Computational and Applied Mathematics (2019) 38:113
https://doi.org/10.1007/s40314-019-0883-8

Caputo–Fabrizio operator in terms of integer derivatives: memory or distributed lag?

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Received: 27 February 2018 / Revised: 18 December 2018 / Accepted: 13 May 2019 /
Published online: 23 May 2019

Abstract
In this paper, we prove that linear and nonlinear equations with the Caputo–Fabrizio operators can be represented as systems of differential equations with derivatives of integer orders. The order of these equations is not more than one with respect to the integer part of the highest order of the Caputo–Fabrizio operators. We state that the Caputo–Fabrizio operators with exponential kernel cannot describe nonlocality and memory (temporal nonlocality) in processes and systems. Using the principle of nonlocality for fractional derivatives of noninteger orders ("No nonlocality. No fractional derivative"), we can state that the Caputo–Fabrizio operators cannot be considered as a fractional derivative. A general physical and economic interpretation (meaning) of the Caputo–Fabrizio operator is proposed. We state that physical and economic meaning of the Caputo–Fabrizio operators is continuously (exponentially) distributed lags.

Keywords Fractional derivative · Nonlocality · Memory · Caputo–Fabrizio operator · Distributed lag

Mathematics Subject Classification 26A33 · 34A08

1 Introduction

Theory of derivatives and integrals of noninteger (fractional) orders has a long history of more than 300 years (Samko et al. 1993). The first appearance of derivative of noninteger order is found in a letter written to G.F.A. de l'Hopital by G.W. Leibniz in 1695 (see (Leibniz an de l'Hospital 1853) and (Leibniz an de l'Hospital 2005, p. 510)). The existence of operators and some applications have been given by J. Liouville.

Communicated by Delfim F. M. Torres.

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Equations with fractional derivatives of noninteger orders are a powerful tool for describing processes and systems with spatial nonlocality and memory. There are many different types of fractional derivatives that have been proposed by well-known mathematicians such as Liouville, Riemann, Letnikov, Sonine, Weyl, Riesz, Hadamard, Marchaud, and others (Samko et al. 1993; Kiryakova 1994; Podlubny 1998; Kilbas et al. 2006). Recently, some papers began to propose various differential operators, which the authors called fractional derivatives. In the paper (Tarasov 2018), we suggest a principle of nonlocality for fractional derivatives of noninteger orders, which states that if the differential equation with considered (tested) fractional operator can be represented by differential equations with a finite number of integer order derivatives, then this operator cannot be considered as a derivative of noninteger order. One such operator is the Caputo–Fabrizio operator (Caputo and Fabrizio 2015, 2016; Losada and Nieto 2015; Hristov 2018; Ortigueira and Tenreiro Machado 2018). In paper (Tarasov 2018), we consider only simple example of the equation with the Caputo—Fabrizio operator. A possibility of representing linear and nonlinear differential equations with the Caputo–Fabrizio operators by differential equations of integer orders is not discussed in (Tarasov 2018).

In the present paper, we prove that linear and nonlinear equations with the Caputo–Fabrizio operators can be represented as a system of differential equations with derivatives of integer orders. We prove that these operators cannot be used to described memory and nonlocality, but the Caputo–Fabrizio operator can be used for modeling processes and systems with the continuously (exponentially) distributed lags.

2 Definition of Caputo–Fabrizio operator

The Caputo–Fabrizio operator has been suggested as a fractional derivative with nonsingular (exponential) kernel in (Caputo and Fabrizio 2015, 2016) and then were considered in various works. The Caputo–Fabrizio operator $D^{(\alpha)}_{\mathrm{CF}}$ of the noninteger order $\alpha \in (0, 1)$ is defined (see equation 2.2 of (Caputo and Fabrizio 2015 p. 74)) by the equation:

$$\left(D^{(\alpha)}_{\mathrm{CF}} X\right)(t) = \frac{1}{1-\alpha} \cdot \int_{t_0}^{t} \exp\left\{-\frac{\alpha}{1-\alpha} \cdot (t-\tau)\right\} \cdot X^{(1)}(\tau) \mathrm{d}\tau, \tag{1}$$

where $X^{(1)}(\tau) = \mathrm{d}X(\tau)/\mathrm{d}\tau$ is the standard derivative of first order with respect to τ. For $n > 1$, the Caputo–Fabrizio operator of the order $\alpha + n \in (n, n+1)$ is defined (see, equation 2.8 of (Caputo and Fabrizio 2015 p. 76)) by the expression:

$$\left(D^{(\alpha+n)}_{\mathrm{CF}} X\right)(t) = \left(D^{(\alpha)}_{\mathrm{CF}} X^{(n)}\right)(t) = \frac{1}{1-\alpha} \cdot \int_{t_0}^{t} \exp\left\{-\frac{\alpha}{1-\alpha} \cdot (t-\tau)\right\} \cdot X^{(n+1)}(\tau) \mathrm{d}\tau, \tag{2}$$

where $\alpha \in (0, 1)$ and $X^{(n)}(\tau) = \mathrm{d}^n X(\tau)/\mathrm{d}\tau^n$ are the standard derivatives of integer order $n \in \mathbb{N}$ with respect to τ. Using Eq. (2), the Caputo–Fabrizio operator of the order $\alpha \in (n, n+1)$ can be represented as follows:

$$\left(D^{(\alpha)}_{\mathrm{CF}} X\right)(t) = \frac{1}{n-\alpha+1} \cdot \int_{t_0}^{t} \exp\left\{-\frac{\alpha-n}{n-\alpha+1} \cdot (t-\tau)\right\} \cdot X^{(n+1)}(\tau) \mathrm{d}\tau, \tag{3}$$

where $\alpha - n = \{\alpha\}$ and $n = [\alpha]$. We can see that the Caputo–Fabrizio operators are integro-differential operators.

