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Perturbation bounds for DMP and CMP inverses of tensors via Einstein product

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Abstract
In this paper, the DMP and CMP inverses of tensors via Einstein product are defined. Some characterizations, representations, and properties for these generalized inverses are investigated. The perturbation bounds related to the DMP and CMP inverses are also developed.

Keywords DMP inverse · CMP inverse · Perturbation · Tensor · Einstein product

Mathematics Subject Classification 15A09 · 65F20

1 Introduction

Baksalary and Trenkler (2010) introduced a new pseudoinverse of a matrix named as the core inverse. Malik and Thome (2014) extended this definition and defined a new generalized inverse of a square matrix of an arbitrary index. They used the Drazin inverse (D) and the Moore–Penrose (MP) inverse, and therefore, this new generalized inverse is called the DMP inverse. The DMP inverse is analyzed from both algebraic as well as geometrical approaches establishing the equivalence between them. DMP inverse extends the notion of core inverse. Recently, Mehdipour and Salemi (2018) introduced another new inverse of a square matrix A, named after CMP inverse. The Drazin inverse, Moore–Penrose inverse, the weighted Moore–Penrose inverse, core and core-EP inverse, and outer inverse via Einstein product can be found in Behera and Mishra (2017), Behera et al. (2019), Ji and Wei (2017), Ji and Wei

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(2018), Jin et al. (2017), Liang and Zheng (2019), Liang et al. (2019), Ma (2018), Ma et al. (2019), Sahoo et al. (2019), Stanimirović et al. (2018), and Sun et al. (2016). DMP inverse and CMP inverse via Einstein product of tensors provide a new class of generalized inverses of tensors. Recently, Miao et al. (2019a, b) investigated the tensor functions via the tensor singular value decomposition and tensor Jordan canonical form based on the T-product for the tensor Moore–Penrose inverse and the tensor Drazin inverse, respectively. The monographs on the theory and computation of tensors and generalized inverses can be found in Ding and Wei (2016), Wei et al. (2018).

For a positive integer N, let $\mathbf{I}_1, \ldots, \mathbf{I}_N$ be positive integers. An order N tensor $\mathcal{A} = (\mathcal{A}_{i_1,i_2,\ldots,i_N})_{1\le i_j\le \mathbf{I}_j}, (j = 1, \ldots, N)$ is a multidimensional array with $\mathfrak{I} = \mathbf{I}_1\mathbf{I}_2\cdots\mathbf{I}_N$ entries, where $\mathbf{I}_1, \ldots, \mathbf{I}_N$ are positive integers. Let $\mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ (resp. $\mathbb{R}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$) be the set of the order N tensors of dimension $\mathbf{I}_1 \times\cdots\times \mathbf{I}_N$ over complex numbers $\mathbb{C}$ (resp. real numbers $\mathbb{R}$).

For a tensor $\mathcal{A} = (\mathcal{A}_{i_1,\ldots,i_N,i_1,\ldots,i_N}) \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$, if there exists a tensor $\mathcal{X}$, such that $\mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} = \mathcal{I}$, then tensor $\mathcal{A}$ is invertible. In this case, $\mathcal{X}$ is called the inverse of $\mathcal{A}$ and denoted by $\mathcal{A}^{-1}$. For a tensor $\mathcal{A} = (\mathcal{A}_{i_1,\ldots,i_M,j_1,\ldots,j_N}) \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_M\times\mathbf{J}_1\times\cdots\times\mathbf{J}_N}$, the tensor $\mathcal{A}^{\mathrm{T}} = (\mathcal{A})_{i_1,\ldots,i_M,j_1,\ldots,j_N} \in \mathbb{C}^{\mathbf{J}_1\times\cdots\times\mathbf{J}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_M}$ is the transpose of $\mathcal{A}$. The conjugate transpose of a tensor $\mathcal{A}$ is denoted by $\mathcal{A}^*$ and elementwise defined as $(\mathcal{A}^*)_{j_1,\ldots,j_N,i_1,\ldots,i_M} = (\overline{\mathcal{A}})_{i_1,\ldots,i_M,j_1,\ldots,j_N} \in \mathbb{C}^{\mathbf{J}_1\times\cdots\times\mathbf{J}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_M}$ where the overline means the conjugate operator.

The Einstein product of tensors is defined in Einstein (2007) by the operation $*_N$ via:

$$(\mathcal{A} *_N \mathcal{B})_{i_1\ldots i_N j_1\ldots j_M} = \sum_{k_1\ldots k_N} \mathcal{A}_{i_1\ldots i_N k_1\ldots k_N}\mathcal{B}_{k_1\ldots k_N j_1\ldots, j_M}, \tag{1.1}$$

where $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$, $\mathcal{B} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N\times\mathbf{J}_1\times\cdots\times\mathbf{J}_M}$ and $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{J}_1\times\cdots\times\mathbf{J}_M}$.

The associative law of this tensor product holds. In the above formula, when $\mathcal{B} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$, then

$$(\mathcal{A} *_N \mathcal{B})_{i_1 i_2\ldots i_N} = \sum_{k_1,\ldots,k_N} \mathcal{A}_{i_1\ldots i_N k_1\ldots k_N}\mathcal{B}_{k_1\ldots k_N},$$

where $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$.

Definition 1.1 (Sun et al. 2016) Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$. The tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ which satisfies:

$$\begin{array}{ll}(1)\quad \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}; & (2)\quad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}; \\ (3)\quad (\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}; & (4)\quad (\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}\end{array}$$

is called the Moore–Penrose inverse of $\mathcal{A}$, abbreviated by MP inverse, denoted by $\mathcal{A}^{\dagger}$. If the equation (i) of the above Eqs. (1)–(4) holds, $\mathcal{X}$ is called an $(i)-$inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{(i)}$.

Definition 1.2 (Ji and Wei 2017) For $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$, the range $\mathcal{R}(\mathcal{A})$ and the null space $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ are defined by:

$$\begin{aligned}\mathcal{R}(\mathcal{A}) &= \{\mathcal{Y} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N} : \mathcal{Y} = \mathcal{A} *_N \mathcal{X},\ \mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N}\}\\ \mathcal{N}(\mathcal{A}) &= \{\mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N} : \mathcal{A} *_N \mathcal{X} = \mathcal{O}\},\end{aligned}$$

where $\mathcal{O}$ is an appropriate zero tensor.

Definition 1.3 (Ji and Wei 2018, Lemma 2.1) Let $\mathcal{X} \in \mathbb{C}^{\mathbf{N}_1\times\cdots\times\mathbf{N}_K}$. The spectral norm $\|\mathcal{X}\|_2$ is defined as:

$$\|\mathcal{X}\|_2 = \sqrt{\lambda_{\max}(\mathcal{X}^* *_N \mathcal{X})},$$

where $\lambda_{\max}(\mathcal{X}^* *_N \mathcal{X})$ denotes the largest eigenvalue of $\mathcal{X}^* *_N \mathcal{X}$.

