



Asymptotical stability of numerical methods for semi-linear impulsive differential equations

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Abstract

This paper is concerned with asymptotical stability of a class of semi-linear impulsive ordinary differential equations. First of all, sufficient conditions for asymptotical stability of the exact solutions of semi-linear impulsive differential equations are provided. Under the sufficient conditions, some explicit exponential Runge–Kutta methods can preserve asymptotically stability without additional restriction on stepsizes. Moreover, it is proved that some explicit Runge–Kutta methods can preserve asymptotical stability without additional restriction on stepsizes under stronger conditions.

Keywords Impulsive differential equations · Runge–Kutta method · Exponential Runge–Kutta method · Asymptotically stable

Mathematics Subject Classification 65L05 · 65L06 · 65L07 · 65L20

1 Introduction

The impulsive differential equations (IDEs) are widely applied in numerous fields of science and technology. Recently, the theory of IDEs has been an object of active research. Especially, the exact solutions of semi-linear IDEs have been widely studied (see Abada et al. 2009; Benchohra et al. 2006; Cardinali and Rubbioni 2008; Fan 2010; Friedli 1978; Fan and Li 2010; Liang et al. 2009). However, many IDEs cannot be solved analytically or their solving is more complicated. Hence, taking numerical methods is a good choice.

In recent years, the stability of numerical methods for IDEs has attracted more and more attention (see Liu et al. 2007, 2014; Liang et al. 2011; Ran et al. 2008; Wen and Yu 2011;

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Zhang and Liang 2014 etc). Stability of Runge–Kutta (RK) methods with the constant step-size for scalar linear IDEs has been studied by Ran et al. (2008). RK methods with variable stepsizes for multidimensional linear IDEs have been investigated in Liu et al. (2007). Collocation methods for linear nonautonomous IDEs has been considered in Zhang and Liang (2014). An improved linear multistep method for linear IDEs has been investigated in Liu et al. (2012). Stability of the exact and numerical solutions of nonlinear IDEs has been studied by the Lyapunov method in Liang et al. (2011). Stability of RK methods for a special kind of nonlinear IDEs has been investigated by the properties of the differential equations without impulsive perturbations in Liu et al. (2014). Asymptotic stability of implicit Euler method for stiff IDEs in Banach space has been studied in Wen and Yu (2011). There are a lot of meaningful work on the numerical solution of impulsive differential equations, for example Ding et al. (2006, 2010), Liang et al. (2014), Liu and Zeng (2018), Wu et al. (2010), Wu and Ding (2012a, b) and Wu et al. (2019a, b). But, up to our knowledge, the numerical methods for impulsive semi-linear ordinary differential equations are still a blank field.

Exponential RK methods are widely applied to semi-linear ordinary differential equations without impulsive perturbations have been widely studied (see Calvo et al. 2000; Lawson 1967; Maset and Zennaro 2009; Ostermann and Daele 2007; Ostermann et al. 2006). Now the exponential RK methods are applied to the following equation:

$$\begin{cases} x'(t) = Lx(t) + f(t, x(t)), & t > t_0, t \neq \tau_k, k \in \mathbb{Z}^+, \\ x(\tau_k^+) = I_k(x(\tau_k)), & k \in \mathbb{Z}^+, \\ x(0^+) = x_0, \end{cases} \quad (1.1)$$

where $L \in \mathbb{R}^{d \times d}$, $f : (t_0, +\infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $I_k : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathbb{Z}^+ = \{1, 2, \dots\}$ and τ_k is a positive real constant and $\tau_k < \tau_{k+1}$ for arbitrary $k \in \mathbb{Z}^+$.

Outline of the rest of the paper is provided as follows. In Sect. 2, sufficient conditions for asymptotical stability of the exact solution of (1.1) are provided.

In Sect. 3, explicit Euler method, implicit Euler method and explicit exponential Euler are applied for two semi-linear IDEs to explore the stability of numerical methods. From the two numerical examples, the following two key questions which will be worked out in this article are presented. Are there some exponential RK methods which can preserve asymptotical stability of (1.1) without additional restrictions on stepsizes under the sufficient conditions which are provided in Sect. 2? Are there some RK methods which can preserve asymptotical stability of (1.1) without additional restrictions on stepsizes under stronger sufficient conditions?

In Sect. 4, it is proved that a class of explicit exponential RK methods can preserve asymptotical stability semi-linear IDEs without restriction on the stepsizes except that $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$, under the sufficient conditions which are provided in Sect. 2.

In Sect. 5, stronger sufficient conditions for asymptotical stability of the exact solution of (1.1) are provided first. Under these stronger sufficient conditions, asymptotical stability of RK methods for (1.1) is studied.

In Sect. 6, some conclusions are drafted.

2 Stability of IDEs

Assume that $y(t)$ is the solution of the following equation:

$$\begin{cases} y'(t) = Ly(t) + f(t, y(t)), & t \neq \tau_k, k \in \mathbb{Z}^+, \\ y(\tau_k^+) = I_k(y(\tau_k)), & k \in \mathbb{Z}^+, \\ y(0^+) = y_0, \end{cases} \tag{2.1}$$

and the functions f and I_k satisfy the following properties, respectively:

$$\|f(t, x) - f(t, y)\| \leq \lambda \|x - y\|, \quad \forall t \in \mathbb{R}^+, \forall x, y \in \mathbb{R}^d, \tag{2.2}$$

$$\|I_k(x) - I_k(y)\| \leq \gamma_k \|x - y\|, \quad \forall x, y \in \mathbb{R}^d. \tag{2.3}$$

Definition 2.1 (Bainov and Simeonov 1989; Samoilenko et al. 1995) The exact solution $x(t)$ of (1.1) is said to be

1. Stable if, for an arbitrary $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that, for the solution $y(t)$ of (2.1), $\|x_0 - y_0\| < \delta$ implies

$$\|x(t) - y(t)\| < \epsilon, \quad \forall t > t_0;$$

2. Asymptotically stable, if it is stable and $\lim_{t \rightarrow \infty} \|x(t) - y(t)\| = 0$.

Theorem 2.2 Assume that there exists a positive constant \hat{C} such that $\tau_k - \tau_{k-1} \leq \hat{C}, k \in \mathbb{Z}^+$. The exact solution of (1.1) is asymptotically stable if there is a positive constant C such that

$$\gamma_k e^{(\mu(L)+\lambda)(\tau_k - \tau_{k-1})} \leq C < 1, \tag{2.4}$$

for arbitrary $k \in \mathbb{Z}^+$.

Proof Define $\delta(t) = \|x(t) - y(t)\|^2$. Obviously, we have

$$\begin{aligned} \delta'(t) &= 2 \langle x'(t) - y'(t), x(t) - y(t) \rangle \\ &= 2 \langle Lx(t) + f(t, x(t)) - Ly(t) - f(t, y(t)), x(t) - y(t) \rangle \\ &\leq 2\mu(L)\delta(t) + 2\|f(t, x(t)) - f(t, y(t))\| \|x(t) - y(t)\| \\ &\leq 2\mu(L)\delta(t) + 2\lambda \|x(t) - y(t)\| \|x(t) - y(t)\| \end{aligned}$$

Since $x(t)$ and $y(t)$ are left continuous, we get

$$\delta(t) \leq e^{2(\mu(L)+\lambda)(t-\tau_k)} \delta(\tau_k^+), \quad t \in (\tau_k, \tau_{k+1}],$$

that is

$$\|x(t) - y(t)\| \leq e^{(\mu(L)+\lambda)(t-\tau_k)} \|x(\tau_k^+) - y(\tau_k^+)\|, \quad t \in (\tau_k, \tau_{k+1}],$$

which implies

$$\begin{aligned} \|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| &= \|I_{k+1}(x(\tau_{k+1})) - I_{k+1}(y(\tau_{k+1}))\| \\ &\leq \gamma_{k+1} \|x(\tau_{k+1}) - y(\tau_{k+1})\| \\ &\leq \|x(\tau_k^+) - y(\tau_k^+)\| \gamma_{k+1} e^{(\mu(L)+\lambda)(\tau_{k+1}-\tau_k)}. \end{aligned}$$