The Caputo–Fabrizio operators (3) of order $\alpha \in (n, n+1)$ with $t_0 = 0$ can be represented in the form:

$$\left(D_{\mathrm{CF}}^{(\alpha)} X\right)(t) = \left(K * X^{(n+1)}\right)(t), \tag{4}$$

where $K * X^{(n)}$ is the Laplace convolution of the derivative $X^{(n)}(t)$ and the exponential kernel:

$$K(t) = \frac{1}{n-\alpha+1} \cdot \exp\left\{-\frac{\alpha-n}{n-\alpha+1} \cdot t\right\}. \tag{5}$$

The Laplace convolution of two functions $X(t)$ and $Y(t)$ is defined for $t \in \mathbb{R}_+$ by the equation:

$$(X * Y)(t) = \int_0^t X(\tau) \cdot Y(t-\tau) \mathrm{d}\tau, \tag{6}$$

where the commutative property $(X * Y)(t) = (Y * X)(t)$ holds.

3 Equation with Caputo–Fabrizio operators as differential equations of integer orders

Let us consider the nonhomogeneous linear integro-differential equation with the Caputo—Fabrizio operators (3) and constant coefficients in the form:

$$\sum_{k=1}^{m} A_k \left(D_{\mathrm{CF}}^{(\alpha_k)} X\right)(t) + A_0 X(t) = f(t), \tag{7}$$

where $t > t_0 = 0$; $m \in \mathbb{N}$; $0 < \alpha_1 < \cdot s < \alpha_m < \infty$; $A_0, A_1, \ldots, A_m \in \mathbb{R}$; $A_0 \neq 0$ and involving the Caputo–Fabrizio operators $D_0^{(\alpha_k)} X$ $(k = 1, \ldots, m)$ given by (3), where $t_0 = 0$, $\alpha_k = \{\alpha_k\} + n_k \in (n_k, n_k + 1)$, and $n_k = [\alpha_k]$.

Theorem 1 *The integro-differential Eq.* (7) *with the Caputo–Fabrizio operators* (7) of orders $\alpha_k \in (n_k, n_k + 1)$ can be represented as the system of differential equation with derivatives of integer orders $n_k + 1$ that has the form:

$$\begin{cases} \sum\limits_{j=1}^{m} \frac{A_j}{A_0} \cdot Y_j^{(n_k+1)} + (1 - \{\alpha_k\}) Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = \frac{1}{A_0} \cdot f^{(n_k+1)}(t), \\ X(t) = \frac{1}{A_0} \cdot f(t) - \sum\limits_{k=1}^{m} \frac{A_k}{A_0} \cdot Y_k(t), \end{cases} \tag{8}$$

where $k = 1, \ldots, m$, $n_k = [\alpha_k]$, and $\{\alpha_k\} = \alpha_k - n_k$.

Proof The Laplace transform $\mathcal{L}$ of a function $X(t)$ of a real variable $t \in (0, \infty)$ is defined by the equation:

$$(\mathcal{L}X)(s) = \int_0^\infty X(t) \cdot \exp\{-s \cdot t\} \mathrm{d}t. \tag{9}$$

The integral (9) convergences for $\mathrm{Re}(s) > s_{\inf}$, where $s_{\inf}$ is abscissa of convergence. Using equation 4.1.20 of Bateman (1954 p. 131) that describes the property of the Laplace convolution (6) in the form

$$(\mathcal{L}(X * Y))(s) = (\mathcal{L}X)(s) \cdot (\mathcal{L}Y)(s), \tag{10}$$

we get the Laplace transform $\mathcal{L}$ of the Caputo–Fabrizio operators (4) as follows:

$$\left(\mathcal{L}D_{CF}^{(\alpha)}X\right)(s) = \left(\mathcal{L}\left(K * X^{(n+1)}\right)\right)(s) = (\mathcal{L}K)(s) \cdot \left(\mathcal{L}X^{(n+1)}\right)(s), \tag{11}$$

where $n = [\alpha]$. Then, we can use equation 4.5.1. (Bateman 1954 p. 143) in the form:

$$\begin{aligned}(\mathcal{L}K)(s) &= \frac{1}{n-\alpha+1} \cdot \left(\mathcal{L}\exp\left\{-\frac{\alpha-n}{n-\alpha+1} \cdot t\right\}\right) \\ &= \frac{1}{n-\alpha+1} \cdot \left(s + \frac{\alpha-n}{n-\alpha+1}\right)^{-1} = \frac{1}{s \cdot (n-\alpha+1) + (\alpha-n)},\end{aligned} \tag{12}$$

and equation 4.1.8. (Bateman 1954 p. 129) or equation 1.4.9 of (Kilbas et al. 2006 p. 19) in the form:

$$\left(\mathcal{L}X^{(n+1)}\right)(s) = s^{n+1} \cdot (\mathcal{L}X)(s) - \sum_{j=0}^{n} s^{n-j} X^{(j)}(0), \tag{13}$$

where $X^{(j)}(0)$ are the standard derivatives of integer order $j \in \mathbb{N}$ at $t = 0$, and

$$\mathrm{Re}(s) > -\frac{\alpha-n}{n-\alpha+1}. \tag{14}$$

Using (11), (12), and (13), the Laplace transform $\mathcal{L}$ of $\left(D_{\mathrm{CF}}^{\alpha_k}X\right)(t)$ with $\alpha_k \in (n_k, n_k+1)$ and $n_k = [\alpha_k]$ is given by the following:

$$\left(\mathcal{L}D_{CF}^{\alpha_k}X\right)(s) = \frac{1}{s \cdot (1-\{\alpha_k\}) + \{\alpha_k\}} \cdot \left(s^{n_k+1} \cdot (\mathcal{L}X)(s) - \sum_{j=0}^{n_k} s^{n_k-j} X^{(j)}(0)\right), \tag{15}$$

where $\{\alpha_k\} = \alpha_k - n_k$ and $k = 1, \ldots, m$. Equation (15) allows us to write the Laplace transform $\mathcal{L}$ of Eq. (7) in the form:

$$\begin{aligned}\sum_{k=1}^{m} \frac{A_k}{s \cdot (1-\{\alpha_k\}) + \{\alpha_k\}} \cdot \left(s^{n_k+1} \cdot (\mathcal{L}X)(s) - \sum_{j=0}^{n_k} s^{n_k-j} X^{(j)}(0)\right) \\ +A_0 \cdot (\mathcal{L}X)(s) = (\mathcal{L}f)(s).\end{aligned} \tag{16}$$

Using Eq. (13), we can rewrite Eq. (16) in the form:

$$\sum_{k=1}^{m} \frac{A_k}{s \cdot (1-\{\alpha_k\}) + \{\alpha_k\}} \cdot \left(\mathcal{L}X^{(n_k+1)}\right)(s) + A_0 \cdot (\mathcal{L}X)(s) = (\mathcal{L}f)(s), \tag{17}$$

where $n_k \in \mathbb{N}$ for $k = 1, \ldots, m$.

