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Complex fuzzy sets with applications in signals

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Abstract
A complex fuzzy set is characterized by a membership function, whose range is not limited to [0, 1], but extended to the unit circle in the complex plane. In this paper, we introduce some new operations and laws of a complex fuzzy set such as disjunctive sum, simple difference, bounded difference, distributive law of union over intersection and intersection over union, equivalence formula, symmetrical difference formula, involution law, absorption law, and idempotent law. We introduce some basic results on complex fuzzy sets with respect to standard complex fuzzy intersection, union, and complement functions corresponding to the same functions for determining the phase term, and we give particular examples of these operations. We use complex fuzzy sets in signals and systems, because its behavior is similar to a Fourier transform in certain cases. Moreover, we develop a new algorithm using complex fuzzy sets for applications in signals and systems by which we identify a reference signal out of large number of signals detected by a digital receiver. We use the inverse discrete Fourier transform of a complex fuzzy set for incoming signals and a reference signal. Thus, a method for measuring the exact values of two signals is provided by which we can identity the reference signal.

Keywords Discrete Fourier transform matrix · Signal processing · Complex fuzzy sets · Complex fuzzy operations · Complex fuzzy intersections · Complex fuzzy union · Complex valued grades of memberships · Complex fuzzy complement

Mathematics Subject Classification 30E10 · 03E72 · 94A12

1 Introduction

Models representing real-life phenomenon with only choices of truth and falsehood are insufficient to represent the actual reality of the problems. The reason for this is that there are several complexities which exist in the models, that is, why a system is needed to be developed

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to handle such ill-defined situations of the models. Now, there are two ways to handle these types of situations: one is to find the numerical solutions of the problems and second is to develop a model which is numerical. In both cases, we get numerical solutions of the problems. The second one is dealt with fuzzy set theory which includes probability theory, fuzzy soft set theory, intuitionistic fuzzy sets theory, and, most importantly, neutrosophic set theory. The later one is more generalized theory for handling problems involving complexities. One of the suitable examples of these theories is the theory of fuzzy differential equations which is more general than the differential equations to solve daily life problems with more accuracy.

Zadeh (1965) gave the concept of a fuzzy set (FS) which is similar to a probability function. A fuzzy set plays a vital role in models of real-world problems in various branches of sciences. Fuzzy set theory has a lot of applications in operation research, medicine, psychology, engineering design, decision-making, quantum physics, mathematical chemistry, non-equilibrium thermodynamics image processing, biological classification, and economics (see Alcantud and Calle 2017; El Allaoui et al. 2017; Dubois and Prade 2000; Li and Yen 1995; Ngan et al. 2018; Nguyen and Walker 2006; Nisren et al. 2017; Pedrycz and Gomide 1998; Poodeh 2017). For basic and recent work, one can refer to Hu et al. (2017), Naz and Akram (2019), Nguyen et al. (1998), Ramot et al. (2003), Peng et al. (2017, 2018), Peng and Dai (2018a, b), Peng and Garg (2018), Peng and Selvachandran (2017), Singh (2017), Tamir et al. (2011), and Yazdanbakhsh and Dick (2018).

The idea of a complex fuzzy set (CFS) was first given by Ramot et al. (2002). As the set of complex numbers is a generalization of the set of real numbers initiated by Gauss in 1795. Accordingly, a CFS is an extension of the fuzzy set, whose range is extended from closed interval [0, 1] to a disk of radius one in a complex plane. The membership function of CFS S is denoted as $\mu_S(x)$ and defined on the universal U as: for any $x \in U$ a complex value in the disk of radius one in a complex plane. Thus, all values of $\mu_S(x)$ lie inside a circle of radius one in complex plane and $\mu_S(x) = r_S(x)e^{i\omega_S(x)}$; where $i = \sqrt{-1}$. The term $r_S(x)$ is said to be amplitude term; $\omega_S(x)$ is said to be phase term. Both these terms are real valued with $r_S(x) \in [0, 1]$. The CFS S is represented as $\{(x, \mu_S(x))|x \in U\}$.

Fuzzy sets and intuitionistic fuzzy sets cannot handle imprecise, inconsistent, and incomplete information of periodic nature. These theories are applicable to different areas of science, but there is one major deficiency in both sets, that is, a lack of capability to model two-dimensional phenomena. To overcome this difficulty, Ramot introduce a complex fuzzy set. The phase term of CFS plays a vital role in defining the feature of the complex fuzzy set model. This term distinguish a CFS model from all other models available in literature. The potential of a complex fuzzy set for representing two-dimensional phenomena makes it superior to handle ambiguous and intuitive information that are prevalent in time-periodic phenomena. CFSs and their classes have an important role in applications, such as prediction of periodic events and advanced control systems. A CFS is quite similar to a Fourier transform; in fact, it is the specific form of the Fourier transform by restricting the range of Fourier transform to a complex unit disk; since Fourier transform has a lot of applications in various fields such as in signals and systems, communication, astronomy, geology, optics, etc. Therefore, a CFS can be used in certain models like Fourier transform. Several other real-life phenomena are ambiguous and cannot be modeled using one-dimensional variables. For example, items can be expressed by a set of measurements in pattern recognition and are seen as vectors in a multidimensional space. These multidimensional variables can not be expressed via a simple combination of variables, especially by consideration of fuzzy sets. These types of sets can be expressed via complex classes. A CFS is very useful for periodic phenomena. Ramot et al. recommended that the periodic problems or recurring problems phenomena can be modeled more accurately using the phase component of a complex fuzzy set memberships, such as expressing the effect of financial indicators of two countries on each other with the passage of time. He proposed that signal processing is another field of desirable application for a CFS. Moreover, it is used for expressing solar activity (solar maximum and solar minimum) through the average number of sunspot (Ramot et al. 2002). Dick proposed that one of the desirable applications of the CFSs is to use it for representing phenomena with relatively periodic behavior (Dick 2005). The traffic congestions in a big city is the periodic phenomena that never repeat themselves. Thus, complex fuzzy logic can be used to solve certain classes of problems more efficiently and accurately rather than a fuzzy logic.

The set theoretic operations on a CFS such as intersection, union, complement, rotation, and reflection were first introduced by Ramot et al. (2002). Also De Morgan Laws for a CFS and CF relation are presented in Ramot et al. (2002).

In this paper, we define some new operations and laws for a complex fuzzy set such as distributive property, idempotent property, absorption law, equivalence formula, simple difference, symmetric difference formula, etc. with respect to standard complex fuzzy union, intersection, and complement function. We present some basic results of CFS regarding CF union, CF intersection, and CF complement, and also discuss particular examples of these operations. Moreover, we develop an algorithm using the discrete Fourier transform matrix introduced in Selesnick and Schuller (2001). Then, we apply it to a problem in signals and systems.

2 Preliminaries

We will discuss here the basic set theoretic operations and laws of CFS, and also discuss particular examples of these operations and laws.

Definition 1 (Garrido 2007) A review of traditional fuzzy simple difference: For any two fuzzy sets R and S, the simple difference is defined as $R \setminus S = R \cap S^c$, where "$c$" represents the standard complement.

Definition 2 Consider two CFSs R and S, $\mu_R(x)$, $\mu_S(x)$ denote the membership functions R and S. The simple difference $R \setminus S$ of these two CFS R and S is defined as:

$$R \setminus S = R \cap S^c = \mu_R(x) \star \mu_{S^c}(x),$$

where $\star$ represents the t-norm (Dubois and Prade 2000) and $\mu_{S^c}(x)$ denotes the membership function and x belongs to S^c.

Example 1 Let

$$R = \frac{0.5e^{i\pi}}{u} + \frac{0.8e^{i\frac{\pi}{2}}}{v} + \frac{0.4e^{i\frac{3\pi}{2}}}{w}$$

and

$$S = \frac{0.6e^{i2\pi}}{u} + \frac{1e^{i\frac{5\pi}{2}}}{v} + \frac{0.3e^{i2\pi}}{w}$$

be two CFSs. Using standard intersection and standard complement function with max function for determining the phase term, the simple difference is:

$$R \setminus S = R \cap S^c = \left[\frac{0.5e^{i\pi}}{u} + \frac{0.8e^{i\frac{\pi}{2}}}{v} + \frac{0.4e^{i\frac{3\pi}{2}}}{w}\right] \star \left[\frac{0.4e^{i2\pi}}{u} + \frac{0e^{i\frac{5\pi}{2}}}{v} + \frac{0.7e^{i2\pi}}{w}\right]$$

$$= \frac{0.4e^{i2\pi}}{u} + \frac{0e^{i\frac{5\pi}{2}}}{v} + \frac{0.4e^{i2\pi}}{w}.$$

Definition 3 (Garrido 2007) For any two fuzzy sets R and S, the bounded difference is defined as:

$$\mu_{R\diamond S}(x) = Max[0, \mu_R(x) \diagdown \mu_S(x)],$$

where $\mu_R(x)$ and $\mu_S(x)$ denotes the membership function to which x is a member of R and S.

Definition 4 Let R and S be two CFSs, and let μ_R, μ_S denote the membership functions of R and S. The bounded difference of these two CFSs R and S is defined as:

$$\mu_{R\diamond S}(x) = Max[0, \mu_R(x) \diagdown \mu_S(x)] = Max[0, (r_R(x) \diagdown r_S(x))e^{i\omega_{R\diagdown S}(x)}].$$

Here, $[\mu_R(x) \diagdown \mu_S(x)]$ is same as the bounded difference of traditional fuzzy set. However, the main problem is to find the phase term. As the functions defined in "complex fuzzy set"(Ramot et al. 2002) for determining the phase term of complex fuzzy union and complex fuzzy intersection, here, these functions are also applicable for finding the phase term, so for calculating the phase term, we use the following functions:

(i) Sum. $\omega_{R\diagdown S} = \omega_R + \omega_S$.
(ii) Max. $\omega_{R\diagdown S} = \max(\omega_R, \omega_S)$.
(iii) Min. $\omega_{R\diagdown S} = \min(\omega_R, \omega_S)$.
(iv) "Winner take all". $\omega_{R\diagdown S} = \begin{cases} \omega_R; & r_R > r_S \\ \omega_S; & r_S > r_R \end{cases}$

The following are also acceptable possibilities for finding the phase term.

(i) Difference: $\omega_{R\diagdown S} = \omega_R - \omega_S$.
(ii) Average: $\omega_{R\diagdown S} = \frac{\omega_R + \omega_S}{2}$.
(iii) Weighted Average: $\omega_{R\diagdown S} = \frac{r_R\omega_R + r_S\omega_S}{r_R + r_S}$.

