



Analysis of the consistency and consensus for group decision-making with interval-valued intuitionistic fuzzy preference relations

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Abstract

This paper aims to offer a new group decision-making (GDM) method based on interval-valued intuitionistic fuzzy preference relations (IVIFPRs). To furnish this goal, a new additive consistency definition of IVIFPRs is first proposed. Then, a programming model is built to check the additive consistency of IVIFPRs. For incomplete IVIFPRs, two programming models are constructed, which aim at maximizing the consistency and minimizing the uncertainty of missing information. To achieve the minimum total adjustment, a goal programming model is established to repair inconsistent IVIFPRs. Considering the consensus, a programming model for improving the consensus degree is established, which permits different IVIFVs to have different adjustments and makes individual IVIFPRs have the smallest total adjustment to remain more original information. Based on these results, a consistency- and consensus-based GDM method is proposed. At length, a practical example for screening new majors of a private college in China is offered to illustrate the feasibility and efficiency of proposed method.

Keywords Group decision-making · IVIFPR · Additive consistency and consensus · Programming model

Mathematics Subject Classification 03E72 · 90B50 · 90Cxx

1 Introduction

As the effectiveness and convenience of preference relations (PRs) for dealing with complex and timely decision-making problems, it has become an important branch in multi-criteria decision making (MCDM) to evaluate the possible alternatives under different criteria (Callejas et al. 2019). A lot of research results about decision-making with PRs have been achieved by scholars, which promote many new developing categories. Among of which

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there are two basic types of PRs: multiplicative PRs (Saaty 1980) and fuzzy PRs (Tanino 1984). Based on this classification, several other kinds of PRs have been proposed, such as interval multiplicative PRs (IMPRs) (Saaty and Vargas 1987), fuzzy interval PRs (FIPRs) (Xu 2001), linguistic PRs (LPRs) (Herrera and Herrera-Viedma 2000), and interval linguistic PRs (ILPRs) (Tapia García et al. 2012). Their common feature is to only consider the membership degree of one compared object over the other.

In view of the limitation of the decision makers (DMs)' mastery knowledge, the judgements may be inconsistent. For example, a DM thinks that the preferred degree of one object over the other is 0.6. However, their non-preferred degree is judged as 0.3 rather than 0.4, namely, the hesitancy degree between them is 0.1. To solve this situation, Atanassov (1986) introduced the definition of intuitionistic fuzzy sets (IFSs) that can express the DMs' preferred and non-preferred information by the membership and non-membership functions.

Based on IFSs, Xu (2007a) proposed the definition of intuitionistic fuzzy preference relation (IFPR), where the elements are intuitionistic fuzzy values (IFVs). Since then, many scholars have investigated decision-making with IFPRs. Based on FIPRs' multiplicative consistency definition, Gong et al. (2009) built a goal programming model to derive the intuitionistic fuzzy priority weight vector. Wan et al. (2016a) gave a TOPSIS-based method for group decision-making (GDM) with IFPRs and researched its application in the radio frequency identification technology selection. Based on the work of Herrera-Viedma et al. (2007), Ureña et al. (2015a) introduced an iterative procedure to determine missing values in IFPRs. Meng et al. (2017b) analyzed the limitations of previous multiplicative consistency definitions for IFPRs and proposed a new one. Moreover, from the perspective of consistency, consensus and prioritization, Xu and Liao (2015) presented a survey on decision-making with IFPRs.

Since IFSs cannot denote the uncertainty of DMs' judgements, Atanassov and Gargov (1989) further introduced the interval-valued intuitionistic fuzzy set (IVIFS). As a generalization of the IFS, IVIFS can denote both the preferred and non-preferred degrees simultaneously by closed subintervals of $[0, 1]$. For simplifying the utilization of IVIFSs, Xu and Chen (2007) proposed interval-valued intuitionistic fuzzy values (IVIFVs) and introduced them into PRs, which are known as interval-valued IFPRs (IVIFPRs). Xu and Cai (2009) studied incomplete IVIFPRs and provided two methods for estimating missing values using the additive and multiplicative consistency concepts for IVIFPRs. According to fuzzy PRs' multiplicative transitivity (Chiclana et al. 2009), Liao et al. (2014) gave a multiplicative consistency concept for IVIFPRs and proposed an iterative algorithm to improve the consistency level of inconsistent IVIFPRs. Similar to the handling of inconsistency, the authors further proposed an iterative algorithm to improve individual IVIFPRs' consensus level. Using multiplicative consistency definition for IFPRs (Liao and Xu 2014) and the induced matrices obtained from IVIFPRs, Wan et al. (2016b) proposed another definition for IVIFPRs' multiplicative consistency. According to the additively consistent FIPRs, Wan et al. (2017) defined an additive consistency concept for IVIFPRs and then investigated a GDM method. In addition, Wang et al. (2009) used the normalized interval weight vector to propose an additive consistency concept for IVIFPRs. Then, the authors gave a method for generating the interval-valued intuitionistic fuzzy priority weight vector (IVIF-PWV) that is based on the offered programming models. By extending the intuitionistic fuzzy aggregation operators (Xia and Xu 2012), Liao et al. (2014) proposed an operator to aggregate individual IVIFPRs into a collective one. Recently, Tang et al. (2018) and Meng et al. (2018) analyzed the limitations of previous additive and multiplicative consistency concepts for IVIFPRs separately. Using 2-tuple preferred fuzzy interval preference relations

(2TPFIPRs) and quasi 2TPFIPRs (Q2TPFIPRs), the authors proposed the definitions for additively consistent and multiplicatively consistent IVIFPRs. Based on these consistency definitions, programming models were constructed to check the consistency and to determine the missing values. Correspondingly, the consistency- and consensus-based algorithms for GDM with IVIFPRs were developed. Moreover, Wu and Chiclana (2012) analyzed the limitations of previous ranking methods for IVIFVs and presented the attitudinal expected score function for ranking objects from IVIFPRs. In addition to IVIFPRs, there are three other types of interval-valued intuitionistic preference relations: interval-valued intuitionistic multiplicative preference relations (IVIMPRs) (Meng et al. 2020), interval-valued intuitionistic linguistic fuzzy preference relations (IVILFPRs) (Tang et al. 2019), and interval-valued intuitionistic multiplicative linguistic preference relations (IVIMLPRs) (Tang et al. 2020). IVIMPRs employ interval-valued intuitionistic multiplicative variables (IVIMVs) to denote the uncertain multiplicative preferred and non-preferred judgements of the DMs, IVILFPRs adopt interval-valued intuitionistic linguistic fuzzy variables (IVILFVs) to express the uncertain preferred and non-preferred qualitative judgments of the DMs, and IVIMLPRs use interval-valued intuitionistic multiplicative linguistic variables (IVIMLVs) to show the asymmetrically uncertain preferred and non-preferred qualitative judgments of the DMs.

Through the literature review, we find that there are some limitations of research about decision-making with IVIFPRs. In the first place, the consistency is dependent on the objects' compared orders, and the contradictory conclusions may be derived from different compared orders (Wang 2014). In the second place, some methods are not based on the consistency and consensus analysis, which can lead to illogical results (Tang et al. 2018). In the third place, the consistency concepts used in some research seem to be not flexible enough and the procedure for calculating the IVIFPWV is complex (Meng et al. 2019). Based on the research status analysis of decision-making with IVIFPRs, this paper continues to investigate GDM with IVIFPRs in a new vision and offers a new approach. The major contributions of this paper include:

- (i) A new additive consistency concept for IVIFPRs is proposed and a programming model is built to check the consistency.
- (ii) For IVIFPRs with incomplete preference information, two programming models are constructed to determine unknown judgements, which achieve two goals: maximizing the consistency level and minimizing the total uncertain degree.
- (iii) To repair inconsistent IVIFPRs, a goal programming model is established for deriving their associated consistent IVIFPRs, which makes the overall adjustment be minimum.
- (iv) A linear programming model is established to improve the consensus level of individual IVIFPRs, which endows different IVIFPRs with different adjustments and achieves the adjusting individual IVIFPR to have the smallest total adjustment to remain more original information.
- (v) A new algorithm for GDM with IVIFPRs is provided, and the example of setting up new majors in the Chinese private college is taken to illustrate the application of the algorithm.

This paper is arranged into seven sections. Section 2 first reviews several basic concepts related to the following research. Then, several existing additive consistency definitions of IVIFPRs are analyzed. Section 3 proposes a new additive consistency concept for IVIFPRs that avoids the issues of existing ones. Meanwhile, a programming model is built to judge

IVIFPRs’ additive consistency. Section 4 tackles incomplete and inconsistent IVIFPRs and constructs two programming models for determining missing values. Furthermore, a goal programming model for repairing inconsistent IVIFPRs is presented. Section 5 discusses GDM with incomplete and inconsistent IVIFPRs. To achieve this goal, it first offers two formulae to derive the weights of the DMs and measure the consensus degree. To meet the given consensus threshold, a programming model for improving the consensus level is provided. After that, the concrete procedure for GDM with IVIFPRs is put forward. Section 6 offers two illustrative examples to show the application of proposed method and shows the comparison of the new method with previous ones. Conclusions and future remarks are conducted in Sect. 7.

2 Basic concepts

Since the new method is based on the transformation from IVIFPRs to FIPRs, and IVIFPRs can be viewed as an extension of IFPRs, this section first introduces the concepts of FIPRs and IFPRs.

Definition 1 (FIPR) (Xu 2004): A FIPR $\bar{A} = (\bar{a}_{ij})_{n \times n}$ on the finite set $X = \{x_1, x_2, \dots, x_n\}$ is an interval fuzzy binary relation, characterized by an interval fuzzy subset of $X \times X$, i.e., $\bar{\mu}_{\bar{A}} : X \times X \rightarrow D[0, 1]$ such that $D[0, 1]$ is the set of all possible intervals in $[0, 1]$, where the interval $\bar{a}_{ij} = \bar{\mu}_{\bar{A}}(x_i, x_j)$ is the interval preferred degree of the object x_i over x_j . Furthermore, its elements satisfy

$$\begin{cases} \bar{a}_{ij} = [a_{ij}^L, a_{ij}^U] \subseteq [0, 1] \\ a_{ij}^L \leq a_{ij}^U \\ a_{ij}^L + a_{ji}^U = a_{ij}^U + a_{ji}^L = 1 \end{cases}, \tag{1}$$

where $i, j = 1, 2, \dots, n$.

Through comparing and analyzing the previous additive consistency concepts, Meng et al. (2017a) introduced the below FIPRs’ additive consistency concept.

Definition 2 (Additively consistent FIPR) (Meng et al. 2017a): Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be a FIPR. \bar{A} is additively consistent if there is an associated additively consistent quasi FIPR (QFIPR) $\bar{B} = (\bar{b}_{ij})_{n \times n}$, namely,

$$\bar{b}_{ij} \oplus [0.5, 0.5] = \bar{b}_{ik} \oplus \bar{b}_{kj}, \tag{2}$$

for all $i, k, j = 1, 2, \dots, n$, where $\begin{cases} \bar{b}_{ij} = [a_{ij}^L, a_{ij}^U] \\ \bar{b}_{ji} = [a_{ji}^U, a_{ji}^L] \end{cases}$ or $\begin{cases} \bar{b}_{ij} = [a_{ij}^U, a_{ij}^L] \\ \bar{b}_{ji} = [a_{ji}^L, a_{ji}^U] \end{cases}$ for all $i, j = 1, 2, \dots, n$.

Definition 2 shows that Meng et al.’s concept requires the interval judgements’ endpoints to satisfy the additive transitivity. In contrast to Definition 2, Krejčí (2017) offered another additive consistency concept based on intervals’ constrained operations (Lodwick and Jenkins 2013) that does not restrict to the endpoints of interval judgements.

Definition 3 (Additively consistent FIPR) (Krejčí 2017): Let $\bar{A} = (\bar{a}_{ij})_{n \times n}$ be a FIPR. \bar{A} is additively consistent if

$$\forall a_{ij} \in \bar{a}_{ij}, \exists a_{ik} \in \bar{a}_{ik} \wedge a_{kj} \in \bar{a}_{kj} \Rightarrow a_{ij} = a_{ik} + a_{kj} - 0.5 \tag{3}$$

is true for all $i, k, j = 1, 2, \dots, n$.

Based on Definition 3, one can check that formula (3) is equivalent to

$$\begin{cases} a_{ij}^L \geq a_{ik}^L + a_{kj}^L - 0.5 \\ a_{ij}^U \leq a_{ik}^U + a_{kj}^U - 0.5 \end{cases} \tag{4}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

From formulae (2) and (4), one can find that Definition 2 can be seen as a special case of Definition 3. Just as Meng et al. (2019) noted, Definition 3 is more reasonable and flexible than Definition 2.

To express both preferred and non-preferred judgements simultaneously, Xu (2007a) introduced intuitionistic fuzzy PRs (IFPRs) on the basis of IFVs.

Definition 4 (IFPR) (Xu 2007a): An IFPR R on the set $X = \{x_1, x_2, \dots, x_n\}$ is presented by a matrix $R = (r_{ij})_{n \times n}$ such that $r_{ij} = (\mu_{ij}, \nu_{ij})$ is an IFV denoting the intuitionistic fuzzy preference of the object x_i over x_j , $i, j = 1, 2, \dots, n$. Furthermore, its elements are IFVs that satisfy the following characteristics:

$$\begin{cases} \mu_{ij}, \nu_{ij} \geq 0 \\ \mu_{ij} + \nu_{ij} \leq 1 \\ \mu_{ji} = \nu_{ij}, \nu_{ji} = \mu_{ij} \\ \mu_{ii} = \nu_{ii} = 0.5 \end{cases}, \tag{5}$$

where $i, j = 1, 2, \dots, n$.