Let us define the auxiliary variables $Y_k(t)$ ($k = 1, \ldots, m$), such that $Y_k(0) = 0$ and

$$(\mathcal{L}Y_k)(s) = \frac{1}{s \cdot (1-\{\alpha_k\}) + \{\alpha_k\}} \cdot \left(\mathcal{L}X^{(n_k+1)}\right)(s). \tag{18}$$

Equation (18) can be rewritten in the form:

$$(s \cdot (1-\{\alpha_k\}) + \{\alpha_k\}) \cdot (\mathcal{L}Y_k)(s) = \left(\mathcal{L}X^{(n_k+1)}\right)(s). \tag{19}$$

Using the relation

$$\left(\mathcal{L}\left(a \cdot Y_k^{(1)} + bY_k\right)\right)(s) = (a \cdot s + b) \cdot (\mathcal{L}Y_k)(s) - a \cdot Y_k(0) \tag{20}$$

in the form

$$(1 - \{\alpha_k\}) \cdot \left(\mathcal{L}Y_k^{(1)}\right)(s) + \{\alpha_k\} \cdot (\mathcal{L}Y_k)(s) = (s \cdot (1 - \{\alpha_k\}) + \{\alpha_k\}) \cdot (\mathcal{L}Y_k)(s) - (1 - \{\alpha_k\}) \cdot Y_k(0), \tag{21}$$

we can state that condition (19) is a Laplace transform of the equation:

$$(1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) + (1 - \{\alpha_k\}) \cdot Y_k(0) \cdot \delta(t) = X^{(n_k+1)}(t). \tag{22}$$

Using the assumption $Y_k(0) = 0$, we have Eq. (22) in the form:

$$(1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = X^{(n_k+1)}(t), \tag{23}$$

where $k = 1, \ldots, m$. Equation (23) can be considered as a definition of the suggested new variables $Y_k(t)$. Using (18), we can consider Eq. (17) as the Laplace transform of equation:

$$\sum_{k=1}^{m} A_k \cdot Y_k(t) + A_0 \cdot X(t) = f(t).$$

As a result, we obtain the system of the differential equations:

$$\begin{cases} (1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = X^{(n_k+1)}(t), \\ \sum\limits_{k=1}^{m} A_k \cdot Y_k(t) + A_0 \cdot X(t) = f(t). \end{cases} \tag{24}$$

If $A_0 \neq 0$, then the last equation of system (24) can be written in the form:

$$X(t) = \frac{1}{A_0} \cdot f(t) - \sum_{j=1}^{m} \frac{A_j}{A_0} \cdot Y_j(t). \tag{25}$$

Substitution of expression (25) into Eq. (23) gives the following:

$$(1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = \frac{1}{A_0} \cdot f^{(n_k+1)}(t) - \sum_{j=1}^{m} \frac{A_j}{A_0} \cdot Y_j^{(n_k+1)}. \tag{26}$$

Equation (26) can be rewritten in the form:

$$\sum_{j=1}^{m} \frac{A_j}{A_0} \cdot Y_j^{(n_k+1)} + (1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = f^{(n_k+1)}(t), \tag{27}$$

where $k = 1, \ldots, m$. This is a system of m differential equations of the integer order $n_m + 1$, where $n_m = [\alpha_m]$. The required function $X(t)$ is defined as the sum:

$$X(t) = \frac{1}{A_0} \cdot f(t) - \sum_{k=1}^{m} \frac{A_k}{A_0} \cdot Y_k(t). \tag{28}$$

As a result, we get a system of $(m + 1)$ equations for $(m + 1)$ unknown variables:

$$\begin{cases} \sum\limits_{j=1}^{m} \frac{A_j}{A_0} \cdot Y_j^{(n_k+1)} + (1 - \{\alpha_k\})Y_k^{(1)}(t) + \{\alpha_k\} \cdot Y_k(t) = \frac{1}{A_0} \cdot f^{(n_k+1)}(t), \\ X(t) = \frac{1}{A_0} \cdot f(t) - \sum\limits_{k=1}^{m} \frac{A_k}{A_0} \cdot Y_k(t). \end{cases} \tag{29}$$

This is the end of proof. Q.E.D.

Remark Theorem 1 does not use the assumption of vanishing the auxiliary variables $Y_j(t)$ at the initial instant of time $t_0 = 0$. It should be note that Eq. (7) and system (8) of equations are solution equivalent if the auxiliary variable $Y_j(t)$ satisfies the initial condition $Y_j(0) = 0$ for all $j = 1, \ldots m$.

Note that system (8) can be used to obtain solution of Eq. (7) for the case $Y_j(0) \neq 0$. It can be realized using the conditions $Y_j(0) = 0$ only in the Laplace transform of the first-order derivatives $Y_j^{(1)}(t)$, i.e., when we use $\left(\mathcal{L}Y_j^{(1)}\right)(s) = s\left(\mathcal{L}Y_j\right)(s)$ instead of $\left(\mathcal{L}Y_j^{(1)}\right)(s) = s\left(\mathcal{L}Y_j\right)(s) - Y_j(0)$. In other words if we consider the Laplace transform of the first equation of (24) with $Y_k(0) = 0$ in the form:

$$(1 - \{\alpha_k\})s(\mathcal{L}Y_k)(s) + \{\alpha_k\}(\mathcal{L}Y_k)(s) = \left(\mathcal{L}\left(X^{(n_k+1)}\right)\right)(s)$$

without using these conditions $Y_k(0) = 0$ in the second equation of the system, i.e., without the assumption:

$$\sum_{k=1}^{m} A_k Y_k(0) + A_0 X(0) = f(0),$$

In this case, we get solution of system (8) that coincides with solution of Eq. (7).