Example 2 Let

$$R = \frac{0e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i\frac{3\pi}{2}}}{w}$$

and

$$S = \frac{1e^{i2\pi}}{u} + \frac{0.6e^{i\frac{3\pi}{2}}}{v} + \frac{0.2e^{i\frac{5\pi}{2}}}{w}$$

be two CFSs. The bounded difference of these two CFSs is:

$$\mu_{R\diamond S}(x) = \frac{0e^{i2\pi}}{u} + \frac{0e^{i\frac{3\pi}{2}}}{v} + \frac{0.6e^{i\frac{5\pi}{2}}}{w}.$$

Definition 5 (Garrido 2007) Let R and S be any two fuzzy sets; the disjoint sum is defined as:

$$\mu_{R\otimes S}(x) = \mid \mu_R(x) \diagdown \mu_S(x) \mid,$$

where $\mu_R(x)$ and $\mu_S(x)$ denote the membership functions to which x is a member of R and S.

Definition 6 Let R and S be two CFSs, and let μ_R, μ_S denote the membership functions of R and S. The disjoint sum of these two CFSs R and S is defined as:

$$\mu_{R \otimes S}(x) =| \mu_R(x) \smallsetminus \mu_S(x) |=| [r_R(x) \smallsetminus r_S(x)] e^{i\omega_{R \smallsetminus S}(x)} | .$$

The term $| r_R(x) \smallsetminus r_S(x) |$ is same as traditional FS. To find the phase term for disjoint sum, the functions defined in bounded difference are also applicable here.

Example 3 Let

$$R = \frac{1e^{i\frac{5\pi}{2}}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0e^{i\pi}}{w}$$

and

$$S = \frac{0.5e^{i\pi}}{u} + \frac{0.6e^{i\frac{3\pi}{2}}}{v} + \frac{0.9e^{i\frac{\pi}{2}}}{w}$$

be two CFSs. Using the max function for calculating the phase term, the disjoint sum of these two fuzzy sets is:

$$\mu_{R \otimes S}(x) = \frac{0.5}{u} + \frac{0.1}{v} + \frac{0.9}{w}.$$

Definition 7 (Garrido 2007) For any two fuzzy sets R and S, the disjunctive sum is defined as:

$$R \Delta S = (R \cap S^c) \cup (R^c \cap S) = (R \star S^c) \oplus (R^c \star S),$$

where $\star$, $\oplus$ and c represent the standard intersection, standard union, and standard complement function.

Definition 8 Let R and S, be any two CFSs and $\mu_R(x)$, $\mu_S(x)$ denote the membership functions of R and S. Let $R \Delta S$ represents the disjunctive sum of CFSs R and S, defined as $R \Delta S = (R \cap S^c) \cup (R^c \cap S)$. The membership function of $R \Delta S$ is:

$$\begin{aligned} \mu_{R \Delta S}(x) &= [\mu_{R \cap S^c}(x) \oplus \mu_{R^c \cap S}(x)] \\ \mu_{R \Delta S}(x) &= [r_R(x) \star r_{S^c}(x)] e^{i\omega_{R \cap S^c}(x)} \oplus [r_{R^c}(x) \star r_S(x)] e^{i\omega_{R^c \cap S}(x)}, \end{aligned} \quad (3)$$

where $\star$, $\oplus$, and c represents the standard intersection, standard union, and standard complement function, respectively, of a CFS.

Example 4 Suppose

$$R = \frac{1e^{i0}}{z_1} + \frac{0.4e^{i\pi}}{z_2} + \frac{0.8e^{i\frac{\pi}{2}}}{z_3}$$

and

$$S = \frac{0.2e^{i\frac{3\pi}{2}}}{z_1} + \frac{0.3e^{i2\pi}}{z_2} + \frac{1e^{i\frac{\pi}{4}}}{z_3}$$

be two CFSs. Then the disjunctive sum of these two CFSs is defined as:

$$\mu_{R \Delta S}(z) = [r_R(z) \star r_{S^c}(z)] e^{i\omega_{R \cap S^c}(z)} \oplus [r_{R^c}(z) \star r_S(z)] e^{i\omega_{R^c \cap S}(z)}.$$

Using standard intersection,standard union, and standard complement function with the same function for determining the phase term, we have:

$$\mu_{R\Delta S}(z) = \left[\frac{0.8e^{i\frac{3\pi}{2}}}{z_1} + \frac{0.4e^{i2\pi}}{z_2} + \frac{0e^{i\frac{\pi}{2}}}{z_3}\right] \oplus \left[\frac{0e^{i\frac{3\pi}{2}}}{z_1} + \frac{0.3e^{i2\pi}}{z_2} + \frac{0.2e^{i\frac{\pi}{2}}}{z_3}\right]$$

$$\mu_{R\Delta S}(x) = \frac{0.8e^{i\frac{3\pi}{2}}}{z_1} + \frac{0.4e^{i2\pi}}{z_2} + \frac{0.2e^{i\frac{\pi}{2}}}{z_3}.$$

Definition 9 For any two CFSs R and S, the equivalence formula is:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Using the standard union and standard intersection function with the same function for determining the phase term along with standard complex fuzzy complement, the equivalence formula is hold. The L.H.S and R.H.S of equivalence formula for CFSs R and S are given by:

$$\begin{aligned}
\mu_{R^c\cup S}(y) \cap \mu_{R\cup S^c}(y) &= [\mu_{R^c}(y) \oplus \mu_S(y)]\star[\mu_R(y) \oplus \mu_{S^c}(y)] \\
&= [r_{R^c}(y)e^{i\omega_{R^c}(y)} \oplus r_S(y)e^{i\omega_S(y)}]\star[r_R(y)e^{i\omega_R(y)} \oplus r_{S^c}(y)e^{i\omega_{S^c}(y)}] \\
\mu_{R^c\cap S^c}(y) \cup \mu_{R\cap S}(y) &= [\mu_{R^c}(y)\star\mu_{S^c}(y)] \oplus [\mu_R(y)\star\mu_S(y)] \\
&= [r_{R^c}(y)e^{i\omega_{R^c}(y)}\star r_{S^c}(y)e^{i\omega_{S^c}(y)}] \oplus [r_R(y)e^{i\omega_R(y)}\star r_S(y)e^{i\omega_S(y)}],
\end{aligned}$$

where $\star$, $\oplus$, and c represent the standard intersection, standard union, and standard complement functions, respectively.

Example 5 Suppose

$$R = \frac{0.8e^{i\frac{\pi}{4}}}{u} + \frac{0.6e^{i\frac{\pi}{6}}}{v} + \frac{0.9e^{i\frac{\pi}{2}}}{w} + \frac{1e^{i2\pi}}{t}$$

and

$$S = \frac{0.5e^{i\frac{3\pi}{2}}}{y_1} + \frac{0.7e^{i\pi}}{y_2} + \frac{0.1e^{i0}}{y_3} + \frac{0.3e^{i\frac{3\pi}{4}}}{y_4}$$

be two complex fuzzy sets. The equivalence formula is hold using standard union, standard intersection, and standard complement function with the same function for determining the phase term. The equivalence formula is:

$$\begin{aligned}
&(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S) \\
&(R^c \cup S) \cap (R \cup S^c) = \left[\frac{0.5e^{i\frac{3\pi}{2}}}{y_1} + \frac{0.7e^{i\pi}}{y_2} + \frac{0.1e^{i\frac{\pi}{2}}}{y_3} + \frac{0.3e^{i2\pi}}{y_4}\right] \\
&\qquad \star \left[\frac{0.8e^{i\frac{3\pi}{2}}}{y_1} + \frac{0.6e^{i\pi}}{y_2} + \frac{0.9e^{i\frac{\pi}{2}}}{y_3} + \frac{1e^{i2\pi}}{y_4}\right]
\end{aligned}$$

$$(R^c \cup S) \cap (R \cup S^c) = \frac{0.5e^{i\frac{3\pi}{2}}}{y_1} + \frac{0.6e^{i\pi}}{y_2} + \frac{0.1e^{i\frac{\pi}{2}}}{y_3} + \frac{0.3e^{i2\pi}}{y_4}$$

$$(R^c \cap S^c) \cup (R \cap S) = \left[\frac{0.2e^{i\frac{3\pi}{2}}}{y_1} + \frac{0.3e^{i\pi}}{y_2} + \frac{0.1e^{i\frac{\pi}{2}}}{y_3} + \frac{0e^{i2\pi}}{y_4}\right]$$

$$\oplus\left[\frac{0.5e^{i\frac{3\pi}{2}}}{y_1}+\frac{0.6e^{i\pi}}{y_2}+\frac{0.1e^{i\frac{\pi}{2}}}{y_3}+\frac{0.3e^{i2\pi}}{y_4}\right] \tag{1}$$

$$(R^c\cap S^c)\cup(R\cap S)=\frac{0.5e^{i\frac{3\pi}{2}}}{y_1}+\frac{0.6e^{i\pi}}{y_2}+\frac{0.1e^{i\frac{\pi}{2}}}{y_3}+\frac{0.3e^{i2\pi}}{y_4}. \tag{2}$$

From (1) and (2), we have:

$$(R^c\cup S)\cap(R\cup S^c)=(R^c\cap S^c)\cup(R\cap S).$$

Definition 10 Symmetrical difference formula for two CFSs R and S is given by:

$$(R^c\cap S)\cup(R\cap S^c)=(R^c\cup S^c)\cap(R\cup S).$$

Using standard union and standard intersection function along with standard complement function with the same function for determining the phase term, then the symmetrical difference formula is hold for complex fuzzy sets. The L.H.S and R.H.S of symmetrical difference formula for CFSs R and S are given by:

$$\begin{aligned}
[\mu_{R^c\cap S}(x)\cup\mu_{R\cap S^c}(x)] &= [\mu_{R^c}(x)\star\mu_S(x)]\oplus[\mu_R(x)\star\mu_{S^c}(x)]\\
&= [r_{R^c}(x)e^{i\omega_{R^c}(x)}\star r_S(x)e^{i\omega_S(x)}]\oplus[r_R(x)e^{i\omega_R(x)}\star r_{S^c}(x)e^{i\omega_{S^c}(x)}]\\
[\mu_{R^c\cup S^c}(x)\cap\mu_{R\cup S}(x)] &= [\mu_{R^c}(x)\oplus\mu_{S^c}(x)]\star[\mu_R(x)\oplus\mu_S(x)]\\
&= [r_{R^c}(x)e^{i\omega_{R^c}(x)}\oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}]\star[r_R(x)e^{i\omega_R(x)}\oplus r_S(x)e^{i\omega_S(x)}],
\end{aligned}$$

where $\star$, $\oplus$, and c represent standard intersection, standard union, and standard complement function.

