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New numerical studies for Darcy's problem coupled with the heat equation

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Abstract
In this article, we consider the heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature. We recall two numerical schemes and introduce a new non-stabilized one, we show the existence and uniqueness of the solutions and we establish an a priori error estimates using the Brezzi–Rappaz–Raviart theorem. Numerical investigations are preformed and showed.

Keywords Darcy's equations · Heat equation · Brezzi–Rappaz–Raviart theorem · Finite element method · A priori error estimates

Mathematics Subject Classification 35K05 · 35B45 · 74S05 · 76M10

1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded simply connected open domain, with a Lipschitz-continuous boundary Γ. This work treats the temperature distribution of a fluid in a porous medium modelled by a convection–diffusion equation coupled with Darcy's law. The system of equations is the following:

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$$
(\mathrm{P})\begin{cases}
\nu(T(\mathbf{x}))\mathbf{u}(\mathbf{x})+\nabla p(\mathbf{x}) & =\mathbf{f}(\mathbf{x}) \quad \text{in } \Omega, \\
(\operatorname{div}\mathbf{u})(\mathbf{x}) & =0 \quad \text{in } \Omega, \\
-\alpha\Delta T(\mathbf{x})+(\mathbf{u}\cdot\nabla T)(\mathbf{x}) & =g(\mathbf{x}) \quad \text{in } \Omega, \\
(\mathbf{u}\cdot\mathbf{n})(\mathbf{x}) & =0 \quad \text{on } \Gamma, \\
T(\mathbf{x}) & =0 \quad \text{on } \Gamma,
\end{cases}
$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. The unknowns are the velocity $\mathbf{u}$, the pressure p and the temperature T of the fluid. The function $\mathbf{f}$ represents an external density force and g an external heat source. The viscosity ν depends on the temperature (Hooman and Gurgenci 2007 or Rashad 2014) while the parameter α is a positive constant that corresponds to the diffusion coefficient.

The heat equation coupled with the Navier–Stokes system has been treated by many works (see for instance Bernardi et al. 1995; Deteix et al. 2014, or Gaultier and Lezaun 1989). The coupling of Darcy's system with the heat equation where the viscosity is constant but the exterior force depends on the temperature has been analyzed by Bernardi et al. (2016) or Boussinesq (1903) and discretized with a spectral method. For the time-dependent convection–diffusion-reaction equation coupled with Darcy's law, we can refer to Feng (1995), Chen and Ewing (1999), Beatrice et al. (2011) and Jizhou et al. (2015).

In Bernardi et al. (2018), we study theoretically and numerically the system (P) which corresponds to the heat equation coupled with Darcy's law by a nonlinear viscosity depending on the temperature. We propose and analyze two numerical schemes (called $(V_{h,1})$ and $(V_{h,2})$) based on finite element methods. The discrete formulation $(V_{h,2})$ is stabilised by the term $\in \frac{1}{2}(\mathrm{div}\mathbf{u}_h\, T_h, S_h)$. For each discrete formulation, existence of a solution is derived without restriction on the data by Galerkin's method and Brouwer's Fixed Point and global uniqueness is established when the solution is slightly smoother and the data are suitably restricted. We also derive an optimal a priori error estimate for each numerical scheme under the smallness condition of the data, study the convergence of the successive approximation algorithm and finally show numerical investigations for $d = 2$.

In this work, we study the same coupled problem, we consider the scheme $(V_{h,1})$ and we introduce a new numerical scheme called $(V_{a,h})$, which is similar to $(V_{h,2})$, but without the stabilized term $\frac{1}{2}(\mathrm{div}\mathbf{u}_h\, T_h, S_h)$. We apply Brezzi–Rappaz–Raviart theorem to conclude the existence, the uniqueness and the a priori error estimates for all the discrete schemes. In fact, we show the details of the proofs for $(V_{a,h})$ and for $(V_{h,1})$, it is a simple consequence with slight modifications. The main differences between this work and Bernardi et al. (2018) are that we show in this paper the existence and uniqueness of the solution, and the a priori error estimate without the smallness condition on the exact and numerical solutions, but when the mesh step h is smaller then a given positive real number h_0, which means that the numerical solution $(\mathbf{u}_h, p_h, T_h)$ is in a neighbourhood of the exact solution $(\mathbf{u}, p, T)$. Another advantage is about the numerical computation of the stabilized term, in $(V_{h,2})$, which is skipped in $(V_{a,h})$. Finally, we show in this paper numerical investigations corresponding to an iterative scheme associated with $(V_{a,h})$ and we compare with those introduced in Bernardi et al. (2018). In a future work, we will study the properties (existence and uniqueness of the solution, convergence, etc.) of the successive algorithm (Vhi) introduced in the last section of this paper. In fact, the difficulties of this studies are related to the omitted term of stabilisation.

This article is organized as follows:

- Section 2 is devoted to the analysis of the corresponding variational formulation.
- In Sect. 3, we introduce the discrete problems.

- In Sect. 4, we show the existence and the uniqueness of the discrete solutions. Hence, an a priori error estimate was also proved.
- Section 5 is devoted to numerical investigations.

2 Analysis of the model

2.1 Notation

Let $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_d)$ be a d-uple of non negative integers, set $|\alpha| = \sum_{i=1}^{d} \alpha_i$, and define the partial derivative ∂^α by

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.$$

Then, for any positive integer m and number $p \geq 1$, we recall the classical Sobolev space (Adams 1975 or Necas 1967)

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega);\ \forall\, |\alpha| \leq m,\ \partial^\alpha v \in L^p(\Omega)\},$$

equipped with the seminorm

$$|v|_{W^{m,p}(\Omega)} = \left\{ \sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^p \,\mathrm{d}\mathbf{x} \right\}^{\frac{1}{p}}$$

and the norm

$$\|v\|_{W^{m,p}(\Omega)} = \left\{ \sum_{0 \leq k \leq m} |v|^p_{W^{k,p}(\Omega)} \right\}^{\frac{1}{p}}.$$

When $p = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{v}$ be a vector valued function; we set

$$\|\mathbf{v}\|_{L^p(\Omega)^d} = \left(\int_\Omega |\mathbf{v}|^p \,\mathrm{d}\mathbf{x} \right)^{\frac{1}{p}},$$

where $|\cdot|$ denotes the Euclidean vector norm.

For vanishing boundary values, we define

$$H_0^1(\Omega) = \{v \in H^1(\Omega);\ v_{|\Gamma} = 0\}$$

and, for any integer $q \geq 2$,

$$W_0^{1,q}(\Omega) = \{v \in W^{1,q}(\Omega);\ v_{|\Gamma} = 0\}.$$

We shall often use the following Sobolev imbeddings: for any real number $p \geq 1$ when $d = 2$, or $1 \leq p \leq \frac{2d}{d-2}$ when $d \geq 3$, there exist constants S_p and S_p^0 such that

$$\forall\, v \in H^1(\Omega),\ \|v\|_{L^p(\Omega)} \leq S_p \|v\|_{H^1(\Omega)}$$

and

$$\forall\, v \in H_0^1(\Omega),\ \|v\|_{L^p(\Omega)} \leq S_p^0 |v|_{H^1(\Omega)}. \tag{2.1}$$

When $p = 2$, (2.1) reduces to Poincaré's inequality.