To denote preferred and non-preferred uncertain memberships of DMs' judgements, Atanassov and Gargov (1989) further introduced IVIFs, and Xu and Chen (2007) and Xu (2007b) proposed IVIFVs for facilitating the application. Later, Xu (2007a) offered the concept of IVIFPRs using IVIFVs.

Definition 5 (IVIFPR) (Xu 2007a): An IVIFPR \tilde{R} on the set X is represented by a matrix $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ such that $\tilde{r}_{ij} = (\bar{\mu}_{ij}, \bar{\nu}_{ij})$ is the interval-valued intuitionistic fuzzy preference of the object x_i over x_j , $i, j = 1, 2, \dots, n$. Furthermore, its elements are IVIFVs that satisfy the following characteristics:

$$\begin{cases} \bar{\mu}_{ij} = [\mu_{l,ij}, \mu_{u,ij}] \subseteq [0, 1] \\ \bar{\nu}_{ij} = [\nu_{l,ij}, \nu_{u,ij}] \subseteq [0, 1] \\ \bar{\mu}_{ji} = \bar{\nu}_{ij}, \bar{\nu}_{ji} = \bar{\mu}_{ij} \\ \mu_{u,ij} + \nu_{u,ij} \leq 1 \\ \bar{\mu}_{ii} = \bar{\nu}_{ii} = [0.5, 0.5] \end{cases}, \tag{6}$$

where $i, j = 1, 2, \dots, n$.

To our knowledge, there are four main additive consistency concepts for IVIFPRs. By directly using the addition operation on IVIFVs, Xu and Cai (2009) gave the below additive consistency concept:

Definition 6 (*Additively consistent IVIFPR*) (Xu and Cai 2009): let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR. It is additively consistent if the formula

$$\begin{aligned} \tilde{r}_{ij} &= \frac{1}{2}(\tilde{r}_{ik} \oplus \tilde{r}_{kj}) \\ &= \left(\left[1 - \left((1 - \mu_{ik}^L)(1 - \mu_{kj}^L) \right)^{\frac{1}{2}}, 1 - \left((1 - \mu_{ik}^U)(1 - \mu_{kj}^U) \right)^{\frac{1}{2}} \right], \left[\left(v_{ik}^L v_{kj}^L \right)^{\frac{1}{2}}, \left(v_{ik}^U v_{kj}^U \right)^{\frac{1}{2}} \right] \right) \end{aligned} \tag{7}$$

is true for all $i, j, k = 1, 2, \dots, n$.

Different from Definition 6 that takes IVIFVs to define IVIFPRs' additive consistency, Wang et al. (2009) adopted the FIPRs' normalized interval weight vector to offer the below additive consistency concept:

Definition 7 (*additively consistent IVIFPR*) (Wang et al. 2009): let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR. If there is a normalized interval weight vector $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_n)$ such that

$$\begin{cases} \mu_{l,ij} \leq 0.5(w_i^L - w_j^L + 1) \leq 1 - v_{u,ij} \\ \mu_{u,ij} \leq 0.5(w_i^U - w_j^U + 1) \leq 1 - v_{l,ij} \end{cases}, \tag{8}$$

for all $i, j = 1, 2, \dots, n$ with $i < j$, then \tilde{R} is additively consistent, where

$$\bar{w}_i = [w_i^L, w_i^U] \subseteq [0, 1] \text{ and } \begin{cases} w_i^U + \sum_{j=1, i \neq j}^n w_j^L \leq 1 \\ w_i^L + \sum_{j=1, i \neq j}^n w_j^U \geq 1 \end{cases} \text{ for all } i = 1, 2, \dots, n.$$

Using the transformation formula between IVIFVs and intervals (Bustince 1994), Wan et al. (2017) proposed the below concept of additively consistent IVIFPRs that is based on Definition 7.

Definition 8 (*Additively consistent IVIFPR*) (Wan et al. 2017): An IVIFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ is additively consistent if there is an additively consistent FIPR $\bar{R} = (\bar{r}_{ij})_{n \times n}$ following Definition 7, where $\bar{r}_{ij} = [r_{ij}^L, r_{ij}^U]$ satisfies $\mu_{l,ij} \leq r_{ij}^L \leq 1 - v_{u,ij}$ and $\mu_{u,ij} \leq r_{ij}^U \leq 1 - v_{l,ij}$ for all $i, j = 1, 2, \dots, n$.

Later, Tang et al. (2018) analyzed these additive consistency concepts and listed their limitations in some aspects. For example, an IVIFPR is additively consistent following Definition 6 if and only if all of its IVIFVs are equal. According to Definition 7, we cannot derive the priority vector from additively consistent IVIFPRs in some cases, the IVIFPR listed in Example 1 (Wang et al. 2009) is additively consistent following Definition 7. However, it derives $x_3 > x_4$ from $\tilde{r}_{34} = ([0.35, 0.45], [0.15, 0.25])$, $x_4 > x_2$ from $\tilde{r}_{42} = ([0.35, 0.45], [0.15, 0.25])$ and $x_2 > x_3$ from $\tilde{r}_{23} = ([0.55, 0.65], [0.15, 0.25])$. Therefore, we get $x_3 > x_4 > x_2 > x_3$. This example concretely shows that it is unreasonable to define IVIFPRs' additive consistency using Definition 7. From the relationship between Definitions 7 and 8, one can check that Definition 8 has the same issues as that in Definition 7.

In contrast to the above three additive consistency concepts, Tang et al. (2018) defined 2TPFIPRs and Q2TPFIPRs, by which the below additive consistency concept is derived.

Definition 9 (*2TPFIPR*) (Tang et al. 2018): let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR, where $\tilde{r}_{ij} = ([\mu_{l,ij}, \mu_{u,ij}], [v_{l,ij}, v_{u,ij}])$ for all $i, j = 1, 2, \dots, n$. $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ is called a 2TPFIPR, where $\tilde{p}_{ij} = ([\mu_{l,ij}, 1 - v_{l,ij}], [\mu_{u,ij}, 1 - v_{u,ij}])$ denotes the interval possible preferred degree of the object x_i over x_j for all $i, j = 1, 2, \dots, n$.

According to 2TPFIPRs, the concept of Q2TPFIPRs is offered as follows:

Definition 10 (*Q2TPFIPR*) (Tang et al. 2018): let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR, let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be its 2TPFIPR as shown in Definition 9. $\tilde{S} = (\tilde{s}_{ij})_{n \times n}$ is called a Q2TPFIPR if its elements satisfy one of the below four cases:

$$\begin{aligned}
 & \text{(i)} \left\{ \begin{array}{l} \tilde{s}_{ij} = ([\mu_{l,ij}, 1 - v_{l,ij}], [\mu_{u,ij}, 1 - v_{u,ij}]) \\ \tilde{s}_{ji} = ([1 - \mu_{l,ij}, v_{l,ij}], [1 - \mu_{u,ij}, v_{u,ij}]) \end{array} \right\}, \quad \text{(ii)} \left\{ \begin{array}{l} \tilde{s}_{ij} = ([1 - v_{l,ij}, \mu_{l,ij}], [\mu_{u,ij}, 1 - v_{u,ij}]) \\ \tilde{s}_{ji} = ([v_{l,ij}, 1 - \mu_{l,ij}], [1 - \mu_{u,ij}, v_{u,ij}]) \end{array} \right\} \\
 & \text{(iii)} \left\{ \begin{array}{l} \tilde{s}_{ij} = ([\mu_{l,ij}, 1 - v_{l,ij}], [1 - v_{u,ij}, \mu_{u,ij}]) \\ \tilde{s}_{ji} = ([1 - \mu_{l,ij}, v_{l,ij}], [v_{u,ij}, 1 - \mu_{u,ij}]) \end{array} \right\}, \quad \text{(iv)} \left\{ \begin{array}{l} \tilde{s}_{ij} = ([1 - v_{l,ij}, \mu_{l,ij}], [1 - v_{u,ij}, \mu_{u,ij}]) \\ \tilde{s}_{ji} = ([v_{l,ij}, 1 - \mu_{l,ij}], [v_{u,ij}, 1 - \mu_{u,ij}]) \end{array} \right\}, \tag{9}
 \end{aligned}$$

for all $i, j = 1, 2, \dots, n$.

From Definition 10, it can be found that the Q2TPFIPR $\tilde{S} = (\tilde{s}_{ij})_{n \times n}$ is composed by two QFIPRs $\tilde{\eta} = (\tilde{\eta}_{ij})_{n \times n}$ and $\tilde{\lambda} = (\tilde{\lambda}_{ij})_{n \times n}$, where

$$\left\{ \begin{array}{l} \tilde{\eta}_{ij} = [\mu_{l,ij}, 1 - v_{l,ij}] \\ \tilde{\eta}_{ji} = [1 - \mu_{l,ij}, v_{l,ij}] \end{array} \right\} \vee \left\{ \begin{array}{l} \tilde{\eta}_{ij} = [1 - v_{l,ij}, \mu_{l,ij}] \\ \tilde{\eta}_{ji} = [v_{l,ij}, 1 - \mu_{l,ij}] \end{array} \right\} \text{ and } \left\{ \begin{array}{l} \tilde{\lambda}_{ij} = [\mu_{u,ij}, 1 - v_{u,ij}] \\ \tilde{\lambda}_{ji} = [1 - \mu_{u,ij}, v_{u,ij}] \end{array} \right\} \vee \left\{ \begin{array}{l} \tilde{\lambda}_{ij} = [1 - v_{u,ij}, \mu_{u,ij}] \\ \tilde{\lambda}_{ji} = [v_{u,ij}, 1 - \mu_{u,ij}] \end{array} \right\}, \tag{10}$$

for all $i, j = 1, 2, \dots, n$.

Based on this fact and Definition 2, Tang et al. (2018) defined the additive consistency of Q2TPFIPRs as below:

Definition 11 (*Additively consistent Q2TPFIPR*) (Tang et al. 2018): Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR and let $\tilde{S} = (\tilde{s}_{ij})_{n \times n}$ be its Q2TPFIPR. \tilde{S} is additively consistent if the QFIPRs $\tilde{\eta} = (\tilde{\eta}_{ij})_{n \times n}$ and $\tilde{\lambda} = (\tilde{\lambda}_{ij})_{n \times n}$ as shown in formula (10) are both additively consistent, namely,

$$\left\{ \begin{array}{l} \tilde{\eta}_{ij} \oplus [0.5, 0.5] = \tilde{\eta}_{ik} \oplus \tilde{\eta}_{kj} \\ \tilde{\lambda}_{ij} \oplus [0.5, 0.5] = \tilde{\lambda}_{ik} \oplus \tilde{\lambda}_{kj} \end{array} \right\}, \tag{11}$$

for all $i, k, j = 1, 2, \dots, n$.

Based on Q2TPFIPRs' additive consistency, Tang et al. (2018) gave the below concept:

Definition 12 (*additively consistent IVIFPR*) (Tang et al. 2018): Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR, and let $\tilde{S} = (\tilde{s}_{ij})_{n \times n}$ be its Q2TPFIPR. If \tilde{S} is additively consistent following Definition 11, then \tilde{R} is additively consistent.

From Definition 12, one can find that Tang et al.'s additive consistency concept is based on Definition 2. The advantage of Definition 12 is to be able to avoid the limitations in Definitions 6, 7 and 8, while the main issue of Definition 12 is not flexible enough. The rationality of restriction on the IVIFVs' endpoints is not discussed.

3 A new additive consistency concept for IVIFPRs

It should be noted that consistency is an important research topic in decision-making with PRs to ensure that DMs are neither random nor illogical when providing their preferences (Li et al. 2019; Cabrerizo et al. 2018). To obtain the reasonable conclusions from IVIFPRs, many scholars have devoted themselves into investigating the consistency of IVIFPRs. Considering the limitations of previous additive consistency concepts for IVIFPRs, this section introduces a new one based on Krejčí’s additive consistency concept for FIPRs.

Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR, and let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be its 2TPFIPR as shown in Definition 9, where $\tilde{p}_{ij} = ([\mu_{l,ij}, 1 - v_{l,ij}], [\mu_{u,ij}, 1 - v_{u,ij}])$, $i, j = 1, 2, \dots, n$.

One can find that the 2TPFIPR $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ is composed by the matrices $\bar{P}^L = (\bar{p}_{ij}^L)_{n \times n}$ and $\bar{P}^U = (\bar{p}_{ij}^U)_{n \times n}$, where

$$\begin{cases} \bar{p}_{ij}^L = [\mu_{l,ij}, 1 - v_{l,ij}] \\ \bar{p}_{ij}^U = [\mu_{u,ij}, 1 - v_{u,ij}] \end{cases} \tag{12}$$

for all $i, j = 1, 2, \dots, n$.

Following the concept of IVIFPRs and formula (12), we have

$$\begin{cases} \bar{p}_{ij}^L = [\mu_{l,ij}, 1 - v_{l,ij}] \subseteq [0, 1] \\ \bar{p}_{ji}^L = [\mu_{l,ji}, 1 - v_{l,ji}] \subseteq [0, 1] \\ \mu_{l,ji} + 1 - v_{l,ij} = v_{l,ij} + 1 - v_{l,ij} = 1 \\ \mu_{l,ij} + 1 - v_{l,ji} = \mu_{l,ij} + 1 - \mu_{l,ij} = 1 \end{cases} \text{ and } \begin{cases} \bar{p}_{ij}^U = [\mu_{u,ij}, 1 - v_{u,ij}] \subseteq [0, 1] \\ \bar{p}_{ji}^U = [\mu_{u,ji}, 1 - v_{u,ji}] \subseteq [0, 1] \\ \mu_{u,ji} + 1 - v_{u,ij} = v_{u,ij} + 1 - v_{u,ij} = 1 \\ \mu_{u,ij} + 1 - v_{u,ji} = \mu_{u,ij} + 1 - \mu_{u,ij} = 1 \end{cases}, \tag{13}$$

for all $i, j = 1, 2, \dots, n$. Therefore, $\bar{P}^L = (\bar{p}_{ij}^L)_{n \times n}$ and $\bar{P}^U = (\bar{p}_{ij}^U)_{n \times n}$ are FIPRs.