Example 6 Let

$$R=\frac{0.5e^{i\pi}}{u}+\frac{0.8e^{i\frac{\pi}{6}}}{v}+\frac{1e^{i\frac{3\pi}{2}}}{w},$$

and

$$S=\frac{0.2e^{i2\pi}}{u}+\frac{0.6e^{i\pi}}{v}+\frac{0.4e^{i\frac{\pi}{4}}}{w}$$

be two complex fuzzy sets. Using standard union and standard intersection and standard complement function with the same function for determining the phase term, we have:

$$(R^c\cap S)\cup(R\cap S^c)=\left[\frac{0.2e^{i2\pi}}{u}+\frac{0.2e^{i\pi}}{v}+\frac{0e^{i\frac{3\pi}{2}}}{w}\right]\oplus\left[\frac{0.5e^{i2\pi}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.6e^{i\frac{3\pi}{2}}}{w}\right]$$

$$(R^c\cap S)\cup(R\cap S^c)=\frac{0.5e^{i2\pi}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.6e^{i\frac{3\pi}{2}}}{w} \tag{1}$$

$$(R^c\cup S^c)\cap(R\cup S)=\left[\frac{0.8e^{i2\pi}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.6e^{i\frac{3\pi}{2}}}{w}\right]\star\left[\frac{0.5e^{i2\pi}}{u}+\frac{0.8e^{i\pi}}{v}+\frac{1e^{i\frac{3\pi}{2}}}{w}\right]$$

$$(R^c\cup S^c)\cap(R\cup S)=\frac{0.5e^{i2\pi}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.6e^{i\frac{3\pi}{2}}}{w}. \tag{2}$$

From (1) and (2), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Definition 11 Complex fuzzy sets satisfy distributive laws using standard union and standard intersection function with the same function for determining the phase term.

Let R, S, and T be three CFSs. Then, the distributive laws are:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$$

and

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

These two laws are called distributive law of union over intersection and distributive law of intersection over union.

If $R = r_R(x)e^{i\omega_R(x)}$, $S = r_S(x)e^{i\omega_S(x)}$ and $T = r_T(x)e^{i\omega_T(x)}$, then the L.H.S and R.H.S distributive law of union over intersection become:

$$\begin{aligned}
[\mu_R(x) \oplus (\mu_S(x) \star \mu_T(x))] &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)})] \\
[\mu_R(x) \oplus \mu_S(x)] \star [\mu_R(x) \oplus \mu_T(x)] &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\
&\quad \star [r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}].
\end{aligned}$$

Also, the L.H.S and R.H.S of distributive law of intersection over union are:

$$\begin{aligned}
[\mu_R(x) \star (\mu_S(x) \oplus \mu_R(x))] &= [r_R(x)e^{i\omega_R(x)} \star (r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})] \\
[\mu_R(x) \star \mu_S(x)] \oplus [\mu_R(x) \star \mu_T(x)] &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\
&\quad \oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}],
\end{aligned}$$

where $\star$ and $\oplus$ represent the standard intersection and standard union function, respectively.

Example 7 Let

$$R = \frac{1e^{i0}}{u} + \frac{0.4e^{i\pi}}{v} + \frac{0.8e^{i\frac{\pi}{2}}}{w},$$

$$S = \frac{0.2e^{i\frac{3\pi}{4}}}{u} + \frac{0.3e^{i2\pi}}{v} + \frac{1e^{i\frac{\pi}{6}}}{w}$$

and

$$T = \frac{0.6e^{i\frac{3\pi}{2}}}{u} + \frac{0.4e^{i\frac{\pi}{4}}}{v} + \frac{0.5e^{i\frac{\pi}{5}}}{w}$$

be three CFSs. To satisfies the distributive law of union over intersection and intersection over union, using standard union and standard intersection function with the same function for determining the phase term.

Now, the distributive law of union over intersection is:

$$R\cup(S\cap T)=\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\oplus\left(\left[\frac{0.2e^{i\frac{3\pi}{4}}}{u}+\frac{0.3e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{6}}}{w}\right]\right.$$
$$\left.\star\left[\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i\frac{\pi}{4}}}{v}+\frac{0.5e^{i\frac{\pi}{5}}}{w}\right]\right)$$
$$=\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\oplus\left[\frac{0.2e^{i\frac{3\pi}{2}}}{u}+\frac{0.3e^{i2\pi}}{v}+\frac{0.5e^{i\frac{\pi}{5}}}{w}\right]$$
$$R\cup(S\cap T)=\frac{1e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i2\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w} \tag{1}$$

$$(R\cup S)\cap(R\cup T)=\left(\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\oplus\left[\frac{0.2e^{i\frac{3\pi}{4}}}{u}+\frac{0.3e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{6}}}{w}\right]\right)$$
$$\star\left(\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\oplus\left[\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i\frac{\pi}{4}}}{v}+\frac{0.5e^{i\frac{\pi}{5}}}{w}\right]\right)$$
$$=\left[\frac{1e^{i\frac{3\pi}{4}}}{u}+\frac{0.4e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{2}}}{w}\right]\star\left[\frac{1e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]$$
$$(R\cup S)\cap(R\cup T)=\frac{1e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i2\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}. \tag{2}$$

From (1) and (2), we have:

$$R\cup(S\cap T)=(R\cup S)\cap(R\cup T).$$

Now, the distributive property of intersection over union is:

$$R\cap(S\cup T)=\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\star\left(\left[\frac{0.2e^{i\frac{3\pi}{4}}}{u}+\frac{0.3e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{6}}}{w}\right]\right.$$
$$\left.\oplus\left[\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i\frac{\pi}{4}}}{v}+\frac{0.5e^{i\frac{\pi}{5}}}{w}\right]\right)$$
$$=\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\star\left[\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{5}}}{w}\right]$$
$$R\cap(S\cup T)=\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i2\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w} \tag{3}$$

$$(R\cap S)\cup(R\cap T)=\left(\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\star\left[\frac{0.2e^{i\frac{3\pi}{4}}}{u}+\frac{0.3e^{i2\pi}}{v}+\frac{1e^{i\frac{\pi}{6}}}{w}\right]\right)$$
$$\oplus\left(\left[\frac{1e^{i0}}{u}+\frac{0.4e^{i\pi}}{v}+\frac{0.8e^{i\frac{\pi}{2}}}{w}\right]\star\left[\frac{0.6e^{i\frac{3\pi}{2}}}{u}+\frac{0.4e^{i\frac{\pi}{4}}}{v}+\frac{0.5e^{i\frac{\pi}{5}}}{w}\right]\right)$$

$$= \left[\frac{0.2e^{i\frac{3\pi}{4}}}{u} + \frac{0.3e^{i2\pi}}{v} + \frac{0.8e^{i\frac{\pi}{2}}}{w}\right] \oplus \left[\frac{0.6e^{i\frac{3\pi}{2}}}{u} + \frac{0.4e^{i\pi}}{v} + \frac{0.5e^{i\frac{\pi}{2}}}{w}\right]$$

$$(R \cap S) \cup (R \cap T) = \frac{0.6e^{i\frac{3\pi}{2}}}{u} + \frac{0.4e^{i\pi}}{v} + \frac{0.8e^{i\frac{\pi}{2}}}{w}. \tag{4}$$

From (3) and (4), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Definition 12 For any CFS R, idempotent laws hold using standard union and standard intersection function.

The idempotent law of union for a CFS R is $R \cup R = R$ and the idempotent law of intersection is $R \cap R = R$.

If a grade value of R is $\mu_R(x) = r_R(x)e^{i\omega_R(x)}$,the idempotent law of union becomes:

$$\mu_R(x) = \mu_{R\cup R}(x).$$

To prove this, we have:

$$\mu_{R\cup R}(x) = [\mu_R(x) \oplus \mu_R(x)] = [r_R(x)e^{i\omega_R(x)} \oplus r_R(x)e^{i\omega_R(x)}] = r_R(x)e^{i\omega_R(x)} = \mu_R(x).$$

Also, the idempotent law of intersection is:

$$\mu_R(x) = \mu_{R\cap R}(x).$$

Now, to prove this, we have:

$$\mu_{R\cap R}(x) = [\mu_R(x)\star\mu_R(x)] = [r_R(x)e^{i\omega_R(x)}\star r_R(x)e^{i\omega_R(x)}] = r_R(x)e^{i\omega_R(x)} = \mu_R(x).$$

Here, $\star$ and $\oplus$ represent the standard intersection and standard union function.

Example 8 Let

$$R = \frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w}$$

be a complex fuzzy set. Using standard union and standard intersection function, the idempotent law of union is:

$$R \cup R = \left[\frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w}\right] \oplus \left[\frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w}\right]$$

$$= \frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w} = R.$$

The idempotent law of intersection is:

$$R \cap R = \left[\frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w}\right] \star \left[\frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w}\right]$$

$$= \frac{1e^{i\pi}}{u} + \frac{0.5e^{i\frac{\pi}{2}}}{v} + \frac{0.8e^{i0}}{w} = R.$$

The idempotent law of union and intersection holds for a CFS R.

Definition 13 CFS satisfied the involution law using standard complement function.

The involution law for a CFS R is $(R^c)^c = R$

If a grade value of a CFS R is $\mu_R(x) = r_R(x)e^{i\omega_R(x)}$,the involution law is:

$$r_R(x)e^{i\omega_R(x)} = r_{R^{c^c}}(x)e^{i\omega_{R^{c^c}}(x)},$$

where c represents standard complement function.

Example 9 Let

$$R = \frac{0.1e^{i0}}{u} + \frac{0.5e^{i\pi}}{v} + \frac{1e^{i\frac{\pi}{2}}}{w} + \frac{0.8e^{i\frac{3\pi}{2}}}{t}$$

be a CFS. The involution law is satisfied using standard complement function. Now

$$R^c = \mu_{R^c}(x) = \frac{0.9e^{i0}}{u} + \frac{0.5e^{i\pi}}{v} + \frac{0e^{i\frac{\pi}{2}}}{w} + \frac{0.2e^{i\frac{3\pi}{2}}}{t}$$

$$R^{c^c} = \mu_{R^{c^c}}(x) = \frac{0.1e^{i0}}{u} + \frac{0.5e^{i\pi}}{v} + \frac{1e^{i\frac{\pi}{2}}}{w} + \frac{0.8e^{i\frac{3\pi}{2}}}{t} = R.$$

3 Main results

Proposition 1 *The bounded difference of two CFSs is always a fuzzy set.*

Proof Let R and S be two CFSs. The bounded difference of these two CFSs is:

$\mu_{R\otimes S}(x) = \mid \mu_R(x) \smallsetminus \mu_S(x) \mid = \mid [r_R(x) \smallsetminus r_S(x)]e^{i\omega_{R\smallsetminus S}(x)} \mid = \mid r_R(x) \smallsetminus r_S(x) \mid\mid e^{i\omega_{R\smallsetminus S}(x)} \mid .$

As $\mid e^{i\omega_{R\smallsetminus S}(x)} \mid = 1$. Thus

$$\mu_{R\otimes S}(x) = \mid r_R(x) \smallsetminus r_S(x) \mid\mid e^{i\omega_{R\smallsetminus S}(x)} \mid = \mid r_R(x) \smallsetminus r_S(x) \mid .$$

□

Proposition 2 *For CFSs R and S over a crisp set U, the standard union, standard intersection, and standard complement function with the same function for determining the phase term satisfy the symmetrical difference formula.*