According to mathematical induction, we can get that, for $\forall t \in (\tau_k, \tau_{k+1}]$,

$$\|x(t) - y(t)\| \leq \|x_0 - y_0\| (\gamma_1 e^{(\mu(L)+\lambda)(\tau_1 - \tau_0)})$$

$$\begin{aligned} & (\gamma_2 e^{(\mu(L)+\lambda)(\tau_2-\tau_1)})(\gamma_k e^{(\mu(L)+\lambda)(\tau_k-\tau_{k-1})})e^{(\mu(L)+\lambda)(t-\tau_k)} \\ & \leq C^k \|x_0 - y_0\| e^{(\mu(L)+\lambda)(t-\tau_k)} \\ & \leq C^k \|x_0 - y_0\| \hat{C}_1, \end{aligned}$$

where $\hat{C}_1 = \max\{1, e^{(\mu(L)+\lambda)\hat{C}}\}$. Hence, for an arbitrary $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{\hat{C}_1}$ such that $\|x_0 - y_0\| < \delta$ implies

$$\|x(t) - y(t)\| \leq C^k \|x_0 - y_0\| \hat{C}_1 \leq \|x_0 - y_0\| \hat{C}_1 < \epsilon$$

for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$. And at the moments of impulsive effect $t = \tau_{k+1}$, $k = 0, 1, 2, \dots$, we can also obtain that

$$\|x(\tau_{k+1}) - y(\tau_{k+1})\| \leq C^k \|x_0 - y_0\| \hat{C}_1 < \epsilon,$$

and

$$\|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| \leq C^{k+1} \|x_0 - y_0\| < \epsilon.$$

Therefore, the exact solution of (1.1) is stable. Obviously, for arbitrary $t \in (\tau_k, \tau_{k+1}]$, $k = 0, 1, 2, \dots$,

$$\|x(t) - y(t)\| \leq C^k \|x_0 - y_0\| \hat{C}_1 \rightarrow 0, \quad k \rightarrow \infty.$$

At the moments of impulsive effect $t = \tau_{k+1}$, $k = 0, 1, 2, \dots$, we can also obtain that

$$\|x(\tau_{k+1}) - y(\tau_{k+1})\| \leq C^k \|x_0 - y_0\| \hat{C}_1 \rightarrow 0, \quad k \rightarrow \infty,$$

and

$$\|x(\tau_{k+1}^+) - y(\tau_{k+1}^+)\| \leq C^{k+1} \|x_0 - y_0\| \rightarrow 0, \quad k \rightarrow \infty.$$

Consequently, the exact solutions of (1.1) is asymptotically stable under the condition (2.4). □

3 Two numerical examples

In this section, explicit Euler method, implicit Euler method and explicit exponential Euler are applied to semi-linear IDEs to explore the stability of numerical methods.

Example 3.1 Consider one-dimension semi-linear IDE:

$$\begin{cases} x'(t) = -5x(t) + \sin(x(t)), & t > 0, t \neq k, k \in \mathbb{Z}^+, \\ x(k^+) = 18x(k), & k \in \mathbb{Z}^+, \\ x(0^+) = x_0, \end{cases} \tag{3.1}$$

and the same equation with another initial value

$$\begin{cases} y'(t) = -5y(t) + \sin(y(t)), & t > 0, t \neq k, k \in \mathbb{Z}^+, \\ y(k^+) = 18y(k), & k \in \mathbb{Z}^+, \\ y(0^+) = y_0, \end{cases} \tag{3.2}$$

where x_0 and y_0 are real constants and $x_0 \neq y_0$. By Theorem 2.2, we can obtain that the exact solution of (3.1) is asymptotically stable.

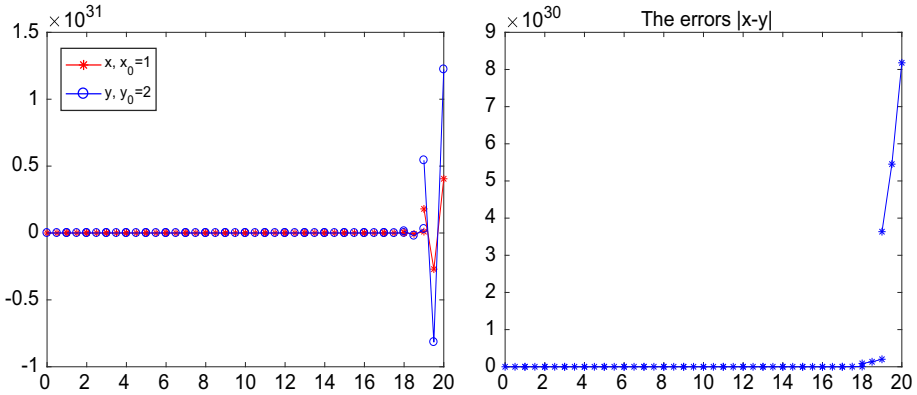


Fig. 1 Explicit Euler method for (3.1) and (3.2)

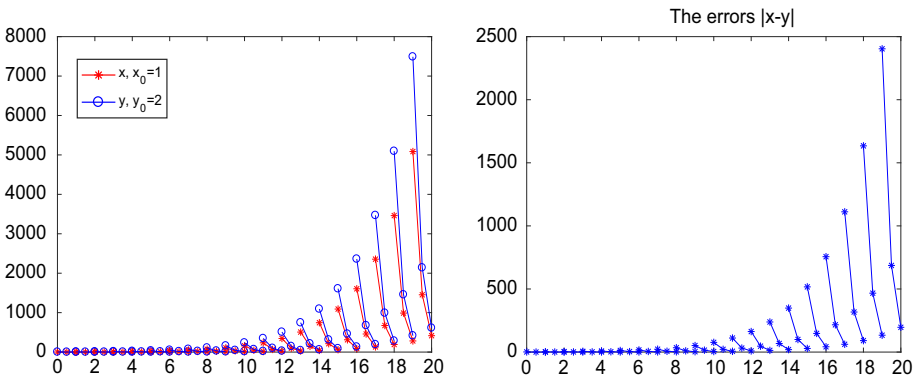


Fig. 2 Implicit Euler method for (3.1) and (3.2)

Figures 1, 2 and 3 are the numerical solutions and the absolute value of these errors between two different numerical solutions, which are obtained from explicit Euler method, implicit Euler method and explicit exponential Euler, respectively, when $x_0 = 1, y_0 = 2, h_k = h = \frac{1}{2}, k \in \mathbb{N}$.

First of all, we claim that the absolute value of these errors between two different numerical solutions obtained from explicit Euler method for (3.1) and (3.2) tend to infinite when the stepsize $h = \frac{1}{2}$ (see Fig. 1).

To prove this, let us consider the explicit Euler method for (3.1):

$$\begin{cases} x_{k,l+1} = x_{k,l} + h(-5x_{k,l} + \sin(x_{k,l})), & k \in \mathbb{N}, l = 0, 1, \dots, m - 1, \\ x_{k+1,0} = 18x_{k,m}, & k \in \mathbb{N}, \\ x_{0,0} = x_0, \end{cases} \tag{3.3}$$

and explicit Euler method for (3.2):

$$\begin{cases} y_{k,l+1} = y_{k,l} + h(-5y_{k,l} + \sin(y_{k,l})), & k \in \mathbb{N}, l = 0, 1, \dots, m - 1, \\ y_{k+1,0} = 18y_{k,m}, & k \in \mathbb{N}, \\ y_{0,0} = y_0, \end{cases} \tag{3.4}$$

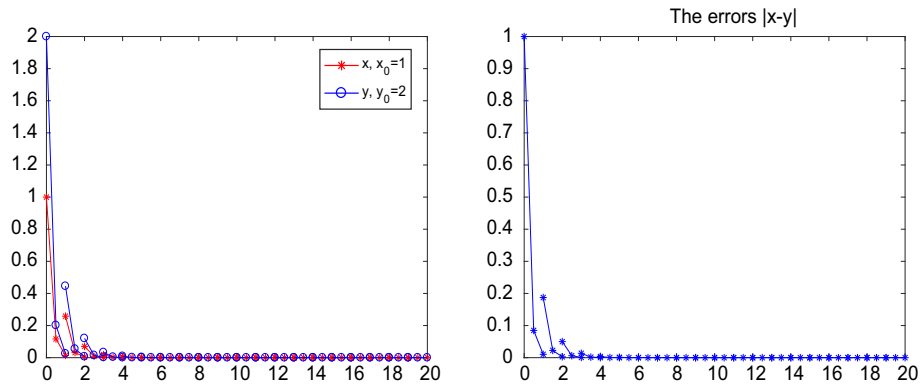


Fig. 3 Explicit exponential Euler method for (3.1) and (3.2)

where $h = \frac{1}{m}$, $m \in \mathbb{Z}^+$, $x_{k,l}$ is an approximation to the exact solution $x(k + lh)$ and $x_{k,m}$ is an approximation to $x(k + 1^-)$, $k \in \mathbb{N}$, $l = 0, 1, \dots, m - 1$.