Recall the standard spaces for Darcy's equations

$$L^2_m(\Omega) = \left\{ v \in L^2(\Omega);\ \int_\Omega v \, d\mathbf{x} = 0 \right\}.$$

Finally, we recall the inf-sup condition between $H^1(\Omega) \cap L^2_m(\Omega)$ and $L^2(\Omega)^d$ (see for instance Proposition 1.19, Chap XIII in Bernardi et al. 2004),

$$\inf_{q \in H^1(\Omega) \cap L^2_m(\Omega)} \sup_{\mathbf{v} \in L^2(\Omega)^d} \frac{\int_\Omega \mathbf{v} \cdot \nabla q \, d\mathbf{x}}{\|\mathbf{v}\|_{L^2(\Omega)^d} |q|_{H^1(\Omega)}} \geq 1. \tag{2.2}$$

2.2 Variational formulation

We suppose that $\nu \in W^{1,\infty}(\Omega)$; then the function ν is Lipschitz-continuous with Lipschitz constant λ, i.e.,

$$\forall s, t \in \mathbb{R}, \quad |\nu(s) - \nu(t)| \leq \lambda |s - t|. \tag{2.3}$$

In addition, before introducing the variational formulation of (P), we precise the following assumption on the function ν:

Assumption 2.1 ν is bounded and there exist two positive constants ν_1 and ν_2 such that for any $s \in \mathbb{R}$

$$\nu_1 \leq \nu(s) \leq \nu_2. \tag{2.4}$$

We recall the following variational formulation introduced in Bernardi et al. (2018) equivalent to (P):

$$(V_a) \begin{cases} \text{Find } (\mathbf{u}, p, T) \in L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_m(\Omega)) \times H^1_0(\Omega) \text{ such that} \\ \forall \mathbf{v} \in L^2(\Omega)^d, \quad \int_\Omega \nu(T) \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega \nabla p \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \\ \forall q \in H^1(\Omega) \cap L^2_m(\Omega), \quad \int_\Omega \nabla q \cdot \mathbf{u} \, d\mathbf{x} = 0, \\ \forall S \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad \alpha \int_\Omega \nabla T \cdot \nabla S \, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla T) S \, d\mathbf{x} = \int_\Omega g \, S \, d\mathbf{x}. \end{cases}$$

We refer to Bernardi et al. (2018) for the equivalence between the variational formulation (V_a) and the problem (P), and for the existence and uniqueness of the solution of (V_a).

Remark 2.2 In Bernardi et al. (2018), we have introduced the following variational formulation:

$$(V) \begin{cases} \text{Find } (\mathbf{u}, p, T) \in H_0(\text{div}, \Omega)^d \times L^2_m(\Omega) \times H^1_0(\Omega) \text{ such that} \\ \forall \mathbf{v} \in H_0(\text{div}, \Omega)^d, \quad \int_\Omega \nu(T) \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} - \int_\Omega p (\text{div}(\mathbf{v})) \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \\ \forall q \in L^2_m(\Omega), \quad \int_\Omega q (\text{div}\, \mathbf{u}) \, d\mathbf{x} = 0, \\ \forall S \in H^1_0(\Omega) \cap L^\infty(\Omega), \quad \alpha \int_\Omega \nabla T \cdot \nabla S \, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \nabla T) S \, d\mathbf{x} = \int_\Omega g \, S \, d\mathbf{x}, \end{cases}$$

where

$$H(\text{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^d;\ \text{div}\, \mathbf{v} \in L^2(\Omega)\}, \tag{2.5}$$

$$H_0(\text{div}, \Omega) = \{\mathbf{v} \in H(\text{div}, \Omega);\ (\mathbf{v} \cdot \mathbf{n})|_\Gamma = 0\}, \tag{2.6}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{H(\mathrm{div},\Omega)} = \|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\mathrm{div}\,\mathbf{v}\|^2_{L^2(\Omega)}. \tag{2.7}$$

We refer to Bernardi et al. (2018) for the properties and the studies of (V).

2.3 Regularity of the solution

We are looking to establish a certain regularity for the solution of Problem (P).

Definition 2.3 The domain Ω is called of class $\mathcal{D}^q$, $2 < q < \infty$, if the equation

$$\Delta u = \mathrm{div}\,\mathbf{f}$$

admits a unique solution u in $W^{1,q}_0(\Omega)$, for all $\mathbf{f} \in L^q(\Omega)$, such that

$$\|u\|_q \leq K_q \|\mathbf{f}\|_q,$$

where K_q is a constant independent of $\mathbf{f}$.

Theorem 2.4 *We assume that Ω is of class $\mathcal{D}^q$ and of class $C^{1,1}$ or polygonal or polyhedral convex and that $\mathbf{f} \in L^\infty(\Omega)^d$. So the solution of Problem (P) satisfies the following regularity:*

$$(\mathbf{u}, p, T) \in (L^r(\Omega)^d, W^{1,r}(\Omega), H^1_0(\Omega) \cap W^{2,s}(\Omega)), \text{ where } r > 2, s = \frac{2r}{r+2} > 1.$$

Proof We first start by writing the first equation of Problem (P) as follows:

$$\mathbf{u} + \frac{\nabla p}{\nu(T)} = \frac{\mathbf{f}}{\nu(T)}.$$

As $\mathrm{div}\,\mathbf{u} = 0$, we get the equation

$$\mathrm{div}\left(\frac{\nabla p}{\nu(T)}\right) = \mathrm{div}\left(\frac{\mathbf{f}}{\nu(T)}\right),$$

where $\frac{1}{\nu_2} \leq \frac{1}{\nu(T)} \leq \frac{1}{\nu_1}$ (Assumption 2.1). We denote by $\mathbf{g} = \frac{\mathbf{f}}{\nu(T)}$, so $\mathbf{g} \in L^\infty(\Omega)^d$ as we have $\mathbf{f} \in L^\infty(\Omega)^d$. According to Meyers (1963), there exist a number $r > 2$ depending on ν_2, on the norm of $\frac{1}{\nu(T)}$ in $L^\infty(\Omega)$ and on the domain Ω and its dimension, such that $p \in W^{1,r}(\Omega)$ and $\mathbf{u} \in L^r(\Omega)^d$.

Since $\mathbf{u} \in L^r(\Omega)^d$ and $\nabla T \in L^2(\Omega)$, we get $\mathbf{u} \cdot \nabla T \in L^s(\Omega)$ where $s = \frac{2r}{r+2} > 1$. We deduce by using the heat equation of System (P) that $T \in W^{2,s}(\Omega)$, as Ω is of class $C^{1,1}$ or polygonal or polyhedral convex (see for instance Amrouche and Rodríguez-Bellido 2018). □

3 Discretization

Thus, we assume that Ω is a polygon when $d = 2$ or polyhedron when $d = 3$. So it can be completely meshed. Now, we describe the discretization in space. We consider a regular (see Ciarlet 1991) family of triangulations $(\mathcal{T}_h)_h$ of Ω which is a set of closed non degenerate triangles for $d = 2$ or tetrahedra for $d = 3$, called elements, satisfying,

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{T}_h$;
- the intersection of two distinct elements of $\mathcal{T}_h$ is either empty, a common vertex, or an entire common edge (or face when $d = 3$);
- the ratio of the diameter of an element K in $\mathcal{T}_h$ to the diameter of its inscribed circle when $d = 2$ or ball when $d = 3$ is bounded by a constant independent of h.