Proof Suppose R and S be two CFSs. To prove the symmetrical difference formula $(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S)$, eight cases arise here. Using Max function for determining the phase term. □

Case 1

$$r_R(x) \leq r_S(x), \omega_R(x) \leq \omega_S(x), r_{R^c}(x) \leq r_S(x), r_{S^c}(x) \leq r_R(x), r_{S^c}(x) \leq r_{R^c}(x)$$

$$\begin{aligned}(R^c \cap S) \cup (R \cap S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_{R^c}(x)e^{i\omega_S(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}\end{aligned}$$

$$(R^c \cap S) \cup (R \cap S^c) = r_{R^c}(x)e^{i\omega_{S^c}(x)} \tag{1}$$

$$\begin{aligned}(R^c \cup S^c) \cap (R \cup S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{S^c}(x)} \star r_S(x)e^{i\omega_S(x)}\end{aligned}$$

$$(R^c \cup S^c) \cap (R \cup S) = r_{R^c}(x)e^{i\omega_{S^c}(x)}. \tag{2}$$

From (1) and (2), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 2

$$\begin{aligned} & r_R(x) \le r_S(x), \omega_R(x) \le \omega_S(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{S^c}(x) \le r_{R^c}(x) \\ & (R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ & \quad = r_S(x)e^{i\omega_S(x)} \oplus r_R(x)e^{i\omega_{S^c}(x)} \\ & (R^c \cap S) \cup (R \cap S^c) = r_S(x)e^{i\omega_S(x)} \end{aligned} \tag{3}$$

$$\begin{aligned} (R^c \cup S^c) \cap (R \cup S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{S^c}(x)} \star r_S(x)e^{i\omega_S(x)} \\ (R^c \cup S^c) \cap (R \cup S) &= r_S(x)e^{i\omega_S(x)}. \end{aligned} \tag{4}$$

From (3) and (4) we have

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 3

$$\begin{aligned} & r_R(x) \le r_S(x), \omega_S(x) \le \omega_R(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{S^c}(x) \le r_{R^c}(x) \\ & (R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ & \quad = r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_R(x)} \\ & (R^c \cap S) \cup (R \cap S^c) = r_{R^c}(x)e^{i\omega_{R^c}(x)} \end{aligned} \tag{5}$$

$$\begin{aligned} (R^c \cup S^c) \cap (R \cup S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)} \\ (R^c \cup S^c) \cap (R \cup S) &= r_{R^c}(x)e^{i\omega_{R^c}(x)}. \end{aligned} \tag{6}$$

From (5) and (6), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 4

$$\begin{aligned} & r_R(x) \le r_S(x), \omega_S(x) \le \omega_R(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{S^c}(x) \le r_{R^c}(x) \\ & (R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ & \quad = r_S(x)e^{i\omega_{R^c}(x)} \oplus r_R(x)e^{i\omega_R(x)} (R^c \cap S) \cup (R \cap S^c) = r_S(x)e^{i\omega_R(x)} \end{aligned} \tag{7}$$

$$\begin{aligned} (R^c \cup S^c) \cap (R \cup S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_R(x)} \\ (R^c \cup S^c) \cap (R \cup S) &= r_S(x)e^{i\omega_R(x)}. \end{aligned} \tag{8}$$

From (7) and (8), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 5

$$\begin{aligned}
&r_S(x) \le r_R(x), \omega_S(x) \le \omega_R(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{R^c}(x) \le r_{S^c}(x)\\
&(R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}]\\
&\qquad = r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_R(x)}\\
&(R^c \cap S) \cup (R \cap S^c) = r_{S^c}(x)e^{i\omega_R(x)} \qquad (9)
\end{aligned}$$

$$\begin{aligned}
&(R^c \cup S^c) \cap (R \cup S) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\\
&\qquad = r_{S^c}(x)e^{i\omega_{R^c}(x)} \star r_R(x)e^{i\omega_R(x)}\\
&(R^c \cup S^c) \cap (R \cup S) = r_{S^c}(x)e^{i\omega_R(x)}. \qquad (10)
\end{aligned}$$

From (9) and (10), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 6

$$\begin{aligned}
&r_S(x) \le r_R(x), \omega_S(x) \le \omega_R(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{R^c}(x) \le r_{S^c}(x)\\
&(R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}]\\
&\qquad = r_S(x)e^{i\omega_{R^c}(x)} \oplus r_R(x)e^{i\omega_R(x)}\\
&(R^c \cap S) \cup (R \cap S^c) = r_R(x)e^{i\omega_R(x)} \qquad (11)\\
&(R^c \cup S^c) \cap (R \cup S) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\\
&\qquad = r_{S^c}(x)e^{i\omega_{R^c}(x)} \star r_R(x)e^{i\omega_R(x)}\\
&(R^c \cup S^c) \cap (R \cup S) = r_R(x)e^{i\omega_R(x)}. \qquad (12)
\end{aligned}$$

From (11) and (12), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 7

$$\begin{aligned}
&r_S(x) \le r_R(x), \omega_R(x) \le \omega_S(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{R^c}(x) \le r_{S^c}(x)\\
&(R^c \cap S) \cup (R \cap S^c) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}]\\
&\qquad = r_{R^c}(x)e^{i\omega_S(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}\\
&(R^c \cap S) \cup (R \cap S^c) = r_{S^c}(x)e^{i\omega_{S^c}(x)} \qquad (13)
\end{aligned}$$

$$\begin{aligned}
&(R^c \cup S^c) \cap (R \cup S) = [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\\
&\qquad = r_{S^c}(x)e^{i\omega_{S^c}(x)} \star r_R(x)e^{i\omega_S(x)}\\
&(R^c \cup S^c) \cap (R \cup S) = r_{S^c}(x)e^{i\omega_{S^c}(x)}. \qquad (14)
\end{aligned}$$

From (13) and (14), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Case 8

$$r_S(x) \le r_R(x), \omega_R(x) \le \omega_S(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{R^c}(x) \le r_{S^c}(x)$$

$$\begin{aligned}(R^c \cap S) \cup (R \cap S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_S(x)e^{i\omega_S(x)} \oplus r_R(x)e^{i\omega_{S^c}(x)} \\ (R^c \cap S) \cup (R \cap S^c) &= r_R(x)e^{i\omega_S(x)} \end{aligned} \tag{15}$$

$$\begin{aligned}(R^c \cup S^c) \cap (R \cup S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\ &= r_{S^c}(x)e^{i\omega_{S^c}(x)} \star r_R(x)e^{i\omega_S(x)} \\ (R^c \cup S^c) \cap (R \cup S) &= r_R(x)e^{i\omega_S(x)}. \end{aligned} \tag{16}$$

From (15) and (16), we have:

$$(R^c \cap S) \cup (R \cap S^c) = (R^c \cup S^c) \cap (R \cup S).$$

Thus, in all cases, the symmetrical difference formula holds.

Proposition 3 *For CFSs R and S over a crisp set U, the standard union, standard intersection, and standard complement function with the same function for determining the phase term satisfy the equivalence formula.*

Proof Suppose R and S be two CFSs. To prove the equivalence formula $(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S)$, eight cases arise here. Using Max function for determining the phase term. □

Case 1

$$r_R(x) \le r_S(x), \omega_R(x) \le \omega_S(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{S^c}(x) \le r_{R^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_S(x)e^{i\omega_S(x)} \star r_R(x)e^{i\omega_{S^c}(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_R(x)e^{i\omega_S(x)} \end{aligned} \tag{1}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &= r_{S^c}(x)e^{i\omega_{S^c}(x)} \oplus r_R(x)e^{i\omega_S(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_R(x)e^{i\omega_S(x)}. \end{aligned} \tag{2}$$

From (1) and (2), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 2

$$r_R(x) \le r_S(x), \omega_R(x) \le \omega_S(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{S^c}(x) \le r_{R^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_{R^c}(x)e^{i\omega_S(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_{S^c}(x)e^{i\omega_{S^c}(x)} \end{aligned} \tag{3}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &= r_{S^c}(x)e^{i\omega_{S^c}(x)} \oplus r_R(x)e^{i\omega_S(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_{S^c}(x)e^{i\omega_{S^c}(x)}. \end{aligned} \tag{4}$$

From (3) and (4), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 3

$$r_R(x) \le r_S(x), \omega_S(x) \le \omega_R(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{S^c}(x) \le r_{R^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_S(x)e^{i\omega_{R^c}(x)} \star r_R(x)e^{i\omega_R(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_R(x)e^{i\omega_R(x)} \qquad (5)\end{aligned}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &= r_{S^c}(x)e^{i\omega_{R^c}(x)} \oplus r_R(x)e^{i\omega_R(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_R(x)e^{i\omega_R(x)}. \qquad (6)\end{aligned}$$

From (5) and (6), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 4

$$r_R(x) \le r_S(x), \omega_S(x) \le \omega_R(x), r_S(x) \le r_{R^c}(x), r_R(x) \le r_{S^c}(x), r_{S^c}(x) \le r_{R^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_R(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_{S^c}(x)e^{i\omega_R(x)} \qquad (7)\end{aligned}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &= r_{S^c}(x)e^{i\omega_{R^c}(x)} \oplus r_R(x)e^{i\omega_R(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_{S^c}(x)e^{i\omega_R(x)}. \qquad (8)\end{aligned}$$

From (7) and (8), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 5

$$r_S(x) \le r_R(x), \omega_S(x) \le \omega_R(x), r_{R^c}(x) \le r_S(x), r_{S^c}(x) \le r_R(x), r_{R^c}(x) \le r_{S^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_S(x)e^{i\omega_{S^c}(x)} \star r_R(x)e^{i\omega_R(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_S(x)e^{i\omega_{R^c}(x)} \qquad (9)\end{aligned}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_R(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_S(x)e^{i\omega_{R^c}(x)}. \qquad (10)\end{aligned}$$

From (9) and (10), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 6

$$r_S(x) \leq r_R(x), \omega_S(x) \leq \omega_R(x), r_S(x) \leq r_{R^c}(x), r_R(x) \leq r_{S^c}(x), r_{R^c}(x) \leq r_{S^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}]\star[r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)}\star r_{S^c}(x)e^{i\omega_R(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_{R^c}(x)e^{i\omega_R(x)} \end{aligned} \tag{11}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)}\star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_{R^c}(x)e^{i\omega_R(x)}. \end{aligned} \tag{12}$$

From (11) and (12), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 7

$$r_S(x) \leq r_R(x), \omega_R(x) \leq \omega_S(x), r_{R^c}(x) \leq r_S(x), r_{S^c}(x) \leq r_R(x), r_{R^c}(x) \leq r_{S^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}]\star[r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_S(x)e^{i\omega_S(x)}\star r_R(x)e^{i\omega_{S^c}(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_S(x)e^{i\omega_S(x)} \end{aligned} \tag{13}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)}\star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{S^c}(x)} \oplus r_S(x)e^{i\omega_S(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_S(x)e^{i\omega_S(x)}. \end{aligned} \tag{14}$$

From (13) and (14), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Case 8

$$r_S(x) \leq r_R(x), \omega_R(x) \leq \omega_S(x), r_S(x) \leq r_{R^c}(x), r_R(x) \leq r_{S^c}(x), r_{R^c}(x) \leq r_{S^c}(x)$$

$$\begin{aligned}(R^c \cup S) \cap (R \cup S^c) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)} \oplus r_S(x)e^{i\omega_S(x)}]\star[r_R(x)e^{i\omega_R(x)} \oplus r_{S^c}(x)e^{i\omega_{S^c}(x)}] \\ &= r_{R^c}(x)e^{i\omega_S(x)}\star r_{S^c}(x)e^{i\omega_{S^c}(x)} \\ (R^c \cup S) \cap (R \cup S^c) &= r_{R^c}(x)e^{i\omega_S(x)} \end{aligned} \tag{15}$$

$$\begin{aligned}(R^c \cap S^c) \cup (R \cap S) &= [r_{R^c}(x)e^{i\omega_{R^c}(x)}\star r_{S^c}(x)e^{i\omega_{S^c}(x)}] \oplus [r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}] \\ &= r_{R^c}(x)e^{i\omega_{S^c}(x)} \oplus r_S(x)e^{i\omega_S(x)} \\ (R^c \cap S^c) \cup (R \cap S) &= r_{R^c}(x)e^{i\omega_S(x)}. \end{aligned} \tag{16}$$

From (15) and (16), we have:

$$(R^c \cup S) \cap (R \cup S^c) = (R^c \cap S^c) \cup (R \cap S).$$

Thus, in all cases, the equivalence formula is hold.

