



Numerical analysis of a type III thermo-porous-elastic problem with microtemperatures

Noelia Bazarra¹ · José R. Fernández¹ · Ramón Quintanilla²

Received: 6 October 2019 / Revised: 18 May 2020 / Accepted: 5 July 2020 / Published online: 11 August 2020
© SBMAC - Sociedade Brasileira de Matemática Aplicada e Computacional 2020

Abstract

In this work, we consider, from the numerical point of view, a poro-thermoelastic problem. The thermal law is the so-called of type III and the microtemperatures are also included into the model. The variational formulation of the problem is written as a linear system of coupled first-order variational equations. Then, fully discrete approximations are introduced by using the classical finite-element method and the implicit Euler scheme. A discrete stability property and an a priori error estimates result are proved, from which the linear convergence of the algorithm is derived under suitable additional regularity conditions. Finally, some one- and two-dimensional numerical simulations are presented to show the accuracy of the approximation and the behavior of the solution.

Keywords Type III thermoelasticity with voids · Microtemperatures · Numerical approximation · Error estimates · Numerical solutions

Mathematics Subject Classification 65M60 · 65M12 · 74F05 · 74B05

1 Introduction

The most useful model to describe the heat conduction is based on the Fourier law that proposes a linear relation between the heat flux vector and the gradient of temperature. If we combine this equation with the usual energy equation, we obtain the existence of thermal

Communicated by Cassio Oishi.

✉ José R. Fernández
jose.fernandez@uvigo.es

Noelia Bazarra
nbazarra@uvigo.es

Ramón Quintanilla
ramon.quintanilla@upc.edu

¹ Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñaría de Telecomunicación Campus as Lagoas Marcosende s/n, 36310 Vigo, Spain

² Departament de Matemàtiques, Universitat Politècnica de Catalunya, C. Colom 11, 08222 Terrassa, Barcelona, Spain

waves propagating with an unbounded speed. That is, a thermal perturbation at one point is instantaneously felt at any other point of the space for every distance. It is clear that this effect contradicts the *causality principle*. For this reason, a big deal trying to overcome this paradox has been developed in the last and current centuries. It seems that the first works in this aspect correspond to Cattaneo and Maxwell (Cattaneo 1958). They proposed the introduction of a relaxation time in the Fourier law. Recently, in the 1990s decade, Green and Naghdi proposed several alternative models (Green and Naghdi 1992, 1993). In fact, they proposed these new theories in the context of the thermoelasticity and the main difference concerning the classical theory corresponds to the thermal effects. The most general is the so-called type III and it contains the Fourier model as a limit case. It is also worth recalling the type II which is also called *without energy dissipation*. It also corresponds to another limit case of the type III theory.

A big interest has also been developed to understand models with microstructure. In fact, Eringen (1999) contributed in an important way in this sense in the last century. An interesting case for these models corresponds to those where microtemperatures are taken into account. That is, among the microstructure effects, we can consider the microtemperatures. First contribution on this kind of materials came back to the one by Grot (1969) and some people used them to study several problems (Riha 1975, 1976; Verma et al. 1979). We recall the contribution (Iesan and Quintanilla 2000) as a new reborn of the interest for this kind of questions, because many works have studied this kind of problems recently (see (Casas and Quintanilla 2005a; Chirita et al. 2013; Ciarletta et al. 2010; Ieşan 2007; Iesan 2018; Iesan and Quintanilla 2018; Magaña and Quintanilla 2018; Quintanilla 2011, 2013) among others). In the last 20 years, there has been a big deal of people interested in the study of elastic materials with microtemperatures.

Cowin (1985), Cowin and Nunziato (1983), Nunziato and Cowin (1979) proposed a mathematical theory to model elastic materials with voids. Since these contributions, many people have been interested in the study of thermoelastic materials with voids and the quantity of contributions involving this model is huge (Bazarra and Fernández 2018; Casas and Quintanilla 2005b; Feng and Apalara 2019; Feng and Yin 2019; Fernández and Masid 2016, 2017a, b; Ieşan and Quintanilla 2014; Kumar and Vohra 2017, 2019; Kumar et al. 2016; Magaña and Quintanilla 2006, 2017; Ramos et al. 2020). It is worth noting that the model has become useful to understand the behavior of elastic materials with small distributed pores and we can find them in the study of biological materials as bones or in the study of soils, woods, ceramics, or rocks. It is also worth noting the structural similarity (from the mathematical point of view) of the system of equations for the poro-elasticity with the equations of the Timoshenko beam (see, for instance, Almeida Júnior and Ramos 2017).

In the present paper, we want to joint these three basic ideas: on one side, we consider the type III theory; on the second aspect, we consider microtemperatures; and on the third side, we consider porous aspects. First contribution concerning the three aspects at the same time can be seen at Magaña and Quintanilla (2020). There, the authors consider the system of equations that we can obtain from the studies (Aouadi et al. 2018; Ieşan 2007; Ieşan and Quintanilla 2009). Here, we continue the research started in Magaña and Quintanilla (2020), introducing a fully discrete approximation based on the finite-element method and the implicit Euler scheme, proving a discrete stability property and a priori error estimates, and performing some one- and two-dimensional numerical simulations to demonstrate the accuracy and the behavior of the discrete solutions.

We think that it is relevant to point out that the behavior of the thermoelastic materials in the context of the type III theory has been revealed different from the classical theory based on the Fourier law. We can cite several contributions (Leseduarte et al. 2010; Magaña and

Quintanilla 2018; Miranville and Quintanilla 2020, 2019; Magaña and Quintanilla 2020) where we have detected relevant differences in the behavior of the solutions corresponding to this kind of materials. The main reason is that, when we consider type III theory, new coupling terms appear which are not present when we consider the theory based on the classical Fourier law. At the same time, when we consider microtemperatures, there are also more new coupling terms which are not present in the case of the Fourier theory with microtemperatures. Therefore, our system is more complex from the mathematical point of view, and then, new and strong difficulties could appear when we consider the new theory. Furthermore, what we will develop here cannot be a direct extension of the classical theory, but we could consider new aspects in our study.

2 Mathematical and variational formulations

First, we describe the problem (see Magaña and Quintanilla 2020 for further details). Let Ω be a bounded domain in \mathbb{R}^d (for $d = 1, 2, 3$) with boundary smooth enough to allow the application of the divergence theorem. We will use the standard notation where “ \cdot, i ” means the partial derivative with respect to the variable x_i , a superposed dot represents time derivative and summation on repeated indices is assumed. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest.

Let us denote $\mathbf{u} = (u_i)_{i=1}^d, \varphi, \theta$, and $\mathbf{M} = (M_i)_{i=1}^d$ the displacement, the volume fraction, the temperature, and the microtemperatures, respectively.

Since we are interested in the thermoelastic theory of type III with voids and microtemperatures, the corresponding thermo-mechanical problem is the following (see Aouadi et al. 2018; Iesan and Quintanilla 2000; Ieşan and Quintanilla 2009; Magaña and Quintanilla 2020):

Problem P. Find the displacement $\mathbf{u} : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$, the volume fraction $\varphi : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}$, the thermal displacement $\tau : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}$, and the microthermal displacement $\mathbf{R} : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$, such that:

$$\begin{aligned} \rho \ddot{u}_i &= (A_{ijkl} u_{k,l} - a_{ij} \theta + \zeta_{ij} \varphi + B_{ijkl} R_{k,l})_{,j} \text{ in } \Omega \times (0, T) \\ &\text{for } i = 1, \dots, d, \end{aligned} \tag{1}$$

$$\begin{aligned} J \ddot{\varphi} &= (A_{ij} \varphi_{,j} - \alpha_{ij} \dot{R}_i + H_{ij} \tau_{,i})_{,j} - \zeta_{ij} u_{i,j} + \kappa \dot{\tau} - F_{ij} R_{i,j} - \xi \varphi \\ &\text{in } \Omega \times (0, T), \end{aligned} \tag{2}$$

$$\begin{aligned} c \ddot{\tau} &= -a_{ij} \dot{u}_{i,j} + (H_{ij} \varphi_{,i})_{,j} - (d_{ij} \dot{R}_i)_{,j} + (K_{ij} \tau_{,i} + K_{ij}^* \dot{\tau}_{,i})_{,j} - b_{ij} \dot{R}_{i,j} \\ &\quad - \kappa \dot{\varphi} + (A_{ij}^1 M_i)_{,j} \text{ in } \Omega \times (0, T), \end{aligned} \tag{3}$$

$$\begin{aligned} c_{ij} \ddot{R}_j &= (B_{kl ij} u_{k,l} - b_{ij} \dot{\tau} + F_{ij} \varphi + C_{ijkl} R_{k,l})_{,j} - d_{ij} \dot{\tau}_{,j} + (C_{ijkl}^* \dot{R}_{k,l})_{,j} \\ &\quad - \alpha_{ij} \dot{\varphi}_{,j} - A_{ij}^2 \theta_{,j} - A_{ij}^3 M_j \text{ in } \Omega \times (0, T) \text{ for } i = 1, \dots, d, \end{aligned} \tag{4}$$

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \varphi(\mathbf{x}, 0) = \varphi^0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \tag{5}$$

$$\dot{\varphi}(\mathbf{x}, 0) = e^0(\mathbf{x}), \quad \tau(\mathbf{x}, 0) = \tau^0(\mathbf{x}), \quad \dot{\tau}(\mathbf{x}, 0) = \theta^0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \tag{6}$$

$$R_i(\mathbf{x}, 0) = R_i^0(\mathbf{x}), \quad \dot{R}_i(\mathbf{x}, 0) = M_i^0(\mathbf{x}) \text{ for } \mathbf{x} \in \Omega, \tag{7}$$

$$u_i(\mathbf{x}, t) = \varphi(\mathbf{x}, t) = \tau(\mathbf{x}, t) = R_i(\mathbf{x}, t) = 0 \text{ for } \mathbf{x} \in \partial\Omega, \quad t \in [0, T]. \tag{8}$$

Here, τ is the thermal displacement introduced by Green and Naghdi and $\mathbf{R} = (R_i)_{i=1}^d$ are the microthermal displacements, defined, respectively, by:

$$\tau(\mathbf{x}, t) = \tau^0(\mathbf{x}) + \int_0^t \theta(\mathbf{x}, s) \, ds, \quad R_i(\mathbf{x}, t) = R_i^0(\mathbf{x}) + \int_0^t M_i(\mathbf{x}, s) \, ds.$$

As usual, ρ denotes the mass density, J the product of the mass density by the equilibrated inertia, and c the thermal capacity. A_{ijkl} is the elastic tensor, and a_{ij} , ζ_{ij} , and B_{ijkl} are, respectively, the coupling tensors between the displacement and the temperature, the displacement and the volume fraction, and the displacement and the microtemperatures. A_{ij} , $A_{ij}^{(1)}$, $A_{ij}^{(2)}$, $A_{ij}^{(3)}$, α_{ij} , H_{ij} , F_{ij} , d_{ij} , and b_{ij} are other coupling tensors between the variables. K_{ij} is the tensor introduced by Green and Naghdi, and it is usually called rate conductivity, K_{ij}^* is the thermal conductivity tensor, c_{ij} is a typical tensor of the theories with microtemperatures, and, finally, C_{ijkl} and C_{ijkl}^* are the specific type III tensors with microtemperatures.

The following symmetries are assumed (see Aouadi et al. 2018; Magaña and Quintanilla 2020):

$$\left. \begin{aligned} A_{ijkl} &= A_{klij}, & A_{ij} &= A_{ji}, & K_{ij} &= K_{ji}, & K_{ij}^* &= K_{ji}^*, & C_{ijkl} &= C_{klij}, \\ C_{ijkl}^* &= C_{klij}^*, & c_{ij} &= c_{ji}, & A_{ij}^3 &= A_{ji}^3, & H_{ij} &= H_{ji}, & B_{ijkl} &= B_{klij}. \end{aligned} \right\} \quad (9)$$

From the second law of thermodynamics the following inequality must be satisfied (see Iesan and Quintanilla 2000):

$$K_{i,j}^* \xi_i \xi_j + (A_{ij}^1 + A_{ij}^2) \eta_i \xi_j + A_{ij}^3 \eta_i \eta_j + C_{ijkl}^* \eta_{ij} \eta_{kl} \geq K_0 (\xi_i \xi_i + \eta_i \eta_i + \eta_{ij} \eta_{ij}), \quad (10)$$

for a positive constant K_0 and for each pair of vectors ξ_i and η_i and for each tensor η_{ij} .

We will also impose some assumptions over the constitutive coefficients. For each vector ξ_i , each pair of tensors ξ_{ij} and η_{ij} and each real number l , the following inequalities are assumed:

$$\left. \begin{aligned} &A_{ijkl} \xi_{ij} \xi_{kl} + 2B_{ijkl} \xi_{ij} \eta_{kl} + C_{ijkl} \eta_{ij} \eta_{kl} + 2\zeta_{ij} \xi_{ij} l + \xi l^2 \\ &\geq C_0 (\xi_{ij} \xi_{ij} + \eta_{ij} \eta_{ij} + l^2), \\ &A_{ij} \xi_i \xi_j + 2H_{ij} \xi_i \eta_j + K_{ij} \eta_i \eta_j \geq C_1 (\xi_i \xi_i + \eta_i \eta_j), \\ &c_{ij} \xi_i \xi_j \geq C_2 \xi_i \xi_i, \quad \rho \geq \rho_0 > 0, \quad J \geq J_0 > 0, \quad c \geq c_0 > 0, \end{aligned} \right\} \quad (11)$$

for positive constants J_0, c_0, C_0, C_1, C_2 , and ρ_0 . The first two conditions proposed here can be interpreted with the help of the stability theory for thermoelastic materials. The physical meaning of the assumptions in the third line of (11) is clear.

First, we show that the energy of the system is dissipative.

Proposition 1 *Let us define the energy of the system $\mathcal{E}(t)$ as follows:*

$$\begin{aligned} \mathcal{E}(t) = & \frac{1}{2} \left\{ \rho (v_i(t), v_i(t))_Y + J \|e(t)\|_Y^2 + c \|\theta(t)\|_Y^2 + (c_{ij} M_j(t), M_i(t))_Y \right. \\ & + (A_{ijkl} u_{i,j}(t), u_{k,l}(t))_Y + (B_{ijkl} u_{i,j}(t), R_{k,l}(t))_Y + \xi \|\varphi(t)\|_Y^2 \\ & + (C_{ijkl} R_{i,j}(t), R_{k,l}(t))_Y + (\zeta_{ij} u_{i,j}(t), \varphi(t))_Y + (A_{ij} \varphi_i(t), \varphi_j(t))_Y \\ & \left. + (H_{ij} \varphi_i(t), \tau_j(t))_Y + (K_{ij} \tau_i(t), \tau_j(t))_Y \right\}, \end{aligned} \quad (12)$$

where we have used the notation $Y = L^2(\Omega)$ and $(\cdot, \cdot)_Y$ for the usual scalar product in this space. Then, this energy is dissipative.

Proof We note that, from the previous definition after a direct calculation, we find that:

$$\begin{aligned} \mathcal{E}'(t) = & - \int_{\Omega} \left(K_{ij}^* \theta_{,i} \theta_{,j} + (A_{ij}^1 + A_{ij}^2) M_i \theta_{,j} + A_{ij}^3 M_i M_j \right) dv \\ & - \int_{\Omega} C_{ijkl}^* M_{i,j} M_{k,l} dv, \end{aligned}$$

and using assumption (10), we then conclude that the energy is always dissipative.

Now, we recall the following existence and uniqueness result (Magaña and Quintanilla 2020).

Theorem 1 *Under assumptions (9)–(11), if the following regularity on the initial conditions hold:*

$$\mathbf{u}^0, \mathbf{v}^0, \mathbf{R}^0, \mathbf{M}^0 \in [H^2(\Omega)]^d, \quad \varphi^0, e^0, \tau^0, \theta^0 \in H^2(\Omega),$$

then there exists a unique solution to Problem P with the regularity:

$$\mathbf{u}, \mathbf{R} \in C^1([0, T]; V) \cap C^2([0, T]; H), \quad \varphi, \tau \in C^1([0, T]; E) \cap C^2([0, T]; Y).$$

To obtain the exponential decay of the solutions to Problem P, we will assume that, for every tensor ξ_{ij} and every vector ζ_i :

$$B_{klj} \xi_{kl} \xi_{ij} \geq C \xi_{ij} \xi_{ij}, \quad H_{ij} \zeta_i \zeta_j \geq C^* \zeta_i \zeta_i, \tag{13}$$

for two positive constants C and C^* .

