e\|_\infty$ Wongsaijai and Poochinapan (2014)	$\|e\|_\infty$ Present
0.5	2.94337E+00	2.73398E+00	1.08501E+00	1.07489E+00
0.25	8.05629E−01	5.53416E−01	3.00424E−01	2.18886E−01
0.125	2.05276E−01	1.32228E−01	7.66547E−02	5.22006E−02
0.0625	5.15696E−02	3.26985E−02	1.92614E−02	1.28993E−02

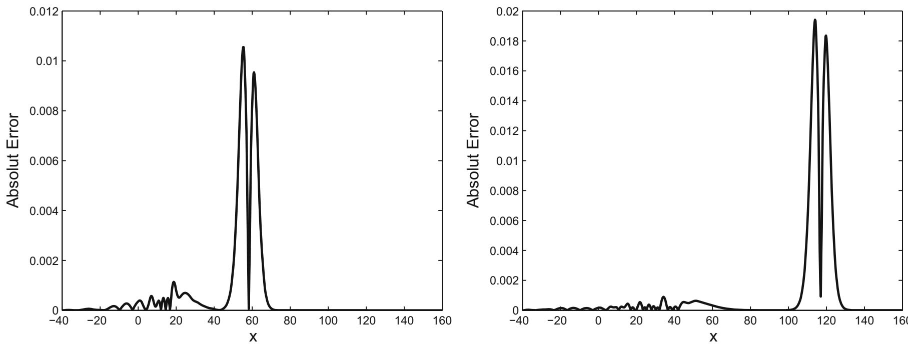

Fig. 7 Absolute error distribution at $T = 30$ (left) and $T = 60$ (right) with $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 160$

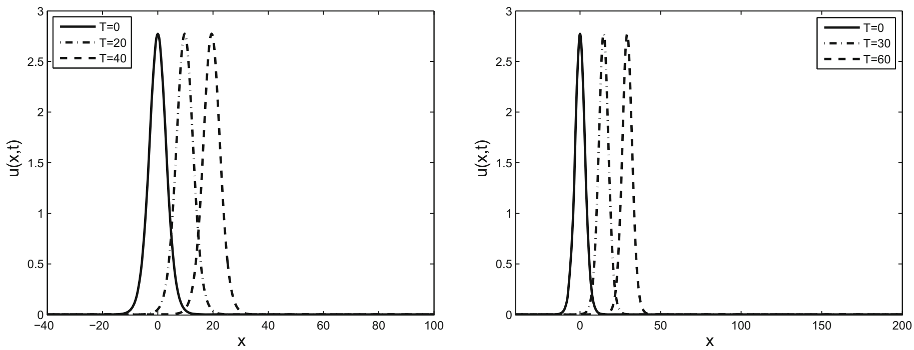

Fig. 8 Numerical solutions of the Rosenau–KdV–RLW equation with $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 100$ (left) and $\beta = 200$ (right)

solutions computed by the present scheme in with $h = 0.25$, $\tau = h^2$, $\alpha = -40$, $\beta = 100$, 200 are given in Fig. 8, and we can see that the waves at $T = 20 \sim 60$ agree with the corresponding waves at $T = 0$ quite well.

5 Conclusion

A new conservative finite difference scheme for the generalized Rosenau–KdV–RLW equation is introduced and analyzed. The scheme is unconditionally stable and convergent with order of $O(\tau^2 + h^4)$. Numerical experiments confirm well the theoretical analysis and show that the present scheme is efficient and reliable.

Acknowledgements The first author was supported in part by Fujian Province Science Foundation for Middle-aged and Young Teachers (no. JAT190368). The authors also thank the reviewers and editors for their very helpful comments and suggestions which greatly improved the quality of this paper.

References

Cai W, Sun Y, Wang Y (2015) Variational discretizations for the generalized Rosenau-type equations. Appl Math Comput 271:860–873

Coclite G, Ruvob L (2017) On the convergence of the modified Rosenau and the modified Benjamin-Bona-Mahony equations. Comput Math Appl 74(5):899–919

Esfahani A (2011) Solitary wave solutions for generalized Rosenau-KdV equation. Commun Theor Phys 55(3):396–398

Ghiloufi A, Omrani K (2017) New conservative difference schemes with fourth-order accuracy for some model equation for nonlinear dispersive waves. Numer Methods Partial Differ E 34(2):451–500

Hu B, Xu Y, Hu J (2008) Crank-Nicolson finite difference scheme for the Rosenau-Burgers equation. Appl Math Comput 204(1):311–316