Proposition 4 *The standard union and standard intersection of any finite number of CFSs is always a CFS.*

Proof (i) Let $R_1, R_2, \ldots, R_M$ be any M CFSs and $r_{R_1}(x)e^{i\omega_{R_1}(x)}, r_{R_2}(x)e^{i\omega_{R_2}(x)}, \ldots, r_{R_M}(x)e^{i\omega_{R_M}(x)}$ denote the membership functions of these complex fuzzy sets, respectively.

Suppose $r'_R(x) = \max[r_{R_1}(x), r_{R_2}(x), \ldots, r_{R_M}(x)]$ and $\omega'_R(x) = \max[\omega_{R_1}(x), \omega_{R_2}(x), \ldots, \omega_{R_M}(x)]$.

Now

$$\begin{aligned} R_1 \cup R_2 \cup \cdots \cup R_M &= r_{R_1}(x)e^{i\omega_{R_1}(x)} \oplus r_{R_2}(x)e^{i\omega_{R_2}(x)} \oplus \cdots \oplus r_{R_M}(x)e^{i\omega_{R_M}(x)} \\ &= [r_{R_1}(x) \oplus r_{R_2}(x) \oplus \cdots \oplus r_{R_M}(x)]e^{i[\omega_{R_1}(x)\oplus\omega_{R_2}(x)\oplus\cdots\oplus\omega_{R_M}(x)]} \\ &= r'_R(x)e^{i\omega'_R(x)} = R', \end{aligned}$$

which is also a CFS.

(ii) Now, we show that the finite intersection of any CFSs is always a complex fuzzy set. Let $R_1, R_2, \ldots, R_M$ be any M CFSs and $r_{R_1}(x)e^{i\omega_{R_1}(x)}, r_{R_2}(x)e^{i\omega_{R_2}(x)}, \ldots, r_{R_M}(x)e^{i\omega_{R_M}(x)}$ denote the membership functions of these complex fuzzy sets, respectively.

Suppose $r'_R(x) = \min[r_{R_1}(x), r_{R_2}(x), \ldots, r_{R_M}(x)]$ and $\omega'_R(x) = \max[\omega_{R_1}(x), \omega_{R_2}(x), \ldots, \omega_{R_M}(x)]$.

Now

$$\begin{aligned} R_1 \cap R_2 \cap \cdots \cap R_M &= r_{R_1}(x)e^{i\omega_{R_1}(x)} \star r_{R_2}(x)e^{i\omega_{R_2}(x)} \star \cdots \star r_{R_M}(x)e^{i\omega_{R_M}(x)} \\ &= [r_{R_1}(x) \star r_{R_2}(x) \star \cdots \star r_{R_M}(x)]e^{i[\omega_{R_1}(x)\star\omega_{R_2}(x)\star\cdots\star\omega_{R_M}(x)]} \\ &= r'_R(x)e^{i\omega'_R(x)} = R', \end{aligned}$$

which is also a CFS. □

Proposition 5 *For any two CFSs R and S, the standard union and standard intersection function with the same function for determining the phase term satisfy:*

$$\sum_{i=1, x_i \in U}^{M} \mid \mu_{R\cap S}(x_i) \mid \leq \sum_{i=1, x_i \in U}^{M} \mid \mu_{R\cup S}(x_i) \mid .$$

Proof The standard union and intersection functions are defined by the respective expressions:

$$\mu_{R\cup S}(x) = \max[\mu_R(x), \mu_S(x)]$$

and

$$\mu_{R\cap S}(x) = \min[\mu_R(x), \mu_S(x)].$$

As

$$\begin{aligned} &\mid \mu_{R\cap S}(u) \mid \leq \mid \mu_{R\cup S}(u) \mid \\ &\mid \mu_{R\cap S}(v) \mid \leq \mid \mu_{R\cup S}(v) \mid . \end{aligned}$$

Continuing this process up to M, we have:

$$\mid \mu_{R\cap S}(x_M) \mid \leq \mid \mu_{R\cup S}(x_M) \mid .$$

Adding all these inequalities, we get:

$$\sum_{i=1,x\in U}^{M} \mid \mu_{R\cap S}(x_i) \mid \leq \sum_{i=1,x\in U}^{M} \mid \mu_{R\cup S}(x_i) \mid .$$

□

Proposition 6 *For any CFSs R, S and T, the standard intersection and standard union functions with the same function for determining the phase term satisfy the distributive law.*

Proof To prove the distributive laws for any CFSs R, S and T, six cases arise here. □

First, we prove the distributive law of union over intersection.

Case 1

$$\begin{aligned} r_R(x) \leq r_S(x) \leq r_T(x), &\quad \omega_R(x) \leq \omega_S(x) \leq \omega_T(x) \\ R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_T(x)} \\ R \cup (S \cap T) &= r_S(x)e^{i\omega_T(x)} \end{aligned} \tag{1}$$

$$\begin{aligned} (R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}] \\ &= r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)} \\ (R \cup S) \cap (R \cup T) &= r_S(x)e^{i\omega_T(x)}. \end{aligned} \tag{2}$$

From (1) and (2), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Case 2

$$\begin{aligned} r_S(x) \leq r_T(x) \leq r_R(x), &\quad \omega_S(x) \leq \omega_T(x) \leq \omega_R(x) \\ R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_T(x)} \\ R \cup (S \cap T) &= r_R(x)e^{i\omega_R(x)} \end{aligned} \tag{3}$$

$$\begin{aligned} (R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \star [r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}] \\ &= r_R(x)e^{i\omega_R(x)} \star r_R(x)e^{i\omega_R(x)} \\ (R \cup S) \cap (R \cup T) &= r_R(x)e^{i\omega_R(x)}. \end{aligned} \tag{4}$$

From (3) and (4), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Case 3

$$\begin{aligned} r_R(x) \leq r_T(x) \leq r_S(x), &\quad \omega_R(x) \leq \omega_T(x) \leq \omega_S(x) \\ R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_S(x)} \end{aligned}$$

$$R \cup (S \cap T) = r_T(x)e^{i\omega_S(x)} \tag{5}$$

$$\begin{aligned}(R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\star[r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}]\\ &= r_S(x)e^{i\omega_S(x)}\star r_T(x)e^{i\omega_T(x)}\\ (R \cup S) \cap (R \cup T) &= r_T(x)e^{i\omega_S(x)}. \end{aligned} \tag{6}$$

From (5) and (6), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Case 4

$$\begin{aligned} r_T(x) \le r_S(x) \le r_R(x), \quad \omega_T(x) \le \omega_S(x) \le \omega_R(x)\\ R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)}\star r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}\\ R \cup (S \cap T) &= r_R(x)e^{i\omega_R(x)} \end{aligned} \tag{7}$$

$$\begin{aligned}(R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\\ &\quad\star[r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}]\\ &= r_R(x)e^{i\omega_R(x)}\star r_R(x)e^{i\omega_R(x)}\\ (R \cup S) \cap (R \cup r) &= r_R(x)e^{i\omega_R(x)}. \end{aligned} \tag{8}$$

From (7) and (8), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Case 5

$$\begin{aligned} r_S(x) \le r_R(x) \le r_T(x), \quad \omega_S(x) \le \omega_R(x) \le \omega_T(x)\\ R \cup (S \cap r) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)}\star r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_T(x)}\\ R \cup (S \cap T) &= r_R(x)e^{i\omega_T(x)} \end{aligned} \tag{9}$$

$$\begin{aligned}(R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\\ &\quad\star[r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}]\\ &= r_R(x)e^{i\omega_R(x)}\star r_T(x)e^{i\omega_T(x)}\\ (R \cup S) \cap (R \cup T) &= r_R(x)e^{i\omega_T(x)}. \end{aligned} \tag{10}$$

From (9) and (10), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Case 6

$$\begin{aligned} r_T(x) \le r_R(x) \le r_S(x), \quad \omega_T(x) \le \omega_R(x) \le \omega_S(x)\\ R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)}\star r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_S(x)} \end{aligned}$$

$$R \cup (S \cap T) = r_R(x)e^{i\omega_S(x)} \tag{11}$$

$$\begin{aligned}(R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}]\star[r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}]\\ &= r_S(x)e^{i\omega_S(x)}\star r_R(x)e^{i\omega_R(x)}\\ (R \cup S) \cap (R \cup T) &= r_R(x)e^{i\omega_S(x)}.\end{aligned} \tag{12}$$

From (11) and (12), we have:

$$R \cup (S \cap T) = (R \cup S) \cap (R \cup T).$$

Hence, in all cases, this law is hold.

To prove the distributive law of intersection over union, there are also six cases.