Obviously, when $h = \frac{1}{2}$, we can obtain that, for $l = 0, 1$,

$$\begin{aligned}
 |x_{k,l+1} - y_{k,l+1}| &= |(1 - 5h)(x_{k,l} - y_{k,l}) + h(\sin(x_{k,l}) - \sin(y_{k,l}))| \\
 &\geq |(1 - 5h)(x_{k,l} - y_{k,l})| - h|\sin(x_{k,l}) - \sin(y_{k,l})| \\
 &\geq |(1 - 5h)(x_{k,l} - y_{k,l})| - h|x_{k,l} - y_{k,l}| \\
 &= \frac{3}{2} \cdot |x_{k,l} - y_{k,l}| - \frac{1}{2} \cdot |x_{k,l} - y_{k,l}| \\
 &= |x_{k,l} - y_{k,l}|,
 \end{aligned}$$

which implies that, for any $l = 0, 1, 2$,

$$\begin{aligned}
 |x_{k,l} - y_{k,l}| &\geq 18|x_{k-1,l} - y_{k-1,l}| \\
 &\geq 18^k |x_{0,l} - y_{0,l}| \\
 &\geq 18^k |x_0 - y_0| \rightarrow \infty, \quad k \rightarrow \infty.
 \end{aligned}$$

Second, we claim that the absolute value of these errors between two different numerical solutions, which are obtained from implicit Euler method for (3.1) and (3.2), tend to infinite when $h = \frac{1}{2}$ (see Fig. 1).

To prove this, let us consider implicit Euler method for (3.1)

$$\begin{cases} x_{k,l+1} = x_{k,l} + h(-5x_{k,l+1} + \sin(x_{k,l+1})), & k \in \mathbb{N}, \quad l = 0, 1, \dots, m - 1, \\ x_{k+1,0} = 18x_{k,m}, & k \in \mathbb{N}, \\ x_{0,0} = x_0. \end{cases} \tag{3.5}$$

and implicit Euler method for (3.2)

$$\begin{cases} y_{k,l+1} = y_{k,l} + h(-5y_{k,l+1} + \sin(y_{k,l+1})), & k \in \mathbb{N}, \quad l = 0, 1, \dots, m - 1, \\ y_{k+1,0} = 18y_{k,m}, & k \in \mathbb{N}, \\ y_{0,0} = y_0. \end{cases} \tag{3.6}$$

Obviously, when $h = \frac{1}{2}$, we can obtain that

$$\begin{aligned} |x_{k,l+1} - y_{k,l+1}| &= |x_{k,l} - y_{k,l} - 5h(x_{k,l+1} - y_{k,l+1}) + h(\sin(x_{k,l+1}) - \sin(y_{k,l+1}))| \\ &\geq |x_{k,l} - y_{k,l}| - 5h|x_{k,l+1} - y_{k,l+1}| - h|(\sin(x_{k,l+1}) - \sin(y_{k,l+1}))| \\ &\geq |x_{k,l} - y_{k,l}| - 6h|x_{k,l+1} - y_{k,l+1}| \\ &= |x_{k,l} - y_{k,l}| - 3|x_{k,l+1} - y_{k,l+1}| \end{aligned}$$

which implies that

$$|x_{k,l+1} - y_{k,l+1}| \geq \frac{1}{4} \cdot |x_{k,l} - y_{k,l}|.$$

Hence, for any $l = 0, 1, 2,$

$$\begin{aligned} |x_{k,l} - y_{k,l}| &\geq \frac{18}{4^2} \cdot |x_{k-1,l} - y_{k-1,l}| \\ &\geq \left(\frac{9}{8}\right)^k \cdot |x_{0,l} - y_{0,l}| \\ &\geq \left(\frac{9}{8}\right)^k \cdot \frac{1}{4^2} \cdot |x_0 - y_0| \rightarrow \infty, \quad k \rightarrow \infty. \end{aligned}$$

It is different from the above two cases, we can easily prove that exponential explicit Euler method (see Fig. 3) for (3.1) is asymptotically stable for arbitrary $h = \frac{1}{m}, \forall m \in \mathbb{Z}^+$. By Corollary 4.4 of this paper, we also can obtain this point.

Example 3.2 Consider one-dimension semi-linear IDE:

$$\begin{cases} x'(t) = 4x(t) + \frac{\sqrt{x^2(t)+1}}{10}, & t > 0, t \neq k, k \in \mathbb{Z}^+, \\ x(k^+) = \frac{x(k)}{80}, & k \in \mathbb{Z}^+, \\ x(0^+) = x_0, \end{cases} \tag{3.7}$$

and the same equation with another initial value

$$\begin{cases} y'(t) = 4y(t) + \frac{\sqrt{y^2(t)+1}}{10}, & t > 0, t \neq k, k \in \mathbb{Z}^+, \\ y(k^+) = \frac{y(k)}{80}, & k \in \mathbb{Z}^+, \\ y(0^+) = y_0, \end{cases} \tag{3.8}$$

where x_0 and y_0 are real constants and $x_0 \neq y_0$. By Theorem 2.2, we can obtain that the exact solution of (3.7) is asymptotically stable.

Figures 4, 5 and 6 are the numerical solutions and the absolute value of these errors between two different numerical solutions, which are obtained from explicit Euler method, implicit Euler method and explicit exponential Euler, respectively, when $x_0 = 1, y_0 = 2, h = \frac{1}{5}$.

We claim that the absolute value of these errors between two different numerical solutions obtained from implicit Euler method for (3.7) and (3.8) tend to infinite when $h = \frac{1}{5}$.

To prove this, let us consider implicit Euler method for (3.7)

$$\begin{cases} x_{k,l+1} = x_{k,l} + h \left(4x_{k,l+1} + \frac{\sqrt{x_{k,l+1}^2+1}}{10} \right), & k \in \mathbb{N}, l = 0, 1, \dots, m-1, \\ x_{k+1,0} = \frac{x_{k,m}}{80}, & k \in \mathbb{N}, \\ x_{0,0} = x_0. \end{cases} \tag{3.9}$$

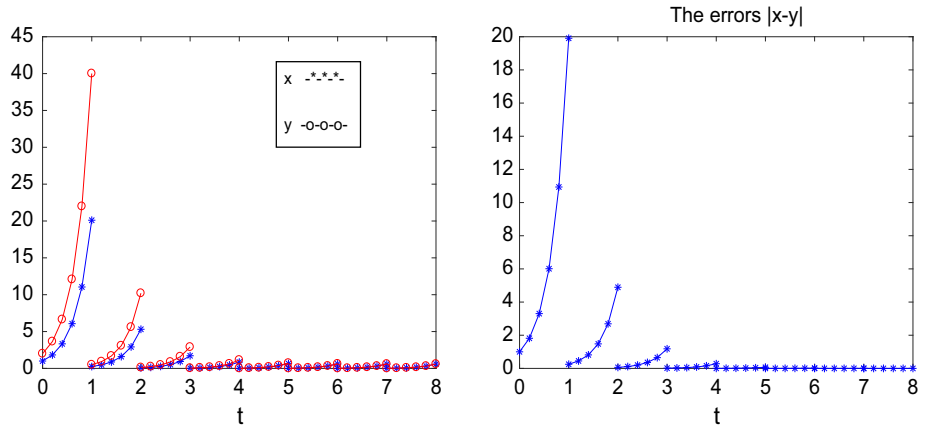


Fig. 4 Explicit Euler method for (3.7) and (3.8)

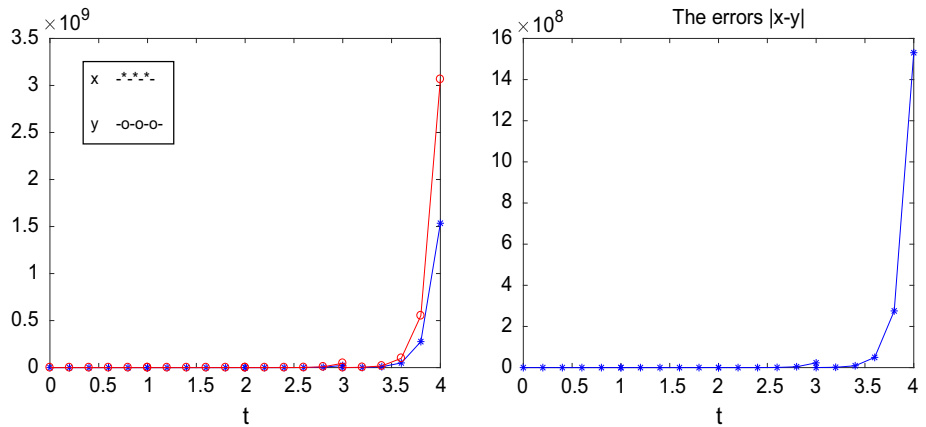


Fig. 5 Explicit Euler method for (3.7) and (3.8)