Based on the above relationship, we can employ the additive consistency of the associated FIPRs to define additively consistent 2TPFIPRs.

Definition 13 (Additively consistent 2TPFIPR): Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be the 2TPFIPR of the IVIFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, \tilde{P} is additively consistent if the FIPRs $\bar{P}^L = (\bar{p}_{ij}^L)_{n \times n}$ and $\bar{P}^U = (\bar{p}_{ij}^U)_{n \times n}$ shown in formula (12) are both additively consistent, namely,

$$\begin{cases} \forall p_{ij}^L \in \bar{p}_{ij}^L, \exists p_{ik}^L \in \bar{p}_{ik}^L \wedge p_{kj}^L \in \bar{p}_{kj}^L \Rightarrow p_{ij}^L = p_{ik}^L + p_{kj}^L - 0.5 \\ \forall p_{ij}^U \in \bar{p}_{ij}^U, \exists p_{ik}^U \in \bar{p}_{ik}^U \wedge p_{kj}^U \in \bar{p}_{kj}^U \Rightarrow p_{ij}^U = p_{ik}^U + p_{kj}^U - 0.5 \end{cases}, \tag{14}$$

for all $i, k, j = 1, 2, \dots, n$.

Following Definition 3, one can easily derive the below theorem:

Theorem 1 Let $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ be the 2TPFIPR of the IVIFPR $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$, \tilde{P} is additively consistent according to Definition 13 if and only if the below conclusions

$$\begin{cases} \mu_{l,ij} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} \geq v_{l,ik} + v_{l,kj} - 0.5 \end{cases} \text{ and } \begin{cases} \mu_{u,ij} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\ v_{u,ij} \geq v_{u,ik} + v_{u,kj} - 0.5 \end{cases}, \tag{15}$$

hold for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

Theorem 1 shows that the additive consistency of 2TPFIPRs is based on the additive transitivity of the endpoints of IVIFVs in associated IVIFPRs. Thus, we can adopt Definition 13 to further define the additive consistency of IVIFPRs.

Definition 14 (*additively consistent IVIFPR*): Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR. It is additively consistent if its 2TPFIPR $\tilde{P} = (\tilde{p}_{ij})_{n \times n}$ is additively consistent based on Definition 13, namely, its elements satisfy formula (15).

Thus, formula (15) provides an effective tool to determine whether an IVIFPR satisfies the additive consistency. Moreover, according to the independence of Definition 3 for the compared orders, we can easily derive that Definition 14 is invariant under the permutation of objects.

Now, we study the relationship between Definitions 12 and 14 to show the flexibility of the new concept.

Theorem 2 *Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR. When \tilde{R} is additively consistent following Definition 12, then it is additively consistent following Definition 14. However, the opposite is not true, that is, when \tilde{R} is additively consistent by Definition 14, we cannot conclude that it is additively consistent from Definition 12.*

Proof Sufficiency: when $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ is additively consistent following Definition 12, by formulae (9) and (10) in the literature (Tang et al. 2019), we derive

$$(i) \begin{cases} \mu_{l,ij} + 0.5 = \mu_{l,ik} + \mu_{l,kj} \\ v_{l,ij} + 0.5 = v_{l,ik} + v_{l,kj} \end{cases}, (ii) \begin{cases} \mu_{l,kj} + 0.5 = \mu_{l,ki} + \mu_{l,ij} \\ v_{l,kj} + 0.5 = v_{l,ki} + v_{l,ij} \end{cases}, (iii) \begin{cases} \mu_{l,ik} + 0.5 = \mu_{l,ij} + \mu_{l,jk} \\ v_{l,ik} + 0.5 = v_{l,ij} + v_{l,jk} \end{cases}, \quad (16)$$

and

$$(i) \begin{cases} \mu_{u,ij} + 0.5 = \mu_{u,ik} + \mu_{u,kj} \\ v_{u,ij} + 0.5 = v_{u,ik} + v_{u,kj} \end{cases}, (ii) \begin{cases} \mu_{u,kj} + 0.5 = \mu_{u,ki} + \mu_{u,ij} \\ v_{u,kj} + 0.5 = v_{u,ki} + v_{u,ij} \end{cases}, (iii) \begin{cases} \mu_{u,ik} + 0.5 = \mu_{u,ij} + \mu_{u,jk} \\ v_{u,ik} + 0.5 = v_{u,ij} + v_{u,jk} \end{cases} \quad (17)$$

for all $i, k, j = 1, 2, \dots, n$.

For case (i) in formula (16), we have

$$\begin{cases} \mu_{l,ij} + 0.5 = \mu_{l,ik} + \mu_{l,kj} \\ v_{l,ij} + 0.5 = v_{l,ik} + v_{l,kj} \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} = \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} = v_{l,ik} + v_{l,kj} - 0.5 \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} \geq v_{l,ik} + v_{l,kj} - 0.5 \end{cases}.$$

For case (ii) in formula (16), we have

$$\begin{cases} \mu_{l,kj} + 0.5 = \mu_{l,ki} + \mu_{l,ij} \\ v_{l,kj} + 0.5 = v_{l,ki} + v_{l,ij} \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} \geq -(1 - \mu_{l,ik}) + \mu_{l,kj} + 0.5 \\ v_{l,ij} \geq -(1 - v_{l,ik}) + v_{l,kj} + 0.5 \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} \geq v_{l,ik} + v_{l,kj} - 0.5 \end{cases}.$$

For case (iii) in formula (16), we have

$$\begin{cases} \mu_{l,ik} + 0.5 = \mu_{l,ij} + \mu_{l,jk} \\ v_{l,ik} + 0.5 = v_{l,ij} + v_{l,jk} \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} \geq \mu_{l,ik} - (1 - \mu_{l,kj}) + 0.5 \\ v_{l,ij} \geq v_{l,ik} - (1 - v_{l,kj}) + 0.5 \end{cases} \Rightarrow \begin{cases} \mu_{l,ij} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} \geq v_{l,ik} + v_{l,kj} - 0.5 \end{cases}.$$

Similarly, we can derive $\begin{cases} \mu_{u,ij} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\ v_{u,ij} \geq v_{u,ik} + v_{u,kj} - 0.5 \end{cases}$ following three cases in formula (17).

Thus, $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ is additively consistent according to Definition 14.

Necessity: Considering the IVIFPR,

$$\tilde{R} = \begin{pmatrix} ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.60], [0.30, 0.40]) & ([0.45, 0.60], [0.40, 0.40]) \\ ([0.30, 0.40], [0.40, 0.60]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.50], [0.45, 0.50]) \\ ([0.40, 0.40], [0.45, 0.60]) & ([0.45, 0.50], [0.35, 0.50]) & ([0.50, 0.50], [0.50, 0.50]) \end{pmatrix}.$$

It can be checked that \tilde{R} is additively consistent according to Definition 14. However, \tilde{R} is inconsistent by Definition 12. The proof of Theorem 2 is completed.

Although Definition 14 owns all properties of Definition 3, it is inefficient to directly use formula (15) to judge IVIFPRs' additive consistency. For example, for a n order IVIFPR, we derive two n order FIPRs following formula (12). Then, we need to judge $n(n-1)(n-2)$ triples of (i, k, j) using formula (15) for judging its additive consistency. Therefore, this method is very time consuming. To solve this issue, we next introduce a programming model-based method. Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an IVIFPR. The following programming model can be built to judge its additive consistency:

$$\Gamma^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sum_{k=1, k \neq i, j}^n (\alpha_{l,ikj} + \beta_{l,ikj} + \alpha_{u,ikj} + \beta_{u,ikj}),$$

$$s.t. \begin{cases} \mu_{l,ij} + \alpha_{l,ikj} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} + \beta_{l,ikj} \geq v_{l,ik} + v_{l,kj} - 0.5 \\ \mu_{u,ij} + \alpha_{u,ikj} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\ v_{u,ij} + \beta_{u,ikj} \geq v_{u,ik} + v_{u,kj} - 0.5 \\ i, k, j = 1, 2, \dots, n, k \neq i, j, i < j \\ \alpha_{l,ikj}, \beta_{l,ikj}, \alpha_{u,ikj}, \beta_{u,ikj} \geq 0, i, j = 1, 2, \dots, n, i < j \end{cases}, \tag{18}$$

where the first four constraints are obtained from formula (15) by adding the corresponding non-negative deviation variables $\alpha_{l,ikj}, \beta_{l,ikj}, \alpha_{u,ikj}, \beta_{u,ikj}$.

In solving model (19), when $\Gamma^* = 0$, we know that \tilde{R} satisfies formula (15), which shows the IVIFPR's additive consistency following Definition 14. Otherwise, \tilde{R} is inconsistent.

4 Programming models for dealing incomplete and inconsistent IVIFPRs

For the time pressure, lack of knowledge, the limitation of the DMS' limited expertise, and the incapacity to quantify the preference degree of one object over another, it is not always possible for the DMS to provide complete PRs (Cabrerizo et al. 2020; Ureña et al. 2015b). This section first focuses on incomplete IVIFPRs and builds programming models for determining missing values to make the incomplete IVIFPRs have the highest level of consistency.

Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be an incomplete IVIFPR, namely, there are missing values in \tilde{R} . If it is additively consistent following Definition 14, then formula (15) holds. However, we

usually cannot derive the conclusion that \tilde{R} is consistent. Thus, we relax formula (15) by adding the deviation variables, where

$$\begin{cases} \mu_{l,ij} + \alpha_{l,ikj} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\ v_{l,ij} + \beta_{l,ikj} \geq v_{l,ik} + v_{l,kj} - 0.5 \\ \mu_{u,ij} + \alpha_{u,ikj} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\ v_{u,ij} + \beta_{u,ikj} \geq v_{u,ik} + v_{u,kj} - 0.5 \end{cases} \tag{19}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$, and $\alpha_{l,ikj}, \beta_{l,ikj}, \alpha_{u,ikj}, \beta_{u,ikj}$ are non-negative variables.

In addition, there are following 15 different types of missing values of an IVIFV $\tilde{r}_{ij} = ([\mu_{l,ij}, \mu_{u,ij}], [v_{l,ij}, v_{u,ij}])$.

$$\left\{ \begin{array}{l} \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} ,, \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \notin U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \notin U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \notin U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \notin U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \\ \mu_{l,ij} \in U_{\mu}^{-} \wedge \mu_{u,ij} \in U_{\mu}^{+} \wedge v_{l,ij} \in U_{v}^{-} \wedge v_{u,ij} \in U_{v}^{+} \end{array} \right. \tag{20}$$

where

$$\begin{aligned} U_{\mu}^{-} &= \{ \mu_{l,ij} \text{ is unknown for all } i, j = 1, 2, \dots, n \text{ with } i < j \}, \\ U_{\mu}^{+} &= \{ \mu_{u,ij} \text{ is unknown for all } i, j = 1, 2, \dots, n \text{ with } i < j \}, \\ U_{v}^{-} &= \{ v_{l,ij} \text{ is unknown for all } i, j = 1, 2, \dots, n \text{ with } i < j \}, \\ U_{v}^{+} &= \{ v_{u,ij} \text{ is unknown for all } i, j = 1, 2, \dots, n \text{ with } i < j \}. \end{aligned}$$

Each case in formula (20) corresponds to a constraint c_p , ($i = 1, 2, \dots, 15$) as listed in the following formula:

$$\left\{ \begin{array}{l}
 c_1 : 0 \leq \mu_{l,ij} \leq \mu_{u,ij}, v_{l,ji} = \mu_{l,ij} \\
 c_2 : \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, v_{u,ji} = \mu_{u,ij} \\
 c_3 : 0 \leq v_{l,ij} \leq v_{u,ij}, \mu_{l,ji} = v_{l,ij} \\
 c_4 : v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_5 : 0 \leq \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, v_{l,ji} = \mu_{l,ij}, v_{u,ji} = \mu_{u,ij} \\
 c_6 : 0 \leq \mu_{l,ij} \leq \mu_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij}, v_{l,ji} = \mu_{l,ij}, \mu_{l,ji} = v_{l,ij} \\
 c_7 : 0 \leq \mu_{l,ij} \leq \mu_{u,ij}, v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, v_{l,ji} = \mu_{l,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_8 : \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{l,ji} = v_{l,ij} \\
 c_9 : \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, v_{l,ij} \leq v_{u,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_{10} : 0 \leq v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, \mu_{l,ji} = v_{l,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_{11} : 0 \leq \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij}, v_{l,ji} = \mu_{l,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{l,ji} = v_{l,ij} \\
 c_{12} : 0 \leq \mu_{l,ij} \leq \mu_{u,ij}, v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, v_{l,ji} = \mu_{l,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_{13} : 0 \leq \mu_{l,ij} \leq \mu_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, v_{l,ji} = \mu_{l,ij}, \mu_{l,ji} = v_{l,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_{14} : \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{l,ji} = v_{l,ij}, \mu_{u,ji} = v_{u,ij} \\
 c_{15} : 0 \leq \mu_{l,ij} \leq \mu_{u,ij} \leq 1 - v_{u,ij}, 0 \leq v_{l,ij} \leq v_{u,ij} \leq 1 - \mu_{u,ij}, \\
 v_{l,ji} = \mu_{l,ij}, v_{u,ji} = \mu_{u,ij}, \mu_{l,ji} = v_{l,ij}, \mu_{u,ji} = v_{u,ij}
 \end{array} \right. \tag{21}$$

As well known, the determined values of unknown judgements make the higher the consistency of incomplete IVIFPRs, the more useful the information will be. Let $C = \{c_1, c_2, \dots, c_{15}\}$. We build the following programming model to determine the missing values' optimal results:

$$\begin{aligned}
 f^* &= \min \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sum_{k=1, k \neq i, j}^n (\alpha_{l,ikj} + \beta_{l,ikj} + \alpha_{u,ikj} + \beta_{u,ikj}), \\
 \text{s.t.} \left\{ \begin{array}{l}
 \mu_{l,ij} + \alpha_{l,ikj} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\
 v_{l,ij} + \beta_{l,ikj} \geq v_{l,ik} + v_{l,kj} - 0.5 \\
 \mu_{u,ij} + \alpha_{u,ikj} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\
 v_{u,ij} + \beta_{u,ikj} \geq v_{u,ik} + v_{u,kj} - 0.5 \\
 \alpha_{l,ikj}, \beta_{l,ikj}, \alpha_{u,ikj}, \beta_{u,ikj} \geq 0, i, k, j = 1, 2, \dots, n, k \neq i, j \wedge i < j \\
 c_i \in C, i = 1, 2, \dots, 15
 \end{array} \right. , \tag{22}
 \end{aligned}$$

where the first four constraints can ensure the incomplete IVIFPR \tilde{R} to have the highest consistency level for the determined values of unknown judgements.