Even if the above assumptions are quite natural, we need to impose also two more technical conditions on some of the tensors. Let us suppose that there exist two constants, m_1 and m_2 , such that:

$$a_{ij} = m_1 \zeta_{ij} \quad \text{and} \quad \alpha_{ij} = m_2 \zeta_{ij}. \tag{14}$$

Notice that, for *isotropic and homogeneous materials*, assumptions (14) are satisfied whenever the corresponding constitutive parameter is different from zero, because, in this case, $\zeta_{ij} = \zeta \delta_{ij}$ for a constant $\zeta \neq 0$ (δ_{ij} denotes the Kronecker delta).

Therefore, we have the following (see Magaña and Quintanilla 2020).

Theorem 2 *Under the assumptions of Theorem 1 and (13)–(14), the solution to Problem P is asymptotically stable; that is, there exist two positive constants M and α , such that:*

$$\|\mathcal{E}(t)\| \leq M \|\mathcal{E}(0)\| e^{-\alpha t},$$

where the energy of the system \mathcal{E} was defined in (12).

Finally, to provide the numerical approximation of Problem P in the next section, we will obtain the variational formulation of this problem. Thus, let $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d \times d}$, and denote by $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces $E = H_0^1(\Omega)$ and $V = [H_0^1(\Omega)]^d$.

Then, applying Green’s formula to Eqs. (1)–(4) and using boundary conditions (8), we have the following weak problem.

Problem VP. *Find the velocity $\mathbf{v} : [0, T] \rightarrow V$, the volume fraction speed $e : [0, T] \rightarrow E$, the temperature $\theta : [0, T] \rightarrow E$, and the microtemperatures $\mathbf{M} : [0, T] \rightarrow V$, such that $\mathbf{v}(0) = \mathbf{v}^0, e(0) = e^0, \theta(0) = \theta^0, \mathbf{M}(0) = \mathbf{M}^0$ and, for a.e. $t \in (0, T)$:*

$$\rho(\dot{\mathbf{v}}(t), \mathbf{w}_i)_Y + (A_{ijkl} u_{k,l}(t), \mathbf{w}_{i,j})_Y = (a_{ij} \theta(t), \mathbf{w}_{i,j})_Y - (B_{ijkl} R_{k,l}(t), \mathbf{w}_{i,j})_Y$$

$$-(\zeta_{ij}\varphi(t), w_{i,j})_Y \quad \forall \mathbf{w} = (w_i)_{i=1}^d \in V, \tag{15}$$

$$J(\dot{\theta}(t), r)_Y + (A_{ij}\varphi_{,j}(t), r_{,i})_Y + \xi(\varphi(t), r)_Y = (\alpha_{ij}M_i(t), r_{,j})_Y + \kappa(\theta(t), r)_Y - (H_{ij}\tau_{,i}(t), r_{,j})_Y - (\zeta_{ij}u_{i,j}(t), r)_Y - (F_{ij}R_{i,j}(t), r)_Y \quad \forall r \in E, \tag{16}$$

$$c(\dot{\theta}(t), z)_Y + (K_{ij}^*\theta_{,i}(t), z_{,j})_Y + (K_{ij}\tau_{,i}(t), z_{,j})_Y = (d_{ij}M_i(t), z_{,j})_Y - (A_{ij}^1M_i(t), z_{,j})_Y - (b_{ij}M_{i,j}(t), z)_Y - (a_{ij}v_{i,j}(t), z)_Y - \kappa(e(t), z)_Y - (H_{ij}\varphi_{,i}(t), z_{,j})_Y \quad \forall z \in E, \tag{17}$$

$$(c_{ij}\dot{M}_j(t), \xi_i)_Y + (C_{ijkl}R_{k,l}(t), \xi_{i,j}) + (C_{ijkl}^*M_{k,l}(t), \xi_{i,j})_Y = (b_{ij}\theta(t), \xi_{i,j})_Y - (B_{klij}u_{k,l}(t), \xi_{i,j})_Y - (F_{ij}\varphi(t), \xi_{i,j})_Y - (d_{ij}\theta_{,j}(t), \xi_i)_Y - (\alpha_{ij}e_{,j}(t), \xi_i)_Y - (A_{ij}^2\theta_{,j}(t), \xi_i)_Y - (A_{ij}^3M_j(t), \xi_i)_Y \quad \forall \xi = (\xi_i)_{i=1}^d \in V, \tag{18}$$

where we recall that the displacement, the volume fraction, the thermal displacement, and the microthermal displacements are then recovered from relations:

$$\mathbf{u}(t) = \int_0^t \mathbf{v}(s) \, ds + \mathbf{u}^0, \quad \varphi(t) = \int_0^t e(s) \, ds + \varphi^0, \tag{19}$$

$$\tau(t) = \int_0^t \theta(s) \, ds + \tau^0, \quad \mathbf{R}(t) = \int_0^t \mathbf{M}(s) \, ds + \mathbf{R}^0. \tag{20}$$

3 Fully discrete approximations: an a priori error analysis

In this section, we now consider a fully discrete approximation of Problem VP . This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by \mathcal{T}^h a regular triangulation in the sense of Ciarlet (1993). Thus, we construct the finite-dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by:

$$V^h = \{z^h \in [C(\overline{\Omega})]^d ; z^h|_{Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad z^h = \mathbf{0} \quad \text{on } \partial\Omega\}, \tag{21}$$

$$E^h = \{\eta^h \in C(\overline{\Omega}) ; \eta^h|_{Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad \eta^h = 0 \quad \text{on } \partial\Omega\}, \tag{22}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr , i.e., the finite-element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\mathbf{u}^{0h}, \mathbf{v}^{0h}, \varphi^{0h}, e^{0h}, \tau^{0h}, \theta^{0h}, \mathbf{R}^{0h}$ and \mathbf{M}^{0h} , are given by:

$$\begin{aligned} \mathbf{u}^{0h} &= \mathcal{P}_1^h \mathbf{u}^0, & \mathbf{v}^{0h} &= \mathcal{P}_1^h \mathbf{v}^0, & \varphi^{0h} &= \mathcal{P}_2^h \varphi^0, & e^{0h} &= \mathcal{P}_2^h e^0, \\ \tau^{0h} &= \mathcal{P}_2^h \tau^0, & \theta^{0h} &= \mathcal{P}_2^h \theta^0, & \mathbf{R}^{0h} &= \mathcal{P}_1^h \mathbf{R}^0, & \mathbf{M}^{0h} &= \mathcal{P}_1^h \mathbf{M}^0, \end{aligned} \tag{23}$$

where \mathcal{P}_1^h and \mathcal{P}_2^h are the classical finite-element interpolation operators over V^h and E^h , respectively (see, e.g., Ciarlet 1993).

Second, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. Find the discrete velocity $\mathbf{v}^{hk} = \{\mathbf{v}^{hk,n}\}_{n=0}^N \subset V^h$, the discrete volume fraction speed $e^{hk} = \{e^{hk,n}\}_{n=0}^N \subset E^h$, the temperature $\theta^{hk} = \{\theta^{hk,n}\}_{n=0}^N \subset E^h$, and the microtemperatures $\mathbf{M}^{hk} = \{\mathbf{M}^{hk,n}\}_{n=0}^N \subset V^h$, such that $\mathbf{v}^{hk,0} = \mathbf{v}^{0h}$, $e^{hk,0} = e^{0h}$, $\theta^{hk,0} = \theta^{0h}$, $\mathbf{M}^{hk,0} = \mathbf{M}^{0h}$, and, for $n = 1, \dots, N$:

$$\rho(\delta v_i^{hk,n}, w_i^h)_Y + (A_{ijkl} u_{k,l}^{hk,n}, w_{i,j}^h)_Y = (a_{ij} \theta^{hk,n}, w_{i,j}^h)_Y - (B_{ijkl} R_{k,l}^{hk,n}, w_{i,j}^h)_Y - (\zeta_{ij} \varphi^{hk,n}, w_{i,j}^h)_Y \quad \forall \mathbf{w}^h = (w_i^h)_{i=1}^d \in V^h, \tag{24}$$

$$J(\delta e^{hk,n}, r^h)_Y + (A_{ij} \varphi_{i,j}^{hk,n}, r_i^h)_Y + \xi(\varphi^{hk,n}, r^h)_Y = (\alpha_{ij} M_i^{hk,n}, r_j^h)_Y - (H_{ij} \tau_{i,j}^{hk,n}, r_j^h)_Y - (\zeta_{ij} u_{i,j}^{hk,n}, r^h)_Y + \kappa(\theta^{hk,n}, r^h)_Y - (F_{ij} R_{i,j}^{hk,n}, r^h)_Y \quad \forall r^h \in E^h, \tag{25}$$

$$c(\delta \theta^{hk,n}, z^h)_Y + (K_{ij}^* \theta_{i,j}^{hk,n}, z_j^h)_Y + (K_{ij} \tau_{i,j}^{hk,n}, z_j^h)_Y = (d_{ij} M_i^{hk,n}, z_j^h)_Y - (A_{ij}^1 M_{i,j}^{hk,n}, z_j^h)_Y - (b_{ij} M_{i,j}^{hk,n}, z^h)_Y - (a_{ij} v_{i,j}^{hk,n}, z^h)_Y - \kappa(e^{hk,n}, z^h)_Y - (H_{ij} \varphi_{i,j}^{hk,n}, z_j^h)_Y \quad \forall z^h \in E^h, \tag{26}$$

$$(c_{ij} \delta M_j^{hk,n}, \xi_i^h)_Y + (C_{ijkl} R_{k,l}^{hk,n}, \xi_{i,j}^h)_Y + (C_{ijkl}^* M_{k,l}^{hk,n}, \xi_{i,j}^h)_Y + (A_{ij}^3 M_j^{hk,n}, \xi_i^h)_Y = (b_{ij} \theta^{hk,n}, \xi_{i,j}^h)_Y - (F_{ij} \varphi^{hk,n}, \xi_{i,j}^h)_Y - (d_{ij} \theta_{i,j}^{hk,n}, \xi_i^h)_Y - (\alpha_{ij} e_{i,j}^{hk,n}, \xi_i^h)_Y - (A_{ij}^2 \theta_{i,j}^{hk,n}, \xi_i^h)_Y - (B_{klij} u_{k,l}^{hk,n}, \xi_{i,j}^h)_Y \quad \forall \xi^h = (\xi_i^h)_{i=1}^d \in V^h, \tag{27}$$

where the discrete displacement, the discrete volume fraction, the discrete thermal displacement, and the discrete microthermal displacement are then recovered from relations:

$$\mathbf{u}^{hk,n} = k \sum_{j=1}^n \mathbf{v}^{hk,j} + \mathbf{u}^{0h}, \quad \varphi^{hk,n} = k \sum_{j=1}^n e^{hk,j} + \varphi^{0h}, \tag{28}$$

$$\tau^{hk,n} = k \sum_{j=1}^n \theta^{hk,j} + \tau^{0h}, \quad \mathbf{R}^{hk,n} = k \sum_{j=1}^n \mathbf{M}^{hk,j} + \mathbf{R}^{0h}. \tag{29}$$

The existence of a unique solution to Problem VP^{hk} can be easily proved using Lax–Milgram lemma and taking into account assumptions (11)–(14).

The aim of this section is to provide the numerical analysis of Problem VP. First, we have the following discrete stability result.

Lemma 1 Under the assumptions of Theorem 2, it follows that the sequences $\{\mathbf{u}^{hk}, \mathbf{v}^{hk}, \varphi^{hk}, e^{hk}, \tau^{hk}, \theta^{hk}, \mathbf{R}^{hk}, \mathbf{M}^{hk}\}$ generated by Problem VP^{hk} satisfy the stability estimate:

$$\|\mathbf{v}^{hk,n}\|_H^2 + \|\nabla \mathbf{u}^{hk,n}\|_Q^2 + \|e^{hk,n}\|_Y^2 + \|\nabla \varphi^{hk,n}\|_H^2 + \|\varphi^{hk,n}\|_Y^2 + \|\theta^{hk,n}\|_Y^2 + \|\nabla \tau^{hk,n}\|_H^2 + \|\mathbf{M}^{hk,n}\|_H^2 + \|\nabla \mathbf{R}^{hk,n}\|_Q^2 \leq C,$$

where C is a positive constant assumed to be independent of the discretization parameters h and k .

Proof First, if we take as a test function $w_i^h = v_i^{hk,n}$ in discrete variational equation (24), we find that:

$$\rho(\delta v_i^{hk,n}, v_i^{hk,n})_Y + (A_{ijkl} u_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y = (a_{ij} \theta^{hk,n}, v_{i,j}^{hk,n})_Y$$

$$-(B_{ijkl}R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y - (\zeta_{ij}\varphi^{hk,n}, v_{i,j}^{hk,n})_Y.$$

Thus, taking into account that:

$$(\delta v_i^{hk,n}, v_i^{hk,n})_Y \geq \frac{1}{2k} \left\{ \|v^{hk,n}\|_H^2 - \|v^{hk,n-1}\|_H^2 \right\},$$

we find that:

$$\begin{aligned} & \frac{\rho}{2k} \left\{ \|v^{hk,n}\|_H^2 - \|v^{hk,n-1}\|_H^2 \right\} + \frac{1}{2k} \left\{ (A_{ijkl}u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y - (A_{ijkl}u_{k,l}^{hk,n-1}, u_{i,j}^{hk,n-1})_Y \right. \\ & \quad \left. + (A_{ijkl}(u_{k,l}^{hk,n} - u_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y \right\} \\ & = (a_{ij}\theta^{hk,n}, v_{i,j}^{hk,n})_Y - (B_{ijkl}R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y - (\zeta_{ij}\varphi^{hk,n}, v_{i,j}^{hk,n})_Y. \end{aligned} \tag{30}$$

Second, taking $r^h = e^{hk,n}$ as a test function in (25), we have:

$$\begin{aligned} & J(\delta e^{hk,n}, e^{hk,n})_Y + (A_{ij}\varphi_j^{hk,n}, e_i^{hk,n})_Y + \xi(\varphi^{hk,n}, e^{hk,n})_Y = (\alpha_{ij}M_i^{hk,n}, e_j^{hk,n})_Y \\ & \quad - (H_{ij}\tau_i^{hk,n}, e_j^{hk,n})_Y - (\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y + \kappa(\theta^{hk,n}, e^{hk,n})_Y - (F_{ij}R_{i,j}^{hk,n}, e^{hk,n})_Y, \end{aligned}$$

and using the estimates:

$$J(\delta e^{hk,n}, e^{hk,n})_Y \geq \frac{J}{2k} \left\{ \|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2 \right\},$$

we obtain:

$$\begin{aligned} & \frac{J}{2k} \left\{ \|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (A_{ij}\varphi_j^{hk,n}, \varphi_i^{hk,n})_Y - (A_{ij}\varphi_j^{hk,n-1}, \varphi_i^{hk,n-1})_Y \right. \\ & \quad \left. + (A_{ij}(\varphi_j^{hk,n} - \varphi_j^{hk,n-1}), \varphi_i^{hk,n} - \varphi_i^{hk,n-1})_Y \right\} \\ & \quad + \frac{\xi}{2k} \left\{ \|\varphi^{hk,n}\|_Y^2 - \|\varphi^{hk,n-1}\|_Y^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|_Y^2 \right\} \\ & = (\alpha_{ij}M_i^{hk,n}, e_j^{hk,n})_Y - (H_{ij}\tau_i^{hk,n}, e_j^{hk,n})_Y - (\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y \\ & \quad + \kappa(\theta^{hk,n}, e^{hk,n})_Y - (F_{ij}R_{i,j}^{hk,n}, e^{hk,n})_Y. \end{aligned} \tag{31}$$