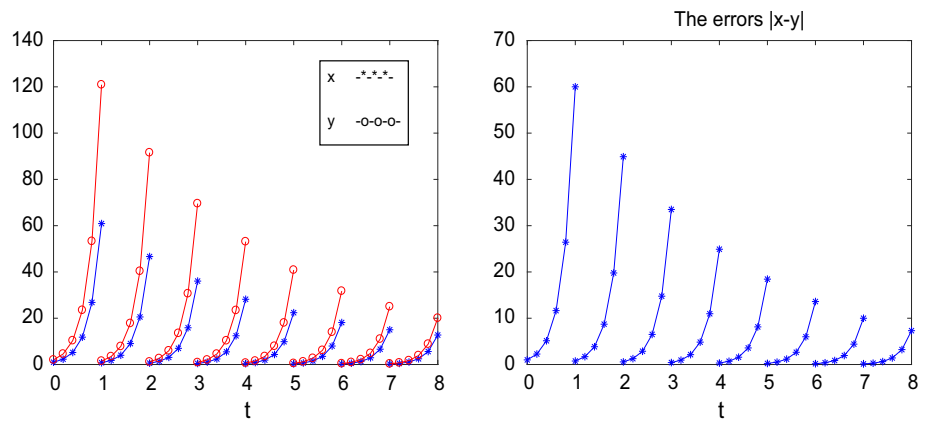


Fig. 6 Explicit exponential Euler method for (3.7) and (3.8)

and implicit Euler method for (3.8)

$$\begin{cases} y_{k,l+1} = y_{k,l} + h \left(4y_{k,l+1} + \frac{\sqrt{y_{k,l+1}^2 + 1}}{10} \right), & k \in \mathbb{N}, l = 0, 1, \dots, m - 1, \\ y_{k+1,0} = \frac{y_{k,m}}{80}, & k \in \mathbb{N}, \\ y_{0,0} = y_0. \end{cases} \tag{3.10}$$

Obviously, when $h = \frac{1}{5}$, we can obtain that

$$\begin{aligned} (1 - 4h)|x_{k,l+1} - y_{k,l+1}| &= |(x_{k,l} - y_{k,l}) + \frac{h(\sqrt{x_{k,l+1}^2 + 1} - \sqrt{y_{k,l+1}^2 + 1})}{10}| \\ &\geq |x_{k,l} - y_{k,l}| - \frac{h|x_{k,l+1} - y_{k,l+1}|}{10} \end{aligned}$$

which implies that

$$|x_{k,l+1} - y_{k,l+1}| \geq \frac{50}{11} \cdot |x_{k,l} - y_{k,l}|.$$

Hence, for any $l = 0, 1, 2,$

$$\begin{aligned} |x_{k,l} - y_{k,l}| &\geq \left(\frac{50}{11}\right)^5 \cdot \frac{1}{80} \cdot |x_{k-1,l} - y_{k-1,l}| \\ &\geq \left(\frac{50}{11}\right)^{5k} \cdot \frac{1}{80^k} \cdot |x_{0,l} - y_{0,l}| \\ &\geq \left(\frac{50}{11}\right)^{5k} \cdot \frac{1}{80^k} \cdot |x_0 - y_0| \rightarrow \infty, k \rightarrow \infty. \end{aligned}$$

On the other hand, it is easy to prove that both explicit Euler method (see Fig. 4) and exponential explicit Euler method (see Fig. 6) for (3.7) are asymptotically stable for arbitrary $h = \frac{1}{m}, \forall m \in \mathbb{Z}^+$. This point can also be obtained by Corollary 4.4 and Theorem 5.2 of this paper.

From Examples 3.1 and 3.2, we can see that explicit exponential Euler method can preserve asymptotically stability of the exact solutions of (3.1) and (3.7) for arbitrary $h = \frac{1}{m}, m \in \mathbb{Z}^+$, but Runge–Kutta methods sometimes cannot preserve asymptotically stability without additional restriction on stepsizes. Naturally, the following two questions arise. Are there some exponential RK methods which can preserve asymptotical stability of (1.1) under the conditions of Theorem 2.2 without restrictions on stepsizes except that $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $m \in \mathbb{Z}^+, k \in \mathbb{N}$? Are there some RK methods which can preserve asymptotical stability of (1.1) without restrictions on stepsizes except that $h_k = \frac{\tau_{k+1} - \tau_k}{m}, m \in \mathbb{Z}^+, k \in \mathbb{N}$, under stronger conditions?

4 Stability of exponential RK methods

In this section, exponential RK methods for (1.1) can be constructed as follows:

$$\begin{cases} X_{k,l}^i = e^{c_i h_k L} x_{k,l} + h_k \sum_{i=1}^v a_{ij}(h_k L) f(t_{k,l}^j, X_{k,l}^j), & k \in \mathbb{N}, i = 1, 2, \dots, v, \\ x_{k,l+1} = e^{h_k L} x_{k,l} + h_k \sum_{i=1}^v b_i(h_k L) f(t_{k,l}^i, X_{k,l}^i), & l = 0, 1, 2, \dots, m - 1, \\ x_{k+1,0} = I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0, \end{cases} \quad (4.1)$$

where $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $t_{k,l} = \tau_k + lh_k$, $t_{k,l}^i = t_{k,l} + c_i h_k$, $x_{k,l}$ is an approximation to the exact solution $x(t_{k,l})$, $x_{k,m}$ is an approximation to $x(\tau_{k+1})$ and $X_{k,l}^i$ is an approximation to the exact solution $x(t_{k,l}^i)$, $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, $l = 0, 1, \dots, m - 1$, $i = 1, 2, \dots, v$, v is referred to as the number of stages. The weights b_i , the abscissae $c_i = \sum_{j=1}^v a_{ij}$ and the matrix $A = [a_{ij}]_{i,j=1}^v$ are denoted by (A, b, c) . Now we introduce, for constant α (may be nonnegative), the $1 \times v$ -vector $\mathbf{b}(\alpha)$ of components

$$\mathbf{b}_i(\alpha) := \sup_{M \in \mathbb{R}^{d \times d}, \mu(M) \leq \alpha} \|b_i(M)\|, \quad i = 1, 2, \dots, v$$

and the $v \times v$ -vector $\mathbf{A}(\alpha)$ whose elements are

$$a_{ij}(\alpha) := \sup_{M \in \mathbb{R}^{d \times d}, \mu(M) \leq \alpha} \|a_{ij}(M)\|, \quad i, j = 1, 2, \dots, v$$

And the exponential RK methods for (2.1):

$$\begin{cases} Y_{k,l}^i = e^{c_i h_k L} y_{k,l} + h_k \sum_{i=1}^v a_{ij}(h_k L) f(t_{k,l}^j, Y_{k,l}^j), & k \in \mathbb{N}, i = 1, 2, \dots, v, \\ y_{k,l+1} = e^{h_k L} y_{k,l} + h_k \sum_{i=1}^v b_i(h_k L) f(t_{k,l}^i, Y_{k,l}^i), & l = 0, 1, 2, \dots, m - 1, \\ y_{k+1,0} = I_{k+1}(y_{k,m}), \\ y_{0,0} = y_0. \end{cases} \quad (4.2)$$

Definition 4.1 The exponential RK methods (4.1) (or RK methods) for impulsive differential equation (1.1) is said to be

1. stable, if $\exists M > 0, m \geq M, h_k = \frac{\tau_{k+1} - \tau_k}{m}, k \in \mathbb{N}$,
 - (i) $I - zA$ is invertible for all $z = \alpha h_k$,
 - (ii) for an arbitrary $\epsilon > 0$, there exists such a positive number $\delta = \delta(\epsilon)$ that, for the numerical solutions of (4.2), $\|x_0 - y_0\| < \delta$ implies

$$\max_{1 \leq l \leq m} \|x_{k,l} - y_{k,l}\| < \epsilon, \quad \forall k \in \mathbb{N},$$

2. asymptotically stable, if it is stable and if there exists such a positive number δ_0 that $\|x_0 - y_0\| < \delta_1$ implies

$$\lim_{k \rightarrow \infty} \max_{1 \leq l \leq m} \|x_{k,l} - y_{k,l}\| = 0.$$

Theorem 4.2 (Asymptotically stable) *If the explicit exponential RK method (4.1) satisfies $\mathbf{S}(\alpha, \beta) \leq e^{\alpha+\beta}$, where*

$$\mathbf{S}(\alpha, \beta) = e^\alpha + \sum_{k=0}^{v-1} \beta^{k+1} \mathbf{b}(\alpha) \mathbf{A}(\alpha)^k e^{c\alpha} e, \quad \alpha \in \mathbb{R}, \beta \geq 0,$$

then the explicit exponential RK method (4.1) for (1.1) is asymptotically stable, for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$.