Lemma 1.1 (Ma et al. 2019) *Let $\mathcal{E} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. Suppose that $\|\mathcal{E}\|_2 < 1$. Then, $\mathcal{I}+\mathcal{E}$ is nonsingular and*

$$\|(\mathcal{I}+\mathcal{E})^{-1}\|_2 \le \frac{1}{1-\|\mathcal{E}\|_2}.$$

Lemma 1.2 (Ma et al. 2019) *Let $\mathcal{E} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_K\times\mathbf{I}_1\times\cdots\times\mathbf{I}_K}$. If $\|\mathcal{E}\|_2 < 1$, then*

$$(\mathcal{I}-\mathcal{E})^{-1} = \sum_{n=0}^{\infty}\mathcal{E}^n, \tag{1.2}$$

and

$$\|(\mathcal{I}-\mathcal{E})^{-1}-\mathcal{I}\|_2 \le \frac{\|\mathcal{E}\|_2}{1-\|\mathcal{E}\|_2}. \tag{1.3}$$

Definition 1.4 (Behera et al. 2019; Ji and Wei 2018) Assume that $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. Define

$$\mathcal{A}^0 = \mathcal{I} \ and \ \mathcal{A}^p = \mathcal{A}^{p-1} *_N \mathcal{A}, \ \ for \quad p \ge 2.$$

It is easy to see that

$$\{0\} \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}^{p+1}) \subseteq \mathcal{R}(\mathcal{A}^p) \subseteq \cdots \subseteq \mathcal{R}(\mathcal{A}^2) \subseteq \mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{I}) = \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$$

and

$$\{0\} = \mathcal{N}(\mathcal{I}) \subseteq \mathcal{N}(\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}^2) \subseteq \cdots \subseteq \mathcal{N}(\mathcal{A}^p) \subseteq \mathcal{N}(\mathcal{A}^{p+1}) \subseteq \cdots \subseteq \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N}.$$

The smallest non-negative integer p, such that $\mathcal{R}(\mathcal{A}^{p+1}) = \mathcal{R}(\mathcal{A}^p)$ (or $\mathcal{N}(\mathcal{A}^{p+1}) = \mathcal{N}(\mathcal{A}^p)$), denoted by $\mathrm{Ind}(\mathcal{A})$, is called the index of $\mathcal{A}$.

Definition 1.5 (Behera et al. 2019; Ji and Wei 2018) Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. The tensor $\mathcal{X} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ which satisfies:

$$(2) \quad \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}; \qquad (5) \quad \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A}; \qquad (1^k) \quad \mathcal{A}^{k+1} *_N \mathcal{X} = \mathcal{A}^k$$

is called the Drazin inverse of $\mathcal{A}$, denoted by $\mathcal{A}^d$. Especially, if $\mathrm{Ind}(\mathcal{A}) = 1$, $\mathcal{X}$ is called the group inverse of $\mathcal{A}$, denoted by $\mathcal{A}_g$.

For a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$, the singular value decomposition (SVD) Brazell et al. (2013), Sun et al. (2016) of $\mathcal{A}$ has the form:

$$\mathcal{A} = \mathcal{U} *_N \mathcal{D} *_N \mathcal{V}^*, \tag{1.4}$$

where $\mathcal{U} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ and $\mathcal{V} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$ are unitary tensors, and the tensor $\mathcal{D} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$ is a diagonal tensor satisfying:

$$\mathcal{D}_{i_1,\ldots,i_N,k_1,\ldots,k_N} = \begin{cases} 0, & (i_1,\ldots,i_N) \neq (k_1,\ldots,k_N), \\ \mu_{i_1\ldots i_N}, & (i_1,\ldots,i_N) = (k_1,\ldots,k_N), \end{cases}$$

where $\mu_{i_1\ldots i_N}$ are the singular values of $\mathcal{A}$.

The diagonal tensor $\mathcal{D}$ can be written as: $\mathcal{D} = \begin{pmatrix} \Sigma & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$, where $\Sigma \in \mathbb{C}^{\mathbf{R}_1\times\cdots\times\mathbf{R}_N\times\mathbf{R}_1\times\cdots\times\mathbf{R}_N}$ is a diagonal tensor of singular values of $\mathcal{A}$. Then, the singular value decomposition of $\mathcal{A}$ can be written as follows (Brazell et al. 2013):

$$\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \Sigma & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{V}^*. \tag{1.5}$$

Multiplying (1.5) by $\mathcal{U} *_N \mathcal{U}^* (= \mathcal{I})$ on the right-hand side, and assuming that unitary $\mathcal{V}^* *_N \mathcal{U}$ is partitioned according to:

$$\mathcal{V}^* *_N \mathcal{U} = \begin{pmatrix} \mathcal{K} & \mathcal{L} \\ \mathcal{M} & \mathcal{N} \end{pmatrix},$$

where $\mathcal{K} \in \mathbb{C}^{\mathbf{R}_1\times\cdots\times\mathbf{R}_N\times\mathbf{R}_1\times\cdots\times\mathbf{R}_N}$.

Hartwig and Spindelböck decomposition (Hartwig and Spindelböck 1983) of tensor arrived at the following result.

Lemma 1.3 *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. Then, there exist unitary $\mathcal{U} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$, such that*

$$\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*, \tag{1.6}$$

where $\Sigma \in \mathbb{C}^{\mathbf{R}_1\times\cdots\times\mathbf{R}_N\times\mathbf{R}_1\times\cdots\times\mathbf{R}_N}$ is a diagonal tensor of singular values of $\mathcal{A}$, and the tensors $\mathcal{K} \in \mathbb{C}^{\mathbf{R}_1\times\cdots\times\mathbf{R}_N\times\mathbf{R}_1\times\cdots\times\mathbf{R}_N}$, $\mathcal{L} \in \mathbb{C}^{\mathbf{R}_1\times\cdots\times\mathbf{R}_N\times(\mathbf{I}_1-\mathbf{R}_1)\times\cdots\times(\mathbf{I}_N-\mathbf{R}_N)}$ satisfy:

$$\mathcal{K} *_N \mathcal{K}^* + \mathcal{L} *_N \mathcal{L}^* = \mathcal{I}. \tag{1.7}$$

From (1.6), the Drazin inverse and the Moore–Penrose inverse of $\mathcal{A}$ are presented as follows:

$$\mathcal{A}^d = \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^d & ((\Sigma *_N \mathcal{K})^d)^2 *_N \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*,$$
$$\mathcal{A}^\dagger = \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \Sigma^{-1} & \mathcal{O} \\ \mathcal{L}^* *_N \Sigma^{-1} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*. \tag{1.8}$$

Theorem 1.1 (Behera et al. 2019) *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. Then, $\mathcal{A}$ can be written as the sum of two tensors $\mathcal{C}_\mathcal{A}$ and $\mathcal{N}_\mathcal{A}$, i.e., $\mathcal{A} = \mathcal{C}_\mathcal{A} + \mathcal{N}_\mathcal{A}$, where* $\mathrm{Ind}(\mathcal{C}_\mathcal{A}) \leq 1$, *$\mathcal{N}_\mathcal{A}$ is nilpotent and $\mathcal{C}_\mathcal{A} *_N \mathcal{N}_\mathcal{A} = \mathcal{N}_\mathcal{A} *_N \mathcal{C}_\mathcal{A} = \mathcal{O}$.*

The tensors $\mathcal{C}_\mathcal{A}$ and $\mathcal{N}_\mathcal{A}$ called the core and nilpotent parts of the tensor $\mathcal{A}$, respectively. We know that if $\mathrm{Ind}(\mathcal{A}) \leq 1$, then $\mathcal{A} = \mathcal{C}_\mathcal{A}$. Also, it is valid that $\mathcal{C}_\mathcal{A} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}$.

Theorem 1.2 (Schur decomposition) (Liang et al. 2019) *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ be a tensor of index k. Then, it can be factorized as the Schur form of $\mathcal{A}$:*

$$\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^*, \tag{1.9}$$

where $\mathcal{U} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ is unitary, $\mathcal{T}_{11}$ is a nonsingular upper triangular tensor, and $\mathcal{T}_{22}$ is a nilpotent tensor with index k.

2 Preliminary results

In this section, we define the DMP and CMP inverses of tensor via Einstein product. Furthermore, we collect a few properties of DMP and CMP inverses.

Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$ have index k and consider the system of equations:

$$\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X},\ \mathcal{X} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A},\ \mathcal{A}^k *_N \mathcal{X} = \mathcal{A}^k *_N \mathcal{A}^{\dagger}. \tag{2.1}$$

Theorem 2.1 *If system (2.1) has a solution, then it is unique.*

Proof Assume that $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy (2.1). Then, using that $\mathcal{A} *_N \mathcal{A}^d = \mathcal{A}^d *_N \mathcal{A}$, we get

$$\begin{aligned}
\mathcal{X}_1 &= \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{X}_1 \\
&= (\mathcal{A}^d *_N \mathcal{A})^k *_N \mathcal{X}_1 = (\mathcal{A}^d)^k *_N \mathcal{A}^k *_N \mathcal{X}_1 \\
&= (\mathcal{A}^d)^k *_N \mathcal{A}^k *_N \mathcal{A}^{\dagger} = (\mathcal{A}^d)^k *_N \mathcal{A}^k *_N \mathcal{X}_2 \\
&= (\mathcal{A}^d *_N \mathcal{A})^k *_N \mathcal{X}_2 = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{X}_2 \\
&= \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2.
\end{aligned}$$

□

Theorem 2.2 *The system of (2.1) is consistent and has a unique solution:* $\mathcal{X} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}$.

Proof It is easy to see that $\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}$ satisfies the three equations in system (2.1). Now, Theorem 2.1 gives the uniqueness. □

Thus, for a given tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$, the tensor $\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}$ is the unique tensor satisfying system of (2.1).

Definition 2.1 Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$ be a tensor of index k. The DMP inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{d,\dagger}$, is defined to be the tensor:

$$\mathcal{A}^{d,\dagger} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}. \tag{2.2}$$

DMP inverse has the following several important properties (Malik and Thome 2014). From (1.8), the DMP inverse of $\mathcal{A}$ is given by:

$$\begin{aligned}
\mathcal{A}^{d,\dagger} &= \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} \\
&= \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^d & ((\Sigma *_N \mathcal{K})^d)^2 *_N \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \\
&\quad *_N \begin{pmatrix} \mathcal{K}^* *_N \Sigma^{-1} & \mathcal{O} \\ \mathcal{L}^* *_N \Sigma^{-1} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^* \\
&= \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^d & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.
\end{aligned} \tag{2.3}$$

Theorem 2.3 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$ *be of the form (1.6). Then:*

$$\mathcal{A}^{d,\dagger} = \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^d & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*. \tag{2.4}$$

Lemma 2.1 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$ *be a tensor of index k has the form (1.6). Then,* $\mathrm{Ind}(\Sigma *_N \mathcal{K}) = k - 1$.

Proof Since

$$\mathcal{A}^k = \mathcal{U} *_N \begin{pmatrix} (\Sigma *_N \mathcal{K})^{k-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix} *_N \mathcal{U}^*$$

with $\mathcal{Y}$, $\mathcal{Z}$ of adequate sizes, such that the tensor $\begin{pmatrix} \Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L} \\ \mathcal{Y} & \mathcal{Z} \end{pmatrix}$ is nonsingular, we have $\mathcal{N}(\mathcal{A}^k) = \mathcal{N}((\Sigma *_N \mathcal{K})^{k-1})$. And then, $\mathcal{N}(\mathcal{A}^{k+1}) = \mathcal{N}((\Sigma *_N \mathcal{K})^k)$. Since $\mathrm{Ind}(\mathcal{A}) = k$, we can obtain that $k-1$ is the smallest non-negative integer satisfying $\mathcal{N}((\Sigma *_N \mathcal{K})^{k-1}) = \mathcal{N}((\Sigma *_N \mathcal{K})^k)$, that is $\mathrm{Ind}(\Sigma *_N \mathcal{K}) = k-1$. □

Theorem 2.4 *The DMP inverse $\mathcal{X} = \mathcal{A}^{d,\dagger}$ of a tensor $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ satisfies the equations:*

$$(1)\ \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{C}_{\mathcal{A}} \text{ and } (2)\ \mathcal{A} *_N \mathcal{X} = {}^{d,\dagger}\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^\dagger,$$

where ${}^{d,\dagger}\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A}$ denotes the DMP core part of $\mathcal{A}$.

Proof (1) Using the Hartwig–Spindelbock decomposition of tensor $\mathcal{A}$ (Lemma 1.3), from $\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}$ and ${}^{d,\dagger}\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A}$, we have:

$$\mathcal{C}_{\mathcal{A}} = \mathcal{U} *_N \begin{pmatrix} \mathcal{C}_{\Sigma *_N \mathcal{K}} & \Sigma *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d *_N \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$$

and

$${}^{d,\dagger}\mathcal{C}_{\mathcal{A}} = \mathcal{U} *_N \begin{pmatrix} \mathcal{C}_{\Sigma *_N \mathcal{K}} & \Sigma *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d *_N \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Thus, $\mathcal{C}_{\mathcal{A}} = {}^{d,\dagger}\mathcal{C}_{\mathcal{A}}$. The core part of $\mathcal{A}$ is its DMP core part. $\mathcal{A}^{d,\dagger}$ is a solution of $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{C}_{\mathcal{A}}$.

(2) From ${}^{d,\dagger}\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^\dagger$, we have

$${}^{d,\dagger}\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^\dagger = \mathcal{A} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A} *_N \mathcal{A}^{d,\dagger}.$$

The proof is completed. □

Theorem 2.5 *If $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ has index k, then the following statements hold:*

(1) *$\mathcal{A} *_N \mathcal{A}^{d,\dagger}$ is a projector onto $\mathcal{R}({}^{d,\dagger}\mathcal{C}_{\mathcal{A}})$ along $\mathcal{N}(\mathcal{A}^d *_N \mathcal{A}^\dagger)$;*
(2) *$\mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A}$ is a projector onto $\mathcal{R}(\mathcal{A}^k)$ along $\mathcal{N}(\mathcal{A}^k)$.*

Proof (1) From (2.1), $\mathcal{A} *_N \mathcal{A}^{d,\dagger}$ is a projection. It is obvious that

$$\begin{aligned}\mathcal{R}(\mathcal{A} *_N \mathcal{A}^{d,\dagger}) &= \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{R}(\mathcal{A} *_N \mathcal{A}^\dagger) = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{R}(\mathcal{A}) \\ &= \mathcal{R}(\mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}) = \mathcal{R}({}^{d,\dagger}\mathcal{C}_{\mathcal{A}})\end{aligned}$$

and

$$\mathcal{N}(\mathcal{A} *_N \mathcal{A}^{d,\dagger}) = \mathcal{N}(\mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger) = \mathcal{N}(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger) = \mathcal{N}(\mathcal{A}^d *_N \mathcal{A}^\dagger).$$

(2) Since $\mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A}$ and $\mathcal{A}^d *_N \mathcal{A}$ is a projection of $\mathcal{A}$, we have:

$$\begin{aligned}\mathcal{R}(\mathcal{A}^d *_N \mathcal{A}) &= \mathcal{R}(\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}) = \cdots \subseteq \mathcal{R}(\mathcal{A}^k) = \mathcal{R}(\mathcal{A}^{k+1} *_N \mathcal{A}^d) \\ &= \mathcal{R}(\mathcal{A}^d *_N \mathcal{A}^{k+1}) \subseteq \mathcal{R}(\mathcal{A}^d *_N \mathcal{A})\end{aligned}$$

and

$$\mathcal{N}(\mathcal{A}^d *_N \mathcal{A}) = \mathcal{N}(\mathcal{A}^k).$$

□

Theorem 2.6 *If $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ has index k, then $\mathcal{A}^{d,\dagger}$ is the unique tensor $\mathcal{X}$ that satisfies:*

$$\mathcal{A}\mathcal{X} = \mathcal{P}_{\mathcal{R}({}^d\mathcal{C}_{\mathcal{A}}),\mathcal{N}(\mathcal{A}^d *_N \mathcal{A}^\dagger)},\ \mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{A}^k). \tag{2.5}$$