As usual, h denotes the maximal diameter of all elements of $\mathcal{T}_h$. For each K in $\mathcal{T}_h$, we denote by $\mathbb{P}_1(K)$ the space of restrictions to K of polynomials in d variables and total degree at most one and by h_K the diameter of K. In what follows, $c, c', C, C', c_1, \ldots$ stand for generic constants which may vary from line to line but are always independent of h.

We also use the following Inverse inequality: for any real number $p \geq 2$, there exists constant C_I such that for any polynomial function $\mathbf{v}_h$ on K

$$\|\mathbf{v}_h\|_{L^p(K)^d} \leq C_I\, h_K^{\frac{d}{p}-\frac{d}{2}} \|\mathbf{v}_h\|_{L^2(K)^d}. \tag{3.1}$$

To reach the Inverse inequality globally, we assume that there exists a positive constant b independent of h such that:

$$b\,h < h_K < h.$$

For a given triangulation $\mathcal{T}_h$, we define the following finite-dimensional spaces:

$$Z_h = \{S_h \in \mathcal{C}^0(\bar{\Omega});\ \forall\, K \in \mathcal{T}_h,\ S_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad X_h = Z_h \cap H_0^1(\Omega).$$

There exists an approximation operator (when $d = 2$, see Bernardi and Girault 1998 or Clément 1975; when $d = 2$ or $d = 3$, see Scott and Zhang 1990) R_h in $\mathcal{L}(W^{1,p}(\Omega); Z_h)$ and in $\mathcal{L}(W^{1,p}(\Omega) \cap H_0^1(\Omega); X_h)$ such that for all K in $\mathcal{T}_h$, $m = 0, 1$, $l = 0, 1$, and all $p \geq 2$,

$$\forall\, S \in W^{l+1,p}(\Omega),\ \ |S - R_h(S)|_{W^{m,p}(K)} \leq C(p, m, l)\, h_K^{l+1-m} |S|_{W^{l+1,p}(\Delta_K)}, \tag{3.2}$$

where Δ_K is the macro element containing the values of S used in defining $R_h(S)$. Let K be an element of $\mathcal{T}_h$ with vertices a_i, $1 \leq i \leq d+1$, and corresponding barycentric coordinates λ_i. We denote by $b_K \in \mathbb{P}_{d+1}(K)$ the basic bubble function

$$b_K(\mathbf{x}) = \lambda_1(\mathbf{x}) \ldots \lambda_{d+1}(\mathbf{x}).$$

We observe that $b_K(\mathbf{x}) = 0$ on ∂K and that $b_K(\mathbf{x}) > 0$ in the interior of K.

Let $(\mathcal{W}_h, M_h)$ be a pair of discrete spaces approximating $L^2(\Omega)^d \times (H^1(\Omega) \cap L_m^2(\Omega))$ defined by

$$\mathcal{W}_h = \{\mathbf{v}_h \in (\mathcal{C}^0(\bar{\Omega}))^d;\ \forall\, K \in \mathcal{T}_h,\ \mathbf{v}_h|_K \in \mathcal{P}(K)^d\},$$

$$\tilde{M}_h = \{q_h \in \mathcal{C}^0(\bar{\Omega});\ \forall\, K \in \mathcal{T}_h,\ q_h|_K \in \mathbb{P}_1(K)\} \quad \text{and} \quad M_h = \tilde{M}_h \cap L_m^2(\Omega),$$

where

$$\mathcal{P}(K) = \mathbb{P}_1(K) \oplus \text{Vect}\{b_K\}.$$

Let $\mathcal{V}_h$ be the kernel of the divergence in $\mathcal{W}_h$,

$$\mathcal{V}_h = \left\{\mathbf{v}_h \in \mathcal{W}_h;\ \forall q_h \in M_h,\ \int_\Omega \nabla q_h \cdot \mathbf{v}_h\, \mathrm{d}\mathbf{x} = 0\right\}.$$

With the discrete spaces, we have (see for instance Bernardi et al. 2018) the following discrete inf-sup condition:

$$\forall\, q_h \in M_h,\quad \sup_{\mathbf{v}_h \in \mathcal{W}_h} \frac{\int_\Omega \nabla q_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}}{\|\mathbf{v}_h\|_{L^2(\Omega)^d}} \geq \beta_2\, |q_h|_{H^1(\Omega)}, \tag{3.3}$$

with a constant $\beta_2 > 0$ independent of h.

Since $\mathcal{W}_h$ contains the polynomials of degree one in each K, we can construct a variant π_h of R_h (cf. Girault and Lions 2001 or Scott and Zhang 1990) in $\mathcal{L}(L^2(\Omega)^d; Z_h)$ that is quasi-locally stable in $L^2(\Omega)$, i.e., for all K in $\mathcal{T}_h$

$$\forall \mathbf{v} \in L^2(\Omega)^d, \quad \|\pi_h(\mathbf{v})\|_{L^2(K)^d} \leq C\|\mathbf{v}\|_{L^2(\Delta_K)^d},$$

and has the same quasi-local approximation properties as R_h for all K in $\mathcal{T}_h$, for $m = 0, 1$ and $1 \leq l \leq 2$,

$$\forall\, \mathbf{v} \in H^l(\Omega)^d,\ |\mathbf{v} - \pi_h(\mathbf{v})|_{H^m(K)^d} \leq C\, h^{l-m} |\mathbf{v}|_{H^l(\Delta_K)^d}. \tag{3.4}$$

Regarding the pressure, since Z_h coincides with $\tilde{M}_h$, an easy modification of R_h yields an operator r_h in $\mathcal{L}(H^1(\Omega); \tilde{M}_h)$ and in $\mathcal{L}(H^1(\Omega) \cap L^2_m(\Omega); M_h)$ (see for instance Abboud et al. 2009), satisfying (3.2). We approximate problem (V_a) by the following discrete scheme:

$$(V_{a,h}) \begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_h \times M_h \times X_h \text{ such as} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_h,\ \displaystyle\int_\Omega \nu(T_h)\mathbf{u}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_\Omega \nabla p_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \\ \forall\, q_h \in M_h,\ \displaystyle\int_\Omega \nabla q_h \cdot \mathbf{u}_h \,\mathrm{d}\mathbf{x} = 0, \\ \forall\, S_h \in X_h,\ \alpha \displaystyle\int_\Omega \nabla T_h \cdot \nabla S_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h \cdot \nabla T_h) S_h \,\mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \,\mathrm{d}\mathbf{x}. \end{cases}$$

Remark 3.1 In Bernardi et al. (2018), we have introduced two other discrete variational formulations:

(1) The first one (called $(V_{h,1})$ in Bernardi et al. 2018) is the following:

$$(V_{h,1}) \begin{cases} \text{Find } (\mathbf{u}_h, p_h, T_h) \in \mathcal{W}_{h,1} \times M_{h,1} \times X_h \text{ such as} \\ \forall\, \mathbf{v}_h \in \mathcal{W}_{h,1},\ \displaystyle\int_\Omega \nu(T_h)\mathbf{u}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} - \int_\Omega p_h (\mathrm{div}\, \mathbf{v}_h) \,\mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}, \\ \forall\, q_h \in M_{h,1},\ \displaystyle\int_\Omega q_h (\mathrm{div}\, \mathbf{u}_h) \,\mathrm{d}\mathbf{x} = 0, \\ \forall\, S_h \in X_h,\ \alpha \displaystyle\int_\Omega \nabla T_h \cdot \nabla S_h \,\mathrm{d}\mathbf{x} + \int_\Omega (\mathbf{u}_h \cdot \nabla T_h) S_h \,\mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \,\mathrm{d}\mathbf{x}, \end{cases}$$

where $\mathcal{W}_{h,1}$ and $M_{h,1}$ are the discrete spaces corresponding to $H_0(\mathrm{div}, \Omega)$ and $L^2_m(\Omega)$ by using RT_0 elements, namely the Raviart-Thomas finite elements for the velocity and the P_0 finite elements for the pressure. We refer to Bernardi et al. (2018) for the details.