To illustrate this remark, we can consider Eq. (7) with $m = 1, n_k = 0$ and $F(t) = 0$, which has the form:

$$A_1\left(D^{(\alpha)}_{\mathrm{CF}} X\right)(t) + A_0 X(t) = 0.$$

This can be represented as the system:

$$\begin{cases} A_1 Y(t) + A_0 X(t) = 0, \\ (\lambda/\{\alpha\}) X^{(1)}(t) = Y^{(1)}(t) + \lambda Y(t), \end{cases}$$

where $\lambda = \{\alpha\}/(1 - \{\alpha\})$. The Laplace transform of the second equation of the system has the form:

$$(\lambda/\{\alpha\})(s(\mathcal{L}X)(s) - X(0)) = s(\mathcal{L}Y)(s) - Y(0) + \lambda(\mathcal{L}Y)(s).$$

Using $Y(0) = 0$ in this equation, when the condition $A_0 X(t) = A_1 Y(t)$ for $t = 0$ is not assumed in the first equation (i.e., $X(0) \neq 0$ in general), we get the following:

$$\begin{cases} A_1 Y(t) + A_0 X(t) = 0, (t > 0) \\ (\lambda/\{\alpha\})(s(\mathcal{L}X)(s) - X(0)) = s(\mathcal{L}Y)(s) - Y(0) + \lambda(\mathcal{L}Y)(s). \end{cases}$$

Using the Laplace transform of the first equation $A_1(\mathcal{L}Y)(s) + A_0(\mathcal{L}X)(s) = 0$, and substituting $(\mathcal{L}Y)(s) = -(A_0/A_1)(\mathcal{L}X)(s)$ into second equation of the system, we get the following:

$$(\lambda/\{\alpha\})(s(\mathcal{L}X)(s) - X(0)) = -s(A_0/A_1)(\mathcal{L}X)(s) - \lambda(A_0/A_1)(\mathcal{L}X)(s).$$

This equation can be rewritten in the form:

$$A_1 \frac{\lambda/\{\alpha\}}{s + \lambda}(s(\mathcal{L}X)(s) - X(0)) + A_0(\mathcal{L}X)(s) = 0,$$

which is equivalent to the Laplace transform of Eq. (7) with $m = 1, n_k = 0$ and $F(t) = 0$.

To generalize Theorem 1 to nonlinear equations, it is necessary to construct a proof without using the Laplace transform. It should be noted that Theorem 1 can be proved without using the Laplace transform. To prove the theorem for nonlinear equations, we separately prove the following auxiliary theorem.

Theorem 2 *The integro-differential equation*

$$Y(t) = \left(D_{\mathrm{CF}}^{(\alpha)} X\right)(t), \tag{30}$$

with the Caputo–Fabrizio operator (3) *can be represented as the differential equation with derivatives of integer orders* $n+1$ *that has the form:*

$$X^{(n+1)}(t) = (1 - \{\alpha\}) \cdot Y^{(1)}(t) + \{\alpha\} \cdot Y(t), \tag{31}$$

where $\{\alpha\} = \alpha - n$ and $n = [\alpha]$.

Proof Equation (3), which defines the Caputo–Fabrizio operator of the order $\alpha \in (n, n+1)$ can rewritten in the form:

$$\left(D_{CF}^{(\alpha)} X\right)(t) = \frac{1}{1-\{\alpha\}} \cdot \exp\left\{-\frac{\{\alpha\}}{1-\{\alpha\}} \cdot t\right\} \int_{t_0}^{t} \exp\left\{\frac{\{\alpha\}}{1-\{\alpha\}} \cdot \tau\right\} \cdot X^{(n+1)}(\tau) d\tau, \tag{32}$$

where $\{\alpha\} = \alpha - n$ and $[\alpha] = n$. Let us define the variable $Z(t)$, such that

$$Z^{(1)}(\tau) = \exp\left\{\frac{\{\alpha\}}{1-\{\alpha\}} \cdot \tau\right\} \cdot X^{(n+1)}(\tau). \tag{33}$$

Using the relation

$$\int_{t_0}^{t} Z^{(1)}(\tau) \mathrm{d}\tau = Z(t) - Z(t_0), \tag{34}$$

substitution of (33) into Eq. (32) gives the following:

$$\left(D_{CF}^{(\alpha)} X\right)(t) = \frac{1}{1-\{\alpha\}} \cdot \exp\left\{-\frac{\{\alpha\}}{1-\{\alpha\}} \cdot t\right\}(Z(t) - Z(t_0)). \tag{35}$$

Let us define the auxiliary variable $Y(t)$, such that

$$Y(t) = \frac{1}{1-\{\alpha\}} \cdot \exp\left\{-\frac{\{\alpha\}}{1-\{\alpha\}} \cdot t\right\}(Z(t) - Z(t_0)). \tag{36}$$

Substitution of (36) into (35) gives $\left(D_{\mathrm{CF}}^{(\alpha)} X\right)(t) = Y(t)$. Differentiation of Eq. (36) gives the following:

$$\begin{aligned} Y^{(1)}(t) = &- \frac{\{\alpha\}}{(1-\{\alpha\})^2} \cdot \exp\left\{-\frac{\{\alpha\}}{1-\{\alpha\}} \cdot t\right\}(Z(t) - Z(t_0)) \\ &+ \frac{1}{1-\{\alpha\}} \cdot \exp\left\{-\frac{\{\alpha\}}{1-\{\alpha\}} \cdot t\right\} Z^{(1)}(t). \end{aligned} \tag{37}$$

Using Eqs. (33) and (36), expression (37) takes the form:

$$Y^{(1)}(t) = -\frac{\{\alpha\}}{1-\{\alpha\}} Y(t) + \frac{1}{1-\{\alpha\}} \cdot X^{(n+1)}(t). \tag{38}$$

Equation (38) can be written in the form (39).

This is the end of proof. Q.E.D.

The Theorem 2 allows us to formulate the following theorem for nonlinear integro-differential equations with the Caputo–Fabrizio operators.