Considering the fact that the larger the uncertain degree is, the less useful the information will be, we further construct the following programming model:

$$\begin{aligned}
 g^* = \min & \sum_{\mu_{l,ij} \in U_{\mu}^- \vee \mu_{u,ij} \in U_{\mu}^+ \vee v_{l,ij} \in U_v^- \vee v_{u,ij} \in U_v^+} (\mu_{u,ij} - \mu_{l,ij} + v_{u,ij} - v_{l,ij}), \\
 \text{s.t.} & \left\{ \begin{aligned}
 & \sum_{i=1}^{n-1} \sum_{j=i+1}^n \sum_{k=1, k \neq i, j}^n (\alpha_{l,ikj} + \beta_{l,ikj} + \alpha_{u,ikj} + \beta_{u,ikj}) = f^* \\
 & \mu_{l,ij} + \alpha_{l,ikj} \geq \mu_{l,ik} + \mu_{l,kj} - 0.5 \\
 & v_{l,ij} + \beta_{l,ikj} \geq v_{l,ik} + v_{l,kj} - 0.5 \\
 & \mu_{u,ij} + \alpha_{u,ikj} \geq \mu_{u,ik} + \mu_{u,kj} - 0.5 \\
 & v_{u,ij} + \beta_{u,ikj} \geq v_{u,ik} + v_{u,kj} - 0.5 \\
 & \alpha_{l,ikj}, \beta_{l,ikj}, \alpha_{u,ikj}, \beta_{u,ikj} \geq 0, i, k, j = 1, 2, \dots, n, k \neq i, j \wedge i < j \\
 & c_i \in C, i = 1, 2, \dots, 15
 \end{aligned} \right. \tag{23}
 \end{aligned}$$

where f^* is the optimal value of model (22), and the other constraints are the same as that in model (22).

Here, we apply an example to show the utilization of models (22) and (23).

Example 1 (Tang et al. 2018) Assume that the incomplete IVIFPR \tilde{R} on object set $X = \{x_1, x_2, x_3, x_4\}$ is offered as follows:

$$\tilde{R} = \left(\begin{array}{cccc}
 ([0.50, 0.50], [0.50, 0.50]) & ([0.45, 0.45], [0.25, 0.35]) & x & x \\
 ([0.25, 0.35], [0.45, 0.45]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.40, 0.60], [0.30, 0.40]) & x \\
 x & ([0.30, 0.40], [0.40, 0.60]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.50], [0.45, 0.50]) \\
 x & x & ([0.45, 0.50], [0.35, 0.50]) & ([0.50, 0.50], [0.50, 0.50])
 \end{array} \right)$$

Based on models (22) and (23), the determined missing values are

$$\tilde{r}_{13} = ([0.55, 0.55], [0.25, 0.25]), \tilde{r}_{14} = ([0.55, 0.55], [0.25, 0.25]), \tilde{r}_{24} = ([0.45, 0.60], [0.40, 0.40]).$$

From $f^* = 0$, we know that this incomplete IVIFPR \tilde{R} is additively consistent.

Following model (24) (Tang et al. 2018), this incomplete IVIFPR \tilde{R} is inconsistent. This example concretely shows that the new concept is more flexible than that of Tang et al.'s.

Next, we consider another case: inconsistent IVIFPRs. Let $\tilde{R} = (\tilde{r}_{ij})_{n \times n}$ be any given IVIFPR, when it is inconsistent, we need to adjust the original judgements offered by the DMs. Meanwhile, the adjustments should be as small as possible for retaining more of the original information. Under these conditions, we can build the following goal programming model:

$$\Phi^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\alpha_{l,ij}^+ + \alpha_{l,ij}^- + \beta_{l,ij}^+ + \beta_{l,ij}^- + \alpha_{u,ij}^+ + \alpha_{u,ij}^- + \beta_{u,ij}^+ + \beta_{u,ij}^-),$$

$$\begin{cases}
 \mu_{l,ij} - \alpha_{l,ij}^+ + \alpha_{l,ij}^- \geq \mu_{l,ik} - \alpha_{l,ik}^+ + \alpha_{l,ik}^- + \mu_{l,kj} - \alpha_{l,kj}^+ + \alpha_{l,kj}^- - 0.5 \\
 v_{l,ij} - \beta_{l,ij}^+ + \beta_{l,ij}^- \geq v_{l,ik} - \beta_{l,ik}^+ + \beta_{l,ik}^- + v_{l,kj} - \beta_{l,kj}^+ + \beta_{l,kj}^- - 0.5 \\
 \mu_{u,ij} - \alpha_{u,ij}^+ + \alpha_{u,ij}^- \geq \mu_{u,ik} - \alpha_{u,ik}^+ + \alpha_{u,ik}^- + \mu_{u,kj} - \alpha_{u,kj}^+ + \alpha_{u,kj}^- - 0.5 \\
 v_{u,ij} - \beta_{u,ij}^+ + \beta_{u,ij}^- \geq v_{u,ik} - \beta_{u,ik}^+ + \beta_{u,ik}^- + v_{u,kj} - \beta_{u,kj}^+ + \beta_{u,kj}^- - 0.5 \\
 i, k, j = 1, 2, \dots, n, k \neq i, j, i < j; \\
 0 \leq \mu_{l,ij} - \alpha_{l,ij}^+ + \alpha_{l,ij}^- \leq \mu_{u,ij} - \alpha_{u,ij}^+ + \alpha_{u,ij}^-, i, j = 1, 2, \dots, n, i < j \\
 0 \leq v_{l,ij} - \beta_{l,ij}^+ + \beta_{l,ij}^- \leq v_{u,ij} - \beta_{u,ij}^+ + \beta_{u,ij}^-, i, j = 1, 2, \dots, n, i < j \\
 \mu_{u,ij} - \alpha_{u,ij}^+ + \alpha_{u,ij}^- + v_{u,ij} - \beta_{u,ij}^+ + \beta_{u,ij}^- \leq 1, i, j = 1, 2, \dots, n, i < j \\
 \alpha_{l,ij}^- - \alpha_{l,ij}^+ + \beta_{l,ji}^+ - \beta_{l,ji}^- = 0, i, j = 1, 2, \dots, n, i < j \\
 \beta_{l,ij}^- - \beta_{l,ij}^+ + \alpha_{l,ji}^+ - \alpha_{l,ji}^- = 0, i, j = 1, 2, \dots, n, i < j \\
 \alpha_{u,ij}^- - \alpha_{u,ij}^+ + \beta_{u,ji}^+ - \beta_{u,ji}^- = 0, i, j = 1, 2, \dots, n, i < j \\
 \beta_{u,ij}^- - \beta_{u,ij}^+ + \alpha_{u,ji}^+ - \alpha_{u,ji}^- = 0, i, j = 1, 2, \dots, n, i < j \\
 \alpha_{l,ij}^+, \alpha_{l,ij}^-, \beta_{l,ij}^+, \beta_{l,ij}^-, \alpha_{u,ij}^+, \alpha_{u,ij}^-, \beta_{u,ij}^+, \beta_{u,ij}^- \geq 0, i, j = 1, 2, \dots, n, i < j
 \end{cases} \quad (24)$$

where the first four constraints are obtained from formula (15) by adding the non-negative deviation variables $\alpha_{l,ij}^+, \alpha_{l,ij}^-, \beta_{l,ij}^+, \beta_{l,ij}^-, \alpha_{u,ij}^+, \alpha_{u,ij}^-, \beta_{u,ij}^+, \beta_{u,ij}^-$, $i, j = 1, 2, \dots, n, i < j$, for each IVIFV in the upper triangular part, the fifth to eleventh constraints are obtained from the construction of IVIFVs in IVIFPRs.

By solving model (24), following the determined deviation values $\alpha_{l,ij}^{*+}, \alpha_{l,ij}^{*-}, \beta_{l,ij}^{*+}, \beta_{l,ij}^{*-}, \alpha_{u,ij}^{*+}, \alpha_{u,ij}^{*-}, \beta_{u,ij}^{*+}, \beta_{u,ij}^{*-}$ for each pair of (i, j) , $i, j = 1, 2, \dots, n, i < j$, we derive the adjusted additively consistent IVIFPR with the smallest total adjustment.

To show the application of model (24), we consider below example:

Example 2 (Wang et al. 2009; Tang et al. 2018) Let the IVIFPR \tilde{R} on object set $X = \{x_1, x_2, x_3, x_4\}$ is offered as:

$$\tilde{R} = \begin{pmatrix}
 ([0.50, 0.50], [0.50, 0.50]) & ([0.25, 0.35], [0.55, 0.65]) & ([0.45, 0.55], [0.25, 0.35]) & ([0.35, 0.45], [0.45, 0.55]) \\
 ([0.55, 0.65], [0.25, 0.35]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.55, 0.65], [0.15, 0.25]) & ([0.15, 0.25], [0.35, 0.45]) \\
 ([0.25, 0.35], [0.45, 0.55]) & ([0.15, 0.25], [0.55, 0.65]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.35, 0.45], [0.15, 0.25]) \\
 ([0.45, 0.55], [0.35, 0.45]) & ([0.35, 0.45], [0.15, 0.25]) & ([0.15, 0.25], [0.35, 0.45]) & ([0.50, 0.50], [0.50, 0.50])
 \end{pmatrix}$$

First, we use model (18) to check whether \tilde{R} meets the additive consistency. Due to $\Gamma^* = 1.8$, we conclude that the IVIFPR \tilde{R} is inconsistent.

Based on model (24), the adjusting additively consistent IVIFPR \tilde{R}^* is obtained as:

$$\tilde{R}^* = \begin{pmatrix}
 ([0.50, 0.50], [0.50, 0.50]) & ([0.25, 0.35], [0.50, 0.60]) & ([0.45, 0.55], [0.25, 0.35]) & ([0.35, 0.45], [0.45, 0.45]) \\
 ([0.50, 0.60], [0.25, 0.35]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.55, 0.65], [0.15, 0.25]) & ([0.35, 0.55], [0.35, 0.35]) \\
 ([0.25, 0.35], [0.45, 0.55]) & ([0.15, 0.25], [0.55, 0.65]) & ([0.50, 0.50], [0.50, 0.50]) & ([0.30, 0.40], [0.40, 0.50]) \\
 ([0.45, 0.45], [0.35, 0.45]) & ([0.35, 0.35], [0.35, 0.55]) & ([0.40, 0.50], [0.30, 0.40]) & ([0.50, 0.50], [0.50, 0.50])
 \end{pmatrix}$$

where the adjusted judgements are shown in red. Moreover, from the IVIFVs: $\tilde{r}_{23} = ([0.55, 0.65], [0.15, 0.25])$, $\tilde{r}_{24} = ([0.35, 0.55], [0.35, 0.35])$ and $\tilde{r}_{34} = ([0.30, 0.40], [0.40, 0.50])$, we conclude that $x_2 > x_4 > x_3$, which avoids the issues in

the literature (Wang et al. 2009). It is noticeable that the method (Tang et al. 2018) is based on the equivalent consistency condition that adjusts all IVIFVs.

5 A new method for GDM with IVIFPRs

To solve the inability of a single DM in dealing with complex decision-making problems, GDM has become an effective technology. To tackle this case, this section discusses GDM with IVIFPRs.

Suppose that m DMs $E = \{e_1, e_2, \dots, e_m\}$ are invited to evaluate n objects $X = \{x_1, x_2, \dots, x_n\}$. Let $\tilde{R}_p = (\tilde{r}_{p,ij})_{n \times n}$ be the individual IVIFPR offered by the DM e_p , where $\tilde{r}_{p,ij} = ([\mu_{l,ij}^p, \mu_{u,ij}^p], [v_{l,ij}^p, v_{u,ij}^p])$ for all $i, j = 1, 2, \dots, n$, and all $p = 1, 2, \dots, m$. To obtain the collective IVIFPR, we need the weights of DMs. Considering the aspect that the higher the consensus degree between DMs' judgements is, the bigger the weights of the DMs will be, we use the distance measure between individual IVIFPRs to reflect this point.