Proof Define

$$\begin{aligned} \delta_{k,l} &:= x_{k,l} - y_{k,l}, \quad k = 0, 1, 2, \dots, \quad l = 0, 1, \dots, m, \\ \Delta_{k,l}^i &:= X_{k,l}^i - Y_{k,l}^i, \quad k = 0, 1, 2, \dots, \quad l = 0, 1, \dots, m, \quad i = 1, 2, \dots, v. \end{aligned}$$

Hence, we have

$$\begin{aligned} \delta_{k,l} &= e^{h_k L} \delta_{k,l-1} + h_k \sum_{i=1}^v b_i(h_k L) (f(t_{k,l}^i, X_{k,l}^i) - f(t_{k,l}^i, Y_{k,l}^i)), \\ \Delta_{k,l}^i &= e^{c_i h_k L} \delta_{k,l-1} + h_k \sum_{j=1}^{i-1} a_{ij}(h_k L) (f(t_{k,l}^i, X_{k,l}^j) - f(t_{k,l}^i, Y_{k,l}^j)), \\ \delta_{k+1,0} &= I_{k+1}(x_{k,m}) - I_{k+1}(y_{k,m}), \end{aligned}$$

which imply that

$$\begin{aligned} \|\delta_{k,l}\| &\leq e^{\mu(h_k L)} \|\delta_{k,l-1}\| + h_k \lambda \sum_{i=1}^v \|b_i(h_k L)\| \|\Delta_{k,l}^i\|, \\ \|\Delta_{k,l}^i\| &\leq e^{c_i \mu(h_k L)} \|\delta_{k,l-1}\| + h_k \lambda \sum_{j=1}^{i-1} \|a_{ij}(h_k L)\| \|\Delta_{k,l}^j\|, \\ \|\delta_{k+1,0}\| &= \|I_{k+1}(x_{k,m}) - I_{k+1}(y_{k,m})\| \leq \gamma_{k+1} \|\delta_{k,m}\|, \end{aligned}$$

where λ and γ_k are the Lipschitz constants of f and I_k , respectively, $\Delta_{k,l}$ is the $v \times 1$ -vector of components $\|\Delta_{k,l}^i\|$, $i = 1, 2, \dots, v$. Since the function \mathbf{b}_i and \mathbf{a}_{ij} , $i, j = 1, 2, \dots, v$, are nondecreasing, we can obtain that

$$\|\delta_{k,l+1}\| \leq \mathbf{S}(h_k \mu(L), h_k \lambda) \|\delta_{k,l}\| \leq e^{(\mu(L)+\lambda)h_k} \|\delta_{k,l}\|,$$

where $\mathbf{S}(\alpha, \beta) = e^\alpha + \sum_{k=0}^{v-1} \beta^{k+1} \mathbf{b}^T(\alpha) \mathbf{A}(\alpha)^k e^{c\alpha} e$, $\alpha \in \mathbb{R}$, $\beta \geq 0$.

Hence, for arbitrary $k = 0, 1, 2, \dots$ and $l = 0, 1, \dots, m$, we have

$$\|\delta_{k,l}\| \leq \|\delta_{k,0}\| e^{(\|L\|+\lambda)lh_k}.$$

Therefore, by the method of introduction and the condition (5.1), we can obtain that

$$\begin{aligned} \|\delta_{k,l}\| &\leq \|x_0 - y_0\| (\gamma_1 e^{(\|L\|+\lambda)(\tau_1-\tau_0)}) (\gamma_2 e^{(\|L\|+\lambda)(\tau_2-\tau_1)}) \dots (\gamma_k e^{(\|L\|+\lambda)(\tau_k-\tau_{k-1})}) e^{(\|L\|+\lambda)\hat{C}} \\ &\leq \|x_0 - y_0\| C^k e^{(\|L\|+\lambda)\hat{C}} \end{aligned}$$

which implies that the exponential RK method for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$. □

Along with the function \mathbf{S} , we consider also the function

$$S(\alpha, \beta) = e^\alpha + \sum_{k=0}^{v-1} \beta^{k+1} b^T(\alpha) A(\alpha)^k e^{c\alpha} e, \quad \alpha \in \mathbb{R}, \beta \geq 0.$$

Therefore, by Theorem 4.2, we immediately get the following corollary.

Corollary 4.3 *If the explicit exponential RK method (4.1) satisfies*

$$S(\alpha, \beta) \leq e^{\alpha+\beta} \tag{4.3}$$

and

$$b(\alpha) = \mathbf{b}(\alpha), \quad \text{and} \quad A(\alpha) = \mathbf{A}(\alpha), \tag{4.4}$$

then the explicit exponential RK method (4.1) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$.

For an IF method with an explicit underlying RK method (A, b, c) , we have

$$S(\alpha, \beta) = e^\alpha \left(1 + \sum_{k=0}^{v-1} \beta^{k+1} b^T A^k e \right), \quad \alpha, \beta \in \mathbb{R}.$$

Condition (4.3) holds if

$$b^T A^k e \leq \frac{1}{(k+1)!}, \quad k \geq \mathbf{k},$$

where $\mathbf{k} = \min\{k : k \in \mathbb{N}, b^T A^k e \neq \frac{1}{(k+1)!}\}$. On the other hand, the condition (4.4) holds if

$$0 = c_1 \leq c_2 \leq \dots \leq c_v \leq 1 \tag{4.5}$$

and

$$b_i \geq 0 \text{ and } a_{ij} \geq 0, \quad i = 1, \dots, i-1. \tag{4.6}$$

Hence, by Corollary 4.3, we can obtain the following corollary.

Corollary 4.4 *If the explicit exponential RK method (4.1) is IF method satisfies (4.5), (4.6) and*

$$b^T A^k e \leq \frac{1}{(k+1)!}, \quad k \geq \mathbf{k},$$

where $\mathbf{k} = \min\{k \in \{0, 1, 2, \dots\} | b^T A^k e \neq \frac{1}{(k+1)!}\}$, then the explicit exponential RK method (4.1) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$.

5 Stability of RK methods

In this section, stronger sufficient conditions for asymptotical stability of the exact solution of (1.1) are provided. Moreover, asymptotical stability of RK methods for (1.1) is studied under these conditions.

Since $\mu(L) \leq \|L\|$, by Theorem 2.2, we can obtain the following corollary.

Corollary 5.1 *Assume that there exists a positive constant \hat{C} such that $\tau_k - \tau_{k-1} \leq \hat{C}, k \in \mathbb{Z}^+$. The exact solution of (1.1) is asymptotically stable if there is a positive constant C such that*

$$\gamma_k e^{(\|L\|+\lambda)(\tau_{k+1}-\tau_k)} \leq C < 1, \tag{5.1}$$

for arbitrary $k \in \mathbb{Z}^+$.

5.1 Explicit RK method

Consider the explicit Euler method for (1.1):

$$\begin{cases} x_{k,l+1} = x_{k,l} + h_k L x_{k,l} + h_k f(t_{k,l}, x_{k,l}), & k \in \mathbb{N}, l = 1, 2, \dots, m - 1, \\ x_{k+1,0} = x_{k,m} + I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0. \end{cases} \tag{5.2}$$

Theorem 5.2 *Under the conditions of Corollary 5.1, the explicit Euler method (5.2) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}, k \in \mathbb{N}, \forall m \in \mathbb{Z}^+$.*

Proof For arbitrary $k = 0, 1, 2, \dots$ and $l = 0, 1, \dots, m - 1$, we can obtain

$$\begin{aligned} \|\delta_{k,l+1}\| &= \|x_{k,l} - y_{k,l} + h_k L(x_{k,l} - y_{k,l}) + h_k f(t_{k,l}, x_{k,l}) - h_k f(t_{k,l}, y_{k,l})\| \\ &\leq (1 + h_k \|L\|)\|\delta_{k,l}\| + h_k \|f(t_{k,l}, x_{k,l}) - f(t_{k,l}, y_{k,l})\| \\ &\leq (1 + \|L\|h_k + \lambda h_k)\|\delta_{k,l}\| \\ &\leq e^{(\|L\|+\lambda)h_k}\|\delta_{k,l}\|. \end{aligned}$$

Hence, for arbitrary $k = 0, 1, 2, \dots$ and $l = 0, 1, \dots, m$, we have

$$\|\delta_{k,l}\| \leq \|\delta_{k,0}\| \cdot e^{(\|L\|+\lambda)lh_k}.$$

Consequently, by the method of introduction and the condition (5.1), we can obtain that

$$\begin{aligned} \|\delta_{k,l}\| &\leq \|x_0 - y_0\| \cdot (\gamma_1 e^{(\|L\|+\lambda)(\tau_1-\tau_0)}) \dots (\gamma_k e^{(\|L\|+\lambda)(\tau_k-\tau_{k-1})}) e^{(\|L\|+\lambda)lh_k} \\ &\leq \|x_0 - y_0\| C^k e^{(\|L\|+\lambda)\hat{C}k} \end{aligned}$$

which implies that the explicit Euler method (5.2) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}, k \in \mathbb{N}, \forall m \in \mathbb{Z}^+$. □

The two-stage explicit Runge–Kutta methods have a Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ c_2 & a_{21} & 0 \\ \hline & b_1 & b_2 \end{array}$$

For brevity, the proofs of the following two theorems are omitted.