Theorem 3 *The nonlinear integro-differential equation*

$$N\Big[X(t),\, \Big(D^{(\alpha_k)}_{CF} X\Big)(t),\, k = 1, \dots m\Big] = 0, \tag{39}$$

with the Caputo–Fabrizio operators (3) can be represented as the system of the differential equation with derivatives of integer orders $n_k + 1$ that has the form:

$$\begin{cases} N[X(t), Y_k(t), k = 1, \dots m] = 0 \\ X^{(n_k+1)}(t) = (1 - \{\alpha_k\}) \cdot Y^{(1)}_k(t) + \{\alpha_k\} \cdot Y_k(t). \end{cases} \tag{40}$$

where $k = 1, \dots m$, and $\{\alpha_k\} = \alpha_k - n_k$, $[\alpha_k] = n_k$.

Proof Let us defined the variables $Z_k(t)$, such that

$$Z^{(1)}_k(\tau) = \exp\left\{\frac{\{\alpha_k\}}{1 - \{\alpha_k\}} \cdot \tau\right\} \cdot X^{(n+1)}(\tau), \tag{41}$$

and the auxiliary variables $Y_k(t)$, such that

$$Y_k(t) = \frac{1}{1 - \{\alpha_k\}} \cdot \exp\left\{-\frac{\{\alpha_k\}}{1 - \{\alpha_k\}} \cdot t\right\}(Z_k(t) - Z_k(t_0)), \tag{42}$$

where $k = 1, \dots m$. Using the Theorem 2, differentiation of Eq. (42) gives the following:

$$Y^{(1)}_k(t) = -\frac{\{\alpha_k\}}{1 - \{\alpha_k\}} Y_k(t) + \frac{1}{1 - \{\alpha_k\}} \cdot X^{(n_k+1)}(t). \tag{43}$$

As a result, Eq. (39) can be written in the form (40), since $\Big(D^{(\alpha_k)}_{CF} X\Big)(t) = Y_k(t)$.

This is the end of proof. Q.E.D.

A remarkable feature of the system (40) is that all differential equations of integer orders are linear. Note that if the variable $X(t)$ can be expressed from an algebraic equation, then we can obtain a system of differential equations for auxiliary variables. In this case, the differential equations can be nonlinear in general. For the linear equations with Caputo–Fabrizio operators, the variable $X(t)$ can be expressed from an algebraic equation, and we can get system, in which differential equations contain only auxiliary variables (see Theorem 1). Note that, in Theorems 2 and 3, we did not use the assumptions of Theorem 1 about the linearity of the equation with Caputo–Fabrizio operators, vanishing the auxiliary variables $Y^{(1)}_k(t)$ at the initial instant of time t_0, and the equality to zero of the initial time moment t_0. However, the solutions of the unknown equation and the resulting system of the equation will coincide if the auxiliary variables at the starting point are equal to zero (see Remark to Theorem 1).

As a result, we proved that equations with the Caputo–Fabrizio operators cannot be used to describe nonlocality and memory, since these can be represented by the differential equations with derivatives of integer orders only, which are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point.

We should note that the representation of integro-differential equations with the Caputo—Fabrizio operators of noninteger orders in the form of the standard differential equations of integer orders is also considered in (Al-Salti et al. 2015; Karimov and Pirnafasov 2017). In paper (Ortigueira and Tenreiro 2018), it was shown that the Caputo–Fabrizio operator implements an integer order high-pass filter and that this operator is neither fractional, nor a derivative.

Using the principle of nonlocality (Tarasov 2018) for fractional derivatives of noninteger orders, we can state that the Caputo–Fabrizio operators cannot be considered as fractional derivatives of noninteger orders, since differential equations with these operators can be represented by the differential equations with derivatives of integer orders only.

An important property of fractional derivatives of noninteger orders, which is related with nonlocality and memory, is their representation in the form of an infinite series of derivatives of integer orders (for example, see Lemma 15.3 of Samko et al. (1993 p. 278) and Tarasov (2016, 2017)). It is important to emphasize here that fractional operators cannot be represented as a finite sum in the general case. The possibility of representing in the form of an infinite sum is not a sufficient condition for operators to describe nonlocality and memory. The fact that an operator cannot be specified as a finite sum of derivatives of integer orders in general is the characteristic property of fractional derivatives of noninteger orders (Tarasov 2018) together with a violation of the standard Leibniz and chain rules (Tarasov 2013, 2016).

4 Examples of representation by differential equations of integer orders

Let us give some examples of the application of Theorems 1 and 2, such that the linear and nonlinear equations with the Caputo–Fabrizio operators are represented by differential equations with standard derivatives of integer orders only.

Example 1 Let us consider the linear integro-differential equation:

$$\left(D^{(\alpha)}_{CF} X\right)(t) + A_0 \cdot X(t) = f(t), \tag{44}$$

where $n < \alpha < n+1, n = [\alpha]$. For $A_0 = -\lambda$ and $0 < \alpha < 1$, Eq. (44) takes the form of Eq. (9) of Losada and Nieto (2015 p. 89). Using Theorem 3, Eq. (44) can be represented as the system of the differential equation with derivatives of integer orders $n+1$ that has the form:

$$\begin{cases} Y(t) + A_0 \cdot X(t) = f(t), \\ X^{(n+1)}(t) = (1 - \{\alpha\}) \cdot Y^{(1)}(t) + \{\alpha\} \cdot Y(t). \end{cases} \tag{45}$$

Expressing the variable $X(t)$ from the first equation of the system, and substituting into the second equation, we obtain the system:

$$\begin{cases} \frac{1}{A_0} \cdot Y^{(n+1)} + (1 - \{\alpha\}) \cdot Y^{(1)}(t) + \{\alpha\} \cdot Y(t) = \frac{1}{A_0} \cdot f^{(n+1)}(t), \\ X(t) = \frac{1}{A_0} \cdot f(t) - \frac{1}{A_0} \cdot Y(t). \end{cases} \tag{46}$$