Let $\tilde{R}^* = (\tilde{r}^*)_{n \times n}$ be the additively consistent IVIFPR of \tilde{R}_p , where $\tilde{r}^*_{p,ij} = ([\mu_{l,ij}^{*p}, \mu_{u,ij}^{*p}], [v_{l,ij}^{*p}, v_{u,ij}^{*p}])$, $i, j = 1, 2, \dots, n$, and $p = 1, 2, \dots, m$. On the basis of the Hamming distance between IVIFVs defined by Xu and Yager (2009), the following formula is used to determine the DMs' weights:

$$w_p = \frac{1 / \sum_{t=1, t \neq p}^m D(\tilde{R}_p^*, \tilde{R}_t^*)}{\sum_{p=1}^m \left(1 / \sum_{t=1, t \neq p}^m D(\tilde{R}_p^*, \tilde{R}_t^*) \right)}, \tag{25}$$

Where

$$D(\tilde{R}_p^*, \tilde{R}_t^*) = \sum_{i,j=1, i < j}^n \left(\left| \mu_{l,ij}^{*p} - \mu_{l,ij}^{*t} \right| + \left| \mu_{u,ij}^{*p} - \mu_{u,ij}^{*t} \right| + \left| v_{l,ij}^{*p} - v_{l,ij}^{*t} \right| + \left| v_{u,ij}^{*p} - v_{u,ij}^{*t} \right| + \left| \pi_{l,ij}^{*p} - \pi_{l,ij}^{*t} \right| + \left| \pi_{u,ij}^{*p} - \pi_{u,ij}^{*t} \right| \right)$$

such that $\pi_{l,ij}^* = 1 - \mu_{l,ij}^* - v_{u,ij}^*$ and $\pi_{u,ij}^* = 1 - \mu_{u,ij}^* - v_{l,ij}^*$.

Let $w = (w_1, w_2, \dots, w_m)$ be the weight vector derived from formula (25). Suppose that $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n \times n}$ is the collective IVIFPR integrated from \tilde{R}_p^* , $p = 1, 2, \dots, m$, using IVIF weighted averaging operator (Xu and Yager 2009), where

$$\tilde{r}_{C,ij}^* = \left([\mu_{l,ij}^{*C}, \mu_{u,ij}^{*C}], [v_{l,ij}^{*C}, v_{u,ij}^{*C}] \right) = \left(\left[\sum_{p=1}^m w_p \mu_{l,ij}^{*p}, \sum_{p=1}^m w_p \mu_{u,ij}^{*p} \right], \left[\sum_{p=1}^m w_p v_{l,ij}^{*p}, \sum_{p=1}^m w_p v_{u,ij}^{*p} \right] \right) \tag{26}$$

$$i, j = 1, 2, \dots, n.$$

Theorem 3 Let $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n \times n}$ be the additively consistent IVIFPR of \tilde{R}_p , $p = 1, 2, \dots, m$. Then, the collective IVIFPR \tilde{R}_C^* integrated as formula (26) is also additively consistent.

Proof To prove the additive consistency of \tilde{R}_C^* , it only needs to show the following condition:

$$\begin{cases} \mu_{l,ij}^{*C} \geq \mu_{l,ik}^{*C} + \mu_{l,kj}^{*C} - 0.5 \\ v_{l,ij}^{*C} \geq v_{l,ik}^{*C} + v_{l,kj}^{*C} - 0.5 \\ \mu_{u,ij}^{*C} \geq \mu_{u,ik}^{*C} + \mu_{u,kj}^{*C} - 0.5 \\ v_{u,ij}^{*C} \geq v_{u,ik}^{*C} + v_{u,kj}^{*C} - 0.5 \end{cases} \tag{27}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$.

Following the additive consistency of \tilde{R}_p^* , $p = 1, 2, \dots, m$, we have

$$\begin{cases} \mu_{l,ij}^{*p} \geq \mu_{l,ik}^{*p} + \mu_{l,kj}^{*p} - 0.5 \\ v_{l,ij}^{*p} \geq v_{l,ik}^{*p} + v_{l,kj}^{*p} - 0.5 \\ \mu_{u,ij}^{*p} \geq \mu_{u,ik}^{*p} + \mu_{u,kj}^{*p} - 0.5 \\ v_{u,ij}^{*p} \geq v_{u,ik}^{*p} + v_{u,kj}^{*p} - 0.5 \end{cases} \tag{28}$$

for all $i, k, j = 1, 2, \dots, n$ with $k \neq i, j \wedge i < j$, and $p = 1, 2, \dots, m$.

Combining formulae (26) and (28), it is easy to derive formula (27).

Consensus refers to the unanimity of individual opinions that reflects the option of the group. To obtain the representative ranking, the consensus analysis is necessary (Herrera-Viedma et al. 2014; Chiclana et al. 2008). Next, we study the consensus for GDM with IVIFPRs.

Definition 15 (*Group consensus index*): Let $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n \times n}$ be the additively consistent IVIFPR of $\tilde{R}_p = (\tilde{r}_{p,ij})_{n \times n}$, $p = 1, 2, \dots, m$, and let $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n \times n}$ be the collective IVIFPR shown as formula (26). Then, the group consensus index of the individual IVIFPR \tilde{R}_p^* is defined as:

$$GCI(\tilde{R}_p^*) = 1 - \frac{1}{2n(n-1)} \sum_{i,j=1,i < j}^n \left(\left| \mu_{l,ij}^{*p} - \mu_{l,ij}^{*C} \right| + \left| \mu_{u,ij}^{*p} - \mu_{u,ij}^{*C} \right| + \left| v_{l,ij}^{*p} - v_{l,ij}^{*C} \right| + \left| v_{u,ij}^{*p} - v_{u,ij}^{*C} \right| \right) \tag{29}$$

Let θ^* be the threshold of the consensus. If $GCI(\tilde{R}_p^*) < \theta^*$ for some $p = 1, 2, \dots, m$, we need to improve its consensus level. To do this, we construct the following programming model:

$$\begin{aligned}
 \zeta^* &= \max \sum_{i,j=1,i < j}^n (\alpha_{l,ij} + \alpha_{u,ij} + \beta_{l,ij} + \beta_{u,ij}) \\
 &\left\{ \begin{aligned}
 &\left| \sum_{i,j=1,i < j}^n \left(\left(\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} \right) - \left(w_p (\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t \mu_{l,ij}^{*t} \right) \right) \right| \\
 &+ \left| \left(\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} \right) - \left(w_p (\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t \mu_{u,ij}^{*t} \right) \right| \\
 &+ \left(\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} \right) - \left(w_p (\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t v_{l,ij}^{*t} \right) \Big| \\
 &+ \left| \left(\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} \right) - \left(w_p (\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t v_{u,ij}^{*t} \right) \right| \Bigg\} \\
 &\leq 2n(n-1)(1-\theta^*) \\
 &i, j = 1, 2, \dots, n, i < j \\
 \text{s.t. } &\left\{ \begin{aligned}
 &\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} \geq (\alpha_{l,ik} \mu_{l,ik}^{*p} + (1 - \alpha_{l,ik}) \mu_{l,ik}^{*C}) + (\alpha_{l,kj} \mu_{l,kj}^{*p} + (1 - \alpha_{l,kj}) \mu_{l,kj}^{*C}) - 0.5 \\
 &\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} \geq (\alpha_{u,ik} \mu_{u,ik}^{*p} + (1 - \alpha_{u,ik}) \mu_{u,ik}^{*C}) + (\alpha_{u,kj} \mu_{u,kj}^{*p} + (1 - \alpha_{u,kj}) \mu_{u,kj}^{*C}) - 0.5 \\
 &\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} \geq (\beta_{l,ik} v_{l,ik}^{*p} + (1 - \beta_{l,ik}) v_{l,ik}^{*C}) + (\beta_{l,kj} v_{l,kj}^{*p} + (1 - \beta_{l,kj}) v_{l,kj}^{*C}) - 0.5 \\
 &\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} \geq (\beta_{u,ik} v_{u,ik}^{*p} + (1 - \beta_{u,ik}) v_{u,ik}^{*C}) + (\beta_{u,kj} v_{u,kj}^{*p} + (1 - \beta_{u,kj}) v_{u,kj}^{*C}) - 0.5 \\
 &0 \leq \alpha_{l,ij}, \alpha_{l,ik}, \alpha_{l,kj}, \alpha_{u,ij}, \alpha_{u,ik}, \alpha_{u,kj}, \beta_{l,ij}, \beta_{l,ik}, \beta_{l,kj}, \beta_{u,ij}, \beta_{u,ik}, \beta_{u,kj} \leq 1 \\
 &i, k, j = 1, 2, \dots, n, k \neq i, j \wedge i < j \\
 &\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} \leq \alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}, i, j = 1, 2, \dots, n, i < j \\
 &\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} \leq \beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}, i, j = 1, 2, \dots, n, i < j \\
 &\alpha_{l,ij} = \beta_{l,ij}, \alpha_{u,ij} = \beta_{u,ij}, \beta_{l,ij} = \alpha_{l,ij}, \beta_{u,ij} = \alpha_{u,ij}, i, j = 1, 2, \dots, n, i < j
 \end{aligned} \right. \tag{30}
 \end{aligned}$$

where the first constraint is derived from formula (29) that can ensure the adjusted IVIFPR to meet the consensus requirement, the second to fifth constraints ensure the adjusted IVIFPR to be additively consistent, the seventh and eighth constraints ensure the adjusted elements to be IVIFVs, and the last constraint guarantees the endpoints of corresponding IVIFVs to have the same adjustment.

To facilitate the solution of model (30), we further introduce a linear programming model. Let

$$\left\{ \begin{aligned}
 &\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} - \left(w_p (\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t \mu_{l,ij}^{*t} \right) - \tau_{ij}^+ + \tau_{ij}^- = 0 \\
 &\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} - \left(w_p (\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t \mu_{u,ij}^{*t} \right) - \eta_{ij}^+ + \eta_{ij}^- = 0 \\
 &\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} - \left(w_p (\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t v_{l,ij}^{*t} \right) - \phi_{ij}^+ + \phi_{ij}^- = 0 \\
 &\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} - \left(w_p (\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}) + \sum_{t=1,t \neq p}^m w_t v_{u,ij}^{*t} \right) - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0
 \end{aligned} \right. \tag{31}$$

where $\tau_{ij}^+, \tau_{ij}^-, \eta_{ij}^+, \eta_{ij}^-, \phi_{ij}^+, \phi_{ij}^-, \varepsilon_{ij}^+, \varepsilon_{ij}^-$ are all non-negative variables for all $i, j = 1, 2, \dots, n, i < j$ such that $\tau_{ij}^+ \times \tau_{ij}^- = \eta_{ij}^+ \times \eta_{ij}^- = \phi_{ij}^+ \times \phi_{ij}^- = \varepsilon_{ij}^+ \times \varepsilon_{ij}^- = 0$.

Therefore, we have

$$\begin{cases} \left| \alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} - \left(w_p (\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t \mu_{l,ij}^{*t} \right) \right| = \tau_{ij}^+ + \tau_{ij}^- \\ \left| \alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} - \left(w_p (\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t \mu_{u,ij}^{*t} \right) \right| = \eta_{ij}^+ + \eta_{ij}^- \\ \left| \beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} - \left(w_p (\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t v_{l,ij}^{*t} \right) \right| = \phi_{ij}^+ + \phi_{ij}^- \\ \left| \beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} - \left(w_p (\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t v_{u,ij}^{*t} \right) \right| = \varepsilon_{ij}^+ + \varepsilon_{ij}^- \end{cases} \quad (32)$$

Following formula (32), we derive below linear programming model:

$$\begin{aligned} \zeta^* = \max \sum_{i,j=1, i < j}^n (\alpha_{l,ij} + \alpha_{u,ij} + \beta_{l,ij} + \beta_{u,ij}) \\ \begin{cases} \sum_{i,j=1, i < j}^n (\tau_{ij}^+ + \tau_{ij}^- + \eta_{ij}^+ + \eta_{ij}^- + \phi_{ij}^+ + \phi_{ij}^- + \varepsilon_{ij}^+ + \varepsilon_{ij}^-) \leq 2n(n-1)(1-\theta^*) \\ \alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} - \left(w_p (\alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t \mu_{l,ij}^{*t} \right) - \tau_{ij}^+ + \tau_{ij}^- = 0 \\ \alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} - \left(w_p (\alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t \mu_{u,ij}^{*t} \right) - \eta_{ij}^+ + \eta_{ij}^- = 0 \\ \beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} - \left(w_p (\beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t v_{l,ij}^{*t} \right) - \phi_{ij}^+ + \phi_{ij}^- = 0 \\ \beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} - \left(w_p (\beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}) + \sum_{t=1, t \neq p}^m w_t v_{u,ij}^{*t} \right) - \varepsilon_{ij}^+ + \varepsilon_{ij}^- = 0 \\ \tau_{ij}^+ \times \tau_{ij}^- = \eta_{ij}^+ \times \eta_{ij}^- = \phi_{ij}^+ \times \phi_{ij}^- = \varepsilon_{ij}^+ \times \varepsilon_{ij}^- = 0 \\ \tau_{ij}^+, \tau_{ij}^-, \eta_{ij}^+, \eta_{ij}^-, \phi_{ij}^+, \phi_{ij}^-, \varepsilon_{ij}^+, \varepsilon_{ij}^- \geq 0, i, j = 1, 2, \dots, n, i < j \\ \text{s.t.} \begin{cases} \alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} \geq (\alpha_{l,ik} \mu_{l,ik}^{*p} + (1 - \alpha_{l,ik}) \mu_{l,ik}^{*C}) + (\alpha_{l,kj} \mu_{l,kj}^{*p} + (1 - \alpha_{l,kj}) \mu_{l,kj}^{*C}) - 0.5 \\ \alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C} \geq (\alpha_{u,ik} \mu_{u,ik}^{*p} + (1 - \alpha_{u,ik}) \mu_{u,ik}^{*C}) + (\alpha_{u,kj} \mu_{u,kj}^{*p} + (1 - \alpha_{u,kj}) \mu_{u,kj}^{*C}) - 0.5 \\ \beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} \geq (\beta_{l,ik} v_{l,ik}^{*p} + (1 - \beta_{l,ik}) v_{l,ik}^{*C}) + (\beta_{l,kj} v_{l,kj}^{*p} + (1 - \beta_{l,kj}) v_{l,kj}^{*C}) - 0.5 \\ \beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C} \geq (\beta_{u,ik} v_{u,ik}^{*p} + (1 - \beta_{u,ik}) v_{u,ik}^{*C}) + (\beta_{u,kj} v_{u,kj}^{*p} + (1 - \beta_{u,kj}) v_{u,kj}^{*C}) - 0.5 \\ 0 \leq \alpha_{l,ij}, \alpha_{l,ik}, \alpha_{l,kj}, \alpha_{u,ij}, \alpha_{u,ik}, \alpha_{u,kj}, \beta_{l,ij}, \beta_{l,ik}, \beta_{l,kj}, \beta_{u,ij}, \beta_{u,ik}, \beta_{u,kj} \leq 1 \\ i, k, j = 1, 2, \dots, n, k \neq i, j \wedge i < j \\ \alpha_{l,ij} \mu_{l,ij}^{*p} + (1 - \alpha_{l,ij}) \mu_{l,ij}^{*C} \leq \alpha_{u,ij} \mu_{u,ij}^{*p} + (1 - \alpha_{u,ij}) \mu_{u,ij}^{*C}, i, j = 1, 2, \dots, n, i < j \\ \beta_{l,ij} v_{l,ij}^{*p} + (1 - \beta_{l,ij}) v_{l,ij}^{*C} \leq \beta_{u,ij} v_{u,ij}^{*p} + (1 - \beta_{u,ij}) v_{u,ij}^{*C}, i, j = 1, 2, \dots, n, i < j \\ \alpha_{l,ij} = \beta_{l,ji}, \alpha_{u,ij} = \beta_{u,ji}, \beta_{l,ij} = \alpha_{l,ji}, \beta_{u,ij} = \alpha_{u,ji}, i, j = 1, 2, \dots, n, i < j \end{cases} \end{cases} \quad (33)$$