Case 1

$$\begin{aligned}&r_R(x) \le r_S(x) \le r_T(x), \quad \omega_R(x) \le \omega_S(x) \le \omega_T(x)\\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)}\star(r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)}\star r_T(x)e^{i\omega_T(x)}\\ R \cap (S \cup T) &= r_R(x)e^{i\omega_T(x)}\end{aligned} \tag{1}$$

$$\begin{aligned}(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)}\star r_T(x)e^{i\omega_T(x)}]\\ &= r_R(x)e^{i\omega_S(x)} \oplus r_R(x)e^{i\omega_T(x)}\\ (R \cap S) \cup (R \cap T) &= r_R(x)e^{i\omega_T(x)}.\end{aligned} \tag{2}$$

From (1) and (2), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Case 2

$$\begin{aligned}&r_S(x) \le r_T(x) \le r_R(x), \quad \omega_S(x) \le \omega_T(x) \le \omega_R(x)\\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)}\star(r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)}\star r_T(x)e^{i\omega_T(x)}\\ R \cap (S \cup T) &= r_T(x)e^{i\omega_R(x)}\end{aligned} \tag{3}$$

$$\begin{aligned}(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)}\star r_T(x)e^{i\omega_T(x)}]\\ &= r_S(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_R(x)}\\ (R \cap S) \cup (R \cap T) &= r_T(x)e^{i\omega_R(x)}.\end{aligned} \tag{4}$$

From (3) and (4), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Case 3

$$\begin{aligned}&r_R(x) \le r_T(x) \le r_S(x), \quad \omega_R(x) \le \omega_T(x) \le \omega_S(x)\\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)}\star(r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})]\\ &= r_R(x)e^{i\omega_R(x)}\star r_S(x)e^{i\omega_S(x)}\end{aligned}$$

$$R \cap (S \cup T) = r_R(x)e^{i\omega_S(x)} \tag{5}$$

$$\begin{aligned}(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\ &\oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}] \\ &= r_R(x)e^{i\omega_S(x)} \oplus r_R(x)e^{i\omega_T(x)} \\ (R \cap S) \cup (R \cap T) &= r_R(x)e^{i\omega_S(x)}. \end{aligned} \tag{6}$$

From (5) and (6), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Case 4

$$\begin{aligned} r_T(x) \le r_S(x) \le r_R(x), &\quad \omega_T(x) \le \omega_S(x) \le \omega_R(x) \\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)} \star (r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)} \\ R \cap (S \cup r) &= r_S(x)e^{i\omega_R(x)} \end{aligned} \tag{7}$$

$$\begin{aligned}(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}] \\ &= r_S(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_R(x)} \\ (R \cap S) \cup (R \cap T) &= r_S(x)e^{i\omega_R(x)}. \end{aligned} \tag{8}$$

From (7) and (8), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Case 5

$$\begin{aligned} r_S(x) \le r_R(x) \le r_T(x), &\quad \omega_S(x) \le \omega_R(x) \le \omega_T(x) \\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)} \star (r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_r(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)} \\ R \cap (S \cup T) &= r_R(x)e^{i\omega_T(x)} \end{aligned} \tag{9}$$

$$\begin{aligned}(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}] \\ &= r_S(x)e^{i\omega_R(x)} \oplus r_R(x)e^{i\omega_T(x)} \\ (R \cap S) \cup (R \cap T) &= r_R(x)e^{i\omega_T(x)}. \end{aligned} \tag{10}$$

From (9) and (10), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Case 6

$$\begin{aligned} r_T(x) \le r_R(x) \le r_S(x), &\quad \omega_T(x) \le \omega_R(x) \le \omega_S(x) \\ R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)} \star (r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})] \\ &= r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)} \\ R \cap (S \cup T) &= r_R(x)e^{i\omega_S(x)} \end{aligned} \tag{11}$$

$$
\begin{aligned}
(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\
&\quad \oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}] \\
&= r_R(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_R(x)} \\
(R \cap S) \cup (R \cap T) &= r_R(x)e^{i\omega_S(x)}. \qquad (12)
\end{aligned}
$$

From (11) and (12), we have:

$$R \cap (S \cup T) = (R \cap S) \cup (R \cap T).$$

Hence, in all cases, the distributive law of intersection over union is hold.

Proposition 7 *For any complex fuzzy set R, the standard union, standard intersection, and standard complement function with the same function for determining the phase term satisfy the following:*

(i) $R \cup R^c = R$ *or* $R \cup R^c = R^c$.
(ii) $R \cap R^c = R$ *or* $R \cap R^c = R^c$.

Proof To prove (1) and (2), two cases arise here.
(i) □

Case 1

$$
\begin{aligned}
&r_{R^c}(x) \le r_R(x) \\
&R \cup R^c = r_R(x)e^{i\omega_R(x)} \oplus r_{R^c}(x)e^{i\omega_{R^c}(x)} = r_R(x)e^{i\omega_R(x)} = R.
\end{aligned}
$$

Case 2

$$
\begin{aligned}
&r_R(x) \le r_{R^c}(x) \\
&R \cup R^c = r_R(x)e^{i\omega_R(x)} \oplus r_{R^c}(x)e^{i\omega_{R^c}(x)} = r_{R^c}(x)e^{i\omega_{R^c}(x)} = R^c.
\end{aligned}
$$

(ii)
Case 1

$$
\begin{aligned}
&r_{R^c}(x) \le r_R(x) \\
&R \cap R^c = r_R(x)e^{i\omega_R(x)} \star r_{R^c}(x)e^{i\omega_{R^c}(x)} = r_{R^c}(x)e^{i\omega_{R^c}(x)} = R^c.
\end{aligned}
$$

Case 2

$$
\begin{aligned}
&r_R(x) \le r_{R^c}(x) \\
&R \cap R^c = r_R(x)e^{i\omega_R(x)} \star r_{R^c}(x)e^{i\omega_{R^c}(x)} = r_R(x)e^{i\omega_R(x)} = R.
\end{aligned}
$$

Proposition 8 *For any complex fuzzy sets R and S over a crisp set U, the standard union and standard intersection function with the* max *function for determining the phase term does not satisfy the absorption law.*

Proof The absorption laws for crisp set are $R \cap (R \cup S) = R$ and $R \cup (R \cap S) = R$. □

Here, we show that for any CFS R and S, the absorption laws do not hold. If

$$
\begin{aligned}
&r_R(x) \le r_S(x), \quad \omega_R(x) \le \omega_S(x) \\
R \cap (R \cup S) &= [r_R(x)e^{i\omega_R(x)} \star (r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)})] \\
&= r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}
\end{aligned}
$$

$$
\begin{aligned}
&= {}^{r}_{R}(x)e^{i\omega_S(x)} \neq R \\
R \cup (R \cap S) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)})] \\
&= r_R(x)e^{i\omega_R(x)} \oplus r_R(x)e^{i\omega_S(x)} \\
&= r_R(x)e^{i\omega_S(x)} \neq R.
\end{aligned}
$$

Also if

$$
\begin{aligned}
r_S(x) \le r_R(x), &\quad \omega_R(x) \le \omega_S(x) \\
R \cap (R \cup S) &= [r_R(x)e^{i\omega_R(x)} \star (r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)})] \\
&= r_R(x)e^{i\omega_R(x)} \star r_R(x)e^{i\omega_S(x)} \\
&= r_R(x)e^{i\omega_S(x)} \neq R \\
R \cup (R \cap S) &= [r_R(x)e^{i\omega_R(x)} \oplus \{r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}\}] \\
&= r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)} \\
&= r_R(x)e^{i\omega_S(x)} \neq R.
\end{aligned}
$$

Hence, the absorption law does not hold for any complex fuzzy sets.

Proposition 9 *For any CFSs R, S, and T, the standard complement, standard intersection, and standard union function with different function for determining the phase term does not satisfy the distributive laws.*

Proof We use here"$taqe\ wiMMer\ all$"and "min" function for determining the phase term. The distributive law of union over intersection is $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$.For $r.H.S$, we use "$take\ winner\ all$" function; for L.H.S, we use "min" function for determining the phase term. ◻

If

$$
\begin{aligned}
r_R(x) \le r_S(x) \le r_T(x), &\quad \omega_R(x) \le \omega_S(x) \le \omega_T(x) \\
R \cup (S \cap T) &= [r_R(x)e^{i\omega_R(x)} \oplus (r_S(x)e^{i\omega_S(x)} \star r_T(x)e^{i\omega_T(x)})] \\
&= r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_T(x)} \\
R \cup (S \cap T) &= r_S(x)e^{i\omega_T(x)} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
(R \cup S) \cap (R \cup T) &= [r_R(x)e^{i\omega_R(x)} \oplus r_S(x)e^{i\omega_S(x)}] \\
&\quad \star [r_R(x)e^{i\omega_R(x)} \oplus r_T(x)e^{i\omega_T(x)}] \\
&= r_S(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_R(x)} \\
(R \cup S) \cap (R \cup T) &= r_S(x)e^{i\omega_R(x)}. \qquad (2)
\end{aligned}
$$

From (1) and (2), we have:

$$R \cup (S \cap T) \neq (R \cup S) \cap (R \cup T).$$

Now, distributive law of intersection over union:

$$
\begin{aligned}
R \cap (S \cup T) &= [r_R(x)e^{i\omega_R(x)} \star (r_S(x)e^{i\omega_S(x)} \oplus r_T(x)e^{i\omega_T(x)})] \\
&= r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)} \\
R \cap (S \cup T) &= r_R(x)e^{i\omega_T(x)} \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
(R \cap S) \cup (R \cap T) &= [r_R(x)e^{i\omega_R(x)} \star r_S(x)e^{i\omega_S(x)}] \\
&\quad \oplus [r_R(x)e^{i\omega_R(x)} \star r_T(x)e^{i\omega_T(x)}] \\
&= r_R(x)e^{i\omega_R(x)} \oplus r_R(x)e^{i\omega_R(x)} \\
(R \cap S) \cup (R \cap T) &= r_R(x)e^{i\omega_R(x)}. \qquad (4)
\end{aligned}
$$

From (3) and (4), we have:

$$R \cap (S \cup T) \neq (R \cap S) \cup (R \cap T).$$

4 Applications

In this section, we will discuss the applications of CFS in signals and systems.

Definition 14 The Mth inverse discrete Fourier transform (IDFT) coefficient of a length M sequence $\{x(M)\}$ is defined as:

$$x(q) = \frac{1}{M}\sum_{q=0}^{M-1} x'(M)e^{i\frac{2\pi}{M}Mq}, \quad q \in \{0, 1, 2, \ldots, M-1\},$$

where $x(M)$ has different values (Selesnick and Schuller 2001).

We take a particular case that is $U[M] = x'(M)$ is restricted to a closed interval [0, 1], because in CFS, the amplitude term has all the values in the closed interval [0, 1].