Proof We know that $\mathcal{A} *_N \mathcal{A}^{d,\dagger}$ is idempotent from Theorem 2.5. Moreover:

$$\mathcal{R}(\mathcal{A}^{d,\dagger}) = \mathcal{R}(\mathcal{A} *_N \mathcal{A} *_N \mathcal{A}^\dagger) \subseteq \mathcal{R}(\mathcal{A} *_N \mathcal{A}) = \mathcal{R}(\mathcal{A}^k).$$

Assume that $\mathcal{X}_1$, $\mathcal{X}_2$ satisfy (2.5). Then, $\mathcal{A} *_N \mathcal{X}_1 = \mathcal{A} *_N \mathcal{X}_2 = \mathcal{P}_{\mathcal{R}({}^d\mathcal{C}_{\mathcal{A}}),\mathcal{N}(\mathcal{A}^d *_N \mathcal{A}^\dagger)}$, $\mathcal{R}(\mathcal{X}_1) \subseteq \mathcal{R}(\mathcal{A}^k)$ and $\mathcal{R}(\mathcal{X}_2) \subseteq \mathcal{R}(\mathcal{A}^k)$. Since $\mathcal{A}(\mathcal{X}_1 - \mathcal{X}_2) = 0$, we get $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq \mathcal{N}(\mathcal{A})$. From $\mathcal{R}(\mathcal{X}_1) \subseteq \mathcal{R}(\mathcal{A}^k)$ and $\mathcal{R}(\mathcal{X}_2) \subseteq \mathcal{R}(\mathcal{A}^k)$, we get $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq \mathcal{R}(\mathcal{A}^k)$; that is $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq \mathcal{N}(\mathcal{A}^k) \cap \mathcal{R}(\mathcal{A}^k) = \{0\}$, since $\mathcal{A}$ has index k. Thus, there is only one $\mathcal{X}$ satisfying conditions. □

Proposition 2.1 *Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ be a tensor of index k. Then:*

(a) $\mathcal{A}^{d,\dagger} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger$.
(b) $\mathcal{A}^{d,\dagger}$ *is an outer inverse of* $\mathcal{A}$.
(c) $(\mathcal{A}^{d,\dagger})^n = \begin{cases} (\mathcal{A}^d *_N \mathcal{A}^\dagger)^{\frac{n}{2}}, & if\ n\ is\ even, \\ \mathcal{A} *_N (\mathcal{A}^d *_N \mathcal{A}^\dagger)^{\frac{n+1}{2}}, & if\ n\ is\ odd. \end{cases}$
(d) $(\mathcal{A}^{d,\dagger})^\dagger = ((\mathcal{A} *_N \mathcal{A} *_N \mathcal{A}^\dagger)^\dagger$.
(e) $((\mathcal{A}^{d,\dagger})^d)^d = \mathcal{A}^{d,\dagger}$.
(f) $\mathcal{A}^{d,\dagger} = \mathcal{O}$ *if and only if* $\mathcal{A}$ *is nilpotent or* $\mathcal{A} = \mathcal{O}$.

Proof (a) and (b) We can obtain (a) and (b) from definition and properties of the Moore–Penrose and Drazin inverses.

(c) We calculate $(\mathcal{A}^{d,\dagger})^1 = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}^\dagger$ and $(\mathcal{A}^{d,\dagger})^2 = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^d *_N (\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A}) *_N \mathcal{A}^d *_N \mathcal{A}^\dagger = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}^\dagger = \mathcal{A}^d *_N \mathcal{A}^\dagger$. Then, we can obtain the formula.

(d) and (e) We can proof them through (2.4).

(f) Suppose that $\mathcal{A} \neq \mathcal{O}$ and $\mathcal{A}^{d,\dagger} = \mathcal{O}$. We can obtain two cases if $\mathcal{A}$ has the form (1.6) and $\mathcal{A}^{d,\dagger}$ has the form (2.4).

(i) $\Sigma *_N \mathcal{K} \neq \mathcal{O}$. In this case, according to $\mathcal{A}^{d,\dagger} = \mathcal{O}$, we obtain $(\Sigma *_N \mathcal{K})^d = \mathcal{O}$. Therefore, $\Sigma *_N \mathcal{K}$ is nilpotent. Hence, $\mathcal{A}$ must be nilpotent.

(ii) $\Sigma *_N \mathcal{K} = \mathcal{O}$. In this case, the tensor $\mathcal{A} = \mathcal{U} *_N \begin{pmatrix} \mathcal{O} & \Sigma *_N \mathcal{L} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*$ is clearly nilpotent. The converse is evident because in both $\mathcal{A} = \mathcal{O}$ and $\mathcal{A}$ nilpotent cases its Drazin inverse is the null tensor. □

By $\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}$, we obtain the following:

$$\begin{aligned} \mathcal{A}^{c,\dagger} &= \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{K}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d & \mathcal{O} \\ \mathcal{L}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*. \end{aligned} \tag{2.6}$$

Theorem 2.7 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{K}_1\times\cdots\times\mathbf{K}_N}$. *The tensor* $\mathcal{X} \in \mathbb{C}^{\mathbf{K}_1\times\cdots\times\mathbf{K}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ *is the unique tensor that satisfies the following system of equations:*

$$\begin{aligned}&\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X},\ \ \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{C}_{\mathcal{A}},\ \ \mathcal{A} *_N \mathcal{X} = \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger},\\ &\mathcal{X} *_N \mathcal{A} = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}}.\end{aligned} \tag{2.7}$$

Proof Easy computation shows that the tensor $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger}$ is a solution of this system.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two tensors satisfying (2.7). Then:

$$\begin{aligned}\mathcal{C}_{\mathcal{A}} *_N \mathcal{X}_1 &= \mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger}\\ &= \mathcal{A} *_N \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{C}_{\mathcal{A}} *_N \mathcal{X}_2.\end{aligned}$$

Thus:

$$\mathcal{X}_1 = \mathcal{X}_1 *_N \mathcal{A} *_N \mathcal{X}_1 = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{X}_1 = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{X}_2 = \mathcal{X}_2 *_N \mathcal{A} *_N \mathcal{X}_2 = \mathcal{X}_2.$$

The result holds. □

CMP inverse has several important properties (Mehdipour and Salemi 2018).

Proposition 2.2 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ *with core-nilpotent decomposition* $\mathcal{A} = \mathcal{C}_{\mathcal{A}} + \mathcal{N}_{\mathcal{A}}$. *Then, the following holds:*

(1) $\mathcal{A}^{c,\dagger} = \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}$, *where* $\mathcal{P}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^{\dagger}$ *and* $\mathcal{Q}_{\mathcal{A}} = \mathcal{A}^{\dagger} *_N \mathcal{A}$ *are orthogonal projections onto* $\mathcal{R}(\mathcal{A})$ *and* $\mathcal{R}(\mathcal{A}^*)$, *respectively;*

(2) $\mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{c,\dagger}$ *and* $\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}$;

(3) $\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} = \mathcal{A} *_N \mathcal{A}^{c,\dagger}$ *and* $\mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} = \mathcal{A}^{c,\dagger} *_N \mathcal{A}$.