Third, choosing $z^h = \theta^{hk,n}$ as a test function in (26), it follows that:

$$\begin{aligned} & c(\delta\theta^{hk,n}, \theta^{hk,n})_Y + (K_{ij}^*\theta_i^{hk,n}, \theta_j^{hk,n})_Y + (K_{ij}\tau_i^{hk,n}, \theta_j^{hk,n})_Y = (d_{ij}M_i^{hk,n}, \theta_j^{hk,n})_Y \\ & \quad - (A_{ij}^1M_i^{hk,n}, \theta_j^{hk,n})_Y - (b_{ij}M_{i,j}^{hk,n}, \theta^{hk,n})_Y - (a_{ij}v_{i,j}^{hk,n}, \theta^{hk,n})_Y \\ & \quad - (H_{ij}\varphi_i^{hk,n}, \theta_j^{hk,n})_Y - \kappa(e^{hk,n}, \theta^{hk,n})_Y. \end{aligned}$$

Keeping in mind that

$$c(\delta\theta^{hk,n}, \theta^{hk,n})_Y \geq \frac{c}{2k} \left\{ \|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2 \right\},$$

we find that:

$$\begin{aligned} & \frac{c}{2k} \left\{ \|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2 \right\} + (K_{ij}^*\theta_i^{hk,n}, \theta_j^{hk,n})_Y + \frac{1}{2k} \left\{ (K_{ij}\tau_i^{hk,n}, \tau_j^{hk,n})_Y \right. \\ & \quad \left. - (K_{ij}\tau_i^{hk,n-1}, \tau_j^{hk,n-1})_Y + (K_{ij}(\tau_i^{hk,n} - \tau_i^{hk,n-1}), \tau_j^{hk,n} - \tau_j^{hk,n-1})_Y \right\} \end{aligned}$$

$$\begin{aligned}
 &= (d_{ij}M_i^{hk,n}, \theta_j^{hk,n})_Y - (A_{ij}^1M_i^{hk,n}, \theta_j^{hk,n})_Y - (b_{ij}M_{i,j}^{hk,n}, \theta^{hk,n})_Y - (a_{ij}v_{i,j}^{hk,n}, \theta^{hk,n})_Y \\
 &\quad - (H_{ij}\varphi_{i,j}^{hk,n}, \theta_j^{hk,n})_Y - \kappa(e^{hk,n}, \theta^{hk,n})_Y.
 \end{aligned} \tag{32}$$

Finally, taking $\xi_i^h = M_i^{hk,n}$ as a test function in (27), we obtain:

$$\begin{aligned}
 &(c_{ij}\delta M_j^{hk,n}, M_i^{hk,n})_Y + (C_{ijkl}R_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (C_{ijkl}^*M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y \\
 &\quad + (A_{ij}^3M_j^{hk,n}, M_i^{hk,n})_Y - (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y \\
 &\quad - (b_{ij}\theta^{hk,n}, M_{i,j}^{hk,n})_Y + (F_{ij}\varphi^{hk,n}, M_{i,j}^{hk,n})_Y \\
 &\quad + (d_{ij}\theta_j^{hk,n}, M_i^{hk,n})_Y + (\alpha_{ij}e_{i,j}^{hk,n}, M_i^{hk,n})_Y + (A_{ij}^2\theta_{i,j}^{hk,n}, M_i^{hk,n})_Y = 0,
 \end{aligned}$$

and, since, using (11), it follows that:

$$(c_{ij}\delta M_j^{hk,n}, M_i^{hk,n})_Y \geq \frac{C_2}{2k} \left\{ \|M^{hk,n}\|_H^2 - \|M^{hk,n-1}\|_H^2 \right\},$$

we have:

$$\begin{aligned}
 &\frac{C_2}{2k} \left\{ \|M^{hk,n}\|_H^2 - \|M^{hk,n-1}\|_H^2 \right\} + (C_{ijkl}^*M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + \frac{1}{2k} \left\{ (C_{ijkl}R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y \right. \\
 &\quad \left. - (C_{ijkl}R_{k,l}^{hk,n-1}, R_{i,j}^{hk,n-1})_Y + (C_{ijkl}(R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), R_{i,j}^{hk,n} - R_{i,j}^{hk,n-1})_Y \right\} \\
 &= (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (b_{ij}\theta^{hk,n}, M_{i,j}^{hk,n})_Y - (F_{ij}\varphi^{hk,n}, M_{i,j}^{hk,n})_Y - (d_{ij}\theta_{i,j}^{hk,n}, M_i^{hk,n})_Y \\
 &\quad - (\alpha_{ij}e_{i,j}^{hk,n}, M_i^{hk,n})_Y - (A_{ij}^2\theta_{i,j}^{hk,n}, M_i^{hk,n})_Y.
 \end{aligned} \tag{33}$$

Combining now estimates (30)–(33), after easy algebraic manipulations, we find that:

$$\begin{aligned}
 &\frac{\rho}{2k} \left\{ \|v^{hk,n}\|_H^2 - \|v^{hk,n-1}\|_H^2 \right\} + \frac{1}{2k} \left\{ (A_{ijkl}u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y - (A_{ijkl}u_{k,l}^{hk,n-1}, u_{i,j}^{hk,n-1})_Y \right. \\
 &\quad \left. + (A_{ijkl}(u_{k,l}^{hk,n} - u_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y \right\} \\
 &\quad + \frac{J}{2k} \left\{ \|e^{hk,n}\|_Y^2 - \|e^{hk,n-1}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (A_{ij}\varphi_{i,j}^{hk,n}, \varphi_{i,j}^{hk,n})_Y - (A_{ij}\varphi_{i,j}^{hk,n-1}, \varphi_{i,j}^{hk,n-1})_Y \right. \\
 &\quad \left. + (A_{ij}(\varphi_{i,j}^{hk,n} - \varphi_{i,j}^{hk,n-1}), \varphi_{i,j}^{hk,n} - \varphi_{i,j}^{hk,n-1})_Y \right\} \\
 &\quad + \frac{\xi}{2k} \left\{ \|\varphi^{hk,n}\|_Y^2 - \|\varphi^{hk,n-1}\|_Y^2 + \|\varphi^{hk,n} - \varphi^{hk,n-1}\|_Y^2 \right\} \\
 &\quad + \frac{c}{2k} \left\{ \|\theta^{hk,n}\|_Y^2 - \|\theta^{hk,n-1}\|_Y^2 \right\} + (K_{ij}^*\theta_{i,j}^{hk,n}, \theta_{i,j}^{hk,n})_Y + \frac{1}{2k} \left\{ (K_{ij}\tau_{i,j}^{hk,n}, \tau_{i,j}^{hk,n})_Y \right. \\
 &\quad \left. - (K_{ij}\tau_{i,j}^{hk,n-1}, \tau_{i,j}^{hk,n-1})_Y + (K_{ij}(\tau_{i,j}^{hk,n} - \tau_{i,j}^{hk,n-1}), \tau_{i,j}^{hk,n} - \tau_{i,j}^{hk,n-1})_Y \right\} \\
 &\quad + \frac{C_2}{2k} \left\{ \|M^{hk,n}\|_H^2 - \|M^{hk,n-1}\|_H^2 \right\} + (C_{ijkl}^*M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y \\
 &\quad + (\zeta_{ij}u_{i,j}^{hk,n}, e^{hk,n})_Y + (A_{ij}^3M_j^{hk,n}, M_i^{hk,n})_Y + \frac{1}{2k} \left\{ (C_{ijkl}R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y \right. \\
 &\quad \left. - (C_{ijkl}R_{k,l}^{hk,n-1}, R_{i,j}^{hk,n-1})_Y + (C_{ijkl}(R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), R_{i,j}^{hk,n} - R_{i,j}^{hk,n-1})_Y \right\} \\
 &\quad + (B_{ijkl}R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y + (B_{klij}u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + (\zeta_{ij}\varphi^{hk,n}, v_{i,j}^{hk,n})_Y \\
 &\quad + (H_{ij}\tau_{i,j}^{hk,n}, e_{i,j}^{hk,n})_Y + (H_{ij}\varphi_{i,j}^{hk,n}, \theta_j^{hk,n})_Y + ((A_{ij}^1 + A_{ij}^2)\theta_{i,j}^{hk,n}, M_i^{hk,n})_Y \\
 &\leq C \left(\|e^{hk,n}\|_Y^2 + \|\varphi^{hk,n}\|_Y^2 + \|\nabla R^{hk,n}\|_Q^2 \right) + \epsilon \|\nabla M^{hk,n}\|_Q^2,
 \end{aligned}$$

where $\epsilon > 0$ is a positive constant assumed small enough, and C is a generic constant, whose value may change from line to line, and it is independent of the discretization parameters h and k .

Keeping in mind assumptions (10), we find that:

$$(C_{ijkl}^* M_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y + ((A_{ij}^1 + A_{ij}^2)\theta_{,j}^{hk,n}, M_i^{hk,n})_Y + (K_{ij}^* \theta_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y + (A_{ij}^3 M_j^{hk,n}, M_i^{hk,n})_Y \geq C(\|\nabla \theta^{hk,n}\|_H^2 + \|\nabla \mathbf{M}^{hk,n}\|_Q^2 + \|\mathbf{M}^{hk,n}\|_H^2).$$

Observing that:

$$(B_{ijkl} R_{k,l}^{hk,n}, v_{i,j}^{hk,n})_Y + (B_{klij} u_{k,l}^{hk,n}, M_{i,j}^{hk,n})_Y = \frac{1}{k} \left\{ (B_{ijkl} R_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y - (B_{ijkl} R_{k,l}^{hk,n-1}, u_{i,j}^{hk,n-1})_Y + (B_{ijkl} (R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y \right\},$$

$$(\zeta_{ij} \varphi^{hk,n}, v_{i,j}^{hk,n})_Y + (\zeta_{ij} u_{i,j}^{hk,n}, e^{hk,n})_Y = \frac{1}{k} \left\{ (\zeta_{ij} \varphi^{hk,n}, u_{i,j}^{hk,n})_Y - (\zeta_{ij} \varphi^{hk,n-1}, u_{i,j}^{hk,n-1})_Y + (\zeta_{ij} (\varphi^{hk,n} - \varphi^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y \right\},$$

$$(H_{ij} \tau_{,i}^{hk,n}, e_{,j}^{hk,n})_Y + (H_{ij} \varphi_{,i}^{hk,n}, \theta_{,j}^{hk,n})_Y = \frac{1}{k} \left\{ (H_{ij} \tau_{,i}^{hk,n}, \varphi_{,j}^{hk,n})_Y - (H_{ij} \tau_{,i}^{hk,n-1}, \varphi_{,j}^{hk,n-1})_Y + (H_{ij} (\tau_{,i}^{hk,n} - \tau_{,i}^{hk,n-1}), \varphi_{,j}^{hk,n} - \varphi_{,j}^{hk,n-1})_Y \right\},$$

using assumptions (11), it follows that:

$$(A_{ijkl} (u_{k,l}^{hk,n} - u_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y + 2(B_{ijkl} (R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y + (C_{ijkl} (R_{k,l}^{hk,n} - R_{k,l}^{hk,n-1}), R_{i,j}^{hk,n} - R_{i,j}^{hk,n-1})_Y + 2(\zeta_{ij} (\varphi^{hk,n} - \varphi^{hk,n-1}), u_{i,j}^{hk,n} - u_{i,j}^{hk,n-1})_Y + \xi \|\varphi^{hk,n} - \varphi^{hk,n-1}\|_Y^2 \geq 0,$$

$$(A_{ij} \varphi_{,j}^{hk,n} - \varphi_{,j}^{hk,n-1}), \varphi_{,i}^{hk,n} - \varphi_{,i}^{hk,n-1})_Y + 2(H_{ij} (\tau_{,i}^{hk,n} - \tau_{,i}^{hk,n-1}), \varphi_{,j}^{hk,n} - \varphi_{,j}^{hk,n-1})_Y + (K_{ij} (\tau_{,i}^{hk,n} - \tau_{,i}^{hk,n-1}), \tau_{,j}^{hk,n} - \tau_{,j}^{hk,n-1})_Y \geq 0.$$

Therefore, multiplying the above estimates by k and summing up to n , we have:

$$\rho \|v^{hk,n}\|_H^2 + (A_{ijkl} u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y + J \|e^{hk,n}\|_Y^2 + (A_{ij} \varphi_{,j}^{hk,n}, \varphi_{,i}^{hk,n})_Y + \xi \|\varphi^{hk,n}\|_Y^2 + c \|\theta^{hk,n}\|_Y^2 + (K_{ij} \tau_{,i}^{hk,n}, \tau_{,j}^{hk,n})_Y + C_2 \|\mathbf{M}^{hk,n}\|_H^2 + (C_{ijkl} R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y + 2(B_{ijkl} R_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y + 2(\zeta_{ij} \varphi^{hk,n}, u_{i,j}^{hk,n})_Y + 2(H_{ij} \tau_{,i}^{hk,n}, \varphi_{,j}^{hk,n})_Y \leq Ck \sum_{j=1}^n \left(\|e^{hk,j}\|_Y^2 + \|\varphi^{hk,j}\|_Y^2 + \|\nabla \mathbf{R}^{hk,j}\|_Q^2 \right) + C \left(\|v^{0h}\|_H^2 + \|\nabla u^{0h}\|_Q^2 + \|e^{0h}\|_Y^2 + \|\nabla \varphi^{0h}\|_H^2 + \|\varphi^{0h}\|_Y^2 + \|\theta^{0h}\|_Y^2 + \|\nabla \tau^{0h}\|_H^2 + \|\mathbf{M}^{0h}\|_H^2 + \|\nabla \mathbf{R}^{0h}\|_Q^2 \right).$$

Finally, using again assumptions (11), we obtain:

$$(A_{ijkl} u_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y + 2(B_{ijkl} R_{k,l}^{hk,n}, u_{i,j}^{hk,n})_Y + (C_{ijkl} R_{k,l}^{hk,n}, R_{i,j}^{hk,n})_Y + 2(\zeta_{ij} \varphi^{hk,n}, u_{i,j}^{hk,n})_Y + \xi \|\varphi^{hk,n}\|_Y^2 \geq C \left(\|\nabla u^{hk,n}\|_Q^2 + \|\nabla \mathbf{R}^{hk,n}\|_Q^2 + \|\varphi^{hk,n}\|_Y^2 \right),$$

$$(A_{ij} \varphi_{,j}^{hk,n}, \varphi_{,i}^{hk,n})_Y + 2(H_{ij} \tau_{,i}^{hk,n}, \varphi_{,j}^{hk,n})_Y + (K_{ij} \tau_{,i}^{hk,n}, \tau_{,j}^{hk,n})_Y \geq C \left(\|\nabla \varphi^{hk,n}\|_H^2 + \|\nabla \tau^{hk,n}\|_H^2 \right),$$

and so:

$$\begin{aligned} & \| \mathbf{v}^{hk,n} \|_H^2 + \| \nabla \mathbf{u}^{hk,n} \|_Q^2 + \| e^{hk,n} \|_Y^2 + \| \nabla \varphi^{hk,n} \|_H^2 + \| \varphi^{hk,n} \|_Y^2 + \| \theta^{hk,n} \|_Y^2 + \| \nabla \tau^{hk,n} \|_H^2 \\ & \quad + \| \mathbf{M}^{hk,n} \|_H^2 + \| \nabla \mathbf{R}^{hk,n} \|_Q^2 \\ & \leq Ck \sum_{j=1}^n \left(\| e^{hk,j} \|_Y^2 + \| \varphi^{hk,j} \|_Y^2 + \| \nabla \mathbf{R}^{hk,j} \|_Q^2 \right) + C \left(\| \mathbf{v}^{0h} \|_H^2 + \| \nabla \mathbf{u}^{0h} \|_Q^2 + \| e^{0h} \|_Y^2 \right. \\ & \quad \left. + \| \nabla \varphi^{0h} \|_H^2 + \| \varphi^{0h} \|_Y^2 + \| \theta^{0h} \|_Y^2 + \| \nabla \tau^{0h} \|_H^2 + \| \mathbf{M}^{0h} \|_H^2 + \| \nabla \mathbf{R}^{0h} \|_Q^2 \right). \end{aligned}$$

Therefore, applying a discrete version of Gronwall’s inequality (see, e.g., Campo et al. 2006), we deduce the desired stability property.