We have obtained the system, which directly follows from Theorem 1. For $0 < \alpha < 1$, system (46) takes the form:

$$\begin{cases} \left(1 - \alpha + A_0^{-1}\right) \cdot Y^{(1)}(t) + \alpha \cdot Y(t) = \frac{1}{A_0} \cdot f^{(1)}(t), \\ X(t) = A_0^{-1} \cdot f(t) - A_0^{-1} \cdot Y(t). \end{cases} \tag{47}$$

For the case $A_0 = k/m$, $f(t) = g =$ const, and $X(t) = v(t)$, we get the system for the problem that is considered in Losada and Nieto (2015 p. 91) in the form:

$$\begin{cases} \left(1 - \alpha + m \cdot k^{-1}\right) \cdot Y^{(1)}(t) + \alpha \cdot Y(t) = 0, \\ v(t) = m \cdot g \cdot k^{-1} - m \cdot k^{-1} \cdot Y(t). \end{cases} \tag{48}$$

Expressing the auxiliary variable $Y(t)$ from the second equation of the system, and substituting into the first equation, we obtain the differential equation:

$$\left(1-\alpha+m\cdot k^{-1}\right)\cdot v^{(1)}(t)+\alpha\cdot v(t)-\alpha\cdot m\cdot g\cdot k^{-1}=0, \tag{49}$$

which is the differential equation of the first order.

Example 2 Let us consider the linear integro-differential equation:

$$A_2\left(D_{\mathrm{CF}}^{(\alpha_2)}X\right)(t)+A_1\left(D_{\mathrm{CF}}^{(\alpha_1)}X\right)(t)+A_0X(t)=f(t). \tag{50}$$

For the parameters $\alpha_2=\{\alpha_2\}=0.7$, $\alpha_1=\{\alpha_1\}=0.3$, $n_2=n_1=0$, $A_2=3$, $A_1=2$, $A_0=1$, $f(t)=5\sin(2t)$, Eq. (50) has the form:

$$3\cdot\left(D_{CF}^{(0.7)}X\right)(t)+2\cdot\left(D_{CF}^{(0.3)}X\right)(t)+X(t)=5\cdot\sin(2t). \tag{51}$$

Using Theorem 1, Eq. (50) can be represented as the system of the differential equation with derivatives of integer orders n_k+1 that has the form:

$$\begin{cases}\frac{A_2}{A_0}\cdot Y_2^{(n_1+1)}+\frac{A_1}{A_0}\cdot Y_1^{(n_1+1)}+(1-\{\alpha_1\})Y_1^{(1)}(t)+\{\alpha_1\}\cdot Y_1(t)=\frac{1}{A_0}\cdot f^{(n_1+1)}(t),\\ \frac{A_2}{A_0}\cdot Y_2^{(n_2+1)}+\frac{A_1}{A_0}\cdot Y_1^{(n_2+1)}+(1-\{\alpha_2\})Y_2^{(1)}(t)+\{\alpha_2\}\cdot Y_2(t)=\frac{1}{A_0}\cdot f^{(n_2+1)}(t),\ .\\ X(t)=\frac{1}{A_0}\cdot f(t)-\frac{A_2}{A_0}\cdot Y_2(t)-\frac{A_1}{A_0}\cdot Y_1(t).\end{cases} \tag{52}$$

For considered parameters, Eq. (51) can be represented as the system:

$$\begin{cases}3\cdot Y_2^{(1)}+2.7\cdot Y_1^{(1)}+0.3\cdot Y_1(t)=10\cdot\cos(2t),\\ 3.3\cdot Y_2^{(1)}+2\cdot Y_1^{(1)}+0.7\cdot Y_2(t)=10\cdot\cos(2t),\\ X(t)=5\sin(2t)-3\cdot Y_2(t)-2\cdot Y_1(t).\end{cases} \tag{53}$$

Let us consider the linear integro-differential Eq. (50) $\alpha_2=1.2$ instead of $\alpha_2=0.7$, i.e., $\{\alpha_2\}=0.2$, $\{\alpha_1\}=0.3$, $n_2=1$, $n_1=0$, $A_2=3$, $A_1=2$, $A_0=1$, and $f(t)=5\sin(2t)$. In this case, we have the equation:

$$3\cdot\left(D_{\mathrm{CF}}^{(1.2)}X\right)(t)+2\cdot\left(D_{\mathrm{CF}}^{(0.3)}X\right)(t)+X(t)=5\cdot\sin(2t). \tag{54}$$

Using Theorem 1, Eq. (54) can be represented as the system of the differential equation with derivatives of integer orders n_k+1 that has the form (52). For considered parameters, Eq. (54) can be represented as the system:

$$\begin{cases}3\cdot Y_2^{(1)}+2\cdot Y_1^{(1)}+0.7\cdot Y_1^{(1)}(t)+0.3\cdot Y_1(t)=10\cdot\cos(2t),\\ 3\cdot Y_2^{(2)}+2\cdot Y_1^{(2)}+0.8\cdot Y_2^{(1)}(t)+0.2\cdot Y_2(t)=-20\cdot\sin(2t),\\ X(t)=5\sin(2t)-3\cdot Y_2(t)-2\cdot Y_1(t).\end{cases} \tag{55}$$

Example 3 Let us consider the nonlinear integro-differential equation:

$$\left(D_{CF}^{(0.2)}X\right)(t)=X^2(t)-3\cdot X(t)+4. \tag{56}$$

Using Theorem 3, Eq. (56) can be represented as the system of the differential equation with derivatives of first order in the form:

$$\begin{cases}Y(t)=X^2(t)-3\cdot X(t)+4,\\ X^{(1)}(t)=0.2\cdot Y(t)+0.8\cdot Y^{(1)}(t).\end{cases} \tag{57}$$

In this system (57), we can exclude the auxiliary variable $Y(t)$, and we get the differential equation:

$$(17 - 8 \cdot X(t)) \cdot X^{(1)}(t) = X^2(t) - 3 \cdot X(t) + 4, \tag{58}$$

which represents integro-differential Eq. (50) with the Caputo–Fabrizio operator (3).