(2) The second one (called $(V_{h,2})$ in Bernardi et al. 2018) is similar to $(V_{a,h})$ with exactly the same finite elements, but the third equation is completed with a stabilised term $\frac{1}{2}(\mathrm{div}\mathbf{u}_h\, T_h, S_h)$.

4 Existence and uniqueness of the solution

For each one of the schemes $(V_{h,1})$ and $(V_{h,2})$, and as we mentioned in the introduction, we prove in Bernardi et al. (2018) the existence of a solution by Galerkin's method and

Brouwer's Fixed Point. Furthermore, the global uniqueness is established when the solution is slightly smoother and the data are suitably restricted.

In this section, we will show the existence and the uniqueness of the numerical solution of $(V_{a,h})$, and the corresponding a priori error estimate by using the Brezzi–Rappaz–Raviart theorem. The same steps are valid to show same results for $(V_{h,1})$. We mentioned in the introduction the advantages and disadvantages of this study with respect to that performed in Bernardi et al. (2018).

We introduce the Darcy operator Q, which associates with any datum $\mathbf{f} \in L^2(\Omega)^d$ the solution $(\mathbf{w}, q)$ of the generalized Darcy's problem:

$$\begin{cases} \nu_1 \mathbf{w}(\mathbf{x}) + \nabla q(\mathbf{x}) = \mathbf{f}(\mathbf{x}) & \text{in } \Omega, \\ \operatorname{div} \mathbf{w}(\mathbf{x}) = 0 & \text{in } \Omega, \\ (\mathbf{w}.\mathbf{n})(\mathbf{x}) = 0 & \text{on } \Gamma. \end{cases}$$

For the existence and uniqueness of the solution $(\mathbf{w}, q)$, we can refer for instance to Theorem 1.9 Chapter XIII Bernardi et al. (2004).

We introduce the inverse $\mathcal{L}$ of the Laplace operator which associates with any datum $g \in L^2(\Omega)$, the solution L in $H_0^1(\Omega)$ of the following problem:

$$\begin{cases} -\alpha \Delta L(\mathbf{x}) = g(\mathbf{x}) & \text{dans } \Omega, \\ L(\mathbf{x}) = 0 & \text{sur } \Gamma. \end{cases}$$

In fact, Lax–Milgram theorem implies existence and uniqueness of the solution L.

Remark 4.1 Concerning the operator $\mathcal{L}$:

- $\mathcal{L}$ remains applicable for all $g \in H^{-1}(\Omega)$.
- If $\mathbf{f} \in L^\infty(\Omega)^d$, then according to Theorem 2.4, $\mathcal{L}$ remains applicable for $\tilde{g} = g - \mathbf{u} \cdot \nabla T \in L^p(\Omega)$, $p > 1$. Subsequently, we can define $\mathcal{L}\tilde{g}$.

If $\mathbf{f} \in L^\infty(\Omega)^d$, it is readily checked that, when setting $U = (\mathbf{u}, p, T)$, the problem (P) can be equivalently written as

$$F(U) = U - JG(U) = 0,$$

where $G(U) = (\mathbf{f} - \nu(T)\mathbf{u} + \nu_1 \mathbf{u}, g - \mathbf{u} \cdot \nabla T)$ and $J = \begin{pmatrix} Q & 0 \\ 0 & \mathcal{L} \end{pmatrix}$.

Similarly, let Q_h denote the discrete Darcy operator, i.e, the operator which associates with any datum $\mathbf{f} \in L^2(\Omega)^d$, the solution $(\mathbf{w}_h, q_h) \in \mathcal{W}_h \times M_h$ of the Darcy problem

$$\begin{cases} \forall \mathbf{v}_h \in \mathcal{W}_h, \quad \int_\Omega \nu_1 \mathbf{w}_h \cdot \mathbf{v}_h \, d\mathbf{x} + \int_\Omega \nabla q_h \cdot \mathbf{v}_h \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, d\mathbf{x}, \\ \qquad \forall t_h \in M_h, \quad \int_\Omega \nabla t_h \cdot \mathbf{w}_h \, d\mathbf{x} = 0. \end{cases}$$

Finally, let $\mathcal{L}_h$ denote the operator which associates with any datum $g \in L^2(\Omega)$, the function $L_h \in X_h$ which satisfies

$$\forall S_h \in X_h, \quad \alpha \int_\Omega \nabla L_h \cdot \nabla S_h \, d\mathbf{x} = \int_\Omega g \, S_h \, d\mathbf{x}.$$

We set $U_h = (\mathbf{u}_h, p_h, T_h)$. Problem $(V_{a,h})$ can be equivalently written as follows:

$$F_h(U_h) = U_h - J_h G(U_h) = 0,$$

where $J_h = \begin{pmatrix} Q_h & 0 \\ 0 & \mathcal{L}_h \end{pmatrix}$.

Assumption 4.2 We suppose that $Q\mathbf{f} \in H^1(\Omega)^d \times H^2(\Omega)$ and $\mathcal{L}g \in H^2(\Omega)$.

Assumption 4.3 The solution $U = (\mathbf{u}, p, T)$ of Problem (V_a)

- belongs to $H^1(\Omega)^d \times H^2(\Omega) \times H^2(\Omega)$;
- is such that $DF(\mathbf{u}, p, T)$ is an isomorphism of $L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_m(\Omega)) \times H^1_0(\Omega)$.

From now on, we denote by

$$V_1 = L^2(\Omega)^d \times (H^1(\Omega) \cap L^2_m(\Omega)) \times H^1_0(\Omega) \qquad \text{and} \qquad W_1 = L^2(\Omega)^d \times L^2(\Omega).$$

We are thus in a position to prove the preliminary results which we need for applying the theorem of Brezzi et al. (1980). This requires to introduce the linear and continuous operator P_h from V_1 to $\mathcal{W}_h \times M_h \times X_h$ which satisfies

$$\lim_{h\to 0} \|U - P_h U\|_{V_1} = 0, \tag{4.1}$$

with $P_h U = (\pi_h \mathbf{u}, r_h p, R_h T)$ where π_h, r_h and R_h are the operators defined in Sect. 3, and $(\mathbf{u}, p, T) \in H^1(\Omega)^d \times H^2(\Omega) \times H^2(\Omega)$. To simplify, they are stated in three dimensions, but the two-dimensional analogue is easily derived.