Now, our aim will be to obtain a priori error estimates on the numerical errors from the approximations given in Problem $V P^{hk}$. We have the following.

Theorem 3 *Let the assumptions of Theorem 2 still hold. If we also assume that:*

$$a_{ij}(\mathbf{x}) = a_{ij}, \quad \alpha_{ij}(\mathbf{x}) = \alpha_{ij} \quad \text{for all } \mathbf{x} \in \Omega, \tag{34}$$

and if we denote by $(\mathbf{v}, e, \theta, \mathbf{M})$ and $(\mathbf{v}^{hk}, e^{hk}, \theta^{hk}, \mathbf{M}^{hk})$ the respective solutions to problems VP and VP^{hk} , then we have the following a priori error estimates for all $\mathbf{w}^h = \{\mathbf{w}^{h,n}\}_{n=0}^N$, $\xi^h = \{\xi^{h,n}\}_{n=0}^N \subset V^h$ and $r^h = \{r^{h,n}\}_{n=0}^N$, $z^h = \{z^{h,n}\}_{n=0}^N \subset E^h$:

$$\begin{aligned} & \max_{0 \leq n \leq N} \left\{ \| \mathbf{v}^n - \mathbf{v}^{hk,n} \|_H^2 + \| \nabla (\mathbf{u}^n - \mathbf{u}^{hk,n}) \|_Q^2 + \| e^n - e^{hk,n} \|_Y^2 + \| \nabla (\varphi^n - \varphi^{hk,n}) \|_H^2 \right. \\ & \quad \left. + \| \theta^n - \theta^{hk,n} \|_Y^2 + \| \varphi^n - \varphi^{hk,n} \|_Y^2 + \| \nabla (\tau^n - \tau^{hk,n}) \|_H^2 + \| \mathbf{M}^n - \mathbf{M}^{hk,n} \|_H^2 \right. \\ & \quad \left. + \| \nabla (\mathbf{R}^n - \mathbf{R}^{hk,n}) \|_Q^2 \right\} \\ & \leq Ck \sum_{j=1}^n \left(\| \dot{\mathbf{v}}^j - \delta \mathbf{v}^j \|_H^2 + \| \nabla (\dot{\mathbf{u}}^j - \delta \mathbf{u}^j) \|_Q^2 + \| \nabla (\mathbf{v}^j - \mathbf{w}^{h,j}) \|_Q^2 + \| \mathbf{v}^j - \mathbf{w}^{h,j} \|_H^2 \right. \\ & \quad + \| \dot{e}^j - \delta e^j \|_Y^2 + \| \nabla (\dot{\varphi}^j - \delta \varphi^j) \|_H^2 + \| \nabla (e^j - r^{h,j}) \|_H^2 + \| e^j - r^{h,j} \|_Y^2 \\ & \quad + \| \dot{\theta}^j - \delta \theta^j \|_Y^2 + \| \nabla (\dot{\tau}^j - \delta \tau^j) \|_H^2 \\ & \quad + \| \nabla (\theta^j - z^{h,j}) \|_H^2 \\ & \quad + \| \theta^j - z^{h,j} \|_Y^2 + \| \dot{\mathbf{M}}^j - \delta \mathbf{M}^j \|_H^2 \\ & \quad \left. + \| \nabla (\dot{\mathbf{R}}^j - \delta \mathbf{R}^j) \|_Q^2 + \| \nabla (\mathbf{M}^j - \xi^{h,j}) \|_Q^2 + \| \mathbf{M}^j - \xi^{h,j} \|_H^2 \right) \\ & \quad + \frac{C}{k} \sum_{j=1}^{N-1} \left\{ \| \mathbf{v}^j - \mathbf{w}^{h,j} - (\mathbf{v}^{j+1} - \mathbf{w}^{h,j+1}) \|_H^2 + \| e^j - r^{h,j} - (e^{j+1} - r^{h,j+1}) \|_Y^2 \right. \\ & \quad \left. + \| \mathbf{M}^j - \xi^{h,j} - (\mathbf{M}^{j+1} - \xi^{h,j+1}) \|_H^2 + \| \theta^j - z^{h,j} - (\theta^{j+1} - z^{h,j+1}) \|_Y^2 \right\} \\ & \quad + C \max_{0 \leq n \leq N} \left\{ \| \mathbf{v}^n - \mathbf{w}^{h,n} \|_H^2 + \| e^n - r^{h,n} \|_Y^2 + \| \theta^n - z^{h,n} \|_Y^2 + \| \mathbf{M}^n - \xi^{h,n} \|_H^2 \right\} \\ & \quad + C \left(\| \mathbf{v}^0 - \mathbf{v}^{0h} \|_H^2 + \| \nabla (\mathbf{u}^0 - \mathbf{u}^{0h}) \|_Q^2 + \| e^0 - e^{0h} \|_Y^2 \right. \\ & \quad + \| \nabla (\varphi^0 - \varphi^{0h}) \|_H^2 + \| \xi^0 - \xi^{0h} \|_Y^2 \\ & \quad \left. + \| \theta^0 - \theta^{0h} \|_Y^2 + \| \nabla (\tau^0 - \tau^{0h}) \|_H^2 + \| \mathbf{M}^0 - \mathbf{M}^{0h} \|_H^2 + \| \nabla (\mathbf{R}^0 - \mathbf{R}^{0h}) \|_Q^2 \right), \tag{35} \end{aligned}$$

where $C > 0$ is a positive constant assumed to be independent of the discretization parameters but depending on the continuous solution.

Remark 1 We note that assumption (34) implies that these terms are homogeneous. Such conditions are found, for instance, in the case that the material is homogeneous and isotropic, that is:

$$a_{ij}(\mathbf{x}) = a\delta_{ij}, \quad \alpha_{ij}(\mathbf{x}) = \alpha\delta_{ij} \quad \text{for all } \mathbf{x} \in \Omega,$$

where a and α are constants and δ_{ij} represents the Kronecker symbol.

Proof First, we will derive the estimates for the velocity. Thus, subtracting variational equation (15) at time t_n for a test function $\mathbf{w} = \mathbf{w}^h \in V^h$ and discrete variational equation (24), it follows that, for all $\mathbf{w}^h = (w_i^h)_{i=1}^d \in V^h$:

$$\begin{aligned} \rho(\dot{v}_i^n - \delta v_i^{hk,n}, w_i^h)_Y + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), w_{i,j}^h)_Y - (a_{ij}(\theta^n - \theta^{hk,n}), w_{i,j}^h)_Y \\ + (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), w_{i,j}^h)_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), w_{i,j}^h)_Y = 0, \end{aligned}$$

and so, for all $\mathbf{w}^h = (w_i^h)_{i=1}^d \in V^h$:

$$\begin{aligned} \rho(\dot{v}_i^n - \delta v_i^{hk,n}, v_i^n - v_i^{hk,n})_Y \\ + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ - (a_{ij}(\theta^n - \theta^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ + (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ = \rho(\dot{v}_i^n - \delta v_i^{hk,n}, v_i^n - w_i^h)_Y \\ + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y \\ - (a_{ij}(\theta^n - \theta^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y \\ + (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y \\ + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v_{i,j}^n - w_{i,j}^h)_Y. \end{aligned}$$

Keeping in mind that:

$$\begin{aligned} (v_i^n - \delta v_i^{hk,n}, v_i^n - v_i^{hk,n})_Y &\geq (\dot{v}_i^n - \delta v_i^n, v_i^n - v_i^{hk,n})_Y \\ &+ \frac{1}{2k} \left\{ \|v^n - v^{hk,n}\|_H^2 - \|v^{n-1} - v^{hk,n-1}\|_H^2 \right\}, \\ (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y &= (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \dot{u}_{i,j}^n - \delta u_{i,j}^n)_Y \\ &+ (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y, \\ (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y &= \frac{1}{2k} \left\{ (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \right. \\ &\quad - (A_{ijkl}(u_{k,l}^{n-1} - u_{k,l}^{hk,n-1}), u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})_Y \\ &\quad \left. + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n} - (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \right\}, \end{aligned}$$

where we used the notations $\delta v^n = (v^n - v^{n-1})/k$ and $\delta u^n = (u^n - u^{n-1})/k$, we obtain the following estimates for the velocity:

$$\begin{aligned}
 & \frac{\rho}{2k} \left\{ \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H^2 - \|\mathbf{v}^{n-1} - \mathbf{v}^{hk,n-1}\|_H^2 \right\} + \frac{1}{2k} \left\{ (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \right. \\
 & \quad - (A_{ijkl}(u_{k,l}^{n-1} - u_{k,l}^{hk,n-1}), u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})_Y \\
 & \quad + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n} - (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \left. \right\} \\
 & \quad + (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\
 & \quad - (a_{ij}(\theta^n - \theta^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\
 & \leq C \left(\|\dot{\mathbf{v}}^n - \delta \mathbf{v}^n\|_H^2 + \|\nabla(\dot{\mathbf{u}}^n - \delta \mathbf{u}^n)\|_Q^2 + \|\nabla(\mathbf{v}^h - \mathbf{w}^h)\|_Q^2 + \|\nabla(\mathbf{u}^n - \mathbf{u}^{hk,n})\|_Q^2 \right. \\
 & \quad + \|\mathbf{v}^n - \mathbf{w}^h\|_H^2 + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2 + \|\nabla(\mathbf{R}^n - \mathbf{R}^{hk,n})\|_Q^2 \\
 & \quad \left. + (\delta v_i^n - \delta v_i^{hk,n}, v_i^n - w_i^h)_Y \right) \quad \forall \mathbf{w}^h = (w_i^h)_{i=1}^d \in V^h. \tag{36}
 \end{aligned}$$

Now, we will derive the estimates on the volume fraction speed. Therefore, subtracting variational equation (16) at time t_n for all $r = r^h \in E^h$ and discrete variational equation (25), we get:

$$\begin{aligned}
 & J(\dot{e}^n - \delta e^{hk,n}, r^h)_Y + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), r_{,i}^h)_Y + \xi(\varphi^n - \varphi^{hk,n}, r^h)_Y \\
 & \quad - (\alpha_{ij}(M_i^n - M_i^{hk,n}), r_{,j}^h)_Y + (H_{ij}(\tau_{,i}^n - \tau_{,i}^{hk,n}), r_{,j}^h)_Y + (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), r^h)_Y \\
 & \quad - \kappa(\theta^n - \theta^{hk,n}, r^h)_Y + (F_{ij}(R_{i,j}^n - R_{i,j}^{hk,n}), r^h)_Y = 0 \quad \forall r^h \in E^h,
 \end{aligned}$$

and so, for all $r^h \in E^h$:

$$\begin{aligned}
 & J(\dot{e}^n - \delta e^{hk,n}, e^n - e^{hk,n})_Y + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), e_{,i}^n - e_{,i}^{hk,n})_Y + \xi(\varphi^n - \varphi^{hk,n}, e^n - e^{hk,n})_Y \\
 & \quad - (\alpha_{ij}(M_i^n - M_i^{hk,n}), e_{,j}^n - e_{,j}^{hk,n})_Y + (H_{ij}(\tau_{,i}^n - \tau_{,i}^{hk,n}), e_{,j}^n - e_{,j}^{hk,n})_Y \\
 & \quad + (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), e^n - e^{hk,n})_Y - \kappa(\theta^n - \theta^{hk,n}, e^n - e^{hk,n})_Y \\
 & \quad + (F_{ij}(R_{i,j}^n - R_{i,j}^{hk,n}), e^n - e^{hk,n})_Y \\
 & = J(\dot{e}^n - \delta e^{hk,n}, e^n - r^h)_Y + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), e_{,i}^n - r_{,i}^h)_Y + \xi(\varphi^n - \varphi^{hk,n}, e^n - r^h)_Y \\
 & \quad - (\alpha_{ij}(M_i^n - M_i^{hk,n}), e_{,j}^n - r_{,j}^h)_Y + (H_{ij}(\tau_{,i}^n - \tau_{,i}^{hk,n}), e_{,j}^n - r_{,j}^h)_Y \\
 & \quad + (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), e^n - r^h)_Y - \kappa(\theta^n - \theta^{hk,n}, e^n - r^h)_Y \\
 & \quad + (F_{ij}(R_{i,j}^n - R_{i,j}^{hk,n}), e^n - r^h)_Y.
 \end{aligned}$$

Taking into account that:

$$\begin{aligned}
 & (e^n - \delta e^{hk,n}, e^n - e^{hk,n})_Y \geq (\dot{e}^n - \delta e^n, e^n - e^{hk,n})_Y \\
 & \quad + \frac{1}{2k} \left\{ \|e^n - e^{hk,n}\|_Y^2 - \|e^{n-1} - e^{hk,n-1}\|_Y^2 \right\}, \\
 & (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), e_{,i}^n - e_{,i}^{hk,n})_Y = (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \dot{\varphi}_{,i}^n - \delta \varphi_{,i}^n)_Y \\
 & \quad + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \delta \varphi_{,i}^n - \delta \varphi_{,i}^{hk,n})_Y, \\
 & (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \delta \varphi_{,i}^n - \delta \varphi_{,i}^{hk,n})_Y = \frac{1}{2k} \left\{ (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \varphi_{,i}^n - \varphi_{,i}^{hk,n})_Y \right. \\
 & \quad - (A_{ij}(\varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1}), \varphi_{,i}^{n-1} - \varphi_{,i}^{hk,n-1})_Y \\
 & \quad \left. + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n} - (\varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1})), \varphi_{,i}^n - \varphi_{,i}^{hk,n} - (\varphi_{,i}^{n-1} - \varphi_{,i}^{hk,n-1}))_Y \right\},
 \end{aligned}$$

$$\begin{aligned}
 (\varphi^n - \varphi^{hk,n}, e^n - e^{hk,n})_Y &= (\varphi^n - \varphi^{hk,n}, \dot{\varphi}^n - \varphi^{hk,n})_Y + \frac{1}{2k} \left\{ \|\xi^n - \xi^{hk,n}\|_Y^2 \right. \\
 &\quad \left. - \|\xi^{n-1} - \xi^{hk,n-1}\|_Y^2 + \|\xi^n - \xi^{hk,n} - (\xi^{n-1} - \xi^{hk,n-1})\|_Y^2 \right\},
 \end{aligned}$$

where we used the notations $\delta e^n = (e^n - e^{n-1})/k$ and $\delta\varphi^n = (\varphi^n - \varphi^{n-1})/k$, we have the following estimates for the volume fraction speed, for all $r^h \in E^h$:

$$\begin{aligned}
 &\frac{J}{2k} \left\{ \|e^n - e^{hk,n}\|_Y^2 - \|e^{n-1} - e^{hk,n-1}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (A_{ij}(\varphi_j^n - \varphi_j^{hk,n}), \varphi_i^n - \varphi_i^{hk,n})_Y \right. \\
 &\quad - (A_{ij}(\varphi_j^{n-1} - \varphi_j^{hk,n-1}), \varphi_i^{n-1} - \varphi_i^{hk,n-1})_Y \\
 &\quad \left. + (A_{ij}(\varphi_j^n - \varphi_j^{hk,n} - (\varphi_j^{n-1} - \varphi_j^{hk,n-1})), \varphi_i^n - \varphi_i^{hk,n} - (\varphi_i^{n-1} - \varphi_i^{hk,n-1}))_Y \right\} \\
 &\quad + \frac{\xi}{2k} \left\{ \|\xi^n - \xi^{hk,n}\|_Y^2 - \|\xi^{n-1} - \xi^{hk,n-1}\|_Y^2 + \|\xi^n - \xi^{hk,n} - (\xi^{n-1} - \xi^{hk,n-1})\|_Y^2 \right\} \\
 &\quad - (\alpha_{ij}(M_i^n - M_i^{hk,n}), e_j^n - e_j^{hk,n})_Y + (H_{ij}(\tau_i^n - \tau_i^{hk,n}), e_j^n - e_j^{hk,n})_Y \\
 &\quad + (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), e^n - e^{hk,n})_Y - \kappa(\theta^n - \theta^{hk,n}, e^n - e^{hk,n})_Y \\
 &\quad + (F_{ij}(R_{i,j}^n - R_{i,j}^{hk,n}), e^n - e^{hk,n})_Y \\
 &\leq C \left(\|\dot{e}^n - \delta e^n\|_Y^2 + \|\nabla(\dot{\varphi}^n - \delta\varphi^n)\|_H^2 + \|\nabla(e^h - r^h)\|_H^2 + \|\nabla(\varphi^n - \varphi^{hk,n})\|_H^2 \right. \\
 &\quad + \|e^n - r^h\|_Y^2 + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2 + \|\nabla(\mathbf{R}^n - \mathbf{R}^{hk,n})\|_Q^2 \\
 &\quad + \|\nabla(\mathbf{u}^n - \mathbf{u}^{hk,n})\|_Q^2 + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 + \|\nabla(\tau^n - \tau^{hk,n})\|_H^2 \\
 &\quad \left. + (\delta e^n - \delta e^{hk,n}, e^n - r^h)_Y \right). \tag{37}
 \end{aligned}$$