Model (33) not only guarantees the additive consistency and consensus of the adjusted individual IVIFPR, but also endows different IVIFVs with different adjustments. Furthermore, the adjusted individual IVIFPR has the smallest total adjustment so that more original information can be remained.

Based on the above discussion, this paper develops the following GDM method.

Algorithm.

- Step 1. If all individual IVIFPRs $\tilde{R}_p = (\tilde{r}_{p,ij})_{n \times n}$, $p = 1, 2, \dots, m$, are all complete, go to Step 2. Otherwise, models (22) and (23) are adopted to determine the missing values.
- Step 2. Model (18) is used to judge the additive consistency. When individual IVIFPRs $\tilde{R}_p = (\tilde{r}_{p,ij})_{n \times n}$, $p = 1, 2, \dots, m$, are all additively consistent, go to Step 3. Otherwise,

model (24) is applied to derive the additively consistent IVIFPRs, which can be denoted as $\tilde{R}_p^* = (\tilde{r}_{p,ij}^*)_{n \times n}$, $p = 1, 2, \dots, m$.

Step 3. Formula (25) is used to determine the weights of DMs, and formula (26) is adopted to obtain the collective IVIFPR $\tilde{R}_C^* = (\tilde{r}_{C,ij}^*)_{n \times n}$.

Step 4. Formula (29) is employed to measure the consensus. Let θ^* be the given consensus threshold. If we have $GCI(\tilde{R}_p^*) > \theta^*$ for all $p = 1, 2, \dots, m$, go to Step 5. Otherwise, model (33) is applied to improve the consensus level of the corresponding individual IVIFPR.

Step 5. Calculating the row arithmetic means of \tilde{R}_C^* , where

$$\tilde{r}_i = \left(\left[\frac{1}{n} \sum_{j=1}^n \mu_{l,ij}^{*C}, \frac{1}{n} \sum_{j=1}^n \mu_{u,ij}^{*C} \right], \left[\frac{1}{n} \sum_{j=1}^n v_{l,ij}^{*C}, \frac{1}{n} \sum_{j=1}^n v_{u,ij}^{*C} \right] \right) \tag{34}$$

for all $i = 1, 2, \dots, n$.

Step 6. Using the score and accuracy functions (Xu 2007b), we have

$$S(\tilde{r}_i) = 0.5(\mu_{l,i} + \mu_{u,i} - v_{l,i} - v_{u,i}), \tag{35}$$

$$H(\tilde{r}_i) = 0.5(\mu_{l,i} + \mu_{u,i} + v_{l,i} + v_{u,i}), \tag{36}$$

where $\tilde{r}_i = ([\mu_{l,i}, \mu_{u,i}], [v_{l,i}, v_{u,i}])$ for all $i = 1, 2, \dots, n$.

Then, the ranking of objects x_1, x_2, \dots, x_n is derived following the order relationship of $\tilde{r}_i, i = 1, 2, \dots, n$.

6 Case study and comparison

To illustrate the application of the proposed algorithm and compare the new method with previous ones, this section offers two examples.

Remark 1 Tang et al. (2018) have demonstrated the advantages of their method compared with the previous ones, this paper mainly focuses on the comparison of the proposed method with Tang et al.’s method (Tang et al. 2018) through case study.

Example 3 At present, there are more than 2600 colleges in China, among which over 700 are private that accounts for nearly 1/3. Compared with public colleges, the private ones lack the financial support of the state. Besides the quality of talent training, another goal that deserves the attention is the economic benefits. Tuition as the main source of income largely depends on the number of students. To ensure the scale of students, private colleges must apply to the ministry of education for new enrollment majors in due course. To this end, there are a few factors that need to be considered, such as regional economic development, recruiting company manpower needs, synergy effect with existing majors, and conditions for running school. There are usually a few majors to choose. Therefore, the choice of new majors in private colleges can be classified as a MCDM problem. Moreover, the development of social economy is rapid, which increases the difficulty of tackling the problem. Due to the complexity of this problem, a single decision-making body is unable to cope with it, and GDM is a powerful tool.

Table 1 IVIFPR \tilde{R}_1 offered by the DM e_1

e_1	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.55], [0.30,0.40]]	[[0.40,0.65], [0.15,0.25]]	[[0.60,0.80], [0.05,0.15]]
x_2	[[0.30,0.40], [0.40,0.55]]	[[0.50,0.50], [0.50,0.50]]	[[0.35,0.60], [0.25,0.35]]	[[0.45,0.75], [0.15,0.25]]
x_3	[[0.15,0.25], [0.40,0.65]]	[[0.25,0.35], [0.35,0.60]]	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.60], [0.25,0.30]]
x_4	[[0.05,0.15], [0.60,0.80]]	[[0.15,0.25], [0.45,0.75]]	[[0.25,0.30], [0.40,0.60]]	[[0.50,0.50], [0.50,0.50]]

Table 2 IVIFPR \tilde{R}_2 offered by the DM e_2

e_2	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.25,0.35], [0.40,0.60]]	[[0.40,0.55], [0.30,0.40]]	[[0.40,0.55], [0.35,0.40]]
x_2	[[0.40,0.60], [0.25,0.35]]	[[0.50,0.50], [0.50,0.50]]	[[0.50,0.75], [0.10,0.20]]	[[0.45,0.60], [0.30,0.35]]
x_3	[[0.30,0.40], [0.40,0.55]]	[[0.10,0.20], [0.50,0.75]]	[[0.50,0.50], [0.50,0.50]]	[[0.20,0.35], [0.45,0.60]]
x_4	[[0.35,0.40], [0.40,0.55]]	[[0.30,0.35], [0.45,0.60]]	[[0.45,0.60], [0.20,0.35]]	[[0.50,0.50], [0.50,0.50]]

Qingdao Institute of Technology is a private college in Shandong province of China. With the improvement of running conditions, the College Board plans to expand the enrollment. After preliminary screening, there are four major finalists: energy and power engineering, financial management, aviation logistics management and underground city space engineering, shown as $X = \{x_1, x_2, x_3, x_4\}$. In addition to the factors mentioned above, the quota provided by the competent authority is usually one or two. To guarantee permission, three DMs $E = \{e_1, e_2, e_3\}$ are invited to give their preferences and to rank these majors. Due to the limitations of different professional backgrounds and their preferences, they are allowed to offer the uncertain preferred and non-preferred judgements simultaneously, which comes down to the GDM with IVIFPRs. Suppose that the three DMs' individual IVIFPRs are listed in Tables 1, 2 and 3.

Step 1. Since the individual IVIFPRs are all complete, go to Step 2.

Step 2. Based on the three DMs' additively consistent IVIFPRs and formula (25), the weights of the DMs are derived as $w_1 = 0.337$, $w_2 = 0.291$ and $w_3 = 0.372$.

Step 3. Using model (18), we have $I_1^* = 0.05$, $I_2^* = 0.5$, $I_3^* = 0.05$. Thus, all of them are inconsistent. To derive their additively consistent IVIFPRs, model (24) is adopted, by which the associated results are shown in Tables 4, 5 and 6.

Using formula (26), the collective IVIFPR \tilde{R}_C^* is obtained as shown in Table 7.

Step 4. Let $\theta^* = 0.9$ be the given consensus threshold. Using formula (29), we have

Table 3 IVIFPR \tilde{R}_3 offered by the DM e_3

e_3	x_1	x_2	x_3	x_4
x_1	[(0.50,0.50), [0.50,0.50)]	[(0.30,0.40), [0.40,0.55)]	[(0.35,0.50), [0.30,0.45)]	[(0.55,0.70), [0.15,0.25)]
x_2	[(0.40,0.55), [0.30,0.40)]	[(0.50,0.50), [0.50,0.50)]	[(0.45,0.55), [0.25,0.35)]	[(0.65,0.80), [0.05,0.15)]
x_3	[(0.30,0.45), [0.35,0.50)]	[(0.25,0.35), [0.45,0.55)]	[(0.50,0.50), [0.50,0.50)]	[(0.50,0.65), [0.20,0.30)]
x_4	[(0.15,0.25), [0.55,0.70)]	[(0.05,0.15), [0.65,0.80)]	[(0.20,0.30), [0.50,0.65)]	[(0.50,0.50), [0.50,0.50)]

Table 4 Additively consistent IVIFPR \tilde{R}_1^*

e_1	x_1	x_2	x_3	x_4
x_1	[(0.50,0.50), [0.50,0.50)]	[(0.40,0.55), [0.30,0.40)]	[(0.40,0.65), [0.15,0.25)]	[(0.60,0.80), [0.05,0.15)]
x_2	[(0.30,0.40), [0.40,0.55)]	[(0.50,0.50), [0.50,0.50)]	[(0.35,0.55), [0.25,0.35)]	[(0.45,0.75), [0.15,0.25)]
x_3	[(0.15,0.25), [0.40,0.65)]	[(0.25,0.35), [0.35,0.55)]	[(0.50,0.50), [0.50,0.50)]	[(0.40,0.60), [0.25,0.30)]
x_4	[(0.05,0.15), [0.60,0.80)]	[(0.15,0.25), [0.45,0.75)]	[(0.25,0.30), [0.40,0.60)]	[(0.50,0.50), [0.50,0.50)]

Table 5 Additively consistent IVIFPR \tilde{R}_2^*

e_2	x_1	x_2	x_3	x_4
x_1	[(0.50,0.50), [0.50,0.50)]	[(0.25,0.35), [0.40,0.60)]	[(0.40,0.60), [0.30,0.35)]	[(0.40,0.50), [0.35,0.45)]
x_2	[(0.40,0.60), [0.25,0.35)]	[(0.50,0.50), [0.50,0.50)]	[(0.50,0.75), [0.10,0.20)]	[(0.45,0.60), [0.30,0.35)]
x_3	[(0.30,0.35), [0.40,0.60)]	[(0.10,0.20), [0.50,0.75)]	[(0.50,0.50), [0.50,0.50)]	[(0.20,0.35), [0.45,0.60)]
x_4	[(0.35,0.45), [0.40,0.50)]	[(0.30,0.35), [0.45,0.60)]	[(0.45,0.60), [0.20,0.35)]	[(0.50,0.50), [0.50,0.50)]

$$GCI(\tilde{R}_1^*) = 0.930, GCI(\tilde{R}_2^*) = 0.891, GCI(\tilde{R}_3^*) = 0.941.$$

Because $GCI(\tilde{R}_2^*) < 0.9$, model (33) is applied to improve its consensus level. Furthermore, the associated adjusting IVIFPR $\tilde{R}_2'^*$ is listed in Table 8.

Using formula (25), the weights are $w_1 = 0.334$, $w_2 = 0.304$ and $w_3 = 0.362$, and the associated collective IVIFPR \tilde{R}_C^* is obtained as shown in Table 9.

Using formula (29), we have $GCI(\tilde{R}_1^*) = 0.939$, $GCI(\tilde{R}_2'^*) = 0.918$, and $GCI(\tilde{R}_3^*) = 0.947$, which meets the consensus requirement.