Hu J, Xu Y, Hu B (2013) Conservative linear difference scheme for Rosenau-KdV equation. Adv Math Phys 2:1–7

Kaur D, Mohanty RK (2019) Two-level implicit high order method based on half-step discretization for 1D unsteady biharmonic problems of first kind. Appl Numer Math 139:1–14

Kim H, Bae W, Choi J (2012) Numerical stability of symmetric solitary-wave-like waves of a two-layer fluid-forced modified KdV equation. Math Comput Simul 82:1219–1227

Korteweg D, Vries G (1895) On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos Mag 39:422–443

Lu D, Chen C (2015) Computable analysis of a boundary-value problem for the generalized KdV-Burgers equation. Math Methods Appl Sci 38(11):2243–2249

Mohanty RK, Kaur D (2019) High accuracy two-level implicit compact difference scheme for 1D unsteady biharmonic problem of first kind: application to the generalized Kuramoto-Sivashinsky equation. J Differ Equ Appl 25:243–261

Ozer S, Kutluay S (2005) An analytical–numerical method applied to Korteweg-de Vries equation. Appl Math Comput 164(3):789–797

Pan X, Zhang L (2012) On the convergence of a conservative numerical scheme for the usual Rosenau-RLW equation. Appl Math Model 36(8):3371–3378

Park M (1992) Pointwise decay estimate of solutions of the generalized Rosenau equation. J Korean Math Soc 29(2):261–280

Razborova P, Moraru L, Biswas A (2014) Perturbation of dispersive shallow water wave swith Rosenau-KdV-RLW equation with power law nonlinearity. Rom J Phys 59(7):658–676

Razborova P, Kara A, Biswas A (2015) Additional conservation laws for Rosenau-KdV-RLW equation with power law nonlinearity by lie symmetry. Nonlinear Dyn 79(1):743–748

Rosenau P (1986) A quasi-continuous description of a non-linear transmission line. Phys Scr 34(6):827–829

Rosenau P (1988) Dynamics of dense discrete systems: high order effects, general and mathematical physics. Progr Theor Phys 79:1028–1042

Sanchez P, Ebadi G, Mojaver A, Mirzazadeh M, Eslami M, Biswas A (2015) Solitons and other solutions to perturbed Rosenau-KdV-RLW equation with power law nonlinearity. Acta Phys Pol A 127(6):1577–1586

Shao X, Xue G, Li C (2013) A conservative weighted finite difference scheme for regularized long wave equation. Appl Math Comput 219:9202–9209

Skogestad J, Kalisch H (2009) A boundary value problem for the KdV equation: comparison of finite-difference and Chebyshev methods. Math Comput Simul 80:151–163

Triki H, Biswas A (2013) Perturbation of dispersive shallow water waves. Ocean Eng 63(4):1–7

Wang X, Dai W (2018) A three-level linear implicit conservative scheme for the Rosenau-KdV-RLW equation. J Comput Appl Math 330:295–306

Wang X, Dai W (2018) A new implicit energy conservative difference scheme with fourth-order accuracy for the generalized Rosenau-Kawahara-RLW equation. Comput Appl Math 37:6560–6581

Wang B, Sun T, Liang D (2019) The conservative and fourth-order compact finite difference schemes for regularized long wave equation. J Comput Appl Math 356:98–117

Wongsaijai B, Poochinapan K (2014) A three-level average implicit finite difference scheme to solve equation obtained by coupling the Rosenau-KdV equation and the Rosenau-RLW equation. Appl Math Comput 245:289–304

Wongsaijai B, Poochinapan K, Disyadej T (2014) A compact finite difference method for solving the general Rosenau-RLW equation. Int J Appl Math 44(4):192–199

Wongsaijai B, Mouktonglang T, Sukantamala N, Poochinapan K (2019) Compact structure-preserving approach to solitary wave in shallow water modeled by the Rosenau-RLW equation. Appl Math Comput 340:84–100

Yan J, Zhang Q, Zhang Z (2016) New conservative finite volume element schemes for the modified Korteweg-de Vries equation. Math Methods Appl Sci 39(18):5149–5161
Ye H, Liu F, Anh V (2015) Compact difference scheme for distributed-order time-fractional diffusion-wave equation on bounded domains. J Comput Phys 298:652–660
Zheng M, Zhou J (2014) An average linear difference scheme for the generalized Rosenau-KdV equation. J Appl Math 2:1–9
Zuo J (2009) Solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations. Appl Math Comput 215(2):835–840