Now, we will obtain the error estimates on the temperature. Subtracting variational equation (17) at time $t = t_n$ for a test function $z = z^h \in E^h$ and discrete variational equation, we find that, for all $z^h \in E^h$:

$$\begin{aligned}
 c(\dot{\theta}^n - \delta\theta^{hk,n}, z^h)_Y &+ (K_{ij}^*(\theta_i^n - \theta_i^{hk,n}), z_j^h)_Y + (K_{ij}(\tau_i^n - \tau_i^{hk,n}), z_j^h)_Y \\
 &\quad - (d_{ij}(M_i^n - M_i^{hk,n}), z_j^h)_Y + (A_{ij}^1(M_i^n - M_i^{hk,n}), z_j^h)_Y + (b_{ij}(M_{i,j}^n - M_{i,j}^{hk,n}), z^h)_Y \\
 &\quad + (a_{ij}(v_{i,j}^n - v_{i,j}^{hk,n}), z^h)_Y + (H_{ij}(\varphi_i^n - \varphi_i^{hk,n}), z_j^h)_Y - \kappa(e^n - e^{hk,n}, z^h)_Y = 0,
 \end{aligned}$$

and so, for all $z^h \in V^h$, it follows that:

$$\begin{aligned}
 &c(\dot{\theta}^n - \delta\theta^{hk,n}, \theta^n - \theta^{hk,n})_Y + (K_{ij}^*(\theta_i^n - \theta_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\
 &\quad + (K_{ij}(\tau_i^n - \tau_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\
 &\quad - (d_{ij}(M_i^n - M_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y + (A_{ij}^1(M_i^n - M_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\
 &\quad + (b_{ij}(M_{i,j}^n - M_{i,j}^{hk,n}), \theta^n - \theta^{hk,n})_Y \\
 &\quad + (a_{ij}(v_{i,j}^n - v_{i,j}^{hk,n}), \theta^n - \theta^{hk,n})_Y \\
 &\quad + (H_{ij}(\varphi_i^n - \varphi_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y - \kappa(e^n - e^{hk,n}, \theta^n - \theta^{hk,n})_Y \\
 &= c(\dot{\theta}^n - \delta\theta^{hk,n}, \theta^n - z^h)_Y \\
 &\quad + (K_{ij}^*(\theta_i^n - \theta_i^{hk,n}), \theta_j^n - z_j^h)_Y + (K_{ij}(\tau_i^n - \tau_i^{hk,n}), \theta_j^n - z_j^h)_Y \\
 &\quad - (d_{ij}(M_i^n - M_i^{hk,n}), \theta_j^n - z_j^h)_Y + (A_{ij}^1(M_i^n - M_i^{hk,n}), \theta_j^n - z_j^h)_Y
 \end{aligned}$$

$$\begin{aligned}
 &+(b_{ij}(M_{i,j}^n - M_{i,j}^{hk,n}), \theta^n - z^h)_Y + (a_{ij}(v_{i,j}^n - v_{i,j}^{hk,n}), \theta^n - z^h)_Y \\
 &+(H_{ij}(\varphi_i^n - \varphi_i^{hk,n}), \theta_j^n - z_j^h)_Y - \kappa(e^n - e^{hk,n}, \theta^n - z^h)_Y.
 \end{aligned}$$

Keeping in mind that:

$$\begin{aligned}
 &(\dot{\theta}^n - \delta\theta^{hk,n}, \theta^n - \theta^{hk,n})_Y \geq (\dot{\theta}^n - \delta\theta^n, \theta^n - \theta^{hk,n})_Y \\
 &+\frac{1}{2k} \left\{ \|\theta^n - \theta^{hk,n}\|_Y^2 - \|\theta^{n-1} - \theta^{hk,n-1}\|_Y^2 \right\}, \\
 &(K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \theta_i^n - \theta_i^{hk,n})_Y = (K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \tau_i^n - \delta\tau_i^n)_Y \\
 &+(K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \delta\tau_i^n - \delta\tau_i^{hk,n})_Y, \\
 &(K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \delta\tau_i^n - \delta\tau_i^{hk,n})_Y = \frac{1}{2k} \left\{ (K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \tau_i^n - \tau_i^{hk,n})_Y \right. \\
 &\left. - (K_{ij}(\tau_{i,j}^{n-1} - \tau_{i,j}^{hk,n-1}), \tau_i^{n-1} - \tau_i^{hk,n-1})_Y \right. \\
 &\left. + (K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n} - (\tau_{i,j}^{n-1} - \tau_{i,j}^{hk,n-1})), \tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1}))_Y \right\}, \\
 &(a_{ij}(v_{i,j}^n - v_{i,j}^{hk,n}), \theta^n - z^h)_Y = -(a_{ij}(v_i^n - v_i^{hk,n}), \theta_j^n - z_j^h)_Y,
 \end{aligned}$$

where we used the notations $\delta\theta^n = (\theta^n - \theta^{n-1})/k$ and $\delta\tau^n = (\tau^n - \tau^{n-1})/k$ and assumption (34), we get the following estimates for the temperature, for all $z^h \in E^h$:

$$\begin{aligned}
 &\frac{1}{2k} \left\{ \|\theta^n - \theta^{hk,n}\|_Y^2 - \|\theta^{n-1} - \theta^{hk,n-1}\|_Y^2 \right\} + (K_{ij}^*(\theta_i^n - \theta_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\
 &+\frac{1}{2k} \left\{ (K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n}), \tau_i^n - \tau_i^{hk,n})_Y - (K_{ij}(\tau_{i,j}^{n-1} - \tau_{i,j}^{hk,n-1}), \tau_i^{n-1} - \tau_i^{hk,n-1})_Y \right. \\
 &\left. + (K_{ij}(\tau_{i,j}^n - \tau_{i,j}^{hk,n} - (\tau_{i,j}^{n-1} - \tau_{i,j}^{hk,n-1})), \tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1}))_Y \right\} \\
 &- (d_{ij}(M_i^n - M_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y + (A_{ij}^1(M_i^n - M_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\
 &+(b_{ij}(M_{i,j}^n - M_{i,j}^{hk,n}), \theta^n - \theta^{hk,n})_Y + (a_{ij}(v_{i,j}^n - v_{i,j}^{hk,n}), \theta^n - \theta^{hk,n})_Y \\
 &+(H_{ij}(\varphi_i^n - \varphi_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y - \kappa(e_n - e^{hk,n}, \theta^n - \theta^{hk,n})_Y \\
 &\leq C \left(\|\dot{\theta}^n - \delta\theta^n\|_Y^2 + \|\nabla(\tau^n - \delta\tau^n)\|_H^2 + \|\nabla(\theta^n - z^h)\|_H^2 + \|\nabla(\tau^n - \tau^{hk,n})\|_H^2 \right. \\
 &\left. + \|\theta^n - z^h\|_Y^2 + \|e^n - e^{hk,n}\|_Y^2 + \|\nabla(\varphi^n - \varphi^{hk,n})\|_H^2 + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 \right. \\
 &\left. + \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H^2 + (\delta\theta^n - \delta\theta^{hk,n}, \theta^n - z^h)_Y + \|\theta^n - \theta^{hk,n}\|_Y^2 \right) \\
 &+\epsilon \|\nabla(\theta^n - \theta^{hk,n})\|_H^2 + \epsilon \|\nabla(\mathbf{M}^n - \mathbf{M}^{hk,n})\|_Q^2, \tag{38}
 \end{aligned}$$

where $\epsilon > 0$ is assumed small enough.

Finally, we will obtain the error estimates on the microtemperatures. Therefore, we subtract variational equation (18) at time t_n for a test function $\xi = \xi^h \in V^h$ and discrete variational equation (27) to obtain the following estimates, for all $\xi^h = (\xi_i^h)_{i=1}^d \in V^h$:

$$\begin{aligned}
 &(c_{ij}(\dot{M}_j^n - \delta M_j^{hk,n}), \xi_i^h)_Y + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), \xi_{i,j}^h) + (C_{ijkl}^*(M_{k,l}^n - M_{k,l}^{hk,n}), \xi_{i,j}^h)_Y \\
 &+(A_{ij}^3(M_j^n - M_j^{hk,n}), \xi_i^h)_Y - (b_{ij}(\theta^n - \theta^{hk,n}), \xi_{i,j}^h)_Y \\
 &+(F_{ij}(\varphi^n - \varphi^{hk,n}), \xi_{i,j}^h)_Y + (d_{ij}(\theta_j^n - \theta_j^{hk,n}), \xi_i^h)_Y + (\alpha_{ij}(e_j^n - e_j^{hk,n}), \xi_i^h)_Y \\
 &+(A_{ij}^2(\theta_j^n - \theta_j^{hk,n}), \xi_i^h)_Y + (B_{klij}(u_{k,l}^n - u_{k,l}^{hk,n}), \xi_{i,j}^h)_Y = 0,
 \end{aligned}$$

and so, for all $\xi^h = (\xi_i^h)_{i=1}^d \in V^h$, it follows that:

$$\begin{aligned} & (c_{ij}(\dot{M}_j^n - \delta M_j^{hk,n}), M_i^n - M_i^{hk,n})_Y + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n}) \\ & + (C_{ijkl}^*(M_{k,l}^n - M_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y + (A_{ij}^3(M_j^n - M_j^{hk,n}), M_i^n - M_i^{hk,n})_Y \\ & - (b_{ij}(\theta^n - \theta^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y + (F_{ij}(\varphi^n - \varphi^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ & + (d_{ij}(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - M_i^{hk,n})_Y + (\alpha_{ij}(e_{,j}^n - e_{,j}^{hk,n}), M_i^n - M_i^{hk,n})_Y \\ & + (A_{ij}^2(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - M_i^{hk,n})_Y + (B_{klij}(u_{k,l}^n - u_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ = & (c_{ij}(\dot{M}_j^n - \delta M_j^{hk,n}), M_i^n - \xi_i^h)_Y + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y \\ & + (C_{ijkl}^*(M_{k,l}^n - M_{k,l}^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y + (A_{ij}^3(M_j^n - M_j^{hk,n}), M_i^n - \xi_i^h)_Y \\ & - (b_{ij}(\theta^n - \theta^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y + (F_{ij}(\varphi^n - \varphi^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y \\ & + (d_{ij}(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - \xi_i^h)_Y + (\alpha_{ij}(e_{,j}^n - e_{,j}^{hk,n}), M_i^n - \xi_i^h)_Y \\ & + (A_{ij}^2(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - \xi_i^h)_Y + (B_{klij}(u_{k,l}^n - u_{k,l}^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y. \end{aligned}$$

Keeping in mind that:

$$\begin{aligned} & (c_{ij}(\dot{M}_j^n - \delta M_j^{hk,n}), M_i^n - M_i^{hk,n})_Y \geq (c_{ij}(\dot{M}_j^n - \delta M_j^n), M_i^n - M_i^{hk,n})_Y \\ & + \frac{C_2}{2k} \left\{ \|M^n - M^{hk,n}\|_H^2 - \|M^{n-1} - M^{hk,n-1}\|_H^2 \right\}, \\ & (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y = (C_{ijkl}(R_{k,l}^n - R_{k,l}^n), \dot{R}_{i,j}^n - \delta R_{i,j}^n)_Y \\ & + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), \delta R_{i,j}^n - \delta R_{i,j}^{hk,n})_Y, \\ & (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), \delta R_{i,j}^n - \delta R_{i,j}^{hk,n})_Y = \frac{1}{2k} \left\{ (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), R_{i,j}^n - R_{i,j}^{hk,n})_Y \right. \\ & - (C_{ijkl}(R_{k,l}^{n-1} - R_{k,l}^{hk,n-1}), R_{i,j}^{n-1} - R_{i,j}^{hk,n-1})_Y \\ & \left. + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), R_{i,j}^n - R_{i,j}^{hk,n} - (R_{i,j}^{n-1} - R_{i,j}^{hk,n-1}))_Y \right\}, \\ & (\alpha_{ij}(e_{,j}^n - e_{,j}^{hk,n}), M_i^n - \xi_i^h)_Y = -(\alpha_{ij}(e^n - e^{hk,n}), M_{i,j}^n - \xi_{i,j}^h)_Y, \end{aligned}$$

where we used the notations $\delta M^n = (M^n - M^{n-1})/k$ and $\delta R^n = (R^n - R^{n-1})/k$, and assumption (34), we obtain the following estimates for the microtemperatures, for all $\xi^h = (\xi_i^h)_{i=1}^d \in V^h$:

$$\begin{aligned} & \frac{C_2}{2k} \left\{ \|M^n - M^{hk,n}\|_H^2 - \|M^{n-1} - M^{hk,n-1}\|_H^2 \right\} + (C_{ijkl}^*(M_{k,l}^n - M_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ & + \frac{1}{2k} \left\{ (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), R_{i,j}^n - R_{i,j}^{hk,n})_Y \right. \\ & - (C_{ijkl}(R_{k,l}^{n-1} - R_{k,l}^{hk,n-1}), R_{i,j}^{n-1} - R_{i,j}^{hk,n-1})_Y \\ & \left. + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), R_{i,j}^n - R_{i,j}^{hk,n} - (R_{i,j}^{n-1} - R_{i,j}^{hk,n-1}))_Y \right\} \\ & + (A_{ij}^3(M_j^n - M_j^{hk,n}), M_i^n - M_i^{hk,n})_Y \\ & - (b_{ij}(\theta^n - \theta^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y + (F_{ij}(\varphi^n - \varphi^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ & + (d_{ij}(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - M_i^{hk,n})_Y - (\alpha_{ij}(e^n - e^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ & + (A_{ij}^2(\theta_{,j}^n - \theta_{,j}^{hk,n}), M_i^n - M_i^{hk,n})_Y + (B_{klij}(u_{k,l}^n - u_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \end{aligned}$$

$$\begin{aligned} &\leq C \left(\|\dot{\mathbf{M}}^n - \delta \mathbf{M}^n\|_H^2 + \|\nabla(\dot{\mathbf{R}}^n - \delta \mathbf{R}^n)\|_Q^2 + \|\nabla(\mathbf{M}^h - \xi^h)\|_Q^2 + \|\nabla(\mathbf{u}^n - \mathbf{u}^{hk,n})\|_Q^2 \right. \\ &\quad + \|\mathbf{M}^n - \xi^h\|_H^2 + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2 + \|\nabla(\mathbf{R}^n - \mathbf{R}^{hk,n})\|_Q^2 \\ &\quad + \|e^n - e^{hk,n}\|_Y^2 + (\delta M_i^n - \delta M_i^{hk,n}, M_i^n - \xi_i^h)_Y + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 \Big) \\ &\quad + \epsilon \|\nabla(\theta^n - \theta^{hk,n})\|_H^2 + \epsilon \|\nabla(\mathbf{M}^n - \mathbf{M}^{hk,n})\|_Q^2, \end{aligned} \tag{39}$$

where $\epsilon > 0$ is assumed small enough.