Theorem 5.3 *Assume that all the coefficients of two-stage explicit Runge–Kutta methods are nonnegative, that is $a_{21} \geq 0, b_1 \geq 0$ and $b_2 \geq 0$, and*

$$b_1 + b_2 = 1, \quad b_2 c_2 = \frac{1}{2}.$$

Then the 2-stage explicit Runge–Kutta method for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}, k \in \mathbb{N}, \forall m \in \mathbb{Z}^+$, under the conditions of Corollary 5.1.

The following two-stage second-order explicit Runge–Kutta methods all satisfy the conditions of Theorem 5.3.

$\begin{array}{c cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$	$\begin{array}{c cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$	$\begin{array}{c cc} 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & 0 \\ \hline & \frac{1}{3} & \frac{2}{3} \end{array}$
Modified Euler Method	Heun’s Method, order 2	Ralston’s Method

The three-stage explicit Runge–Kutta methods have a Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 \\ \hline & b_1 & b_2 & b_3 \end{array}$$

Theorem 5.4 *Under the conditions of Corollary 5.1 and all the coefficients of explicit Runge–Kutta methods are nonnegative ($a_{ij} \geq 0$ and $b_i \geq 0, 1 \leq i, j \leq 3$), the three-stage third-order explicit Runge–Kutta methods for (1.1) are asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}, k \in \mathbb{N}, \forall m \in \mathbb{Z}^+$.*

The following three-stage third-order explicit Runge–Kutta methods all satisfy the conditions of Theorem 5.4.

$\begin{array}{c ccc} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$	$\begin{array}{c ccc} 0 & 0 & 0 & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$
Heun’s Method, order 3	Runge – Kutta Method, order 3

Consider the explicit four-stage RK methods for (1.1):

$$\left\{ \begin{array}{l} X_{k,l}^1 = x_{k,l}, \quad k \in \mathbb{N}, l = 1, 2, \dots, m - 1, \\ X_{k,l}^2 = x_{k,l} + h_k a_{21} (LX_{k,l}^1 + f(t_{k,l}^1, X_{k,l}^1)), \\ X_{k,l}^3 = x_{k,l} + h_k \sum_{j=1}^2 a_{3j} (LX_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j)), \\ X_{k,l}^4 = x_{k,l} + h_k \sum_{j=1}^3 a_{4j} (LX_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j)), \\ x_{k,l+1} = x_{k,l} + h_k \sum_{i=1}^4 b_i (LX_{k,l}^i + f(t_{k,l}^i, X_{k,l}^i)), \\ x_{k+1,0} = x_{k,m} + I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0. \end{array} \right. \tag{5.3}$$

Theorem 5.5 *Assume that all the coefficients of explicit Runge–Kutta method (5.3) are non-negative ($a_{ij} \geq 0$ and $b_i \geq 0, 1 \leq i, j \leq 4$) and*

$$\sum_i b_i = b_1 + b_2 + b_3 + b_4 = 1, \tag{5.4}$$

$$\sum_i b_i c_i = b_2 c_2 + b_3 c_3 + b_4 c_4 = \frac{1}{2}, \tag{5.5}$$

$$\sum_{i,j} b_i a_{ij} c_j = b_3 a_{32} c_2 + b_4 a_{42} c_2 + b_4 a_{43} c_3 = \frac{1}{6}, \tag{5.6}$$

$$\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = b_4 a_{43} a_{32} c_2 = \frac{1}{24}. \tag{5.7}$$

Then, under the conditions Corollary 5.1, the four-stage explicit Runge–Kutta method (5.3) for (1.1) is asymptotically stable for $h_k = \frac{t_{k+1} - t_k}{m}$, $k \in \mathbb{N}$, $\forall m \in \mathbb{Z}^+$.

Proof Obviously, for arbitrary $k = 0, 1, 2, \dots$ and $l = 0, 1, \dots, m - 1$,

$$\|\Delta_{k,l}^1\| = \|\delta_{k,l}\|. \tag{5.8}$$

Due to the condition (5.1), $a_{21} \geq 0$ and $c_2 = a_{21}$, we have

$$\begin{aligned} \|\Delta_{k,l}^2\| &= \|x_{k,l} - y_{k,l} + h_k a_{21} (LX_{k,l}^1 - LY_{k,l}^1 + f(t_{k,l}^1, X_{k,l}^1) - f(t_{k,l}^1, Y_{k,l}^1))\| \\ &\leq (1 + \|L\| h_k a_{21} + \lambda h_k a_{21}) \|\delta_{k,l}\| \\ &= (1 + \|L\| h_k c_2 + \lambda h_k c_2) \|\delta_{k,l}\|. \end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned} \|\Delta_{k,l}^3\| &= \|x_{k,l} - y_{k,l} + h_k \sum_{j=1}^2 a_{3j} (LX_{k,l}^j - LY_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j))\| \\ &\leq \|\delta_{k,l}\| + (\|L\| + \lambda) h_k a_{31} \|\Delta_{k,l}^1\| + (\|L\| + \lambda) h_k a_{32} \|\Delta_{k,l}^2\| \\ &\leq (1 + (\|L\| + \lambda) h_k (a_{31} + a_{32}) + (\|L\| + \lambda)^2 h_k^2 a_{32} c_2) \|\delta_{k,l}\| \\ &= (1 + (\|L\| + \lambda) h_k c_3 + (\|L\| + \lambda)^2 h_k^2 a_{32} c_2) \|\delta_{k,l}\| \end{aligned}$$

and

$$\begin{aligned} \|\Delta_{k,l}^4\| &= \|x_{k,l} - y_{k,l} + h_k \sum_{j=1}^3 a_{4j} (LX_{k,l}^j - LY_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j))\| \\ &\leq \|\delta_{k,l}\| + h_k (\|L\| + \lambda) \sum_{j=1}^3 a_{4j} \|\Delta_{k,l}^j\| \\ &\leq (1 + (\|L\| + \lambda) h_k c_4 + (\|L\| + \lambda)^2 h_k^2 (a_{42} c_2 + a_{43} c_3) \\ &\quad + (\|L\| + \lambda)^3 h_k^3 a_{43} a_{32} c_2) \|\delta_{k,l}\|. \end{aligned}$$

By the conditions (5.4)–(5.7), the equality (5.8) and the above three inequalities, we can obtain

$$\begin{aligned} \|\delta_{k,l+1}\| &= \|x_{k,l} - y_{k,l} + h_k \sum_{i=1}^4 b_i (LX_{k,l}^i - LY_{k,l}^i + f(t_{k,l}^i, X_{k,l}^i) - f(t_{k,l}^i, Y_{k,l}^i))\| \\ &\leq \|\delta_{k,l}\| + h_k (\|L\| + \lambda) \sum_{i=1}^4 b_i \|\Delta_{k,l}^i\| \\ &\leq (1 + h_k (\|L\| + \lambda) (b_1 + b_2 + b_3 + b_4) + h_k^2 (\|L\| + \lambda)^2 (b_2 c_2 + b_3 c_3 + b_4 c_4) \\ &\quad + h_k^3 (\|L\| + \lambda)^3 (b_3 a_{32} c_2 + b_4 a_{42} c_2 + b_4 a_{43} c_3) + h_k^4 (\|L\| + \lambda)^4 a_{43} a_{32} c_2) \cdot \|\delta_{k,l}\| \\ &= (1 + (\|L\| + \lambda) h_k + \frac{(\|L\| + \lambda)^2 h_k^2}{2} + \frac{(\|L\| + \lambda)^3 h_k^3}{3!} + \frac{(\|L\| + \lambda)^4 h_k^4}{4!}) \cdot \|\delta_{k,l}\| \\ &\leq e^{(\|L\| + \lambda) h_k} \|\delta_{k,l}\|. \end{aligned}$$

Consequently, the four-stage explicit Runge–Kutta method (5.3) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $\forall m \in \mathbb{Z}^+$. □

Obviously, all the four-stage fourth-order explicit Runge–Kutta methods satisfy conditions (5.4)–(5.7) (see (Butcher 2003, p. 90) or (Hairer et al. 1993b, pp. 135–136)). Consequently, by Theorem 5.5, we immediately obtain the following corollary.