In order to apply Brezzi–Rappaz–Raviart theorem which allows us to show the existence and the uniqueness of the solution, we present the following three theorems:

Theorem 4.4 *Assume that $\nu \in W^{2,\infty}(\Omega)$ and Assumptions 4.2 and 4.3 hold. There exists a positive real $h_0 > 0$, such that for all $h \le h_0$, the operator $DF_h(P_h U)$ is an isomorphism of $\mathcal{W}_h \times M_h \times X_h$ with the norm of its inverse bounded independently of h.*

Proof First we write the expansion

$$DF_h(P_h U) = DF(U) - (J_h - J)DG(U) - J_h(DG(P_h U) - DG(U)). \tag{4.2}$$

Due to the Assumption 4.3, it suffices to check that the last two terms in the right-hand side of (4.2) tend to 0 when h tends to 0.

We begin by proving the zero convergence of the first term. We have

$$\|(J_h - J)DG(U)\|_{\mathcal{L}(V_1)} \le \|J_h - J\|_{\mathcal{L}(W_1,V_1)} \|DG(U)\|_{\mathcal{L}(V_1,W_1)}.$$

Since $Q\mathbf{f} \in H^1(\Omega)^d \times H^2(\Omega)$ and $\mathcal{L}g \in H^2(\Omega)$, we get

$$\lim_{h\to 0} \|J_h - J\|_{\mathcal{L}(W_1,V_1)} = 0.$$

In fact, we start by considering the relation

$$\|J_h - J\|_{\mathcal{L}(W_1,V_1)} = \sup_{(\mathbf{f},g)\in W_1} \frac{\|(J_h - J)(\mathbf{f}, g)\|_{V_1}}{\|(\mathbf{f}, g)\|_{W_1}},$$

where

$$\|(J_h - J)(\mathbf{f}, g)\|_V = \left(\|\mathbf{w} - \mathbf{w}_h\|^2_{L^2(\Omega)^3} + |q - q_h|^2_{H^1(\Omega)} + +|L - L_h|^2_{H^1(\Omega)}\right)^{1/2},$$

and $(\mathbf{w}, q)$ and $(\mathbf{w}_h, q_h)$ verify the following equations:

$$\forall \mathbf{v}_h \in \mathcal{W}_h, \quad \int_\Omega \nu_1 \mathbf{w}_h \cdot \mathbf{v}_h \, d\mathbf{x} + \int_\Omega \nabla q_h . \mathbf{v}_h \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, d\mathbf{x}$$

and

$$\forall \mathbf{v} \in L^2(\Omega)^3, \quad \int_\Omega \nu_1 \mathbf{w} \cdot \mathbf{v} \, d\mathbf{x} + \int_\Omega \nabla q \cdot \mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}. \tag{4.3}$$

We choose $\mathbf{v} = \mathbf{v}_h \in \mathcal{V}_h$, insert $\pi_h \mathbf{w}$ and $r_h q$ in Eq. (4.3), and remark that $\int_\Omega (\nabla q_h - r_h q) \cdot \mathbf{v}_h \mathrm{d}\mathbf{x} = 0$ to get

$$\int_\Omega (\pi_h \mathbf{w} - \mathbf{w}_h) \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x} = \int_\Omega (\pi_h \mathbf{w} - \mathbf{w}) \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x} + \frac{1}{\nu_1} \int_\Omega \nabla (r_h q - q) \cdot \mathbf{v}_h \, \mathrm{d}\mathbf{x} .$$

Then, we take $\mathbf{v}_h = \pi_h \mathbf{w} - \mathbf{w}_h$ to have

$$\|\mathbf{w}_h - \pi_h \mathbf{w}\|_{L^2(\Omega)^3} \leq C\, h\, (|\mathbf{w}|_{H^1(\Omega)^3} + |q|_{H^2(\Omega)}). \tag{4.4}$$

The inf-sup condition (3.3) allows us to get

$$|q_h - r_h q|_{H^1(\Omega)} \leq C\, h\, (|\mathbf{w}|_{H^1(\Omega)^3} + |q|_{H^2(\Omega)}). \tag{4.5}$$

Furthermore, L and L_h satisfy the following equations:

$$\forall\, S_h \in X_h, \quad \alpha \int_\Omega \nabla L_h \cdot \nabla S_h \, \mathrm{d}\mathbf{x} = \int_\Omega g\, S_h \, \mathrm{d}\mathbf{x}$$

and

$$\forall\, S \in H_0^1(\Omega), \quad \alpha \int_\Omega \nabla L \cdot \nabla S \, \mathrm{d}\mathbf{x} = \int_\Omega g\, S \, \mathrm{d}\mathbf{x} . \tag{4.6}$$

We choose $S = S_h$ and insert $R_h L$ in (4.6) to get

$$\int_\Omega \nabla (R_h L - L_h) \cdot \nabla S_h \, \mathrm{d}\mathbf{x} = \int_\Omega \nabla (R_h L - L) \cdot \nabla S_h \, \mathrm{d}\mathbf{x} .$$

We take $S_h = R_h L - L_h$ to obtain

$$|R_h L - L_h|_{H^1(\Omega)} \leq C\, h |L|_{H^2(\Omega)}. \tag{4.7}$$

Then, relations (4.4), (4.5) and (4.7) and properties of operators R_h, r_h and π_h allow us to obtain the following limit:

$$\lim_{h \to 0} \|(J_h - J) DG(U)\|_{\mathcal{L}(V_1)} = 0.$$

Let us now treat the last term of (4.2). We have, for all $W_h = (\mathbf{w}_h, q_h, L_h) \in (\mathcal{W}_h \times M_h \times X_h) \backslash \{0\}$,

$$\begin{aligned} &(DG(U) - DG(P_h U)) \cdot W_h \\ &\quad = \begin{pmatrix} (\nu(R_h T) - \nu(T)) \mathbf{w}_h + (\nu'(R_h T) \pi_h \mathbf{u} - \nu'(T) \mathbf{u}) L_h \\ \mathbf{w}_h \cdot \nabla (R_h T - T) + (\pi_h \mathbf{u} - \mathbf{u}) \cdot \nabla L_h \end{pmatrix}. \end{aligned} \tag{4.8}$$

As ν belongs to $W^{2,\infty}(\Omega)$, its derivative ν' is bounded by a given real positive number ν_2' and is also Lipschitz-continuous with a given real positive Lipschitz constant λ'. Then, by using the inverse inequality (3.1) we obtain:

$$\|J_h (DG(P_h U) - DG(U))\|_{\mathcal{L}(V_1)} \leq C\, h^{-\frac{d}{6}} \|U - P_h U\|_{V_1}. \tag{4.9}$$