Using assumption (10), we find that:

$$\begin{aligned} &(C_{ijkl}^* (M_{k,l}^n - M_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y + ((A_{ij}^1 + A_{ij}^2)(\theta_j^n - \theta_j^{hk,n}), M_i^n - M_i^{hk,n})_Y \\ &\quad + (K_{ij}^* (\theta_i^n - \theta_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y + (A_{ij}^3 (M_j^n - M_j^{hk,n}), M_i^n - M_i^{hk,n})_Y \\ &\quad \geq K_0 (\|\nabla(\theta^n - \theta^{hk,n})\|_H^2 + \|\nabla(\mathbf{M}^n - \mathbf{M}^{hk,n})\|_Q^2 + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2). \end{aligned}$$

Therefore, combining estimates (36)–(39), it follows that:

$$\begin{aligned} &\frac{\rho}{2k} \left\{ \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H^2 - \|\mathbf{v}^{n-1} - \mathbf{v}^{hk,n-1}\|_H^2 \right\} + \frac{1}{2k} \left\{ (A_{ijkl} (u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \right. \\ &\quad - (A_{ijkl} (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1}), u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})_Y \\ &\quad + (A_{ijkl} (u_{k,l}^n - u_{k,l}^{hk,n} - (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \Big\} \\ &\quad + (B_{ijkl} (R_{k,l}^n - R_{k,l}^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y + (B_{klij} (u_{k,l}^n - u_{k,l}^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ &\quad + \frac{J}{2k} \left\{ \|e^n - e^{hk,n}\|_Y^2 - \|e^{n-1} - e^{hk,n-1}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (A_{ij} (\varphi_j^n - \varphi_j^{hk,n}), \varphi_i^n - \varphi_i^{hk,n})_Y \right. \\ &\quad - (A_{ij} (\varphi_j^{n-1} - \varphi_j^{hk,n-1}), \varphi_i^{n-1} - \varphi_i^{hk,n-1})_Y \\ &\quad + (A_{ij} (\varphi_j^n - \varphi_j^{hk,n} - (\varphi_j^{n-1} - \varphi_j^{hk,n-1})), \varphi_i^n - \varphi_i^{hk,n} - (\varphi_i^{n-1} - \varphi_i^{hk,n-1}))_Y \Big\} \\ &\quad + \frac{\xi}{2k} \left\{ \|\xi^n - \xi^{hk,n}\|_Y^2 - \|\xi^{n-1} - \xi^{hk,n-1}\|_Y^2 + \|\xi^n - \xi^{hk,n} - (\xi^{n-1} - \xi^{hk,n-1})\|_Y^2 \right\} \\ &\quad + (H_{ij} (\tau_i^n - \tau_i^{hk,n}), e_j^n - e_j^{hk,n})_Y + (H_{ij} (\varphi_i^n - \varphi_i^{hk,n}), \theta_j^n - \theta_j^{hk,n})_Y \\ &\quad + (\zeta_{ij} (u_{i,j}^n - u_{i,j}^{hk,n}), e^n - e^{hk,n})_Y + (\zeta_{ij} (\varphi^n - \varphi^{hk,n}), v_{i,j}^n - v_{i,j}^{hk,n})_Y \\ &\quad + (F_{ij} (R_{i,j}^n - R_{i,j}^{hk,n}), e^n - e^{hk,n})_Y + (F_{ij} (\varphi^n - \varphi^{hk,n}), M_{i,j}^n - M_{i,j}^{hk,n})_Y \\ &\quad + \frac{1}{2k} \left\{ \|\theta^n - \theta^{hk,n}\|_Y^2 - \|\theta^{n-1} - \theta^{hk,n-1}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (K_{ij} (\tau_j^n - \tau_j^{hk,n}), \tau_i^n - \tau_i^{hk,n})_Y \right. \\ &\quad - (K_{ij} (\tau_j^{n-1} - \tau_j^{hk,n-1}), \tau_i^{n-1} - \tau_i^{hk,n-1})_Y \\ &\quad + (K_{ij} (\tau_j^n - \tau_j^{hk,n} - (\tau_j^{n-1} - \tau_j^{hk,n-1})), \tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1}))_Y \Big\} \\ &\quad + \frac{C_2}{2k} \left\{ \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 - \|\mathbf{M}^{n-1} - \mathbf{M}^{hk,n-1}\|_H^2 \right\} \\ &\quad + \frac{1}{2k} \left\{ (C_{ijkl} (R_{k,l}^n - R_{k,l}^{hk,n}), R_{i,j}^n - R_{i,j}^{hk,n})_Y - (C_{ijkl} (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1}), R_{i,j}^{n-1} - R_{i,j}^{hk,n-1})_Y \right. \\ &\quad + (C_{ijkl} (R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), R_{i,j}^n - R_{i,j}^{hk,n} - (R_{i,j}^{n-1} - R_{i,j}^{hk,n-1}))_Y \Big\} \\ &\leq C \left(\|\dot{\mathbf{v}}^n - \delta \mathbf{v}^n\|_H^2 + \|\nabla(\dot{\mathbf{u}}^n - \delta \mathbf{u}^n)\|_Q^2 + \|\nabla(\mathbf{v}^h - \mathbf{w}^h)\|_Q^2 + \|\nabla(\mathbf{u}^n - \mathbf{u}^{hk,n})\|_Q^2 \right. \\ &\quad + \|\mathbf{v}^n - \mathbf{w}^h\|_H^2 + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2 + \|\nabla(\mathbf{R}^n - \mathbf{R}^{hk,n})\|_Q^2 \end{aligned}$$

$$\begin{aligned}
 & +(\delta v_i^n - \delta v_i^{hk,n}, v_i^n - w_i^h)_Y + \|\dot{e}^n - \delta e^n\|_Y^2 + \|\nabla(\dot{\varphi}^n - \delta \varphi^n)\|_H^2 + \|\nabla(e^h - r^h)\|_H^2 \\
 & +\|\nabla(\varphi^n - \varphi^{hk,n})\|_H^2 + \|e^n - r^h\|_Y^2 + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 + \|\nabla(\tau^n - \tau^{hk,n})\|_H^2 \\
 & +(\delta e^n - \delta e^{hk,n}, e^n - r^h)_Y + \|\dot{\theta}^n - \delta \theta^n\|_Y^2 + \|\nabla(\dot{\tau}^n - \delta \tau^n)\|_H^2 + \|\nabla(\theta^n - z^h)\|_H^2 \\
 & +\|\theta^n - z^h\|_Y^2 + \|e^n - e^{hk,n}\|_Y^2 + \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H^2 + (\delta \theta^n - \delta \theta^{hk,n}, \theta^n - z^h)_Y \\
 & +\|\dot{\mathbf{M}}^n - \delta \mathbf{M}^n\|_H^2 + \|\nabla(\dot{\mathbf{R}}^n - \delta \mathbf{R}^n)\|_Q^2 + \|\nabla(\mathbf{M}^h - \xi^h)\|_Q^2 + \|\mathbf{M}^n - \xi^h\|_H^2 \\
 & +(\delta M_i^n - \delta M_i^{hk,n}, M_i^n - \xi_i^h)_Y \quad \forall w^h, \xi^h \in V^h, r^h, z^h \in E^h. \tag{40}
 \end{aligned}$$

Now, we observe that:

$$\begin{aligned}
 & (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y + (B_{klij}(u_{k,l}^n - u_{k,l}^{hk,n}), \delta R_{i,j}^n - \delta R_{i,j}^{hk,n})_Y \\
 & = \frac{1}{k} \left\{ (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y - (B_{ijkl}(R_{k,l}^{n-1} - R_{k,l}^{hk,n-1}), u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})_Y \right. \\
 & \quad \left. + (B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \right\}, \\
 & (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), \delta \varphi^n - \delta \varphi^{hk,n})_Y + (\zeta_{ij}(\varphi^n - \varphi^{hk,n}), \delta u_{i,j}^n - \delta u_{i,j}^{hk,n})_Y \\
 & = \frac{1}{k} \left\{ (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), \varphi^n - \varphi^{hk,n})_Y - (\zeta_{ij}(u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}), \varphi^{n-1} - \varphi^{hk,n-1})_Y \right. \\
 & \quad \left. + (\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1})), \varphi^n - \varphi^{hk,n} - (\varphi^{n-1} - \varphi^{hk,n-1}))_Y \right\}, \\
 & (H_{ij}(\tau_i^n - \tau_i^{hk,n}), \delta \varphi_{,j}^n - \delta \varphi_{,j}^{hk,n})_Y + (H_{ij}(\varphi_{,i}^n - \varphi_{,i}^{hk,n}), \delta \tau_{,j}^n - \delta \tau_{,j}^{hk,n})_Y \\
 & = \frac{1}{k} \left\{ (H_{ij}(\tau_i^n - \tau_i^{hk,n}), \varphi_{,j}^n - \varphi_{,j}^{hk,n})_Y - (H_{ij}(\tau_i^{n-1} - \tau_i^{hk,n-1}), \varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1})_Y \right. \\
 & \quad \left. + (H_{ij}(\tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1})), \varphi_{,j}^n - \varphi_{,j}^{hk,n} - (\varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1}))_Y \right\},
 \end{aligned}$$

so using again assumptions (11), it follows that:

$$\begin{aligned}
 & (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n} - (u_{k,l}^{n-1} - u_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \\
 & + 2(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \\
 & + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n} - (R_{k,l}^{n-1} - R_{k,l}^{hk,n-1})), R_{i,j}^n - R_{i,j}^{hk,n} - (R_{i,j}^{n-1} - R_{i,j}^{hk,n-1}))_Y \\
 & + 2(\zeta_{ij}(\varphi^n - \varphi^{hk,n} - (\varphi^{n-1} - \varphi^{hk,n-1})), u_{i,j}^n - u_{i,j}^{hk,n} - (u_{i,j}^{n-1} - u_{i,j}^{hk,n-1}))_Y \\
 & + \xi \|\varphi^n - \varphi^{hk,n} - (\varphi^{n-1} - \varphi^{hk,n-1})\|_Y^2 \geq 0, \\
 & (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n} - (\varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1})), \varphi_{,i}^n - \varphi_{,i}^{hk,n} - (\varphi_{,i}^{n-1} - \varphi_{,i}^{hk,n-1}))_Y \\
 & + 2(H_{ij}(\tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1})), \varphi_{,j}^n - \varphi_{,j}^{hk,n} - (\varphi_{,j}^{n-1} - \varphi_{,j}^{hk,n-1}))_Y \\
 & + (K_{ij}(\tau_i^n - \tau_i^{hk,n} - (\tau_i^{n-1} - \tau_i^{hk,n-1})), \tau_{,j}^n - \tau_{,j}^{hk,n} - (\tau_{,j}^{n-1} - \tau_{,j}^{hk,n-1}))_Y \geq 0.
 \end{aligned}$$

Multiplying estimates (40) by k and summing up to n , we have:

$$\begin{aligned}
 & \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H^2 + (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \\
 & + 2(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \\
 & + \|e^n - e^{hk,n}\|_Y^2 + (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \varphi_{,i}^n - \varphi_{,i}^{hk,n})_Y + \|\xi^n - \xi^{hk,n}\|_Y^2 \\
 & + 2(H_{ij}(\tau_i^n - \tau_i^{hk,n}), \varphi_{,j}^n - \varphi_{,j}^{hk,n})_Y + 2(\zeta_{ij}(u_{i,j}^n - u_{i,j}^{hk,n}), e^n - e^{hk,n})_Y \\
 & + (K_{ij}(\tau_i^n - \tau_i^{hk,n}), \tau_{,j}^n - \tau_{,j}^{hk,n})_Y + \|\theta^n - \theta^{hk,n}\|_Y^2 + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H^2 \\
 & + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), R_{i,j}^n - R_{i,j}^{hk,n})_Y \\
 & \leq Ck \sum_{j=1}^n \left(\|\dot{\mathbf{v}}^j - \delta \mathbf{v}^j\|_H^2 + \|\nabla(\dot{\mathbf{u}}^j - \delta \mathbf{u}^j)\|_Q^2 + \|\nabla(\mathbf{v}^j - \mathbf{w}^{h,j})\|_Q^2 + \|\nabla(\mathbf{u}^j - \mathbf{u}^{hk,j})\|_Q^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 & +\|v^j - w^{h,j}\|_H^2 + \|\theta^j - \theta^{hk,j}\|_Y^2 + \|\varphi^j - \varphi^{hk,j}\|_Y^2 + \|\nabla(\mathbf{R}^j - \mathbf{R}^{hk,j})\|_Q^2 \\
 & +(\delta v_i^j - \delta v_i^{hk,j}, v_i^j - w_i^{h,j})_Y + \|\dot{e}^j - \delta e^j\|_Y^2 + \|\nabla(\dot{\varphi}^j - \delta\varphi^j)\|_H^2 + \|\nabla(e^j - r^{h,j})\|_H^2 \\
 & +\|\nabla(\varphi^j - \varphi^{hk,j})\|_H^2 + \|e^j - r^{h,j}\|_Y^2 + \|\mathbf{M}^j - \mathbf{M}^{hk,j}\|_H^2 + \|\nabla(\tau^j - \tau^{hk,j})\|_H^2 \\
 & +(\delta e^j - \delta e^{hk,j}, e^j - r^{h,j})_Y + \|\dot{\theta}^j - \delta\theta^j\|_Y^2 + \|\nabla(\dot{\tau}^j - \delta\tau^j)\|_H^2 + \|\nabla(\theta^j - z^{h,j})\|_H^2 \\
 & +\|\theta^j - z^{h,j}\|_Y^2 + \|e^j - e^{hk,j}\|_Y^2 + \|v^j - v^{hk,j}\|_H^2 + (\delta\theta^j - \delta\theta^{hk,j}, \theta^j - z^{h,j})_Y \\
 & +\|\dot{\mathbf{M}}^j - \delta\mathbf{M}^j\|_H^2 + \|\nabla(\dot{\mathbf{R}}^j - \delta\mathbf{R}^j)\|_Q^2 + \|\nabla(\mathbf{M}^j - \xi^{h,j})\|_Q^2 + \|\mathbf{M}^j - \xi^{h,j}\|_H^2 \\
 & +(\delta M_i^j - \delta M_i^{hk,j}, M_i^j - \xi_i^{h,j})_Y) \\
 & +C\left(\|v^0 - v^{0h}\|_H^2 + \|\nabla(u^0 - u^{0h})\|_Q^2 + \|e^0 - e^{0h}\|_Y^2\right. \\
 & \quad +\|\nabla(\varphi^0 - \varphi^{0h})\|_H^2 + \|\xi^0 - \xi^{0h}\|_Y^2 + \|\theta^0 - \theta^{0h}\|_Y^2 \\
 & \quad +\|\nabla(\tau^0 - \tau^{0h})\|_H^2 + \|\mathbf{M}^0 - \mathbf{M}^{0h}\|_H^2 \\
 & \quad \left. +\|\nabla(\mathbf{R}^0 - \mathbf{R}^{0h})\|_Q^2\right) \quad \forall w^h, \xi^h \in V^h, r^h, z^h \in E^h.
 \end{aligned}$$

Finally, using assumptions (11), we find that:

$$\begin{aligned}
 & (A_{ijkl}(u_{k,l}^n - u_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y + 2(B_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \\
 & + (C_{ijkl}(R_{k,l}^n - R_{k,l}^{hk,n}), R_{i,j}^n - R_{i,j}^{hk,n})_Y + 2(\zeta_{ij}(\varphi^n - \varphi^{hk,n}), u_{i,j}^n - u_{i,j}^{hk,n})_Y \\
 & + \xi \|\varphi^n - \varphi^{hk,n}\|_Y^2 \geq C\left(\|\nabla(u^n - u^{hk,n})\|_Q^2 + \|\nabla(R^n - R^{hk,n})\|_H^2 + \|\varphi^n - \varphi^{hk,n}\|_Y^2\right), \\
 & (A_{ij}(\varphi_{,j}^n - \varphi_{,j}^{hk,n}), \varphi_{,i}^n - \varphi_{,i}^{hk,n})_Y + 2(H_{ij}(\tau_{,i}^n - \tau_{,i}^{hk,n}), \varphi_{,j}^n - \varphi_{,j}^{hk,n})_Y \\
 & + (K_{ij}(\tau_{,i}^n - \tau_{,i}^{hk,n}), \tau_{,j}^n - \tau_{,j}^{hk,n})_Y \geq C\left(\|\nabla(\varphi^n - \varphi^{hk,n})\|_H^2 + \|\nabla(\tau^n - \tau^{hk,n})\|_H^2\right).
 \end{aligned}$$