Proof (1) This part follows from $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}$.

(2) The proof follows from the representations give in (1) and $\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A}$:

$$\mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} = \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}} *_N \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A} *_N \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}.$$

Since

$$\mathcal{A} *_N \mathcal{Q}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}} *_N \mathcal{A} = \mathcal{A},\ \ \mathcal{A}^{d} *_N \mathcal{A} *_N \mathcal{A}^{d} = \mathcal{A}^{d},$$

we obtain

$$\mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} = \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}} = \mathcal{A}^{c,\dagger}.$$

Also, the same method as above shows that $\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} = \mathcal{C}_{\mathcal{A}}$. Hence, the statement is proved.

(3) This part can also be demonstrated by combining $\mathcal{C}_{\mathcal{A}} = \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A}$ and $\mathcal{A}^{c,\dagger} = \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}$.

$$\begin{aligned}\mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{c,\dagger} &= \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A} *_N \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}\\ &= \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}\\ &= \mathcal{A} *_N \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^{d} *_N \mathcal{P}_{\mathcal{A}}\\ &= \mathcal{A} *_N \mathcal{A}^{c,\dagger}.\end{aligned}$$

Similarly:

$$\begin{aligned}\mathcal{A}^{c,\dagger} *_N \mathcal{C}_{\mathcal{A}} &= \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{P}_{\mathcal{A}} *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A}\\ &= \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{A}\\ &= \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{P}_{\mathcal{A}} *_N \mathcal{A}\\ &= \mathcal{A}^{c,\dagger} *_N \mathcal{A}.\end{aligned}$$

□

Theorem 2.8 *Suppose that* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ *is a tensor with* $\mathrm{Ind}(\mathcal{A}) = k$. *Then*

(1) $\mathcal{A}^{k+1} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{k+1} *_N \mathcal{A}^{\dagger}$;
(2) $\mathcal{A}^{c,\dagger} *_N \mathcal{A}^k = \mathcal{A}^{\dagger} *_N \mathcal{A}^k$.

Proof (1) Since $\mathcal{A}^m *_N \mathcal{Q}_{\mathcal{A}} = \mathcal{A}^m$ for every positive integer m. By Proposition 2.2:

$$\begin{aligned}\mathcal{A}^{k+1} *_N \mathcal{A}^{c,\dagger} &= \mathcal{A}^{k+1} *_N \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{P}_{\mathcal{A}}\\ &= \mathcal{A}^{k+1} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger}\\ &= \mathcal{A}^k *_N \mathcal{A} *_N \mathcal{A}^{\dagger}\\ &= \mathcal{A}^{k+1} *_N \mathcal{A}^{\dagger}.\end{aligned}$$

(2) As $\mathcal{P}_{\mathcal{A}} *_N \mathcal{A}^m = \mathcal{A}^m$ for every positive integer m. By Proposition 2.2:

$$\begin{aligned}\mathcal{A}^{c,\dagger} *_N \mathcal{A}^k &= \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{P}_{\mathcal{A}} *_N \mathcal{A}^k\\ &= \mathcal{Q}_{\mathcal{A}} *_N \mathcal{A}^d *_N \mathcal{A}^k\\ &= \mathcal{A}^{\dagger} *_N \mathcal{A}^{k+1} *_N \mathcal{A}^d\\ &= \mathcal{A}^{\dagger} *_N \mathcal{A}^k.\end{aligned}$$

□

Theorem 2.9 *Suppose that* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. *Then,* $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger}$ *if and only if* $\mathrm{Ind}(\mathcal{A}) \leq 1$.

Proof If $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger}$. By (1.8) and (2.6), we get $\mathcal{K}^* *_N \Sigma^{-1} = \mathcal{K}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d$ and $\mathcal{L}^* *_N \Sigma^{-1} = \mathcal{L}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d$. Multiplying both of these equalities by $\mathcal{K}$ and $\mathcal{L}$ on the left-hand side, respectively, and using (1.7), we obtain $\mathcal{K} *_N (\Sigma *_N \mathcal{K})^d = \Sigma^{-1}$. Hence, $\Sigma *_N \mathcal{K}$ is nonsingular. Moreover, by (1.6):

$$\begin{aligned}\mathcal{A}^2 &= \mathcal{U} *_N \begin{pmatrix}\Sigma *_N \mathcal{K} & \mathcal{O}\\ \mathcal{O} & \mathcal{O}\end{pmatrix} *_N \begin{pmatrix}\Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L}\\ \mathcal{O} & \mathcal{O}\end{pmatrix} *_N \mathcal{U}^*\\ &= \mathcal{U} *_N \begin{pmatrix}\Sigma *_N \mathcal{K} & \mathcal{O}\\ \mathcal{O} & \mathcal{I}\end{pmatrix} *_N \begin{pmatrix}\Sigma *_N \mathcal{K} & \Sigma *_N \mathcal{L}\\ \mathcal{O} & \mathcal{O}\end{pmatrix} *_N \mathcal{U}^*.\end{aligned}$$

Obviously, $\mathcal{U} *_N \begin{pmatrix}\Sigma *_N \mathcal{K} & \mathcal{O}\\ \mathcal{O} & \mathcal{I}\end{pmatrix} *_N \mathcal{U}^*$ is invertible. Therefore, $\mathcal{N}(\mathcal{A}) = \mathcal{N}(\mathcal{A}^2)$, and hence, $\mathrm{Ind}(\mathcal{A}) \leq 1$. Conversely, let $\mathcal{A} = \mathcal{C}_{\mathcal{A}} + \mathcal{N}_{\mathcal{A}}$ be the core-nilpotent decomposition of $\mathcal{A}$. If $\mathrm{Ind}(\mathcal{A}) \leq 1$, then $\mathcal{A} = \mathcal{C}_{\mathcal{A}}$, and hence, $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger} = \mathcal{C}_{\mathcal{A}}^{\dagger} *_N \mathcal{C}_{\mathcal{A}} *_N \mathcal{C}_{\mathcal{A}}^{\dagger} = \mathcal{C}_{\mathcal{A}}^{\dagger} = \mathcal{A}^{\dagger}$. This completes the proof. □

Theorem 2.10 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$. *Then, the following conditions are equivalent:*

(1) $\mathcal{A} *_N \mathcal{A}^{c,\dagger} = \mathcal{A} *_N \mathcal{A}^{\dagger}$;
(2) $\mathcal{A}^{c,\dagger} *_N \mathcal{A} = \mathcal{A}^{\dagger} *_N \mathcal{A}$;
(3) $\mathcal{A}^{c,\dagger} = \mathcal{O}$ *if and only if* $\mathcal{A}$ *is nilpotent.*

Proof (1) We know that $\mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{c,\dagger}$. If $\mathcal{A} *_N \mathcal{A}^{c,\dagger} = \mathcal{A} *_N \mathcal{A}^{\dagger}$, pre-multiplying by $\mathcal{A}^{\dagger}$, we get $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger}$. Conversely, if $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger}$, we multiplying $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger}$ by $\mathcal{A}$ on the left-hand side, we obtain that $\mathcal{A} *_N \mathcal{A}^{c,\dagger} = \mathcal{A} *_N \mathcal{A}^{\dagger}$.

(2) Also, by the same method as in (1), the result is obvious.