Corollary 5.6 *Assume that all the coefficients of explicit Runge–Kutta method (5.3) are non-negative ($a_{ij} \geq 0$ and $b_i \geq 0$, $1 \leq i, j \leq 4$). Then, under the conditions Corollary 5.1, the four-stage fourth-order explicit Runge–Kutta method (5.3) for (1.1) is stable for $h_k = \frac{\tau_{k+1}-\tau_k}{m}$, $k \in \mathbb{N}$, $\forall m \in \mathbb{Z}^+$.*

The four-stage explicit Runge–Kutta methods have a Butcher tableau

$$\begin{array}{c|cccc}
 0 & 0 & 0 & 0 & 0 \\
 c_2 & a_{21} & 0 & 0 & 0 \\
 c_3 & a_{31} & a_{32} & 0 & 0 \\
 c_4 & a_{41} & a_{42} & a_{43} & 0 \\
 \hline
 & b_1 & b_2 & b_3 & b_4
 \end{array}$$

The following classical four-stage fourth-order explicit Runge–Kutta method satisfies the conditions of Theorem 5.5 and Corollary 5.6.

$$\begin{array}{c|cccc}
 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
 1 & 0 & 0 & 1 & 0 \\
 \hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
 \end{array}$$

Remark 5.7 In fact, there are many four-stage explicit Runge–Kutta methods which are not of order 4, but they satisfy conditions (5.4)–(5.7). The following method is an example which is not of order 4, however, it satisfies not only conditions (5.4)–(5.7), but also the conditions of Theorem 5.5.

$$\begin{array}{c|cccc}
 0 & 0 & 0 & 0 & 0 \\
 \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
 \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
 \hline
 & \frac{1}{2} & 0 & 0 & \frac{1}{2}
 \end{array}$$

Unfortunately, we can not obtain the p -stage explicit Runge–Kutta methods of order p for $p \geq 5$, because of the Butcher Barriers as follows.

Theorem 5.8 (See Butcher 2003, Theorem 370, pp. 258–259) or (Hairer et al. 1993b, Theorem 5.1 p. 173) *For $p \geq 5$, no explicit Runge–Kutta methods exist of order p with $s = p$ stages.*

5.2 RK methods

Consider the following RK methods for (1.1):

$$\begin{cases} x_{k,l+1} = x_{k,l} + h \sum_{i=1}^v b_i (LY_{k,l}^i + f(t_{k,l}^i, X_{k,l+1}^i)), & k \in \mathbb{N}, l = 0, 1, \dots, m - 1, \\ X_{k,l+1}^i = x_{k,l} + h \sum_{j=1}^v a_{ij} (LX_{k,l}^j + f(t_{k,l}^j, X_{k,l+1}^j)), & k \in \mathbb{N}, l = 0, 1, \dots, m - 1, \\ x_{k+1,0} = I_{k+1}(x_{k,m}), & k \in \mathbb{N}, \\ x_{0,0} = x_0. \end{cases} \tag{5.9}$$

Lemma 5.9 (Butcher 1987; Dekker and Verwer 1984; Hairer et al. 1993b; Wanner et al. 1978) *The (j, k)-Padé approximation to e^z is given by*

$$R(z) = \frac{P_j(z)}{Q_k(z)}, \tag{5.10}$$

where

$$\begin{aligned} P_j(z) &= 1 + \frac{j}{j+k} \cdot z + \frac{j(j-1)}{(j+k)(j+k-1)} \cdot \frac{z^2}{2!} + \dots + \frac{j!k!}{(j+k)!} \cdot \frac{z^j}{j!}, \\ Q_k(z) &= 1 - \frac{k}{j+k} \cdot z + \frac{k(k-1)}{(j+k)(j+k-1)} \cdot \frac{z^2}{2!} + \dots + (-1)^k \cdot \frac{k!j!}{(j+k)!} \cdot \frac{z^k}{k!}, \end{aligned}$$

with error

$$e^z - R(z) = (-1)^k \cdot \frac{j!k!}{(j+k)!(j+k+1)!} \cdot z^{j+k+1} + O(z^{j+k+2}).$$

It is the unique rational approximation to e^z of order j + k, such that the degrees of numerator and denominator are j and k, respectively.

Lemma 5.10 (Song et al. 2005; Yang et al. 2005; Wang and Qiu 2014) *Assume that R(z) is the (j, k)-Padé approximation to e^z. Then R(z) < e^z for all z > 0 if and only if k is even, when z > 0.*

Theorem 5.11 *Assume that R(z) is the stability function of Runge–Kutta method (5.9) i.e.*

$$R(z) = 1 + zb^T(I - zA)^{-1}e = \frac{P_j(z)}{Q_k(z)}.$$

Under the conditions of Corollary 5.1, Runge–Kutta method (4.1) with nonnegative coefficients (a_{ij} ≥ 0 and b_i ≥ 0, 1 ≤ i ≤ v, 1 ≤ j ≤ v) for (1.1) is asymptotically stable for h_k = $\frac{\tau_{k+1} - \tau_k}{m}$, k ∈ ℕ, m ∈ ℤ⁺ and m ≥ M, if k is even, where M = inf{m : I - zA is invertible and (I - z_kA)⁻¹e ≥ 0, z_k = (||L|| + λ)h_k, k ∈ ℕ, m ∈ ℤ⁺}.

Proof Since a_{ij} ≥ 0 and b_i ≥ 0, 1 ≤ i ≤ s, 1 ≤ j ≤ v, we can obtain that

$$\begin{aligned} \|\Delta_{k,l}^i\| &= \|x_{k,l} - y_{k,l} + h_k \sum_{j=1}^v a_{ij} (LX_{k,l}^j - LY_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j))\| \\ &\leq \|\delta_{k,l}\| + h_k \sum_{j=1}^v a_{ij} (\|L\Delta_{k,l}^j\| + \|f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)\|) \end{aligned}$$

$$\leq \|\delta_{k,l}\| + (\|L\| + \lambda)h_k \sum_{j=1}^v a_{ij} \|\Delta_{k,l}^j\|.$$

And when $m \geq M$, $(I - z_k A)^{-1}e \geq 0$, $z_k = (\|L\| + \lambda)h_k$, $k \in \mathbb{Z}^+$, so

$$[\|\Delta_{k,l}^i\|] \leq (I - z_k A)^{-1}e \|\delta_{k,l}\|$$

where $[\|\Delta_{k,l}^i\|] = (\|\Delta_{k,l}^1\|, \|\Delta_{k,l}^2\|, \dots, \|\Delta_{k,l}^v\|)^T$. By Lemmas 5.9 and 5.10, we can obtain

$$\begin{aligned} \|\delta_{k,l+1}\| &= \|x_{k,l} - y_{k,l} + h_k \sum_{j=1}^v b_j (LX_{k,l}^j - LY_{k,l}^j + f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j))\| \\ &\leq \|\delta_{k,l}\| + h_k \sum_{j=1}^v b_j (\|LX_{k,l}^j - LY_{k,l}^j\| + \|f(t_{k,l}^j, X_{k,l}^j) - f(t_{k,l}^j, Y_{k,l}^j)\|) \\ &\leq \|\delta_{k,l}\| + (\|L\| + \lambda)h_k \sum_{j=1}^v b_j \|\Delta_{k,l}^j\| \\ &= \|\delta_{k,l}\| + z_k b^T [\|\Delta_{k,l}^i\|] \\ &\leq (1 + z_k b^T (I - z_k A)^{-1}e) \|\delta_{k,l}\| \\ &= \mathbf{R}(\|L\| + \lambda)h_k \|\delta_{k,l}\| \\ &\leq e^{(\|L\| + \lambda)h_k} \|\delta_{k,l}\|. \end{aligned}$$

Therefore, RK method (5.9) for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$. □

5.3 θ -methods

Consider θ -methods for (1.1):

$$\begin{cases} x_{k,l+1} = x_{k,l} + h_k(1 - \theta)(Lx_{k,l} + f(t_{k,l}, x_{k,l})) \\ \quad + h_k\theta(Lx_{k,l+1} + f(t_{k,l+1}, x_{k,l+1})) \\ x_{k+1,0} = I_{k+1}(x_{k,m}), \\ x_{0,0} = x_0, \end{cases} \tag{5.11}$$

where $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $m \geq 1$, m is an integer, $k = 0, 1, 2, \dots$

Lemma 5.12 (See Song et al. 2005) *For $m > \|L\| + \lambda$ and $z_k = h_k(\|L\| + \lambda)$, $h = \frac{\tau_k - \tau_{k-1}}{m}$, $m \in \mathbb{Z}^+, k \in \mathbb{N}$, then*

$$\left(1 + \frac{z_k}{1 - z_k\theta}\right)^m \leq e^{h_k(\|L\| + \lambda)}$$

if and only if $0 \leq \theta \leq \varphi(1)$, where $\varphi(x) = \frac{1}{x} - \frac{1}{e^x - 1}$.