Therefore, taking into account that:

$$\begin{aligned}
 k \sum_{j=1}^n (\delta v^j - \delta v^{hk,j}, v^j - w^{h,j})_H &= \sum_{j=1}^n (v^j - v^{hk,j} - (v^{j-1} - v^{hk,j-1}), v^j - w^{h,j})_H \\
 &= (v^n - v^{hk,n}, v^n - w^{h,n})_H + (v^{0h} - v^0, v^1 - w^{h,1})_H \\
 &\quad + \sum_{j=1}^{n-1} (v^j - v^{hk,j}, v^j - w^{h,j} - (v^{j+1} - w^{h,j+1}))_H, \\
 k \sum_{j=1}^n (\delta e^j - \delta e^{hk,j}, e^j - r^{h,j})_Y &= \sum_{j=1}^n (e^j - e^{hk,j} - (e^{j-1} - e^{hk,j-1}), e^j - r^{h,j})_Y \\
 &= (e^n - e^{hk,n}, e^n - r^{h,n})_Y + (e^{0h} - e^0, e^1 - r^{h,1})_Y \\
 &\quad + \sum_{j=1}^{n-1} (e^j - e^{hk,j}, e^j - r^{h,j} - (e^{j+1} - r^{h,j+1}))_Y, \\
 k \sum_{j=1}^n (\delta\theta^j - \delta\theta^{hk,j}, \theta^j - z^{h,j})_Y &= \sum_{j=1}^n (\theta^j - \theta^{hk,j} - (\theta^{j-1} - \theta^{hk,j-1}), \theta^j - z^{h,j})_Y \\
 &= (\theta^n - \theta^{hk,n}, \theta^n - z^{h,n})_Y + (\theta^{0h} - \theta^0, \theta^1 - z^{h,1})_Y
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^{n-1} (\theta^j - \theta^{hk,j}, \theta^j - r^{h,j} - (\theta^{j+1} - r^{h,j+1}))_Y, \\
 k \sum_{j=1}^n (\delta M^j - \delta M^{hk,j}, M^j - \xi^{h,j})_H \\
 & = \sum_{j=1}^n (M^j - M^{hk,j} - (M^{j-1} - M^{hk,j-1}), M^j - \xi^{h,j})_H \\
 & = (M^n - M^{hk,n}, M^n - \xi^{h,n})_H + (M^{0h} - M^0, M^1 - \xi^{h,1})_H \\
 & \quad + \sum_{j=1}^{n-1} (M^j - M^{hk,j}, M^j - \xi^{h,j} - (M^{j+1} - \xi^{h,j+1}))_H,
 \end{aligned}$$

applying again a discrete version of Gronwall’s inequality (see Campo et al. 2006), we derive error estimates (35).

We remark that these a priori error estimates can be used to obtain the convergence order of the approximations given by Problem VP^{hk} . Thus, as an example, we have the following result which states the linear convergence of the algorithm under suitable additional regularity conditions.

Corollary 1 *Let the assumptions of Theorem 2 hold. Therefore, if we assume the following additional regularity:*

$$\begin{aligned}
 \mathbf{u}, \mathbf{R} & \in H^3(0, T; H) \cap W^{1,\infty}(0, T; [H^2(\Omega)]^d) \cap H^2(0, T; V), \\
 \theta, \tau & \in H^2(0, T; Y) \cap L^\infty(0, T; H^2(\Omega)) \cap H^1(0, T; E),
 \end{aligned}$$

it follows that the approximations obtained by Problem VP^{hk} are linearly convergent; that is, there exists a positive constant C , independent of the discretization parameters h and k , such that:

$$\begin{aligned}
 \max_{0 \leq n \leq N} \left\{ \|\mathbf{v}^n - \mathbf{v}^{hk,n}\|_H + \|\nabla(\mathbf{u}^n - \mathbf{u}^{hk,n})\|_Q + \|e^n - e^{hk,n}\|_Y + \|\nabla(\varphi^n - \varphi^{hk,n})\|_H \right. \\
 \left. + \|\theta^n - \theta^{hk,n}\|_Y + \|\varphi^n - \varphi^{hk,n}\|_Y + \|\nabla(\tau^n - \tau^{hk,n})\|_H + \|\mathbf{M}^n - \mathbf{M}^{hk,n}\|_H \right. \\
 \left. + \|\nabla(\mathbf{R}^n - \mathbf{R}^{hk,n})\|_Q \right\} \leq C(h + k).
 \end{aligned}$$

4 Numerical results

In this final section, we will present some numerical results obtained in one- and two-dimensional examples.

In the numerical resolution, we assume that the material is homogeneous and isotropic, and therefore, the tensors defined in Problem P can be simplified. In particular, we will assume the following form for all of them:

$$\begin{aligned}
 A_{ijkl}u_{k,l}w_{i,j} & = (\lambda + \mu)u_{i,i}w_{i,i} + \mu u_{i,j}w_{i,j} \quad \text{for all } \mathbf{u} = (u_i)_{i=1}^d, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 a_{ij}\theta w_{i,j} & = a\theta w_{i,i} \quad \text{for all } \theta \in E, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 B_{ijkl}R_{k,l}w_{i,j} & = BR_{i,j}w_{i,j} \quad \text{for all } \mathbf{R} = (R_i)_{i=1}^d, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 \zeta_{ij}\varphi w_{i,j} & = \zeta\varphi w_{i,i} \quad \text{for all } \varphi \in E, \mathbf{w} = (w_i)_{i=1}^d \in V,
 \end{aligned}$$

$$\begin{aligned}
 A_{ij}\varphi_{,j}r_{,i} &= A\varphi_{,i}r_{,i} \quad \text{for all } \varphi, r \in E, \\
 \alpha_{ij}M_{i,r,j} &= \alpha M_{i,r,i} \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d \in V, r \in E, \\
 H_{ij}z_{,i}r_{,j} &= Hz_{,i}r_{,i} \quad \text{for all } z, r \in E, \\
 F_{ij}R_{i,j}r &= FR_{i,i}r \quad \text{for all } \mathbf{R} = (R_i)_{i=1}^d \in V, r \in E, \\
 K_{ij}^*\theta_{,j}z_{,i} &= K^*\theta_{,i}z_{,i} \quad \text{for all } \theta, z \in E, \\
 K_{ij}\tau_{,j}z_{,i} &= K\tau_{,i}z_{,i} \quad \text{for all } \tau, z \in E, \\
 d_{ij}M_{i,z,j} &= DM_{i,z,i} \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d \in V, z \in E, \\
 A_{ij}^1M_{i,z,j} &= A^1M_{i,z,i} \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d \in V, z \in E, \\
 b_{ij}M_{i,z,j} &= bM_{i,z,i} \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d \in V, z \in E, \\
 C_{ijkl}R_{k,l}w_{i,j} &= C_1R_{i,i}w_{i,i} + C_2R_{i,j}w_{i,j} \quad \text{for all } \mathbf{R} = (R_i)_{i=1}^d, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 C_{ijkl}^*R_{k,l}w_{i,j} &= C_1^*R_{i,i}w_{i,i} + C_2^*R_{i,j}w_{i,j} \quad \text{for all } \mathbf{R} = (R_i)_{i=1}^d, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 A_{ij}^2\theta_{,j}w_i &= A^2\theta_{,i}w_i \quad \text{for all } \theta \in E, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 A_{ij}^3M_jw_i &= A^3M_iw_i \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d, \mathbf{w} = (w_i)_{i=1}^d \in V, \\
 c_{ij}M_j\xi_i &= c^*M_i\xi_i \quad \text{for all } \mathbf{M} = (M_i)_{i=1}^d, \boldsymbol{\xi} = (\xi_i)_{i=1}^d \in V.
 \end{aligned}$$

Hence, using these tensors, discrete problem $V^{P^{hk}}$ leads to a linear system for a variable U in an adequate product space which is solved using classical Cholesky’s method. This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical 1D run ($h = k = 0.01$) took about 0.622 s of CPU time; meanwhile, a typical 2D run took about 3.66 s of CPU time.

4.1 Numerical convergence and asymptotic behavior in a one-dimensional problem

As a simpler one-dimensional case, we will consider the following one-dimensional version of Problem P using the isotropic and homogeneous expressions given above. We note that, in some cases, the coefficients are collected together, because they lead to the same term.

Problem P^{1D} . Find the displacement $u : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, the volume fraction $\varphi : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, the thermal displacement $\tau : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, and the microthermal displacement $R : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$, such that:

$$\begin{aligned}
 \rho\ddot{u} &= \mu u_{xx} + \zeta\varphi_x - a\dot{\tau}_x + BR_{xx} + F_1 \quad \text{in } (0, 1) \times (0, 1), \\
 J\ddot{\varphi} &= A\varphi_{xx} - \zeta u_x - \xi\varphi + \kappa\dot{\tau} + H\tau_{xx} - FR_x - \alpha\dot{R}_x + F_2 \quad \text{in } (0, 1) \times (0, 1), \\
 c\ddot{\tau} &= K\tau_{xx} + H\varphi_{xx} - a\dot{u}_x - \kappa\dot{\varphi} + A^1\dot{R}_x + K^*\dot{\tau}_{xx} - b\dot{R}_x - DR_x + F_3 \\
 &\quad \text{in } (0, 1) \times (0, 1), \\
 c^*\ddot{R} &= CR_{xx} + Bu_{xx} + F\varphi_x - \alpha\dot{\varphi}_x - D\dot{\tau}_x + C^*\dot{R}_{xx} - A^3\dot{R} - b\dot{\tau}_x - A^2\dot{\tau}_x + F_4 \\
 &\quad \text{in } (0, 1) \times (0, 1), \\
 u(0, t) &= u(1, t) = \varphi(0, t) = \varphi(1, t) = 0 \quad \text{for a.e. } t \in (0, 1), \\
 \tau(0, t) &= \tau(1, t) = R(0, t) = R(1, t) = 0 \quad \text{for a.e. } t \in (0, 1), \\
 u(x, 0) &= \dot{u}(x, 0) = \varphi(x, 0) = \dot{\varphi}(x, 0) = x(x - 1) \quad \text{for a.e. } x \in (0, 1), \\
 \tau(x, 0) &= \dot{\tau}(x, 0) = R(x, 0) = \dot{R}(x, 0) = x(x - 1) \quad \text{for a.e. } x \in (0, 1),
 \end{aligned}$$

Table 1 Example 1: Numerical errors for some h and k

$h \downarrow k \rightarrow$	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
$1/2^3$	1.070021	1.068799	1.068409	1.068338	1.068314	1.068303	1.068300
$1/2^4$	0.537092	0.534805	0.534131	0.534026	0.533997	0.533986	0.533984
$1/2^5$	0.272919	0.268519	0.267229	0.267039	0.266989	0.266974	0.266972
$1/2^6$	0.144580	0.136480	0.133980	0.133609	0.133515	0.133488	0.133484
$1/2^7$	0.085436	0.072347	0.067714	0.066989	0.066804	0.066751	0.066744
$1/2^8$	0.060382	0.042813	0.035229	0.033857	0.033494	0.033390	0.033375
$1/2^9$	0.050677	0.030320	0.019950	0.017616	0.016929	0.016724	0.016695
$1/2^{10}$	0.047323	0.025500	0.013302	0.009978	0.008808	0.008420	0.008362
$1/2^{11}$	0.046323	0.023846	0.010663	0.006656	0.004989	0.004322	0.004209
$1/2^{12}$	0.046056	0.023357	0.009722	0.005338	0.003328	0.002360	0.002159
$1/2^{13}$	0.045988	0.023227	0.009432	0.004868	0.002668	0.001481	0.001176

where the artificial volume forces $F_i, i = 1, 2, 3, 4$, are given by: for $(x, t) \in (0, 1) \times (0, 1)$

$$\begin{aligned}
 F_1(x, t) &= e^t (x(x - 1) - 8), \\
 F_2(x, t) &= e^t (8x - 8), \\
 F_3(x, t) &= e^t (4x(x - 1) - 14 + 4x), \\
 F_4(x, t) &= e^t (3x(x - 1) - 14 + 6x),
 \end{aligned}$$

and we used the following data in the simulations:

$$\begin{aligned}
 \rho = 1, \quad \mu = 2, \quad \zeta = 1, \quad a = 1, \quad B = 2, \quad J = 1, \quad A = 1, \quad \xi = 2, \quad \kappa = 3, \\
 H = 1, \quad F = 2, \quad \alpha = 1, \quad c = 1, \quad K = 3, \quad K^* = 3, \quad D = 1, \quad b = 1, \\
 c^* = 2, \quad C = 3, \quad C^* = 2, \quad A^1 = 1, \quad A^3 = 1, \quad A^2 = 2.
 \end{aligned}$$

The exact solution to Problem P^{1D} can be easily calculated and it has the following form, for $(x, t) \in (0, 1) \times (0, 1)$:

$$u(x, t) = \varphi(x, t) = \tau(x, t) = R(x, t) = e^t x(x - 1).$$

The numerical errors, given by:

$$\begin{aligned}
 \max_{0 \leq n \leq N} \left\{ \|v^n - v^{hk,n}\|_Y + \|(u^n - u^{hk,n})_x\|_Y + \|e^n - e^{hk,n}\|_Y + \|(\varphi^n - \varphi^{hk,n})_x\|_Y \right. \\
 \left. + \|\theta^n - \theta^{hk,n}\|_Y + \|\varphi^n - \varphi^{hk,n}\|_Y + \|(\tau^n - \tau^{hk,n})_x\|_Y \right. \\
 \left. + \|M^n - M^{hk,n}\|_Y + \|(R^n - R^{hk,n})_x\|_Y \right\},
 \end{aligned}$$

and obtained for different discretization parameters h and k , are depicted in Table 1. Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in Fig. 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, is achieved.