Step 5. Following $\tilde{R}_C'^*$ and formula (34), we have

Table 6 Additively consistent IVIFPR \tilde{R}_3^*

e_3	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.30,0.40], [0.40,0.55]]	[[0.35,0.50], [0.30,0.45]]	[[0.55,0.70], [0.15,0.25]]
x_2	[[0.40,0.55], [0.30,0.40]]	[[0.50,0.50], [0.50,0.50]]	[[0.45,0.60], [0.25,0.35]]	[[0.65,0.80], [0.05,0.15]]
x_3	[[0.30,0.45], [0.35,0.50]]	[[0.25,0.35], [0.45,0.60]]	[[0.50,0.50], [0.50,0.50]]	[[0.50,0.65], [0.20,0.30]]
x_4	[[0.15,0.25], [0.55,0.70]]	[[0.05,0.15], [0.65,0.80]]	[[0.20,0.30], [0.50,0.65]]	[[0.50,0.50], [0.50,0.50]]

$$\tilde{r}_1 = ([0.431, 0.552], [0.319, 0.394]), \tilde{r}_2 = ([0.455, 0.579], [0.292, 0.376])$$

$$\tilde{r}_3 = ([0.333, 0.438], [0.399, 0.505]), \tilde{r}_4 = ([0.273, 0.321], [0.480, 0.614])$$

Step 6. Using the score function for IVIFVs, we obtain $s(\tilde{r}_1) = 0.135$, $s(\tilde{r}_2) = 0.183$, $s(\tilde{r}_3) = -0.066$, and $s(\tilde{r}_4) = -0.250$.

Therefore, the ranking order is $x_2 > x_1 > x_3 > x_4$.

From the above ranking, we find that energy and power engineering and financial management should be applied for the quota of two majors. If there is only one quota, the priority should be given to financial management.

Next, we briefly explore the analysis process for this example using the method (Tang et al. 2018).

Step 1. For the individual IVIFPRs in Example 3, the individually additively consistent QFIPRs are derived as follows:

Table 7 The collective IVIFPR \tilde{R}_C^*

	x_1	x_2	x_3	x_4
x_1	([0.500,0.500], [0.500,0.500])	([0.319,0.436], [0.366,0.514])	([0.381,0.580], [0.249,0.354])	([0.523,0.676], [0.175,0.275])
x_2	([0.366,0.514], [0.319,0.436])	([0.500,0.500], [0.500,0.500])	([0.431,0.627], [0.206,0.306])	([0.524,0.725], [0.156,0.242])
x_3	([0.249,0.354], [0.381,0.580])	([0.206,0.306], [0.431,0.627])	([0.500,0.500], [0.500,0.500])	([0.379,0.546], [0.290,0.387])
x_4	([0.175,0.275], [0.523,0.676])	([0.156,0.242], [0.524,0.725])	([0.290,0.387], [0.379,0.546])	([0.500,0.500], [0.500,0.500])

Table 8 Adjusting IVIFPR \tilde{R}_2^*

e_2	x_1	x_2	x_3	x_4
x_1	([0.500,0.500], [0.500,0.500])	([0.250,0.406], [0.400,0.518])	([0.400,0.600], [0.300,0.350])	([0.400,0.506], [0.305,0.305])
x_2	([0.400,0.518], [0.250,0.406])	([0.500,0.500], [0.500,0.500])	([0.500,0.664], [0.100,0.306])	([0.450,0.600], [0.255,0.255])
x_3	([0.300,0.350], [0.400,0.600])	([0.100,0.306], [0.500,0.664])	([0.500,0.500], [0.500,0.500])	([0.200,0.406], [0.420,0.420])
x_4	([0.305,0.305], [0.400,0.506])	([0.255,0.255], [0.450,0.600])	([0.420,0.420], [0.200,0.406])	([0.500,0.500], [0.500,0.500])

Table 9 Collective IVIFPR \tilde{R}_C^*

	x_1	x_2	x_3	x_4
x_1	([0.500,0.500], [0.500,0.500])	([0.319,0.452], [0.366,0.490])	([0.381,0.580], [0.249,0.354])	([0.523,0.677], [0.161,0.232])
x_2	([0.366,0.489], [0.319,0.453])	([0.500,0.500], [0.500,0.500])	([0.431,0.602], [0.206,0.337])	([0.524,0.725], [0.144,0.214])
x_3	([0.249,0.353], [0.382,0.581])	([0.204,0.337], [0.432,0.603])	([0.500,0.500], [0.500,0.500])	([0.379,0.562], [0.281,0.335])
x_4	([0.163,0.232], [0.521,0.675])	([0.146,0.216], [0.522,0.722])	([0.283,0.336], [0.375,0.560])	([0.500,0.500], [0.500,0.500])

$$\begin{aligned} \bar{\eta}_1^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.3000, 0.8750] & [0.4500, 0.8500] & [0.6500, 0.8750] \\ [0.7000, 0.1250] & [0.5000, 0.5000] & [0.6500, 0.4750] & [0.8500, 0.5000] \\ [0.5500, 0.1500] & [0.3500, 0.5250] & [0.5000, 0.5000] & [0.7000, 0.5250] \\ [0.3500, 0.1250] & [0.1500, 0.5000] & [0.3000, 0.4750] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}_1^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.6000, 0.5500] & [0.7250, 0.6625] & [0.8750, 0.7875] \\ [0.4000, 0.4500] & [0.5000, 0.5000] & [0.6250, 0.6125] & [0.7750, 0.7375] \\ [0.2750, 0.3375] & [0.3750, 0.3875] & [0.5000, 0.5000] & [0.6500, 0.6250] \\ [0.1250, 0.2125] & [0.2250, 0.2625] & [0.3500, 0.3750] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\eta}_2^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.3625, 0.4313] & [0.6500, 0.5625] & [0.4375, 0.6063] \\ [0.6375, 0.5867] & [0.5000, 0.5000] & [0.7875, 0.6313] & [0.5750, 0.6750] \\ [0.3500, 0.4375] & [0.2125, 0.3687] & [0.5000, 0.5000] & [0.2875, 0.5438] \\ [0.5625, 0.3937] & [0.4250, 0.3250] & [0.7125, 0.4562] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}_2^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.3875, 0.3625] & [0.6500, 0.6000] & [0.5625, 0.4875] \\ [0.6125, 0.6375] & [0.5000, 0.5000] & [0.7625, 0.7375] & [0.6750, 0.6250] \\ [0.3500, 0.4000] & [0.2375, 0.2625] & [0.5000, 0.5000] & [0.4125, 0.3875] \\ [0.4375, 0.5125] & [0.3250, 0.3750] & [0.5875, 0.6125] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}_3^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.4250, 0.4125] & [0.5625, 0.5000] & [0.7625, 0.6875] \\ [0.5750, 0.5875] & [0.5000, 0.5000] & [0.6375, 0.5875] & [0.8375, 0.7750] \\ [0.4375, 0.5000] & [0.3625, 0.4125] & [0.5000, 0.5000] & [0.7000, 0.6875] \\ [0.2375, 0.3125] & [0.1625, 0.2250] & [0.3000, 0.3125] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\eta}_3^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.5125, 0.3500] & [0.7000, 0.3750] & [0.9375, 0.4750] \\ [0.4875, 0.6500] & [0.5000, 0.5000] & [0.6875, 0.5250] & [0.9250, 0.6250] \\ [0.3000, 0.6250] & [0.3125, 0.4750] & [0.5000, 0.5000] & [0.7375, 0.6000] \\ [0.0625, 0.5250] & [0.0750, 0.3750] & [0.2625, 0.4000] & [0.5000, 0.5000] \end{pmatrix} \end{aligned}$$

Step 2. Following model (35) (Tang et al. 2018), the weights are equal to 1/3. Then, the collectively additively consistent QFIPRs are

$$\begin{aligned} \bar{\eta}^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.3917, 0.5521] & [0.6000, 0.5958] & [0.6750, 0.6521] \\ [0.6083, 0.4479] & [0.5000, 0.5000] & [0.7083, 0.5438] & [0.7833, 0.6000] \\ [0.4000, 0.4042] & [0.2917, 0.4562] & [0.5000, 0.5000] & [0.5750, 0.5563] \\ [0.3250, 0.3479] & [0.2167, 0.4000] & [0.4250, 0.4437] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}^* &= \begin{pmatrix} [0.5000, 0.5000] & [0.4708, 0.4417] & [0.6458, 0.5875] & [0.7333, 0.6542] \\ [0.5292, 0.5583] & [0.5000, 0.5000] & [0.6750, 0.6458] & [0.7625, 0.7125] \\ [0.3542, 0.4125] & [0.3250, 0.3542] & [0.5000, 0.5000] & [0.5875, 0.5667] \\ [0.2667, 0.3458] & [0.2375, 0.2875] & [0.4125, 0.4333] & [0.5000, 0.5000] \end{pmatrix}. \end{aligned}$$

Let $\theta^* = 0.9$, the consensus degrees based on formula (29) are $GCI_1 = 0.8990$, $GCI_2 = 0.8962$, and $GCI_3 = 0.9054$. Using the iterative method for improving the consensus level (Tang et al. 2018), the adjusted additively consistent QFIPRs are

$$\begin{aligned} \bar{\eta}'_2 &= \begin{pmatrix} [0.5000, 0.5000] & [0.3729, 0.4555] & [0.6492, 0.5692] & [0.4896, 0.6155] \\ [0.6271, 0.5445] & [0.5000, 0.5000] & [0.7763, 0.6138] & [0.6167, 0.6600] \\ [0.3508, 0.4308] & [0.2237, 0.3862] & [0.5000, 0.5000] & [0.3404, 0.5463] \\ [0.5104, 0.3845] & [0.3833, 0.3400] & [0.6596, 0.4537] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}'_2 &= \begin{pmatrix} [0.5000, 0.5000] & [0.4204, 0.3783] & [0.6492, 0.5975] & [0.5805, 0.5208] \\ [0.5796, 0.6217] & [0.5000, 0.5000] & [0.7288, 0.7192] & [0.6600, 0.6425] \\ [0.3508, 0.4025] & [0.2712, 0.2808] & [0.5000, 0.5000] & [0.4313, 0.4233] \\ [0.4195, 0.4792] & [0.3400, 0.3575] & [0.5684, 0.5767] & [0.5000, 0.5000] \end{pmatrix}. \end{aligned}$$

Furthermore, the corresponding collectively additively consistent QFIPRs are

$$\begin{aligned} \bar{\eta}' &= \begin{pmatrix} [0.5000, 0.5000] & [0.4260, 0.5602] & [0.6456, 0.5981] & [0.7393, 0.6552] \\ [0.5740, 0.4398] & [0.5000, 0.5000] & [0.7195, 0.5379] & [0.8132, 0.5950] \\ [0.3544, 0.4019] & [0.2805, 0.4621] & [0.5000, 0.5000] & [0.5937, 0.5571] \\ [0.2607, 0.3448] & [0.1868, 0.4050] & [0.4063, 0.4429] & [0.5000, 0.5000] \end{pmatrix}, \\ \bar{\lambda}' &= \begin{pmatrix} [0.5000, 0.5000] & [0.4818, 0.4527] & [0.6456, 0.5981] & [0.7393, 0.6710] \\ [0.5182, 0.5473] & [0.5000, 0.5000] & [0.6638, 0.6454] & [0.7575, 0.7183] \\ [0.3544, 0.4019] & [0.3362, 0.3546] & [0.5000, 0.5000] & [0.5937, 0.5729] \\ [0.2607, 0.3290] & [0.2425, 0.2817] & [0.4063, 0.4271] & [0.5000, 0.5000] \end{pmatrix}. \end{aligned}$$

and the associated consensus levels are $GCI_1 = 0.9034$, $GCI_2 = 0.9089$, $GCI_3 = 0.9088$.

Step 3. With respect to the collectively additively consistent QFIPRs $\bar{\eta}'^*$ and $\bar{\lambda}'^*$, the corresponding collectively additively consistent IVIFPR is

$$\bar{R}^* = \begin{pmatrix} ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.4260, 0.4527], [0.4398, 0.5182]) & ([0.5981, 0.5981], [0.3544, 0.3544]) & ([0.6552, 0.6710], [0.2607, 0.2607]) \\ ([0.4398, 0.5182], [0.4260, 0.4527]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.5379, 0.6454], [0.2805, 0.3362]) & ([0.5950, 0.7183], [0.1868, 0.2425]) \\ ([0.3544, 0.3544], [0.5981, 0.5981]) & ([0.2805, 0.3362], [0.5379, 0.6454]) & ([0.5000, 0.5000], [0.5000, 0.5000]) & ([0.5571, 0.5729], [0.4063, 0.4063]) \\ ([0.2607, 0.2607], [0.6552, 0.6710]) & ([0.1868, 0.2425], [0.5950, 0.7183]) & ([0.4063, 0.4063], [0.5571, 0.5729]) & ([0.5000, 0.5000], [0.5000, 0.5000]) \end{pmatrix}.$$

Following \bar{R}^* and the score function for IVIFVs, we obtain $s(\tilde{r}_1) = 0.152$, $s(\tilde{r}_2) = 0.191$, $s(\tilde{r}_3) = -0.092$, and $s(\tilde{r}_4) = -0.251$, by which the ranking order is $x_2 > x_1 > x_3 > x_4$. This result is the same as the ranking derived from the new method.

To see the ranking results obtained from these two methods visually, see Fig. 1.

Example 3 offers a GDM with complete IVIFPRs to show the concrete application of the new method. The following example gives a GDM with incomplete IVIFPRs to show its utilization.

Example 4 (Tang et al. 2018) To effectively promote the construction of “double first-class” university, a university formulates a plan for talents introduction. After the preliminary screening, there are four eligible candidates $X = \{x_1, x_2, x_3, x_4\}$. To rank these candidates, four experts $E = \{e_1, e_2, e_3, e_4\}$ are invited to give their judgements. Because of the difference in professional background, it is difficult for these experts to offer exact evaluations, and they are allowed to offer the preferred and non-preferred judgements simultaneously. Furthermore, when the experts are unable to give judgements for the comparisons of some candidates, the missing values are permitted. The four individual IVIFPRs offered by the experts are listed in Tables 10, 11, 12 and 13.