Example 4 Let us consider the nonlinear integro-differential equation:

$$A \cdot \left(\left(D_{CF}^{(\alpha)} X\right)(t)\right)^q + B \cdot \left(\left(D_{CF}^{(\beta)} X\right)(t)\right)^p = F(X(t), t), \tag{59}$$

where $\alpha, \beta \in \mathbb{R}_+$ and $q, p \in \mathbb{N}$. Using Theorem 3, Eq. (59) can be represented as the system of the differential equation with derivatives of integer orders in the form:

$$\begin{cases} A \cdot Y^q(t) + B \cdot Z^p(t) = F(X(t), t), \\ X^{(n+1)}(t) = (n - \alpha + 1) \cdot Y^{(1)}(t) + (\alpha - n) \cdot Y(t), \\ X^{(m+1)}(t) = (m - \beta + 1) \cdot Z^{(1)}(t) + (\beta - m) \cdot Z(t), \end{cases} \tag{60}$$

where $n = [\alpha]$ and $m = [\beta]$. In the particular case of (59), the nonlinear equation

$$3 \cdot \left(\left(D_{CF}^{(1.2)} X\right)(t)\right)^3 + 2 \cdot \left(D_{CF}^{(0.3)} X\right)(t) = X^3(t) - 3X(t) + 4, \tag{61}$$

can be represented as the system:

$$\begin{cases} 3 \cdot Y^3(t) + 3 \cdot Z(t) = X^3(t) - 3X(t) + 4, \\ X^{(2)}(t) = 0.8 \cdot Y^{(1)}(t) + 0.2 \cdot Y(t), \\ X^{(1)}(t) = 0.7 \cdot Z^{(1)}(t) + 0.3 \cdot Z(t), \end{cases} \tag{62}$$

which contains standard derivatives of integer order.

For $q = 1$, $A = 1$, $B = 0$, and $0 < \alpha < 1$, Eq. (59) takes the form of nonlinear Eq. (11) of Losada and Nieto (2015 p. 90). Let us consider Eq. (59) for the case $q = 1$, $A = 1$, $B = 0$, and $n < \alpha < n + 1$, i.e., the equation

$$\left(D_{CF}^{(\alpha)} X\right)(t) = F(X(t), t). \tag{63}$$

Using Theorem 3, Eq. (63) can be represented as the system of the differential equation with derivatives of integer orders in the form:

$$\begin{cases} Y(t) = F(X(t), t), \\ X^{(n+1)}(t) = (n - \alpha + 1) \cdot Y^{(1)}(t) + (\alpha - n) \cdot Y(t) \end{cases}. \tag{64}$$

Substituting the auxiliary variable $Y(t)$ from the first equation of the system into the second equation, we obtain the differential equation:

$$X^{(n+1)}(t) - (n - \alpha + 1) \cdot F_X^{(1)}(X(t), t) \cdot X^{(1)}(t) - (\alpha - n) \cdot F(X(t), t) = 0, \tag{65}$$

where $F_X^{(1)}(X, t)$ is the partial derivative of the function $F(X, t)$ with respect to a variable X. Equation (64) with $n = 0$ is a special form of the Lienard equation often used in the theory of oscillations and dynamical systems. For the case $F(X, t) = \lambda \cdot X^2$ and $\alpha = 1.2$, Eq. (63) has the form $\left(D_{CF}^{(1.2)} X\right)(t) = \lambda \cdot X^2(t)$ and it can be represented in the form:

$$X^{(2)}(t) - 2 \cdot (2 - \alpha) \cdot \lambda \cdot X(t) \cdot X^{(1)}(t) - (\alpha - 1) \cdot \lambda \cdot X^2 = 0, \tag{66}$$

which is the differential equation of the second order.

Example 5 Let us consider the nonlinear integro-differential equation:

$$3 \cdot \left(\left(D_{CF}^{(0.7)} X \right)(t) \right)^3 + 2 \cdot \left(D_{CF}^{(0.3)} X \right)(t) + X(t) = 0, \tag{67}$$

can be represented as the system:

$$\begin{cases} 3 \cdot Y^3(t) + 2 \cdot Z(t) + X(t) = 0 \\ X^{(1)}(t) = 0.3 \cdot Y^{(1)}(t) + 0.7 \cdot Y(t) \\ X^{(1)}(t) = 0.7 \cdot Z^{(1)}(t) + 0.3 \cdot Z(t) \end{cases} . \tag{68}$$

Using the first equation of the system, in the second and third equations, it is possible to exclude the dependence of these equations on the main variable:

$$\begin{cases} X(t) = -3 \cdot Y^3(t) - 2 \cdot Z(t) \\ \left(0.3 + 3 \cdot Y^2(t)\right) \cdot Y^{(1)}(t) + 0.7 \cdot Y(t) + 2 \cdot Z^{(1)}(t) = 0 \\ 2.7 \cdot Z^{(1)}(t) + 0.3 \cdot Z(t) + 3 \cdot Y^2(t) \cdot Y^{(1)}(t) = 0 \end{cases} , \tag{69}$$

which contains standard derivatives of integer order.

As a result, we demonstrated an application of proposed Theorem 1–3 for obtaining representations of equations with the Caputo–Fabrizio operators in the form of the differential equations with derivatives of integer orders only. Since all these derivatives are determined by the properties of the differentiable function only in an infinitesimal neighborhood of the considered point, we can state that the Caputo–Fabrizio operators cannot be used to describe nonlocality which is space and time.

5 Economic and physical interpretation of Caputo–Fabrizio operator

We proved that the Caputo–Fabrizio operator cannot describe the nonlocality in space and memory (nonlocality in time). In this connection, two following questions arise. The first question is about constructive criteria that will allow us to check the presence of memory in the process and criteria that allow us to check whether the considered operator can describe processes with memory. The second question is about the physical and economic meaning of the Caputo–Fabrizio operator.

To develop a constructive criterion in addition to the general principle of nonlocality, we proposed (Tarasov and Tarasova 2018) criteria for determining which types of operators can describe memory (Tarasova and Tarasov 2018) that is more general than power-law memory. The criteria of memory have been proposed in (Tarasov and Tarasova 2018) using Fourier transforms. It can be reformulated using the Laplace transforms. As a result, one can come to the conclusion that the Caputo–Fabrizio operators cannot describe the memory, because the Laplace transformation of a kernel is not determined by noninteger powers of the variable s.