In fact, we consider the relation

$$\begin{aligned} &\|J_h (DG(U) - DG(P_h U))\|_{\mathcal{L}(V_1)} \\ &\quad = \sup_{(\mathbf{w}_h, q_h, L_h) \in (\mathcal{W}_h \times M_h \times X_h) \backslash \{0\}} \frac{\|J_h (DG(U) - DG(P_h U))(\mathbf{w}_h, q_h, L_h)\|_{V_1}}{\|(\mathbf{w}_h, q_h, L_h)\|_{V_1}}, \end{aligned}$$

where

$$\|J_h(DG(U) - DG(P_hU))(\mathbf{w}_h, q_h, L_h)\|_{V_1} = \|(\tilde{\mathbf{w}}_h, \tilde{q}_h, \tilde{L}_h)\|_{V_1}, \tag{4.10}$$

and $(\tilde{\mathbf{w}}_h, \tilde{q}_h)$ satisfies the equation

$$\begin{aligned}\int_\Omega \nu_1 \tilde{\mathbf{w}}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} + \int_\Omega \nabla \tilde{q}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} &= \int_\Omega (\nu(R_hT) - \nu(T))\mathbf{w}_h \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega \nu'(R_hT)\, L_h(\pi_h\mathbf{u} - \mathbf{u}) \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x} \\ &\quad + \int_\Omega (\nu'(R_hT) - \nu'(T))\, L_h\mathbf{u} \cdot \mathbf{v}_h \,\mathrm{d}\mathbf{x}\end{aligned} \tag{4.11}$$

and $\tilde{L}_h$ satisfies

$$\alpha \int_\Omega \nabla \tilde{L}_h \cdot \nabla S_h \,\mathrm{d}\mathbf{x} = \int_\Omega (\pi_h\mathbf{u} - \mathbf{u}) \cdot \nabla L_h\, S_h \,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{w}_h \cdot \nabla(R_hT - T)S_h \,\mathrm{d}\mathbf{x}. \tag{4.12}$$

Then, by taking $\mathbf{v}_h = \tilde{\mathbf{w}}_h$ in (4.11), we get the bound

$$\begin{aligned}\nu_1\|\tilde{\mathbf{w}}_h\|_{L^2(\Omega)^3} \le C_1\, h^{-\frac{d}{6}}\big(\lambda|T - R_hT|_{H^1(\Omega)}\|\mathbf{w}_h\|_{L^2(\Omega)^3} + \nu_2'\|\mathbf{u} - \pi_h\mathbf{u}\|_{L^2(\Omega)^3}|L_h|_{H^1(\Omega)} \\ + \lambda'\, S_6^0\|\mathbf{u}\|_{L^3(\Omega)^3}|T - R_hT|_{H^1(\Omega)}|L_h|_{H^1(\Omega)}\big).\end{aligned} \tag{4.13}$$

The discrete inf-sup condition (3.3) allows us to bound $|\tilde{q}_h|_{H^1(\Omega)}$ with a similar right-hand side of (4.13).

Next, we choose $S_h = \tilde{L}_h$ in (4.12) to get

$$\alpha|\tilde{L}_h|_{H^1(\Omega)} \le C_2\, h^{-\frac{d}{6}}\big(\|\mathbf{u} - \pi_h\mathbf{u}\|_{L^2(\Omega)^3}|L_h|_{H^1(\Omega)} + |T - R_hT|_{H^1(\Omega)}\|\mathbf{w}_h\|_{L^2(\Omega)^3}\big).$$

Equation (4.1), the properties of the operators R_h, r_h and π_h, and Assumption 4.3 allow us to obtain

$$\lim_{h\to 0}\|J_h(DG(P_hU) - DG(U))\|_{\mathcal{L}(V_1)} = 0.$$

Then, all the above results allow us to deduce that there exists a positive constant $h_0 > 0$ that for all $h \le h_0$, $DF_h(P_hU)$ is an isomorphism.

To close the proof of the theorem, we have to show that the inverse of $DF_h(P_hU)$ is bounded independently of h. In fact, for all $(\mathbf{v}_h, t_h, S_h) \in \mathcal{W}_h \times M_h \times X_h$, we have

$$\begin{aligned}DF_h(P_hU)(\mathbf{v}_h, t_h, S_h) &= DF(U)(\mathbf{v}_h, t_h, S_h) - (J_h - J)DG(U)(\mathbf{v}_h, t_h, S_h) \\ &\quad - J_h(DG(P_hU) - DG(U))(\mathbf{v}_h, t_h, S_h).\end{aligned}$$

$(DF(U))^{-1}$ is an isomorphism in a discrete space, then $(DF(U))^{-1}$ is continuous. We denote by $\gamma = \|(DF(U))^{-1}\|_{\mathcal{L}(V_1)}$ and use the relation

$$\|(\mathbf{v}_h, t_h, S_h)\|_{V_1} \le \|(DF(U))^{-1}\|_{\mathcal{L}(V_1)}\, \|DF(U)(\mathbf{v}_h, t_h, S_h)\|_{V_1}$$

and the formula $-|ab| \ge -|a||b|$ to obtain:

$$\begin{aligned}\|DF_h(P_hU)(\mathbf{v}_h, t_h, S_h)\|_{V_1} \ge \big(\gamma^{-1} - \|J_h - J\|_{\mathcal{L}(W_1,V_1)}\|DG(U)\|_{\mathcal{L}(V_1,W_1)} \\ - \|J_h(DG(P_hU) - DG(U))\|_{\mathcal{L}(V_1)}\big)\|(\mathbf{v}_h, t_h, S_h)\|_{V_1}.\end{aligned}$$

Therefore,

$$\|DF_h(P_hU)(\mathbf{v}_h, t_h, S_h)\|_{V_1} \geq (\gamma^{-1} - \varepsilon(h))\|(\mathbf{v}_h, t_h, S_h)\|_{V_1},$$

where $\lim_{h\to 0} \varepsilon(h) = 0$. Hence the result. □

Theorem 4.5 *Assume that $\nu \in W^{2,\infty}(\Omega)$ and Assumptions 4.3 hold, there exists a neighborhood of P_hU in $\mathcal{W}_h \times X_h$ and a constant $C > 0$ such that the operator DF_h satisfies the following Lipschitz property, for all U_h^* in this neighbourhood:*

$$\|DF_h(U_h^*) - DF_h(P_hU)\|_{\mathcal{L}(V_1)} \leq C\, h^{-\frac{d}{6}} \|U_h^* - P_hU\|_{V_1},$$

where C is a pocitive constant independent of h.

Proof By setting $U_h^* = (\mathbf{u}_h^*, p_h^*, T_h^*)$, we have

$$DF_h(U_h^*) = DF(U) - (J_h - J)DG(U) - J_h(DG(U_h^*) - DG(U)).$$

and

$$DF_h(P_hU) = DF(U) - (J_h - J)DG(U) - J_h(DG(P_hU) - DG(U)).$$

By using the bound (4.9) we deduce the following inequality:

$$\|J_h(DG(P_hU) - DG(U_h^*))\|_{\mathcal{L}(V_1)} \leq C\, h^{-\frac{d}{6}} \|U_h^* - P_hU\|_{V_1}.$$

Hence the result. □

Theorem 4.6 *Assume that $\mathbf{u} \in H^1(\Omega)^3$, $p \in H^2(\Omega)$, $T \in W^{2,6}(\Omega)$, $\nabla T \in L^\infty(\Omega)$ and Assumption 4.3 holds. Then the following estimate is satisfied:*

$$\|F_h(P_hU)\|_V \leq C\, h\big(|\mathbf{u}|_{H^1(\Omega)^3} + |T|_{H^2(\Omega)} + |p|_{H^2(\Omega)} + |\mathbf{u}|_{H^1(\Omega)^3}\|\nabla T\|_{L^\infty(\Omega)} + \|\mathbf{u}\|_{H^1(\Omega)^3}|T|_{W^{2,6}(\Omega)}\big),$$

where C is a positive constant independent of h.