Fig. 1 The associated ranking results

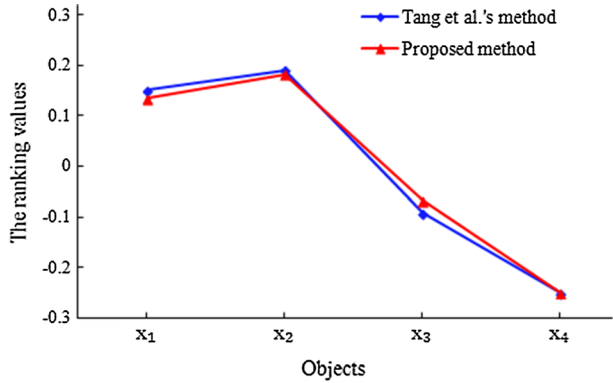


Table 10 IVIFPR \tilde{R}_1 offered by the expert e_1

e_1	x_1	x_2	x_3	x_4
x_1	([0.50,0.50], [0.50,0.50])	([0.40,0.50], [0.20,0.30])	([y, y], [y, y])	([0.55,0.60], [0.30,0.35])
x_2	([0.20,0.30], [0.40,0.50])	([0.50,0.50], [0.50,0.50])	([0.35,0.55], [0.25,0.30])	([y, y], [y, y])
x_3	([y, y], [y, y])	([0.25,0.30], [0.35,0.55])	([0.50,0.50], [0.50,0.50])	([0.40,0.50], [0.30,0.45])
x_4	([0.30,0.35], [0.55,0.60])	([y, y], [y, y])	([0.30,0.45], [0.40,0.50])	([0.50,0.50], [0.50,0.50])

Table 11 IVIFPR \tilde{R}_2 offered by the expert e_2

e_2	x_1	x_2	x_3	x_4
x_1	([0.50,0.50], [0.50,0.50])	([0.30,0.40], [0.40,0.45])	([0.40, y], [y, 0.45])	([y, 0.50], [0.20, y])
x_2	([0.40,0.45], [0.30,0.40])	([0.50,0.50], [0.50,0.50])	([0.40,0.45], [0.35,0.40])	([0.30,0.35], [0.25,0.45])
x_3	([y, 0.45], [0.40, y])	([0.35,0.40], [0.40,0.45])	([0.50,0.50], [0.50,0.50])	([0.30, y], [y, 0.50])
x_4	([0.20, y], [y, 0.50])	([0.25,0.45], [0.30,0.35])	([y, 0.50], [0.30, y])	([0.50,0.50], [0.50,0.50])

Step 1. With respect to each incomplete individual IVIFPR, the missing values following models (22) and (23) are obtained as follows:

Table 12 IVIFPR \tilde{R}_3 offered by the expert e_3

e_3	x_1	x_2	x_3	x_4
x_1	([0.50,0.50], [0.50,0.50])	([0.35,0.40], [0.40,0.50])	([0.30,0.40], [0.40,0.60])	([y, y], [y, y])
x_2	([0.40,0.50], [0.35,0.40])	([0.50,0.50], [0.50,0.50])	([0.40,0.50], [0.35,0.45])	([0.50,0.65], [0.25,0.30])
x_3	([0.40,0.60], [0.30,0.40])	([0.35,0.45], [0.40,0.50])	([0.50,0.50], [0.50,0.50])	([y, y], [y, y])
x_4	([y, y], [y, y])	([0.25,0.30], [0.50,0.65])	([y, y], [y, y])	([0.50,0.50], [0.50,0.50])

Table 13 IVIFPR \tilde{R}_4 offered by the expert e_4

e_4	x_1	x_2	x_3	x_4
x_1	([0.50,0.50], [0.50,0.50])	([y, y], [y, y])	([0.25,0.45], [0.30,0.40])	([0.30,0.45], [0.25, y])
x_2	([y, y], [y, y])	([0.50,0.50], [0.50,0.50])	([y, y], [y, y])	([0.30,0.55], [0.35,0.40])
x_3	([0.30,0.40], [0.25,0.45])	([y, y], [y, y])	([0.50,0.50], [0.50,0.50])	([y, 0.45], [0.25,0.40])
x_4	([0.25, y], [0.30,0.45])	([0.35,0.40], [0.30,0.55])	([0.25,0.40], [y, 0.45])	([0.50,0.50], [0.50,0.50])

$$\begin{aligned} \tilde{r}_{1,13} &= ([0.50, 0.55], [0.25, 0.25]), \tilde{r}_{1,24} = ([0.40, 0.45], [0.30, 0.30]), \\ \mu_{u,13}^2 &= 0.50, v_{l,13}^2 = 0.30, \mu_{l,14}^2 = 0.40, v_{u,14}^2 = 0.45, \mu_{u,34}^2 = 0.40, v_{l,34}^2 = 0.40, \\ \tilde{r}_{3,14} &= ([0.55, 0.55], [0.30, 0.35]), \tilde{r}_{3,34} = ([0.60, 0.65], [0.25, 0.25]), \\ \tilde{r}_{4,12} &= ([0.40, 0.40], [0.40, 0.40]), \tilde{r}_{4,23} = ([0.35, 0.45], [0.40, 0.40]), \\ v_{u,14}^4 &= 0.30, \mu_{l,34}^4 = 0.45. \end{aligned}$$

Step 2. Using model (18), it can be checked that the complete individual IVIFPR \tilde{R}_4 is additively consistent, while the others are not. Model (24) is applied to repair the inconsistency of the first three individual IVIFPRs, and the associated additively consistent IVIFPRs are shown in Tables 14, 15, 16 and 17.

Step 3. Based on the four experts' additively consistent IVIFPRs and formula (25), the weights of the experts are $w_1 = 0.228$, $w_2 = 0.270$, $w_3 = 0.232$, and $w_4 = 0.270$.

Based on formula (26), the collectively additively consistent IVIFPR \tilde{R}_C^* is shown in Table 17.

Step 4. Let $\theta^* = 0.9$ be the given threshold of the consensus. Using formula (29), we have

Table 14 Additively consistent IVIFPR \tilde{R}_1^*

e_1	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.45], [0.20,0.30]]	[[0.50,0.55], [0.25,0.25]]	[[0.55,0.60], [0.30,0.35]]
x_2	[[0.20,0.30], [0.40,0.45]]	[[0.50,0.50], [0.50,0.50]]	[[0.35,0.55], [0.25,0.30]]	[[0.40,0.45], [0.30,0.30]]
x_3	[[0.25,0.25], [0.50,0.55]]	[[0.25,0.30], [0.35,0.55]]	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.40], [0.30,0.45]]
x_4	[[0.30,0.35], [0.55,0.60]]	[[0.30,0.30], [0.40,0.45]]	[[0.30,0.45], [0.40,0.40]]	[[0.50,0.50], [0.50,0.50]]

Table 15 Additively consistent IVIFPR \tilde{R}_2^*

e_2	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.30,0.40], [0.40,0.40]]	[[0.40,0.50], [0.30,0.45]]	[[0.40,0.45], [0.20,0.45]]
x_2	[[0.40,0.40], [0.30,0.40]]	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.45], [0.35,0.40]]	[[0.30,0.35], [0.25,0.45]]
x_3	[[0.30,0.45], [0.40,0.50]]	[[0.35,0.40], [0.40,0.45]]	[[0.50,0.50], [0.50,0.50]]	[[0.30,0.40], [0.40,0.50]]
x_4	[[0.20,0.45], [0.40,0.45]]	[[0.25,0.45], [0.30,0.35]]	[[0.40,0.50], [0.30,0.40]]	[[0.50,0.50], [0.50,0.50]]

Table 16 Additively consistent IVIFPR \tilde{R}_3^*

e_3	x_1	x_2	x_3	x_4
x_1	[[0.50,0.50], [0.50,0.50]]	[[0.35,0.40], [0.40,0.50]]	[[0.30,0.40], [0.40,0.55]]	[[0.55,0.55], [0.30,0.35]]
x_2	[[0.40,0.50], [0.35,0.40]]	[[0.50,0.50], [0.50,0.50]]	[[0.40,0.45], [0.35,0.45]]	[[0.50,0.65], [0.25,0.30]]
x_3	[[0.40,0.55], [0.30,0.40]]	[[0.35,0.45], [0.40,0.45]]	[[0.50,0.50], [0.50,0.50]]	[[0.60,0.65], [0.25,0.25]]
x_4	[[0.30,0.35], [0.55,0.55]]	[[0.25,0.30], [0.50,0.65]]	[[0.25,0.25], [0.60,0.65]]	[[0.50,0.50], [0.50,0.50]]

$$GCI(\tilde{R}_1^*) = 0.933, GCI(\tilde{R}_2^*) = 0.947, GCI(\tilde{R}_3^*) = 0.925 \text{ and } GCI(\tilde{R}_4^*) = 0.960.$$

Step 5. Following \tilde{R}_C^* and formula (34), we have

$$\begin{aligned} \tilde{r}_1 &= ([0.416, 0.473], [0.357, 0.420]), \tilde{r}_2 = ([0.400, 0.467], [0.373, 0.417]), \\ \tilde{r}_3 &= ([0.397, 0.444], [0.384, 0.463]), \tilde{r}_4 = ([0.338, 0.409], [0.436, 0.494]). \end{aligned}$$

Table 17 Collective IVIFPR \tilde{R}_C^*

	x_1	x_2	x_3	x_4
x_1	([0.500,0.500], [0.500,0.500])	([0.361,0.411], [0.354,0.400])	([0.359,0.475], [0.312,0.414])	([0.442,0.507], [0.260,0.364])
x_2	([0.354,0.400], [0.361,0.411])	([0.500,0.500], [0.500,0.500])	([0.375,0.473], [0.341,0.389])	([0.369,0.496], [0.288,0.368])
x_3	([0.312,0.414], [0.359,0.475])	([0.341,0.389], [0.375,0.473])	([0.500,0.500], [0.500,0.500])	([0.433,0.472], [0.302,0.404])
x_4	([0.260,0.364], [0.442,0.507])	([0.288,0.368], [0.369,0.496])	([0.302,0.404], [0.433,0.472])	([0.500,0.500], [0.500,0.500])

Step 6. Using the score function for IVIFVs, we obtain $s(\tilde{r}_1) = 0.056, s(\tilde{r}_2) = 0.039, s(\tilde{r}_3) = -0.003,$ and $s(\tilde{r}_4) = -0.092.$

Thus, the ranking order is $x_1 > x_2 > x_3 > x_4.$

According to the results in the literature (Tang et al. 2018), the ranking values of the four candidates are $s(\tilde{r}_1) = 0.050, s(\tilde{r}_2) = 0.066, s(\tilde{r}_3) = 0.028,$ and $s(\tilde{r}_4) = 0.040,$ and the corresponding ranking order is $x_2 > x_1 > x_4 > x_3,$ which is different from the above ranking.

To see the ranking results obtained from these two methods visually, please see Fig. 2.

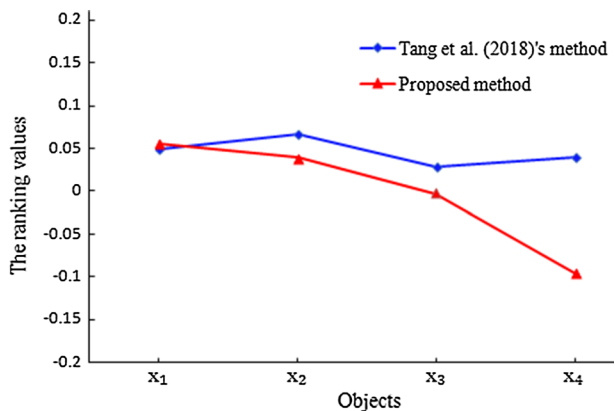
Remark 2 The commonalities between the method (Tang et al. 2018) and the proposed method include:

- (i) both of them are based on the consistency and consensus analysis to solve GDM problems with IVIFPRs, which can ensure the logical ranking results;
- (ii) the using additive consistency definitions are independent of compared objects;
- (iii) both of them can address incomplete and inconsistent IVIFPRs.

Compared with the method (Tang et al. 2018), the proposed method has the following advantages:

- (i) Since there is no need to transform IVIFPRs to QFIPRs, the new method is simpler than Tang et al.'s method.
- (ii) The new method is based on Definition 3 that is more flexible than Tang et al.'s method.
- (iii) Programming models for improving the consistency and consensus level permit different judgements to have different adjustments, while Tang et al.'s method adjusts all judgements without considering their differences. Furthermore, our method can guarantee the minimum total adjustment, while Tang et al.'s method cannot.

Fig. 2 The associated ranking results



7 Conclusion

To promote the application of IVIFPRs in decision making, this paper proposes a new GDM method based on IVIFPRs. First of all, a new additive consistency concept for IVIFPRs is defined, which can avoid the limitations of the previous ones. By constructing programming models, the consistency of an IVIFPR can be checked and the additively consistent IVIFPRs can be obtained. For incomplete IVIFPRs, a programming model is provided to obtain missing information. Afterwards, a linear programming model to improve the individual consensus level is established that ensures the least adjustment. Finally, an additive consistency and consensus based GDM algorithm is put forward and the practical decision-making examples are applied to illustrate the feasibility and efficiency of the new method.

Just as other types of PRs, there are two types of consistency concepts for IVIFPRs: the additive consistency concept and the multiplicative consistency concept. This paper focuses on the former and we shall continue to research IVIFPRs' multiplicative consistency. Furthermore, about the main research contents of GDM with IVIFPRs, there are also some issues need for further investigation, such as setting the consensus threshold, and determining the priority of objects (Wan et al. 2017). Last but not least, in addition to the offered practical decision-making problems, the proposed method can also be similarly extended to other fields, such as the selection of investment strategy (Xian et al. 2018), human resource assessment (He 2019), and social networks (Morente-Molinera et al. 2019).

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