(3) If $\mathcal{A}^{c,\dagger} = \mathcal{O}$, that is $\mathcal{K}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d = \mathcal{O}$ and $\mathcal{L}^* *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d = \mathcal{O}$ by (2.6). Multiplying both of these equalities by $\mathcal{K}$ and $\mathcal{L}$ on the left-hand side, respectively, and using:

$$\mathcal{K} *_N \mathcal{K}^* + \mathcal{L} *_N \mathcal{L}^* = \mathcal{I},$$

we obtain $\mathcal{K} *_N (\Sigma *_N \mathcal{K})^d = \mathcal{O}$. Then $(\Sigma *_N \mathcal{K})^d *_N \Sigma *_N \mathcal{K} *_N (\Sigma *_N \mathcal{K})^d = (\Sigma *_N \mathcal{K})^d = \mathcal{O}$. Moreover, $(\Sigma *_N \mathcal{K})^k = (\Sigma *_N \mathcal{K})^{k+1} *_N (\Sigma *_N \mathcal{K})^d = \mathcal{O}$. Therefore, $\Sigma *_N \mathcal{K}$ is nilpotent, and hence, $\mathcal{A}$ is nilpotent.

Conversely, if $\mathcal{A}$ is nilpotent, let $\mathrm{Ind}(\mathcal{A}) = k$, then $\mathcal{A}^d = (\mathcal{A}^{l+1})_g *_N \mathcal{A}^l = \mathcal{O}$, where $l \geq k$, and hence $\mathcal{A}^{c,\dagger} = \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{O}$. □

3 Main results

In this section, we investigate the perturbations for DMP and CMP inverses. First, we extend the recent results on the DMP inverse from the linear operator (Yu and Deng 2016) to the tensor.

Theorem 3.1 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$ *be a tensor of index* k*. There is a Schur form* (1.9) *of* $\mathcal{A}$*. Then, the Moore–Penrose inverse can be expressed by:*

$$\mathcal{A}^{\dagger} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^*, \tag{3.1}$$

where $\Delta = [\mathcal{T}_{11} *_N \mathcal{T}_{11}^* + \mathcal{T}_{12} *_N (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^*]^{-1}$.

Proof Since $\mathcal{A}$ has the Schur form (1.9) and

$$\mathcal{X} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^*,$$

we have

$$\begin{aligned} &\mathcal{A} *_N \mathcal{X} \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^* *_N \mathcal{U} *_N \\ &\begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} \\ &*_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^*. \end{aligned}$$

In view of Definition 1.1, it is easy to compute the first equation that

$$\begin{aligned}\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{A}.\end{aligned}$$

Furthermore, the second equation follows from:

$$\begin{aligned}&\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} \\ &\quad *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} \\ &\quad *_N \mathcal{U}^* \\ &= \mathcal{X}.\end{aligned}$$

The third equation is verified as:

$$\begin{aligned}(\mathcal{A} *_N \mathcal{X})^* &= \left(\mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^*\right)^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{I}^* & \mathcal{O} \\ \mathcal{O} & (\mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger})^* \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix} *_N \mathcal{U}^* \\ &= \mathcal{A} *_N \mathcal{X},\end{aligned}$$

and the fourth equation can be verified by:

$$\begin{aligned}&(\mathcal{X} *_N \mathcal{A})^* \\ &= \left(\mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta & -\mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta & \mathcal{T}_{22}^{\dagger} - (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N \mathcal{T}_{22}^{\dagger} \end{pmatrix}\right. \\ &\quad \left. *_N \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^*\right)^* \\ &= \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{11} & \mathcal{T}_{11}^* *_N \Delta *_N \mathcal{T}_{12} *_N (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) \\ (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{11} & \mathcal{H} \end{pmatrix} \\ &\quad *_N \mathcal{U}^* \\ &= \mathcal{X} *_N \mathcal{A},\end{aligned}$$

where $\mathcal{H} = \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22} + (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22}) *_N \mathcal{T}_{12}^* *_N \Delta *_N \mathcal{T}_{12} *_N (\mathcal{I} - \mathcal{T}_{22}^{\dagger} *_N \mathcal{T}_{22})$.

The tensor $\mathcal{X}$ satisfies four equations. Let $\mathcal{X}$ and $\mathcal{Y}$ satisfy four equations. To prove the uniqueness:

$$\begin{aligned}\mathcal{X} &= \mathcal{X} *_N (\mathcal{A} *_N \mathcal{X})^* = \mathcal{X} *_N \mathcal{X}^* *_N \mathcal{A}^* \\ &= \mathcal{X} *_N (\mathcal{A} *_N \mathcal{X})^* *_N ((\mathcal{A} *_N \mathcal{Y})^*) = \mathcal{X} *_N \mathcal{A} *_N \mathcal{Y} \\ &= (\mathcal{X} *_N \mathcal{A})^* *_N (\mathcal{Y} *_N \mathcal{A})^* *_N \mathcal{Y} = \mathcal{A}^* *_N \mathcal{Y}^* *_N \mathcal{A} \\ &= (\mathcal{Y} *_N \mathcal{A})^* *_N \mathcal{Y} = \mathcal{Y}.\end{aligned}$$

We know that the conclusion hold. □

Let $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ be a tensor of index k. Let us denote the following condition by $(\mathcal{W})$:

$$\mathcal{B}=\mathcal{A}+\mathcal{E} \ with \ \mathrm{Ind}(\mathcal{A})=k, \ \mathcal{E}=\mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E}=\mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^d, \ and \ \|\mathcal{A}^d *_N \mathcal{E}\|_2<1.$$

Theorem 3.2 (Ji and Wei (2018)) *Suppose that condition* $(\mathcal{W})$ *holds and* $\mathrm{Ind}(\mathcal{B}) = k$. *Then*

$$\mathcal{R}(\mathcal{B}^k) = \mathcal{R}(\mathcal{A}^k), \quad \mathcal{N}(\mathcal{B}^k) = \mathcal{N}(\mathcal{A}^k),$$

and

$$\mathcal{A} *_N \mathcal{A}^d = \mathcal{B} *_N \mathcal{B}^d.$$

Moreover

$$\mathcal{B}^d = (\mathcal{I} + \mathcal{A}^d *_N \mathcal{E})^{-1} *_N \mathcal{A}^d = \mathcal{A}^d *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^d)^{-1}.$$

From (1.9) and (3.1), we have:

$$\mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^\dagger \end{pmatrix} *_N \mathcal{U}^*. \tag{3.2}$$

Now, we develop the perturbation bounds for the DMP inverse of the tensor.

Theorem 3.3 *Let* $\mathcal{A} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$ *be of the form* (1.9) *and of index* k, $\mathcal{B} = \mathcal{A}+\mathcal{E}$. *If the perturbation* $\mathcal{E}$ *satisfies* $\mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{E}$ *and* $\|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2 < 1$, *then*

$$\mathcal{B}^{d,\dagger} = (\mathcal{I} + \mathcal{A}^{d,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{d,\dagger} = \mathcal{A}^{d,\dagger} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{d,\dagger})^{-1} \tag{3.3}$$

and

$$\mathcal{B} *_N \mathcal{B}^{d,\dagger} = \mathcal{A} *_N \mathcal{A}^{d,\dagger}, \ \mathcal{B}^{d,\dagger} *_N \mathcal{B} = \mathcal{A}^{d,\dagger} *_N \mathcal{A}. \tag{3.4}$$

Furthermore:

$$\frac{\|\mathcal{A}^{d,\dagger}\|_2}{1 + \|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2} \le \|\mathcal{B}^{d,\dagger}\|_2 \le \frac{\|\mathcal{A}^{d,\dagger}\|_2}{1 - \|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2}. \tag{3.5}$$

Proof Assume that the perturbation $\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}$ satisfies:

$$\mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{E},$$

then we have:

$$\mathcal{E} = \mathcal{U} *_N \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} *_N \mathcal{U}^*.$$