Definition 15 (Selesnick and Schuller 2001) The DFT for $\{x'(M) : 1 \leq M \leq M\}$ is given by a matrix in product form:

$$
\begin{bmatrix} x'(0) \\ x'(1) \\ x'(2) \\ . \\ . \\ . \\ x'(M-1) \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & e^{i(-\frac{2\pi}{M})} & e^{i(-\frac{2\pi}{M}2)} & \cdots & e^{i(-\frac{2\pi}{M}(M-1))} \\
1 & e^{i(-\frac{2\pi}{M}2)} & e^{i(-\frac{2\pi}{M}4)} & \cdots & e^{i(-\frac{2\pi}{M}2(M-1))} \\
. & . & . & \cdots & . \\
. & . & . & \cdots & . \\
. & . & . & \cdots & . \\
1 & e^{i(-\frac{2\pi}{M}(M-1))} & e^{i(-\frac{2\pi}{M}2(M-1))} & \cdots & e^{i(-\frac{2\pi}{M}(M-1)^2)}
\end{bmatrix}
\times
\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ . \\ x(M-1) \end{bmatrix},
$$

but the IDFT is given by:

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ . \\ x(M-1) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & e^{i\frac{2\pi}{M}} & e^{i\frac{2\pi}{M}2} & . & . & . & e^{i\frac{2\pi}{M}(M-1)} \\ 1 & e^{i\frac{2\pi}{M}2} & e^{i\frac{2\pi}{M}4} & . & . & . & e^{i\frac{2\pi}{M}2(M-1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & e^{i\frac{2\pi}{M}(M-1)} & e^{i\frac{2\pi}{M}2(M-1)} & . & . & . & e^{i\frac{2\pi}{M}(M-1)^2} \end{bmatrix} \begin{bmatrix} x'(0) \\ x'(1) \\ x'(2) \\ . \\ . \\ . \\ x'(M-1) \end{bmatrix}.$$

In the following, we develop an algorithm to use CFS in signals and systems for identification of a particular signal received by a particular receiver.

Let m be different electromagnetic signals, and $u_1(M), u_2(M), u_3(M), \ldots, u_m(M)$ have been received by a particular receiver. Each of these signals is noted at M different $times$. Let $x_m(M)$ be the $m-th$ $(1 \leq m \leq M)$ signal. The discrete Fourier transform of this $m-th$ signal is:

$$u_m(M) = \frac{1}{M}\sum_{q=0}^{M-1} U[M]e^{i\frac{2\pi}{M}Mq}; \quad M, q = 0, 1, 2, \ldots, M-1. \tag{1}$$

We restrict the range of $U[M]$ as $0 \leq U[M] \leq 1$ $(0 \leq q \leq M-1)$. Here, $U[M] = \Theta'_s(q)$ is known as amplitude term and $\frac{2\pi}{M}Mq = \omega_s(q)$ is known as phase term and the first one having range as real numbers and $U[M] \in [0, 1]$:

$$u_m(M) = \frac{1}{M}\sum_{q=0}^{M-1} U[M]e^{i\frac{2\pi}{M}Mq}; \quad M, q = 0, 1, 2, \ldots, M-1. \tag{1 (a)}$$

Thus a general signal representing by Eq. (1) is model for signal representation using a CFS.

We use the CFS in signals and systems using a new kind of matrix to identify a particular signal out of large signals detected by a digital receiver. For this, we have a reference signal r.This reference signal r is noted M times. The DFT of this reference signal r is:

$$r(M) = \frac{1}{M}\sum_{q=0}^{M-1} \Theta'[q]e^{i\frac{2\pi}{M}Mq}; \quad M, q = 0, 1, 2, \ldots, M-1, \tag{2}$$

where $\Theta'[q] \in [0, 1]$; $(0 \leq q \leq M-1)$.

To compare the similarity between two signals, we apply the following method.

Algorithm

Step 1

Expand $u_m(M) = \frac{1}{M}\sum_{q=0}^{M-1} U[M]e^{i\frac{2\pi}{M}Mq}$ for $q = 0, 1, 2, \ldots, M-1$, we get:

$$\begin{aligned} u_m(M) &= \frac{1}{M}[U[0]e^{i\frac{2\pi}{M}M(0)} + U[1]e^{i\frac{2\pi}{M}M(1)} + U[2]e^{i\frac{2\pi}{M}M(2)} \\ &\quad + \cdots + U[M-1]e^{i\frac{2\pi}{M}M(M-1)}] \\ u_m(M) &= \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}M(1)} + U[2]e^{i\frac{2\pi}{M}M(2)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}M(M-1)}]. \end{aligned} \tag{3}$$

From Eq. (3), we get $M - samRles$ by putting $M = 0, 1, 2, 3, \ldots, M-1$.
For $M = 0$, we have:

$$u_m(0) = \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(0)(1)} + U[2]e^{i\frac{2\pi}{M}(0)(2)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}(0)(M-1)}]$$

$$u_m(0) = \frac{1}{M}[U[0].1 + U[1].1 + U[2].1 + \cdots + U[M-1].1]. \tag{3.1}$$

For $M = 1$, we have:

$$u_m(1) = \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(1)(1)} + U[2]e^{i\frac{2\pi}{M}(1)(2)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}(1)(M-1)}]$$

$$u_m(1) = \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(1)} + U[2]e^{i\frac{2\pi}{M}(2)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}(M-1)}]. \tag{3.2}$$

For $M = 2$, we have:

$$u_m(2) = \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(2)(1)} + U[2]e^{i\frac{2\pi}{M}(2)(2)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}(2)(M-1)}]$$

$$u_m(2) = \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(2)} + U[2]e^{i\frac{2\pi}{M}(4)} + \cdots + U[M-1]e^{i\frac{2\pi}{M}2(M-1)}]. \tag{3.3}$$

Continuing this process, for $M = M - 1$, we have:

$$\begin{aligned} u_m(M-1) &= \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(M-1)(1)} + U[2]e^{i\frac{2\pi}{M}(M-1)(2)} \\ &\quad + \cdots + U[M-1]e^{i\frac{2\pi}{M}(M-1)(M-1)}] \\ u_m(M-1) &= \frac{1}{M}[U[0].1 + U[1]e^{i\frac{2\pi}{M}(M-1)} + U[2]e^{i\frac{2\pi}{M}2(M-1)} \\ &\quad + \cdots + U[M-1]e^{i\frac{2\pi}{M}(M-1)^2}]. \end{aligned} \tag{3.4}$$

A similar argument repeats for the reference signal $\Theta(M)$; we get the $M - samRles$ of the reference signal Θ by putting $M = 0, 1, 2, 3, \ldots, M-1$.
Step 2

Now, we develop the matrix form for these $M - samples$ of the signal $u_m(M)$ and the reference signal $\Theta'(M)$ using definition 15; that is, we have:

$$\begin{bmatrix} u_m(0) \\ u_m(1) \\ u_m(2) \\ . \\ . \\ . \\ u_m(M-1) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & e^{i\frac{2\pi}{M}} & e^{i\frac{2\pi}{M}2} & . & . & . & e^{i\frac{2\pi}{M}(M-1)} \\ 1 & e^{i\frac{2\pi}{M}2} & e^{i\frac{2\pi}{M}4} & . & . & . & e^{i\frac{2\pi}{M}2(M-1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & e^{i\frac{2\pi}{M}(M-1)} & e^{i\frac{2\pi}{M}2(M-1)} & . & . & . & e^{i\frac{2\pi}{M}(M-1)^2} \end{bmatrix} \begin{bmatrix} U[0] \\ U[1] \\ U[2] \\ . \\ . \\ . \\ U[M-1] \end{bmatrix}$$

and

$$\begin{bmatrix} \Theta(0) \\ \Theta(1) \\ \Theta(2) \\ . \\ . \\ . \\ \Theta(M-1) \end{bmatrix} = \frac{1}{M} \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 \\ 1 & e^{i\frac{2\pi}{M}} & e^{i\frac{2\pi}{M}2} & . & . & . & e^{i\frac{2\pi}{M}(M-1)} \\ 1 & e^{i\frac{2\pi}{M}2} & e^{i\frac{2\pi}{M}4} & . & . & . & e^{i\frac{2\pi}{M}2(M-1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & e^{i\frac{2\pi}{M}(M-1)} & e^{i\frac{2\pi}{M}2(M-1)} & . & . & . & e^{i\frac{2\pi}{M}(M-1)^2} \end{bmatrix} \begin{bmatrix} \Theta'[0] \\ \Theta'[1] \\ \Theta'[2] \\ . \\ . \\ . \\ \Theta'[M-1] \end{bmatrix}. \tag{A}$$

In Eq. (A), the first matrix on the right hand side is formed from the values of phase term called RhasematΘix, while the second matrix is formed from the values of amplitude term is called amplitude matrix and M denote the number of samples of signal.
Step 3

Multiply these two matrices and dividing by the number of samples M of signal. We get all the values in the disk of radius one in a complex plane. As the order does not hold for complex numbers, so we take absolute of these $M - samRles$ of the signal $x_m(M)$ and the reference signal $\Theta(M)$; that is:

$$\begin{bmatrix} | u_m(0) | \\ | u_m(1) | \\ | u_m(2) | \\ . \\ . \\ . \\ | u_m(M-1) | \end{bmatrix} \text{ and } \begin{bmatrix} | \Theta(0) | \\ | \Theta(1) | \\ | \Theta(2) | \\ . \\ . \\ . \\ | \Theta(M-1) | \end{bmatrix}.$$

These two matrices are called absolute value matrix.
Step 4

Now, we take the maximum value from the absolute value matrix of the signal $u_m(M)$ and reference signal $\Theta(M)$. If these two values are nearly the same, then the signal $u_m(M)$ identifies a reference signal.

Example 10 Assume that four different electromagnetic signals, $u_1(M)$, $u_2(M)$, $u_3(M)$, and $u_4(M)$, have been received by a receiver. Each of these signals is sampled four times. Let $\Theta(M)$ be the reference signal. The discrete Fourier transform of the signal $u_m(M)$; $m = 0, 1, 2, 3$ and reference signal $\Theta(M)$ for $M = 4$ is:

$$u_m(M) = \frac{1}{4}\sum_{q=0}^{3} U_m[M] e^{i\frac{2\pi}{4}Mq}; \quad M, q = 0, 1, 2, 3, \tag{1}$$

where

$$U_m[M] \in [0, 1].$$

Also

$$\Theta(M) = \frac{1}{4}\sum_{q=0}^{3} \Theta'[M] e^{i\frac{2\pi}{4}Mq}; \quad M, q = 0, 1, 2, 3, \tag{2}$$

where

$$\Theta'[M] \in [0, 1].$$

For $M = 0, 1, 2, 3$, Eq. (1) becomes:

$$u_m(M) = \frac{1}{4}[U_m[0].e^{i\frac{2\pi}{4}M(0)} + U_m[1].e^{i\frac{2\pi}{4}M(1)} + U_m[2].e^{i\frac{2\pi}{4}M(2)} + U_m[3].e^{i\frac{2\pi}{4}M(3)}]$$
$$u_m(M) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}M} + U_m[2].e^{i\frac{2\pi}{4}2M} + U_m[3].e^{i\frac{2\pi}{4}3M}]. \tag{3}$$

Now, put $M = 0$ in (3), we have:

$$u_m(0) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(0)} + U_m[2].e^{i\frac{2\pi}{4}2(0)} + U_m[3].e^{i\frac{2\pi}{4}3(0)}]$$
$$u_m(0) = \frac{1}{4}[U_m[0].1 + U_m[1].1 + U_m[2].1 + U_m[3].1]. \tag{3.1}$$