If we assume now that there are not volume forces, and we use the following data:

$$\begin{aligned}
 T = 20, \quad \rho = 0.5, \quad \mu = 7, \quad \zeta = 1, \quad a = 1, \quad B = 2, \quad J = 10, \quad A = 1, \\
 \xi = 5, \quad \kappa = 3, \quad H = 1, \quad F = 0.1, \quad \alpha = 1, \quad c = 1, \quad K = 3, \quad K^* = 3, \\
 D = 1, \quad b = 1, \quad c^* = 2, \quad C = 3, \quad C^* = 5, \quad A^1 = 2, \quad A^3 = 1, \quad A^2 = 2,
 \end{aligned}$$

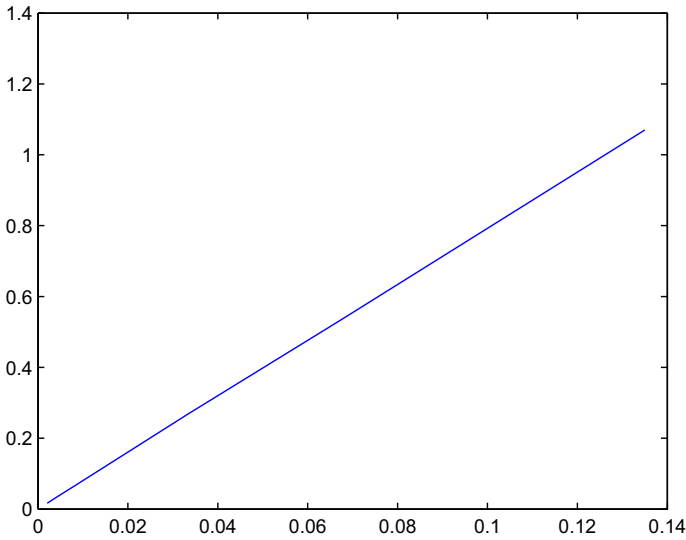


Fig. 1 Example 1: Asymptotic constant error

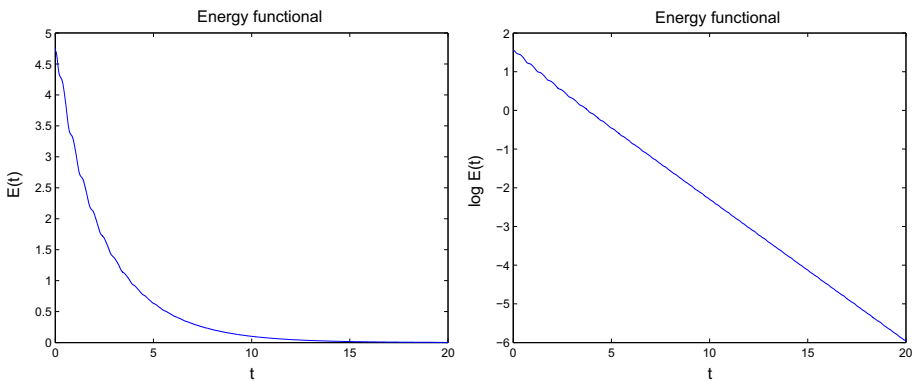


Fig. 2 Example 1: Evolution of the discrete energy in natural and semi-log scales

and the initial conditions:

$$u^0(x) = v^0(x) = R^0(x) = M^0(x) = x(x - 1) \text{ for } x \in (0, 1), \quad \varphi^0 = e^0 = \tau^0 = \theta^0 = 0,$$

taking the discretization parameters $h = k = 10^{-3}$, the evolution in time of the discrete energy $\mathcal{E}^{hk,n}$, defined as:

$$\begin{aligned} \mathcal{E}^{hk,n} = \frac{1}{2} \bigg\{ & \rho \|v^{hk,n}\|_Y^2 + J \|e^{hk,n}\|_Y^2 + c \|\theta^{hk,n}\|_Y^2 + c^* \|M^{hk,n}\|_Y^2 + A^1 \|u_x^{hk,n}\|_Y^2 \\ & + (Bu_x^{hk,n}, R_x^{hk,n})_Y + C \|R_x^{hk,n}\|_Y^2 + (\zeta u_x^{hk,n}, \varphi^{hk,n})_Y + \xi \|\varphi^{hk,n}\|_Y^2 \\ & + A^2 \|\varphi_x^{hk,n}\|_Y^2 + (H\varphi_x^{hk,n}, \tau_x^{hk,n})_Y + K \|\tau_x^{hk,n}\|_Y^2 \bigg\}, \end{aligned}$$

is plotted in Fig. 2. As can be seen, it converges to zero and an exponential decay seems to be achieved.

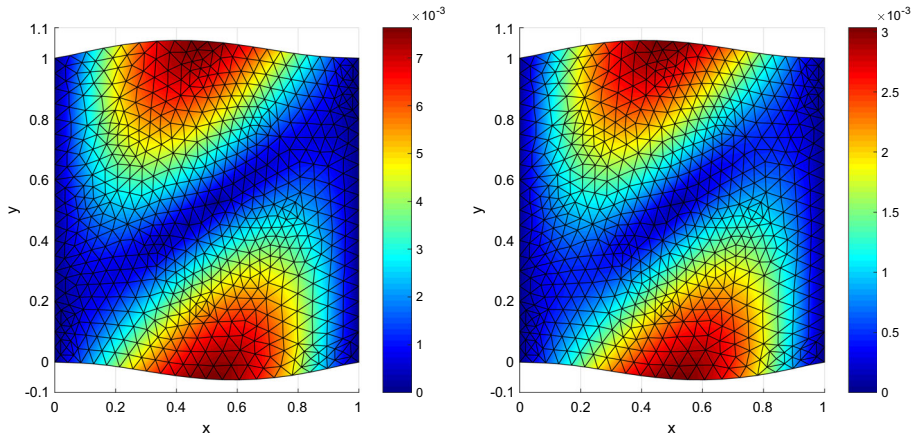


Fig. 3 Example 2: Norms of the displacement (left) and microtemperatures (right) at final time over the deformed mesh (multiplied by 10)

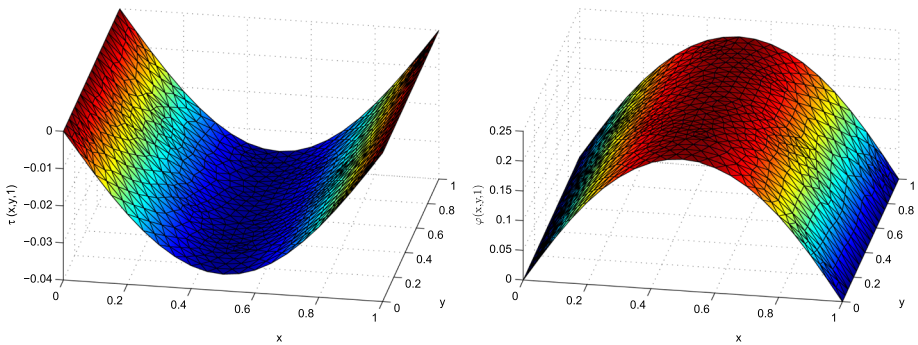


Fig. 4 Example 2: Microthermal displacement and volume fraction at final time

4.2 Numerical results in a two-dimensional problem

For this second example, the square domain $[0, 1] \times [0, 1]$ is considered, assumed to be clamped on its vertical boundaries $\{0, 1\} \times [0, 1]$ and traction-free on the rest of the boundary.

The following data have been employed in this simulation:

$$\begin{aligned} \Omega &= (0, 1) \times (0, 1), \quad T = 1, \quad \rho = 1, \quad \lambda = 1, \quad \mu = 1, \quad \zeta = 1, \quad a = 1, \\ B &= 2, \quad J = 1, \quad A = 1, \quad \xi = 2, \quad \kappa = 3, \quad H = 1, \quad F = 2, \quad \alpha = 1, \quad c = 1, \\ K &= 3, \quad K^* = 3, \quad D = 2, \quad b = 2, \quad c^* = 2, \quad C_1 = 2, \quad C_2 = 2, \quad C_1^* = 1, \\ C_2^* &= 1, \quad A^1 = 2, \quad A^2 = 1, \quad A^3 = 1, \end{aligned}$$

and the initial conditions:

$$\begin{aligned} \mathbf{u}^0 &= \mathbf{v}^0 = \mathbf{R}^0 = \mathbf{M}^0 = \mathbf{0}, \quad \tau^0 = \theta^0 = 0, \\ \varphi^0(x, y) &= e^0(x, y) = x(x - 1) \quad \text{for all } (x, y) \in (0, 1) \times (0, 1). \end{aligned}$$

Taking the time discretization parameter $k = 0.01$, in Fig. 3, we plot the norm of both the displacement (left) and microtemperatures (right) at the final time and over the deformed

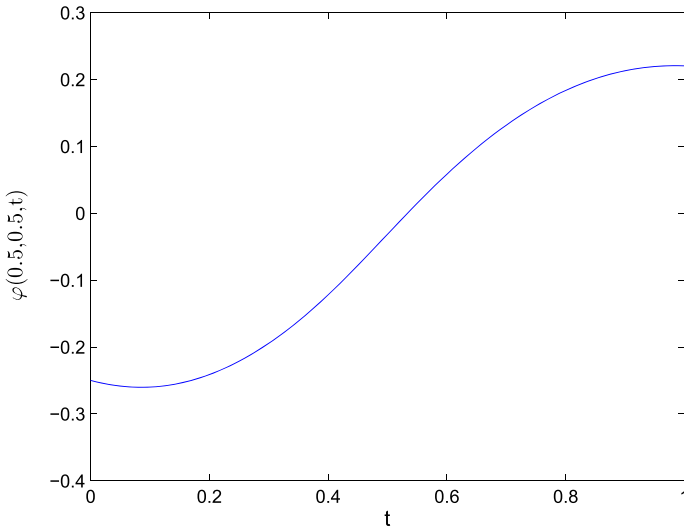


Fig. 5 Example 2: Evolution in time of the volume fraction at point $\mathbf{x} = (0.5, 0.5)$

mesh. As expected, due to the clamping conditions, the displacement and the microtemperatures, which are generated by the volume fraction, have a similar behavior.

Moreover, in Fig. 4 we plot the microthermal displacement (left) and the volume fraction (right) at the final time. We note that the volume fraction, even if it has a quadratic behavior, changes its sign, being now positive. Thus, in Fig. 5, the evolution in time of the volume fraction at middle point $\mathbf{x} = (0.5, 0.5)$ is shown. As we can see, it starts to increase after some time and it seems to converge to a steady state.

Acknowledgements The work of J.R. Fernández was partially supported by Ministerio de Ciencia, Innovación y Universidades under the research project PGC2018-096696-B-I00 (FEDER, UE). The work of R. Quintanilla was supported by projects “Análisis Matemático de Problemas de la Termomecánica” (MTM2016-74934-P), (AEI/FEDER, UE) of the Spanish Ministry of Economy and Competitiveness, and “Análisis matemático aplicado a la termomecánica” (Ref. PID2019-105118GB-I00), (AEI/FEDER, UE) of the Spanish Ministry of Science, Innovation and Universities.

References

- Almeida Júnior DS, Ramos AJA (2017) On the nature of dissipative Timoshenko systems at light of the second spectrum of frequency. *Zeitschrift für Angewandte Mathematik und Physik* 68:145
- Aouadi M, Ciarletta M, Passarella F (2018) Thermoelastic theory with microtemperatures and dissipative thermodynamics. *J Thermal Stress* 41:522–542
- Bazaraa N, Fernández JR (2018) Numerical analysis of a contact problem in poro-thermoelasticity with microtemperatures. *Zeitschrift für Angewandte Mathematik und Mechanik* 98:1190–1209
- Campo M, Fernández JR, Kuttler KL, Shillor M, Viaño JM (2006) Numerical analysis and simulations of a dynamic frictionless contact problem with damage. *Comput Methods Appl Mech Eng* 196(1–3):476–488
- Casas P, Quintanilla R (2005) Exponential stability in thermoelasticity with microtemperatures. *Int J Eng Sci* 43:33–47
- Casas P, Quintanilla R (2005) Exponential decay in one-dimensional porous-thermoelasticity. *Mech Res Commun* 32:652–658
- Cattaneo C (1958) On a form of heat equation which eliminates the paradox of instantaneous propagation. *Comptes Rendus de l’Académie de Sciences de Paris* 247:431–433

- Chirita S, Ciarletta M, D'Apice C (2013) On the theory of thermoelasticity with microtemperatures. *J Math Anal Appl* 397:349–361
- Ciarlet PG (1993) Basic error estimates for elliptic problems. In: *Handbook of Numerical Analysis*, P.G. Ciarlet and J.L. Lions eds., vol II, 17–351
- Ciarletta M, Straughan B, Tibullo V (2010) Structural stability for a rigid body with thermal microstructure. *Int J Eng Sci* 48:592–598
- Cowin SC (1985) The viscoelastic behavior of linear elastic materials with voids. *J Elasticity* 15:185–191
- Cowin SC, Nunziato JW (1983) Linear elastic materials with voids. *J Elasticity* 13:125–147
- Green AE, Naghdi PM (1992) On undamped heat waves in an elastic solid. *J Thermal Stress* 15:253–264
- Eringen AC (1999) *Microcontinuum field theories. I. Foundations and solids*. Springer, New York
- Feng B, Apalara TA (2019) Optimal decay for a porous elasticity system with memory. *J Math Anal Appl* 470:1108–1128
- Feng B, Yin M (2019) Decay of solutions for a one-dimensional porous elasticity system with memory: the case of non-equal wave speeds. *Math Mech Solids* 24:2361–2373
- Fernández JR, Masid M (2016) Analysis of a problem arising in porous thermoelasticity of type II. *J Thermal Stress* 39:513–531
- Fernández JR, Masid M (2017) A porous thermoelastic problem: an a priori error analysis and computational experiments. *Appl Math Comput* 305:117–135
- Fernández JR, Masid M (2017) Numerical analysis of a thermoelastic diffusion problem with voids. *Int J Numer Anal Model* 14:153–174
- Green AE, Naghdi PM (1993) Thermoelasticity without energy dissipation. *J Elast* 31:189–208
- Grot R (1969) Thermodynamics of a continuum with microstructure. *Int J Eng Sci* 7:801–814
- Ieşan D (2007) Thermoelasticity of bodies with microstructure and microtemperatures. *Int J Solids Struct* 44:8648–8653
- Ieşan D (2018) On a theory of thermoelasticity without energy dissipation for solids with microtemperatures. *Zeitschrift für Angewandte Mathematik und Mechanik* 98:870–885
- Ieşan D, Quintanilla R (2000) On a theory of thermoelasticity with microtemperatures. *J Therm Stresses* 23:195–215
- Ieşan D, Quintanilla R (2009) On thermoelastic bodies with inner structure and microtemperatures. *J Math Anal Appl* 354:12–23
- Ieşan D, Quintanilla R (2014) On a theory of thermoelastic materials with double porosity structure. *J Therm Stresses* 37:1017–1036
- Ieşan D, Quintanilla R (2018) Qualitative properties in strain gradient thermoelasticity with microtemperatures. *Math Mech Solids* 23:240–258
- Kothari S, Mukhopadhyay S (2012) Study of harmonic plane waves in rotating thermoelastic media of type III. *Math Mech Solids* 17:824–839
- Kumar R, Vohra R (2017) Effect of hall current in thermoelastic materials with double porosity structure. *Int J Appl Mech Eng* 22:303–319
- Kumar R, Vohra R (2019) Forced vibrations of a thermoelastic double porous microbeam subjected to a moving load. *J Theor Appl Mech* 57:155–166
- Kumar R, Vohra R, Gorla M (2016) Reflection of plane waves in thermoelastic medium with double porosity. *Multidiscip Model Mater Struct* 12:748–778
- Leseduarte MC, Magaña A, Quintanilla R (2010) On the time decay of solutions in porous-thermo-elasticity of type II. *Discr Contin Dyn Syst B* 13:375–391
- Magaña A, Quintanilla R (2006) On the exponential decay of solutions in one-dimensional generalized porous-thermo-elasticity. *Asympt Anal* 49:173–187
- Magaña A, Quintanilla R (2017) On the time decay of solutions in porous-elasticity with quasi-static microvoids. *J Math Anal Appl* 331:617–630
- Magaña A, Quintanilla R (2018) Exponential stability in type III thermoelasticity with microtemperatures. *Z Angew Math Phys* 69(5):129(1)–129(8)
- Magaña A, Quintanilla R (2020) Exponential stability in three dimensional type III thermo-porous-elasticity with microtemperatures. *J Elast* 139:153–161
- Miranville A, Quintanilla R (2019) Exponential decay in one-dimensional type III thermoelasticity with voids. *Appl Math Lett* 94:30–37
- Miranville A, Quintanilla R (2020) Exponential decay in one-dimensional type II thermoviscoelasticity with voids. *J Comput Appl Math* 368:112573
- Nunziato JW, Cowin S (1979) A nonlinear theory of elastic materials with voids. *Arch Ration Mech Anal* 72:175–201
- Quintanilla R (2011) On the growth and continuous dependence in thermoelasticity with microtemperatures. *J Therm Stresses* 34:911–922

- Quintanilla R (2013) On the logarithmic convexity in thermoelasticity with microtemperatures. *J Therm Stresses* 36:378–386
- Ramos AJA, Almeida-Júnior DS, Freitas MM, DosSantos MJ (2020) A new exponential decay result for one-dimensional porous dissipation elasticity from second spectrum viewpoint. *Appl Math Lett* 101:106061
- Riha P (1975) On the theory of heat-conducting micropolar fluids with microtemperatures. *Acta Mech* 23:1–8
- Riha P (1976) On the microcontinuum model of heat conduction in materials with inner structure. *Int J Eng Sci* 14:529–535
- Verma PDS, Singh DV, Singh K (1979) Hagen-Poiseuille flow of microthermopolar fluids in a circular pipe. *Acta Technical CSAV* 24:402–412

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.