Since $\mathcal{B} = \mathcal{A} + \mathcal{E}$, we can obtain:

$$\mathcal{B} = \mathcal{U} *_N \begin{pmatrix} \mathcal{T}_{11} + \mathcal{E}_{11} & \mathcal{T}_{12} + \mathcal{E}_{12} \\ \mathcal{O} & \mathcal{T}_{22} \end{pmatrix} *_N \mathcal{U}^*.$$

From $\mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{A}^d *_N \mathcal{A} = \mathcal{A} *_N \mathcal{A}^d$, we verify the conditions:

$$\mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^{d,\dagger} *_N \mathcal{A} = \mathcal{E}$$

if and only if

$$\mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A} *_N \mathcal{A}^d = \mathcal{E}$$

and

$$\|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2 = \|\mathcal{A}^{d,\dagger} *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E}\|_2 = \|\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E}\|_2 = \|\mathcal{A}^d *_N \mathcal{E}\|_2 \leq 1.$$

Therefore, the $(\mathcal{W})$ condition holds. Then

$$\mathcal{A} *_N \mathcal{A}^d = \mathcal{B} *_N \mathcal{B}^d$$

and

$$\mathcal{B}^d = (\mathcal{I} + \mathcal{A}^d *_N \mathcal{E})^{-1} *_N \mathcal{A}^d = \mathcal{A}^d *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^d)^{-1}.$$

Since $\|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2 < 1$, from Lemma 1.1, we have $\mathcal{I} + \mathcal{A}^{d,\dagger} *_N \mathcal{E}$ is invertible. And $\mathcal{T}_{22}$ is nilpotent tensor with index k. From Theorem 3.1, we can obtain:

$$\mathcal{B}^\dagger = \mathcal{U} *_N \begin{pmatrix} (\mathcal{T}_{11} + \mathcal{E}_{11})^* *_N \Delta & -(\mathcal{T}_{11} + \mathcal{E}_{11})^* *_N \Delta *_N (\mathcal{T}_{12} + \mathcal{E}_{12}) *_N \mathcal{T}_{22}^\dagger \\ (\mathcal{I} - \mathcal{T}_{22}^\dagger *_N \mathcal{T}_{22}) *_N (\mathcal{T}_{12} + \mathcal{E}_{12})^* *_N \Delta & \mathcal{G} \end{pmatrix} *_N \mathcal{U}^*,$$

where $\mathcal{G} = \mathcal{T}_{22}^\dagger - (\mathcal{I} - \mathcal{T}_{22}^\dagger *_N \mathcal{T}_{22}) *_N (\mathcal{T}_{12} + \mathcal{E}_{12})^* *_N \Delta *_N (\mathcal{T}_{12} + \mathcal{E}_{12}) *_N \mathcal{T}_{22}^\dagger$. Through calculations, we can obtain:

$$\mathcal{B} *_N \mathcal{B}^\dagger = \mathcal{U} *_N \begin{pmatrix} \mathcal{I} & \mathcal{O} \\ \mathcal{O} & \mathcal{T}_{22} *_N \mathcal{T}_{22}^\dagger \end{pmatrix} *_N \mathcal{U}^*. \tag{3.6}$$

It is obvious that

$$\mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{B} *_N \mathcal{B}^\dagger. \tag{3.7}$$

We can obtain

$$\begin{aligned}
&(\mathcal{I} + \mathcal{A}^{d,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{d,\dagger} \\
&\quad = (\mathcal{I} + \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E})^{-1} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger \\
&\quad = (\mathcal{I} + \mathcal{A}^d *_N \mathcal{E})^{-1} *_N \mathcal{A}^d *_N \mathcal{B} *_N \mathcal{B}^\dagger \\
&\quad = \mathcal{B}^d *_N \mathcal{B} *_N \mathcal{B}^\dagger \\
&\quad = \mathcal{B}^{d,\dagger}.
\end{aligned}$$

Thus

$$\mathcal{B}^{d,\dagger} = (\mathcal{I} + \mathcal{A}^{d,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{d,\dagger}.$$

Similarly, we can prove that:

$$\mathcal{B}^{d,\dagger} = \mathcal{A}^{d,\dagger} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{d,\dagger})^{-1}$$

and

$$\mathcal{B} *_N \mathcal{B}^{d,\dagger} = \mathcal{A} *_N \mathcal{A}^{d,\dagger}, \; \mathcal{B}^{d,\dagger} *_N \mathcal{B} = \mathcal{A}^{d,\dagger} *_N \mathcal{A}.$$

Moreover, from (3.3), taking norms of both sides, we obtain:

$$\frac{\|\mathcal{A}^{d,\dagger}\|_2}{1 + \|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2} \leq \|\mathcal{B}^{d,\dagger}\|_2 \leq \frac{\|\mathcal{A}^{d,\dagger}\|_2}{1 - \|\mathcal{A}^{d,\dagger} *_N \mathcal{E}\|_2}.$$

The proof is complete. □

Now, we present the perturbation of Moore–Penrose inverse under the two-sided conditions.

Lemma 3.1 *Let* $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$, $\mathcal{B} = \mathcal{A} + \mathcal{E}$. *If the perturbation* $\mathcal{E}$ *satisfies* $\mathcal{E} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^\dagger *_N \mathcal{A}$ *and* $\|\mathcal{A}^\dagger *_N \mathcal{E}\|_2 < 1$, *then*

$$\mathcal{B}^\dagger = (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^\dagger)^{-1}.$$

Proof Since $\mathcal{E} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^\dagger *_N \mathcal{A}$,

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) = (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^\dagger) *_N \mathcal{A}.$$

Since $\|\mathcal{A}^\dagger *_N \mathcal{E}\|_2 < 1$, by using (1.2) of Lemma 1.2, we know that:

$$(\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^\dagger)^{-1}.$$

In view of Definition 1.1, it is easy to compute the first equation that:

$$\begin{aligned}
&\mathcal{B} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger *_N \mathcal{B} \\
&\quad = \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) \\
&\quad = \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) \\
&\quad = \mathcal{B}.
\end{aligned}$$

Furthermore, the second equation follows from:

$$\begin{aligned}
&(\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger *_N \mathcal{B} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger \\
&\quad = (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger \\
&\quad = (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger.
\end{aligned}$$

The third equation is verified as:

$$\begin{aligned}
&(\mathcal{B} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger)^* \\
&\quad = (\mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger)^* \\
&\quad = (\mathcal{A} *_N \mathcal{A}^\dagger)^* \\
&\quad = \mathcal{A} *_N \mathcal{A}^\dagger \\
&\quad = \mathcal{A} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E}) *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger \\
&\quad = \mathcal{B} *_N (\mathcal{I} + \mathcal{A}^\dagger *_N \mathcal{E})^{-1} *_N \mathcal{A}^\dagger,
\end{aligned}$$

and the fourth equation can be verified by:

$$\begin{aligned}
&(\mathcal{A}^\dagger *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger)^{-1} *_N \mathcal{B})^* \\
&= (\mathcal{A}^\dagger *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger)^{-1} *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger) *_N \mathcal{A})^* \\
&= (\mathcal{A}^\dagger *_N \mathcal{A})^* \\
&= \mathcal{A}^\dagger *_N \mathcal{A} \\
&= \mathcal{A}^\dagger *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger)^{-1} *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger) *_N \mathcal{A} \\
&= \mathcal{A}^\dagger *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^\dagger)^{-1} *_N \mathcal{B}.
\end{aligned}$$

Therefore, the result holds. □

We estimate the perturbation bounds for the CMP inverse of the tensor.