Theorem 5.13 *Under the conditions of Corollary 5.1, if $0 \leq \theta \leq \varphi(1)$, there is a positive M such that θ method for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m \geq M$.*

Proof Obviously, we can obtain

$$\begin{aligned} \|\delta_{k,l+1}\| &= \|\delta_{k,l} + (1 - \theta)h_k(Lx_{k,l} - Ly_{k,l} + f(t_{k,l}, x_{k,l}) - f(t_{k,l}, y_{k,l})) \\ &\quad + \theta h_k(Lx_{k,l+1} - Ly_{k,l+1} + f(t_{k,l+1}, x_{k,l+1}) - f(t_{k,l+1}, y_{k,l+1}))\| \\ &\leq (1 + (1 - \theta)(\|L\| + \lambda)h_k)\|\delta_{k,l}\| + \theta h_k\|L\|\|\delta_{k,l+1}\| \\ &\quad + h_k\theta\|f(t_{k,l+1}, x_{k,l+1}) - f(t_{k,l+1}, y_{k,l+1})\| \\ &\leq (1 + (1 - \theta)(\|L\| + \lambda)h_k)\|\delta_{k,l}\| + \theta(\|L\| + \lambda)h_k\|\delta_{k,l+1}\| \end{aligned}$$

which implies

$$\|\delta_{k,l+1}\| \leq \frac{1 + (1 - \theta)(\|L\| + \lambda)h_k}{1 - \theta(\|L\| + \lambda)h_k} \cdot \|\delta_{k,l}\|$$

Therefore, by Lemma 5.12 and the method of introduction, we can obtain that

$$\|\delta_{k,l+1}\| \leq e^{(\|L\| + \lambda)h_k} \|\delta_{k,l}\|$$

Hence, θ -method for (1.1) is asymptotically stable for $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$ and $m > \theta(\|L\| + \lambda)(\tau_{k+1} - \tau_k)$, $k \in \mathbb{N}$, if $0 \leq \theta \leq \varphi(1)$. \square

6 Conclusions

There are three advantages of a class of explicit exponential RK methods (which satisfy the conditions Theorem 4.2 or Corollary 4.4) for semi-linear IDEs. First, they can preserve asymptotical stability of semi-linear IDEs under the conditions of Theorem 2.2 which are weaker than the conditions of Corollary 5.1. Second, they can preserve asymptotical stability semi-linear IDEs without restriction on the stepsizes except that $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$. Third, they are suitable for stiff problems.

On the other hand, classical four-order explicit RK method preserves asymptotical stability of semi-linear IDEs without restriction on the stepsizes except that $h_k = \frac{\tau_{k+1} - \tau_k}{m}$, $k \in \mathbb{N}$, $m \in \mathbb{Z}^+$, under the conditions of Corollary 5.1. And it does not need to compute $e^{h_k L}$ and $e^{c_i h_k L}$, $i = 1, 2, \dots, v$, $k \in \mathbb{N}$.

References

- Abada N, Benchohra M, Hammouche H (2009) Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions. *J Differ Equ* 246:3834–3863
- Bainov DD, Simeonov PS (1989) *Systems with impulsive effect: stability. Theory and applications*, Ellis Horwood, Chichester
- Bainov DD, Simeonov PS (1995) *Impulsive differential equations: asymptotic properties of the solutions*. World Scientific, Singapore
- Benchohra M, Henderson J, Ntouyas S (2006) *Impulsive differential equations and inclusions*. Hindawi Publishing Corporation, London
- Butcher JC (1987) *The numerical analysis of ordinary differential equations: Runge-Kutta and general linear methods*. Wiley, New York
- Butcher JC (2003) *Numerical methods for ordinary differential equations*. Wiley, Chichester
- Calvo M, González-pino S, Montijano JI, (2000) Runge–Kutta methods for the numerical solution of stiff semilinear systems. *BIT* 40:611–639
- Cardinali T, Rubbioni P (2008) Impulsive semilinear differential inclusion: topological structure of the solution set and solutions on non-compact domains. *Nonlinear Anal.* 14:73–84
- Dekker K, Verwer JG (1984) *Stability of Runge-Kutta methods for stiff nonlinear differential equations*. North-Holland, Amsterdam

- Ding XH, Wu KN, Liu MZ (2006) Convergence and stability of the semi-implicit Euler method for linear stochastic delay integro-differential equations. *Int J Comput Math* 83:753–763
- Ding XH, Wu KN, Liu MZ (2010) The Euler scheme and its convergence for impulsive delay differential equations. *Appl Math Comput* 216:1566–1570
- Fan Z (2010) Impulsive problems for semilinear differential equations with nonlocal conditions. *Nonlinear Anal* 72:1104–1109
- Friedli A (1978) Verallgemeinerte Runge-Kutta Verfahren zur Lösung steifer Differentialgleichungssysteme. Numerical treatment of differential equations. *Lect Notes Math* 631:35–50
- Fan Z, Li G (2010) Existence results for semilinear differential equations with nonlocal and impulsive conditions. *J Funct Anal* 258:1709–1727
- Hairer E, Nørsett SP, Wanner G (1993) Solving ordinary differential equations I. Nonstiff problems, Springer, New York
- Hairer E, Nørsett SP, Wanner G (1993) Solving ordinary differential equations II. Stiff and differential algebraic problems, Springer, New York
- Lakshmikantham V, Bainov DD, Simeonov PS (1989) Theory of impulsive differential equations. World Scientific, Singapore
- Lawson J (1967) Generalized Runge-Kutta processes for stable systems with large Lipschitz constants. *SIAM J Numer Anal* 4:372–380
- Liang J, Liu JH, Xiao TJ (2009) Nonlocal impulsive problems for nonlinear differential equations in Banach spaces. *Math Comput Model* 49:798–804
- Liang H, Song MH, Liu MZ (2011) Stability of the analytic and numerical solutions for impulsive differential equations. *Appl Numer Math* 61:1103–1113
- Liang H, Liu MZ, Song MH (2014) Extinction and permanence of the numerical solution of a two-prey one-predator system with impulsive effect. *Int J Comput Math* 88:1305–1325
- Liu X, Zeng YM (2018) Linear multistep methods for impulsive delay differential equations. *Appl Math Comput* 321:555–563
- Liu MZ, Liang H, Yang ZW (2007) Stability of Runge-Kutta methods in the numerical solution of linear impulsive differential equations. *Appl Math Comput* 192:346–357
- Liu X, Song MH, Liu MZ (2012) Linear multistep methods for impulsive differential equations. *Discrete Dyn Nat Soc*. <https://doi.org/10.1155/2012/652928> (article ID 652928)
- Liu X, Zhang GL, Liu MZ (2014) Analytic and numerical exponential asymptotic stability of nonlinear impulsive differential equations. *Appl Numer Math* 81:40–49
- Maset S, Zennaro M (2009) Unconditional stability of explicit exponential Runge-Kutta methods for semilinear ordinary differential equations. *Math Comput* 78:957–967
- Mil'man VD, Myshkis AD (1960) On the stability of motion in the presence of impulses. *Sib Math J* 1:233–237
- Ostermann A, Daele M (2007) Positivity of exponential Runge-Kutta methods. *BIT* 47:419–426
- Ostermann A, Thalhammer M, Wright WM (2006) A class of explicit exponential general linear methods. *BIT* 46:409–431
- Ran XJ, Liu MZ, Zhu QY (2008) Numerical methods for impulsive differential equation. *Math Comput Model* 48:46–55
- Samoilenko AM, Perestyuk NA, Chapovsky Y (1995) Impulsive differential equations. World Scientific, Singapore
- Song MH, Yang ZW, Liu MZ (2005) Stability of θ -methods for advanced differential equations with piecewise continuous arguments. *Comput Math Appl* 49:1295–1301
- Wang Q, Qiu S (2014) Oscillation of numerical solution in the Runge-Kutta methods for equation $x'(t) = ax(t) + a_0x([t])$. *Acta Math Appl Sin (English series)* 30:943–950
- Wanner G, Hairer E, Nørsett SP, (1978) Order stars and stability theorems. *BIT* 18:475–489
- Wen LP, Yu YX (2011) The analytic and numerical stability of stiff impulsive differential equations in Banach space. *Appl Math Lett* 24:1751–1757
- Wu KN, Ding XH (2012) Impulsive stabilization of delay difference equations and its application in Nicholson's blowflies model. *Adv Differ Equ*. <https://doi.org/10.1186/1687-1847-2012-88>
- Wu KN, Ding XH (2012) Stability and stabilization of impulsive stochastic delay differential equations. *Math Probl Eng*. <https://doi.org/10.1155/2012/176375>
- Wu KN, Ding XH, Wang LM (2010) Stability and stabilization of impulsive stochastic delay difference equations. *Discrete Dyn Nat Soc*. <https://doi.org/10.1155/2010/592036>
- Wu KN, Liu XZ, Yang BQ, Lim CC (2019) Mean square finite-time synchronization of impulsive stochastic delay reaction-diffusion systems. *Commun Nonlinear Sci Numer Simul* 79:104899
- Wu KN, Na MY, Wang LM, Ding XH, Wu BY (2019) Finite-time stability of impulsive reaction-diffusion systems with and without time delay. *Appl Math Comput* 363:124591

- Yang ZW, Liu MZ, Song MH (2005) Stability of Runge-Kutta methods in the numerical solution of equation $u'(t) = au(t) + a_0u([t]) + a_1u([t - 1])$. *Appl Math Comput* 162:37–50
- Zhang ZH, Liang H (2014) Collocation methods for impulsive differential equations. *Appl Math Comput* 228:336–348

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