Put $M = 1$ in (3), we have:

$$u_m(1) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(1)} + U_m[2].e^{i\frac{2\pi}{4}2(1)} + U_m[3].e^{i\frac{2\pi}{4}3(1)}]$$
$$u_m(1) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}} + U_m[2].e^{i\frac{2\pi}{4}(2)} + U_m[3].e^{i\frac{2\pi}{4}(3)}]. \tag{3.2}$$

Put $M = 2$ in (3), we have:

$$u_m(2) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(2)} + U_m[2].e^{i\frac{2\pi}{4}2(2)} + U_m[3].e^{i\frac{2\pi}{4}3(2)}]$$
$$u_m(2) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(2)} + U_m[2].e^{i\frac{2\pi}{4}(4)} + U_m[3].e^{i\frac{2\pi}{4}(6)}]. \tag{3.3}$$

Put $M = 3$ in (3), we get:

$$u_m(3) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(3)} + U_m[2].e^{i\frac{2\pi}{4}2(3)} + U_m[3].e^{i\frac{2\pi}{4}3(3)}]$$
$$u_m(3) = \frac{1}{4}[U_m[0].1 + U_m[1].e^{i\frac{2\pi}{4}(3)} + U_m[2].e^{i\frac{2\pi}{4}(6)} + U_m[3].e^{i\frac{2\pi}{4}(9)}]. \tag{3.4}$$

We can write (3.1), (3.2), (3.3), and (3.4) in matrices form as:

$$\begin{bmatrix} u_m(0) \\ u_m(1) \\ u_m(2) \\ u_m(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{4}(1)} & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(3)} \\ 1 & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(4)} & e^{i\frac{2\pi}{4}(6)} \\ 1 & e^{i\frac{2\pi}{4}(3)} & e^{i\frac{2\pi}{4}(6)} & e^{i\frac{2\pi}{4}(9)} \end{bmatrix}\begin{bmatrix} U_1[0] \\ U_1[1] \\ U_1[2] \\ U_1[3] \end{bmatrix}.$$

A similar argument repeats for the reference signal $\Theta(M)$; we get:

$$\begin{bmatrix} \Theta(0) \\ \Theta(1) \\ \Theta(2) \\ \Theta(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{4}(1)} & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(3)} \\ 1 & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(4)} & e^{i\frac{2\pi}{4}(6)} \\ 1 & e^{i\frac{2\pi}{4}(3)} & e^{i\frac{2\pi}{4}(6)} & e^{i\frac{2\pi}{4}(9)} \end{bmatrix}\begin{bmatrix} \Theta[0] \\ \Theta[1] \\ \Theta[2] \\ \Theta[3] \end{bmatrix}.$$

First of all, we find the values of the sample of reference signal $\Theta(M)$. For this, we have:

$$\Theta'[q] = \begin{cases} 0; & q = 0 \\ 0; & q = 1 \\ 0.2; & q = 2 \\ 1; & q = 3 \end{cases}$$

$$\begin{bmatrix} \Theta(0) \\ \Theta(1) \\ \Theta(2) \\ \Theta(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ -0.2 - i \\ -0.8 \\ -0.2 + i \end{bmatrix}.$$

Now, the absolute value matrix of the reference signal is:

$$\begin{bmatrix} \mid \Theta(0) \mid \\ \mid \Theta(1) \mid \\ \mid \Theta(2) \mid \\ \mid \Theta(3) \mid \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.2 \\ 0.3 \end{bmatrix}.$$

The maximum value is 0.3.
Now, for the signal $u_1(M)$; $M = 0, 1, 2, 3$:

$$U_1[q] = \begin{cases} .5; & q = 0 \\ .7; & q = 1 \\ .8; & q = 2 \\ 1; & q = 3 \end{cases}$$

$$\begin{bmatrix} u_1(0) \\ u_1(1) \\ u_1(2) \\ u_1(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} .5 \\ .7 \\ .8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_1(0) \\ u_1(1) \\ u_1(2) \\ u_1(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ -0.3 - 0.3i \\ -0.4 \\ -0.3 + 0.3i \end{bmatrix} = \begin{bmatrix} 0.75 \\ -0.075 - 0.075i \\ -0.1 \\ -0.075 + 0.075i \end{bmatrix}.$$

Now, the absolute value matrix of the signal $u_1(M)$ is:

$$\begin{bmatrix} \mid u_1(0) \mid \\ \mid u_1(1) \mid \\ \mid u_1(2) \mid \\ \mid u_1(3) \mid \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

Here, the maximum value is 0.8.
For $u_2(M)$, we have:

$$U_2[q] = \begin{cases} 0.4; & q = 0 \\ 0.6; & q = 1 \\ 0.8; & q = 2 \\ 1; & q = 3. \end{cases}$$

Now

$$\begin{bmatrix} u_2(0) \\ u_2(1) \\ u_2(2) \\ u_2(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} U_2[0] \\ U_2[1] \\ U_2[2] \\ U_2[3] \end{bmatrix}$$

$$\begin{bmatrix} u_2(0) \\ u_2(1) \\ u_2(2) \\ u_2(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \\ 0.8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_2(0) \\ u_2(1) \\ u_2(2) \\ u_2(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2.8 \\ -0.4 - 0.4i \\ -0.4 \\ -0.4 + 0.4i \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.1 - 0.1i \\ -0.1 \\ -0.1 + 0.1i \end{bmatrix}.$$

Now, the absolute value matrix of the signal $u_2(M)$ is:

$$\begin{bmatrix} \mid u_2(0) \mid \\ \mid u_2(1) \mid \\ \mid u_2(2) \mid \\ \mid u_2(3) \mid \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

Here, the maximum value is 0.7.
Now, for signal $u_3(M)$, we have:

$$U_3[q] = \begin{cases} 0.6; & q = 0 \\ 1; & q = 1 \\ 0.9; & q = 2 \\ 0.8; & q = 3 \end{cases}$$

$$\begin{bmatrix} u_3(0) \\ u_3(1) \\ u_3(2) \\ u_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\frac{2\pi}{4}(1)} & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(3)} \\ 1 & e^{i\frac{2\pi}{4}(2)} & e^{i\frac{2\pi}{4}(4)} & e^{i\frac{2\pi}{4}(6)} \\ 1 & e^{i\frac{2\pi}{4}(3)} & e^{i\frac{2\pi}{4}(6)} & e^{i\frac{2\pi}{4}(9)} \end{bmatrix} \begin{bmatrix} U_3[0] \\ U_3[1] \\ U_3[2] \\ U_3[3] \end{bmatrix}$$

$$\begin{bmatrix} u_3(0) \\ u_3(1) \\ u_3(2) \\ u_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0.6 \\ 1 \\ 0.9 \\ 0.8 \end{bmatrix}$$

$$\begin{bmatrix} u_3(0) \\ u_3(1) \\ u_3(2) \\ u_3(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3.3 \\ -0.3 + 0.2i \\ -0.3 \\ -0.3 - 0.2i \end{bmatrix} = \begin{bmatrix} 0.825 \\ -0.075 + 0.05i \\ -0.075 \\ -0.075 - 0.05i \end{bmatrix}.$$

Now, the absolute value matrix of the signal $u_3(M)$ is:

$$\begin{bmatrix} \mid u_3(0) \mid \\ \mid u_3(1) \mid \\ \mid u_3(2) \mid \\ \mid u_3(3) \mid \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}.$$

Here, the maximum value is 0.8.

Now, for $u_4(M)$, we have:

$$U_4[q] = \begin{cases} 0.8; & q = 0 \\ 0.5; & q = 1 \\ 0; & q = 2 \\ 0; & q = 3 \end{cases}$$

$$\begin{bmatrix} u_4(0) \\ u_4(1) \\ u_4(2) \\ u_4(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} U_4[0] \\ U_4[1] \\ U_4[2] \\ U_4[3] \end{bmatrix}$$

$$\begin{bmatrix} u_4(0) \\ u_4(1) \\ u_4(2) \\ u_4(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_4(0) \\ u_4(1) \\ u_4(2) \\ u_4(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1.3 \\ 0.8 + 0.5i \\ 0.3 \\ 0.8 - 0.5i \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 + 0.125i \\ 0.1 \\ 0.2 - 0.125i \end{bmatrix}.$$

Now, the absolute value matrix of the signal $u_4(M)$ is:

$$\begin{bmatrix} | u_4(0) | \\ | u_4(1) | \\ | u_4(2) | \\ | u_4(3) | \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.1 \\ 0.2 \end{bmatrix}.$$

Here, the maximum value is 0.3.
Now, from the following table of maximum values:

$\Theta(M)$	0.3
$u_1(M)$	0.8
$u_2(M)$	0.7
$u_3(M)$	0.8
$u_4(M)$	0.3

.

The signal $u_4(M)$ identifies as a reference signal.

5 Comparison

The complex fuzzy set has many applications, particularly in signal processing and image restoration as it represents the particular form of a Fourier series. Here, we have presented the application of CFS in signals and systems. In this practical application, one of the main issues is that how to choose a suitable model. We examined this idea in depth and used the CFS in signals and systems by introducing an algorithm using the matrix already introduced in Selesnick and Schuller (2001) and the CFS. In this application, we identified a reference signal out of large interest signals detected by a digital receiver. Ramot et al. (2002) introduced an algorithm to identify the unknown signal received by the digital receiver with reference signal R, and in Zhang et al. (2009), the authors modified the method introduced in Ramot et al. (2002). Furthermore, Ali and Smarandache (2017) work on the same algorithm for complex neutrosophic sets. In all these algorithms, the authors actually tried to find the highest resemblance with the known signal R, while the method which we developed gives the exact value of unknown signal among the all unknown signals received by the digital receiver. We compared it with the known signal R, and observed that both the values are exactly equal. Thus, we identified one unknown signal as a reference signal R among the several signals detected by the receiver. The model that presented in this paper for identifying a reference signal is more effective than the methods previously developed. Here, we used a discrete Fourier transform matrix (DFTM) to develop an algorithm for further use in signal processing. Moreover, through this model, we determined the value of each signal separately, that detected by a digital receiver. In fact, we compared the values of different signals with the reference signal and we easily identified the reference signal. Moreover, it is seen clearly that how the detected signals matched with the reference signal. However, our designed model is not a perfect one; it stucks with a deficiency of theoretical support. The concept of matrix for CFSs may be useful for applications. Therefore, it will be significant for future work.

6 Conclusion

In this paper, we have discussed some new set theoretic operations on a CFS, and properties of a CFS with respect to standard CF union, standard CF intersection, and standard CF complement. We have presented some basic results and examples of these operations and laws under the operations of complex fuzzy union, complex fuzzy intersection, and complex fuzzy complement. Moreover, we have used a complex fuzzy set in signals and systems. We have introduced a new method to develop a new kind of matrix using a complex fuzzy set, through which we identified a reference signal among several signals detected by the digital receiver. In future, the same method can be used for continuous data of signals using continuous Fourier transform. This method can also be applied for identification of signals in geology. Moreover, this work and further study of complex fuzzy sets will give a new direction of applications in different fields of science and engineering.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interests.

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