Theorem 3.4 *Let* $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I}_1\times\cdots\times\mathbf{I}_N\times\mathbf{I}_1\times\cdots\times\mathbf{I}_N}$, $\mathrm{Ind}(\mathcal{A}) = k$ *and* $\mathcal{B} = \mathcal{A}+\mathcal{E}$ *be such that* $\mathcal{E} = \mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A}$ *and* $\|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2 < 1$. *Denote*

$$\mathcal{X} = (\mathcal{I}+\mathcal{A}^{c,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{c,\dagger} *_N (\mathcal{I}+\mathcal{E} *_N \mathcal{A}^{c,\dagger})^{-1}$$

satisfies

$$\begin{aligned}
&\mathcal{B} *_N \mathcal{X} *_N \mathcal{B} = \mathcal{C}_\mathcal{B}, \quad \mathcal{X} *_N \mathcal{B} *_N \mathcal{X} = \mathcal{X}, \quad \mathcal{B} *_N \mathcal{X} = \mathcal{C}_\mathcal{B} *_N \mathcal{B}^\dagger, \\
&\mathcal{X} *_N \mathcal{B} = \mathcal{B}^\dagger *_N \mathcal{C}_\mathcal{B},
\end{aligned}$$

i.e., $X = \mathcal{B}^{c,\dagger}$.

Proof Since $\mathcal{E} = \mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E}$,

$$\mathcal{B} = \mathcal{A}+\mathcal{E} = \mathcal{A} *_N (\mathcal{I}+\mathcal{A}^{c,\dagger} *_N \mathcal{E}). \tag{3.8}$$

By Lemma 1.2, $\mathcal{I}+\mathcal{A}^{c,\dagger} *_N \mathcal{E}$ is invertible and

$$\begin{aligned}
\mathcal{B} *_N \mathcal{X} &= \mathcal{A} *_N (\mathcal{I}+\mathcal{A}^{c,\dagger} *_N \mathcal{E}) *_N (\mathcal{I}+\mathcal{A}^{c,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{c,\dagger} \\
&= \mathcal{A} *_N \mathcal{A}^{c,\dagger} \\
&= \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger \\
&= \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger \\
&= \mathcal{C}_\mathcal{A} *_N \mathcal{A}^\dagger.
\end{aligned} \tag{3.9}$$

Since $\mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E} = \mathcal{C}_\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{E}$, we obtain:

$$\mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{C}_\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{E}.$$

Moreover:

$$\mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{E} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{E} = \mathcal{E}.$$

Similarly:

$$\mathcal{A}^{c,\dagger} *_N \mathcal{A} = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{A}^\dagger *_N \mathcal{C}_\mathcal{A};$$

we obtain

$$\mathcal{E} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A} = \mathcal{E} *_N \mathcal{A}^\dagger *_N \mathcal{C}_\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{E} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{E},$$

and

$$\mathcal{E} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A} = \mathcal{E} *_N \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^{d} *_N \mathcal{A} *_N \mathcal{A}^{\dagger} *_N \mathcal{A} = \mathcal{E} *_N \mathcal{A}^{d} *_N \mathcal{A} = \mathcal{E}$$

which are also true.

Since $\|\mathcal{A}^d *_N \mathcal{E}\|_2 = \|\mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2 \leq \|\mathcal{A}^d *_N \mathcal{A}\|_2 \|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2 < 1$, by Ji and Wei (2018), we know that:

$$\mathcal{A} *_N \mathcal{A}^d = \mathcal{B} *_N \mathcal{B}^d. \tag{3.10}$$

By using the same method, we also get $\|\mathcal{A}^{\dagger} *_N \mathcal{E}\|_2 < 1$, so

$$\mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{B} *_N \mathcal{B}^{\dagger}. \tag{3.11}$$

In view of (3.10) and (3.11):

$$\mathcal{B} *_N \mathcal{X} = \mathcal{C}_{\mathcal{A}} *_N \mathcal{A}^{\dagger} = \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} = \mathcal{B} *_N \mathcal{B}^d *_N \mathcal{B} *_N \mathcal{B}^{\dagger} = \mathcal{C}_{\mathcal{B}} *_N \mathcal{B}^{\dagger}.$$

And

$$\mathcal{B} *_N \mathcal{X} *_N \mathcal{B} = \mathcal{C}_{\mathcal{B}} *_N \mathcal{B}^{\dagger} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{B}^d *_N \mathcal{B} *_N \mathcal{B}^{\dagger} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{B}^d *_N \mathcal{B} = \mathcal{C}_{\mathcal{B}}.$$

Using $\mathcal{B} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{c,\dagger}$ of (3.9), we have

$$\begin{aligned} \mathcal{X} *_N \mathcal{B} *_N \mathcal{X} &= (\mathcal{I} + \mathcal{A}^{c,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N \mathcal{A}^{c,\dagger} \\ &= (\mathcal{I} + \mathcal{A}^{c,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{c,\dagger}. \end{aligned}$$

Finally, Since $\mathcal{E} = \mathcal{E} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A} *_N$,

$$\mathcal{B} = \mathcal{A} + \mathcal{E} = (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{c,\dagger}) *_N \mathcal{A},$$

we have:

$$\begin{aligned} \mathcal{X} *_N \mathcal{B} &= \mathcal{A}^{c,\dagger} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{c,\dagger})^{-1} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{c,\dagger}) *_N \mathcal{A} \\ &= \mathcal{A}^{c,\dagger} *_N \mathcal{A} \\ &= \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} *_N \mathcal{A}^{\dagger} *_N \mathcal{A} \\ &= \mathcal{A}^{\dagger} *_N \mathcal{A} *_N \mathcal{A}^d *_N \mathcal{A} \\ &= \mathcal{B}^{\dagger} *_N \mathcal{B} *_N \mathcal{B}^d *_N \mathcal{B} \\ &= \mathcal{B}^{\dagger} *_N \mathcal{C}_{\mathcal{B}}. \end{aligned}$$

Thus, the proof of the theorem is complete. □

Theorem 3.5 *Let* $\mathcal{A}, \mathcal{E} \in \mathbb{C}^{\mathbf{I}_1 \times \cdots \times \mathbf{I}_N \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_N}$, $\mathrm{Ind}(\mathcal{A}) = k$ *and* $\mathcal{B} = \mathcal{A} + \mathcal{E}$ *be such that* $\mathcal{E} = \mathcal{A} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{E} = \mathcal{E} *_N \mathcal{A}^{c,\dagger} *_N \mathcal{A}$. *If* $\|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2 < 1$, *then*

$$\mathcal{B}^{c,\dagger} = (\mathcal{I} + \mathcal{A}^{c,\dagger} *_N \mathcal{E})^{-1} *_N \mathcal{A}^{c,\dagger} = \mathcal{A}^{c,\dagger} *_N (\mathcal{I} + \mathcal{E} *_N \mathcal{A}^{c,\dagger})^{-1}.$$

Moreover:

$$\frac{\|\mathcal{A}^{c,\dagger}\|_2}{1 + \|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2} \leq \|\mathcal{B}^{c,\dagger}\|_2 \leq \frac{\|\mathcal{A}^{c,\dagger}\|_2}{1 - \|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2}.$$

and

$$\frac{\|\mathcal{B}^{c,\dagger} - \mathcal{A}^{c,\dagger}\|_2}{\|\mathcal{A}^{c,\dagger}\|_2} \leq \frac{\|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2}{1 - \|\mathcal{A}^{c,\dagger} *_N \mathcal{E}\|_2}.$$

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