Proof We consider the relation $F_h(P_hU) = P_hU - J_hG(P_hU)$ and recall that $F(U) = U - JG(U) = 0$. Then we get

$$F_h(P_hU) = (P_hU - U) - J_hG(P_hU) + JG(U). \tag{4.14}$$

We insert $J_hG(U)$ in the right-hand side of Eq. (4.14) and we obtain:

$$F_h(P_hU) = (P_hU - U) + (J - J_h)G(U) + J_h(G(U) - G(P_hU))).$$

We deduce the following inequality:

$$\|F_h(P_hU)\|_{V_1} \leq \|U - P_hU\|_{V_1} + \|(J - J_h)G(U)\|_{V_1} + \|J_h(G(U) - G(P_hU)))\|_{V_1}.$$

Owing to the relations (3.2) and (3.4) , we get

$$\|U - P_hU\|_{V_1} \leq C\, h\left(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^2(\Omega)} + |T|_{H^2(\Omega)}\right).$$

As ν is Lipschitz-continuous in $\mathbb{R}$ and according to Relation (2.4), we get:

$$\|(J - J_h)G(U)\|_{V_1} \leq C_1\, h\big(|\mathbf{u}|_{H^1(\Omega)^3} + |p|_{H^2(\Omega)}\big).$$

We still have to treat the last term $\|J_h(G(U) - G(P_hU)))\|_{V_1}$.

We have

$$G(U) - G(P_hU) = (\nu_1(\mathbf{u} - \pi_h\mathbf{u}) + \nu(R_hT)\pi_h\mathbf{u} - \nu(T)\mathbf{u}, \pi_h\mathbf{u}\cdot\nabla R_hT - \mathbf{u}\cdot\nabla T,$$

then we get:

$$\begin{aligned}\|J_h(G(U) - G(P_hU))\|_{V_1} \leq{}& \|Q_h(\nu_1(\mathbf{u} - \pi_h\mathbf{u}) + \nu(R_hT)\pi_h\mathbf{u} - \nu(T)\mathbf{u})\|_{L^2(\Omega)^3}\\ &+\|\mathcal{L}_h(\pi_h\mathbf{u}\cdot\nabla R_hT - \mathbf{u}\cdot\nabla T.\end{aligned} \tag{4.15}$$

We treat every term of the right-hand side of the last relation.

For the first one, we use the properties of ν to get

$$\begin{aligned}&\|Q(\nu_1(\mathbf{u} - \pi_h\mathbf{u}) + \nu(R_hT)\pi_h\mathbf{u} - \nu(T)\mathbf{u})\|_{L^2(\Omega)^3}\\ &\leq \frac{1}{\nu_1}\|\nu_1(\mathbf{u} - \pi_h\mathbf{u}) + \nu(R_hT)\pi_h\mathbf{u} - \nu(T)\mathbf{u}\|_{L^2(\Omega)^3}\\ &\leq \frac{1}{\nu_1}\|\nu_1(\mathbf{u} - \pi_h\mathbf{u}) + \nu(R_hT)(\pi_h\mathbf{u} - \mathbf{u}) + (\nu(R_hT) - \nu(T))u\\ &\leq C\,h\big[|\mathbf{u}|_{H^1(\Omega)^3} + \|\mathbf{u}\|_{L^3(\Omega)^3}|T|_{H^2(\Omega)}\big].\end{aligned} \tag{4.16}$$

Next, for the second term of the right-hand side (4.15), we insert the terms $\mathbf{u}\cdot\nabla R_hT$ and $\pi_h\mathbf{u}\cdot\nabla T$ to obtain:

$$\pi_h\mathbf{u}\cdot\nabla R_hT - \mathbf{u}\cdot\nabla T = (\pi_h\mathbf{u} - \mathbf{u})\cdot\nabla(R_hT - T) + \mathbf{u}\cdot\nabla(R_hT - T) + (\pi_h\mathbf{u} - \mathbf{u})\cdot\nabla T.$$

We denote by

$$\tilde{L}_h = \mathcal{L}_h(\pi_h\mathbf{u}\cdot\nabla R_hT - \mathbf{u}\cdot\nabla T)$$

which verifies, by using the integration by parts, the relation

$$\begin{aligned}\alpha\int_\Omega \nabla\tilde{L}_h\cdot\nabla S_h ={}& \int_\Omega(\pi_h\mathbf{u} - \mathbf{u})\cdot\nabla(R_hT - T)\,S_h\\ &+\int_\omega \mathbf{u}\cdot\nabla(R_hT - T)\,S_h + \int_\Omega(\pi_h\mathbf{u} - \mathbf{u})\cdot\nabla T\,S_h.\end{aligned}$$

By taking $S_h = \tilde{L}_h$ and using the properties of the operators π_h and R_h, and the inverse inequality (3.1), we have:

$$\|\mathcal{L}_h(\pi_h\mathbf{u}\cdot\nabla R_hT - \mathbf{u}\cdot\nabla T \leq C_2\,h\big(\|\mathbf{u}\|_{H^1(\Omega)^3}|T|_{W^{2,6}(\Omega)} + |\mathbf{u}|_{H^1(\Omega)^3}\|\nabla T\|_{L^\infty(\Omega)}\big).$$

All the above results allow us to deduce the required result. □

The previous three theorems allow us to apply the Brezzi–Rappaz–Raviart theorem and to obtain the following theorem:

Theorem 4.7 *Let $(\mathbf{u}, p, T)$ be a solution of problem (V_a) which satisfies Assumptions 4.2 and 4.3, $\mathbf{u} \in H^1(\Omega)^3$, $p \in H^2(\Omega)$, $T \in W^{2,6}(\Omega)$ and $\nabla T \in L^\infty(\Omega)$. We moreover assume that ν belongs to $W^{2,\infty}(\Omega)$. Then, there exists a positive number $h_0 > 0$ and a neighborhood O of U in V, such that for all $h \leq h_0$, the variational formulation $(V_{a,h})$ has a unique solution $(\mathbf{u}_h, p_h, T_h)$ with $U_h \in O$. Furthermore, we have the a priori error estimate:*

$$|\mathbf{u} - \mathbf{u}_h| \leq Ch|\mathbf{u}|_{2,\Omega},$$

where C is a positive constant independent of h.

Proof Combining Assumptions 4.2–4.3 and Theorems 4.4–4.6 with the Brezzi et al. (1980) yields for h small enough, the local existence and uniqueness of the solution $(\mathbf{u}_h, T_h)$. Moreover, thanks to the discrete inf-sup condition (2.2), we deduce the existence and uniqueness of p_h. □

Remark 4.8 For the scheme $(V_{h,1})$, we can apply the same above steps to show the corresponding results. We use the approximation operator ξ_h^1 instead of π_h for the velocity and ρ_h instead of r_h for the pressure (see Bernardi et al. 2018 for the definition of ξ_h and r_h).

5 Numerical results

To validate the theoretical results, we perform several numerical simulations using Freefem++ (see Hecht 2012). We consider a square domain $\Omega =]0, 3[^2$. Each edge is divided into N equal segments so that Ω is divided into $2N^2$ triangles. We choose same exact solution considered in Bernardi et al. (2018), $(\mathbf{u}, p, T) = (\mathbf{curl}\,\psi, p, T)$ where ψ, p and T are defined by

$$\psi(x, y) = e^{-\beta((x-1)^2+(y-1)^2)}, \tag{5.1}$$

$$p(x, y) = \cos\left(\frac{\pi}{3}x\right)\cos\left(\frac{\pi}{3}y\right), \tag{5.2}$$

and

$$T(x, y) = x^2(x-3)^2y^2(y-3)^2. \tag{5.3}$$

We introduce the following iterative fixed point scheme corresponding to $(V_{a,h})$:

$$(Vhi)\begin{cases} (\nu(T_h^i)\mathbf{u}_h^{i+1}, \mathbf{v}_h)_2 + (\nabla p_h^{i+1}, \mathbf{v}_h)_2 = (\mathbf{f}, \mathbf{v}_h)_2, \\ (\nabla q_h, \mathbf{u}_h^{i+1})_2 = 0, \\ \alpha(\nabla T_h^{i+1}, \nabla S_h)_2 + \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1})(\mathbf{x}) S_h(\mathbf{x})\,\mathrm{d}\mathbf{x} = (g, S_h)_2. \end{cases}$$

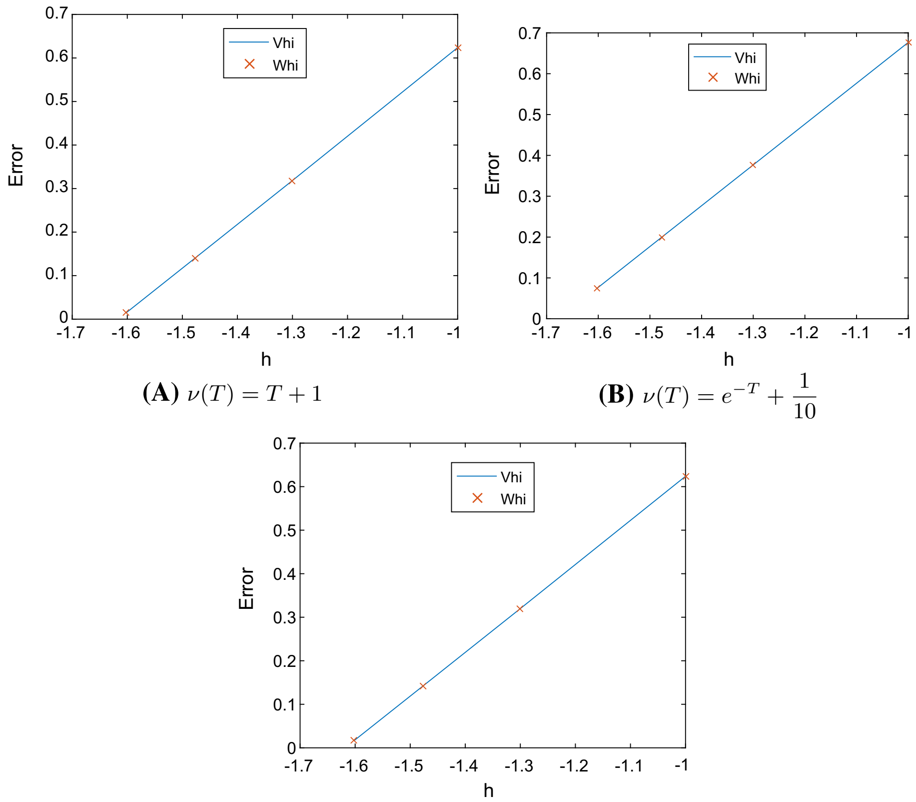
For the numerical computations, we consider $\alpha = 3$, $\beta = 5$ and $N = 100$.

The first two lines of the last iterative algorithm give for each $\mathbf{u}_h^i$ the solution $(\mathbf{u}_h^{i+1}, p_h^{i+1})$. Then, having $\mathbf{u}_h^{i+1}$, we compute T_h^{i+1} by using the third line of the same algorithm. The absence of the stabilised term $\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}_h^{i+1}(\mathbf{x})T_h^{i+1}(\mathbf{x})S_h(\mathbf{x})\,\mathrm{d}\mathbf{x}$ in the third line of the previous algorithm makes the studies difficult (convergence, existence of the solution, etc.). It will be established in a future work.

We will compare (Vhi) with the following similar stabilized iterative algorithm corresponding to $(V_{h,2})$, which is introduced and studied in Bernardi et al. (2018):

$$(Whi)\begin{cases} (\nu(T_h^i)\mathbf{u}_h^{i+1}, \mathbf{v}_h)_2 + (\nabla p_h^{i+1}, \mathbf{v}_h)_2 = (\mathbf{f}, \mathbf{v}_h)_2, \\ (\nabla q_h, \mathbf{u}_h^{i+1})_2 = 0, \\ \alpha(\nabla T_h^{i+1}, \nabla S_h)_2 + \int_\Omega (\mathbf{u}_h^{i+1} \cdot \nabla T_h^{i+1})(\mathbf{x}) S_h(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ \quad +\frac{1}{2}\int_\Omega \operatorname{div}\mathbf{u}_h^{i+1}(\mathbf{x})T_h^{i+1}(\mathbf{x})S_h(\mathbf{x})\,\mathrm{d}\mathbf{x} = (g, S_h)_2. \end{cases}$$

The essential difference between (Vhi) and (Whi) is the stabilised term in the third line which constitutes a supplementary term to compute (supplementary time of computation) in (Whi) but which does not affect the numerical solutions as we will see later. We refer to Bernardi et al. (2018) for the numerical comparison between (Whi) and the iterative scheme corresponding



(A) $\nu(T) = T + 1$

(B) $\nu(T) = e^{-T} + \frac{1}{10}$

(C) $\nu(T) = \sin(T) + 2$

Fig. 1 Error curve for different $\nu(T)$ and for (Vhi)

to $(V_{h,1})$, and we will show in the following the numerical comparisons between (Vhi) and (Whi).

Figure 1 plots the global error curves versus h in logarithmic scales, global in the sense that they depict the sum of the velocity, pressure and temperature errors for the variational formulation. The algorithm is tested when the number N of segments increase from 30 to 120. The slope of the error's curve for (Vhi) is equal to 1.0122 for $\nu(T) = T + 1$, 0.9995 for $\nu(T) = e^{-T} + \frac{1}{10}$ and finally 1.0091 for $\nu(T) = \sin(T) + 2$. Practically, these slops are identical to those obtained in Bernardi et al. (2018) for (Whi).

Remark 5.1 Note that the error curves are consistent with the theoretical results of Sect. 3. □

6 Conclusion

In this work, we introduced The Darcy's Problem coupled with the heat equation. Then, we introduce a discrete non-stabilized problem and then we show the existence, uniqueness and a priori error estimate by using Brezzi–Rappaz–Raviart theorem. Finally, we show several numerical investigations.

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