We can provide an answer to the question about the physical or economic meaning of the Caputo–Fabrizio operators. An interpretation of these operators has been proposed in (Ciancio and Flora 2017) within the framework of the signal theory. The paper (Caputo and Fabrizio 2017) can also be regarded as one of the physical interpretations of the Caputo—Fabrizio operators by the hysteresis. However, these are particular types of interpretation, which point to individual (special) processes. It is important to understand what general type of processes which these operators can describe.

We can give a general physical and economic interpretation (meaning) of the Caputo—Fabrizio operator. It is possible to state that physical and economic meaning of the Caputo–Fabrizio operator is the continuously (exponentially) distributed lags.

In the simple form, the lag can be described by equation $Y(t) = m \cdot X(t-T)$ with a fixed-time delay of $T > 0$ periods (Allen 1960 p. 23). In a more general form, the lag between input and output can be described as continuously distributed lag. The continuously (exponentially) distributed lags are described in section 1.9 of Allen (1960 p. 23–29) and section 5.8 of (Allen 1960 p. 166–170). This approach is often used to describe economic processes with lag. The existence of the time delay (lag) is connected with the fact that the processes take place with a finite speed, and the change of the economic factor (input) does not lead to instant changes of indicator (output) that depends on it. Therefore, continuously distributed lag cannot be considered as a dependence of the state of as process on its history, i.e., it cannot be described as a memory. The lag cannot be considered as a nonlocality in time, i.e., as a memory.

If we define the Caputo–Fabrizio operator of the order $\alpha \in (0, 1)$ in the form:

$$\left(D^{(\alpha)}_{CF} X\right)(t) = \frac{M(\alpha)}{1-\alpha} \cdot \int_{t_0}^{t} \exp\left\{-\frac{\alpha}{1-\alpha} \cdot (t-\tau)\right\} \cdot X^{(1)}(\tau) d\tau \tag{70}$$

with $M(\alpha) = \alpha$ and $t_0 = -\infty$,, then we can write macroeconomic equation of with continuously distributed time lag in the form:

$$Y(t) = v \cdot \left(D^{(\alpha)}_{CF} X\right)(t). \tag{71}$$

Equation (71) describes the economic accelerator with the exponential lag (Allen 1960 p. 62–63), where v is a positive constant that indicating the power of the accelerator. This distributed lag is characterized by the weighting function (Allen 1960 p. 26) in the form:

$$f(t) = K(t) = \frac{\alpha}{1-\alpha} \exp\left\{-\frac{\alpha}{1-\alpha} \cdot t\right\}. \tag{72}$$

This function, which defines the kennel of the integro-differential operator (70), satisfies the normalization condition (see Eq. (8) of (Allen 1960 p. 26)). The parameter

$$\lambda = \frac{\alpha}{1-\alpha} \tag{73}$$

is called the speed of response (Allen 1960 p. 27). As an alternative parameter to the speed of response for the exponential lag, we can consider the time-constant of this lag that is defined as $T = 1/\lambda$. This time-constant is consistent with the term for the fixed-time delay. For exponentially distributed lag, the parameter T is the length of the delay (Allen 1960 p. 27).

The exponential kernel (weighting function) is actively used in macroeconomic models with distributed lag in the framework of the continuous and discrete time approaches (Allen 1960 p. 26). In macroeconomic models, the differential equations of exponentially distributed lad are used instead of equations with integro-differential operators. For example, the economic accelerator with the exponential lag (71) is usually considered (Allen 1960 p. 63) in the form:

$$Y^{(1)}(t) = -\lambda \cdot (Y(t) - v \cdot X(t)). \tag{74}$$

Equation (74) is actively used for macroeconomic models with continuously distributed lag. For example, the Phillips model of multiplier-accelerator that takes into account the exponential lag is described in (Allen 1960 p. 72–74).

In macroeconomic models with the exponential lag, the representation by equivalent differential equations of integer order is usually used instead of the integro-differential operator (70). These equations are called the differential equations of the exponential lag (Allen 1960

p. 27). It is caused by the fact that there are considerable difficulties in handling the integrals in (70). It is known that, under certain conditions, equations with continuously distributed lag are equivalent to differential equations with standard derivatives of integer orders. These differential equations, as a rule, are easier to handle in comparison with the integro-differential equations that describe the distributed lag.

The described interpretation of the operator (70) does not depend on whether we are considering economic, physical, or other processes. Equation (71) describes continuously distributed lag between input (action, external force, and exogenous variable) and output (response and endogenous variable). The lag (delay) caused by the fact that the process has a finite speed, and the changes of input do not lead to instant changes of output. From physics, it is well known that the finite speed of the process does not mean that there is memory in the process. Mathematically, this manifests itself in the possibility of describing a process by equations containing only a finite number of derivatives of integer orders.

In physics, hysteresis is described as a lag between input and output. Hysteresis cannot be considered as a dependence of the state of a system on its history, i.e., it cannot be described as a memory. The application of the Caputo–Fabrizio operators to describe hysteresis has been proposed in (Caputo and Fabrizio 2017). In fact, this paper gives a physical interpretation of the Caputo–Fabrizio operator by hysteresis that is consistent with the proposed interpretation (meaning) of the Caputo–Fabrizio operator by the continuously (exponentially) distributed lags.

As a result, the Caputo–Fabrizio operators cannot be used to describe processes with memory or spatial nonlocality, but it can be applied to describe processes with continuously distributed lag, such as exponential lag. The Caputo–Fabrizio operators can be applied for modeling processes with distributed lag in physics and economics. A general physical and economic meaning of the Caputo–Fabrizio operator is the continuously (exponentially) distributed lags between input and output. The generalizations of the Caputo–Fabrizio operator have been proposed in (Tarasov and Tarasova 2019) to describe different types of probability distributions of delay time and power-law fading memory.

6 Conclusion

It was proved that the Caputo–Fabrizio operators cannot be use to describe processes with memory or spatial nonlocality, since linear and nonlinear equations with these operators can be represented as differential equations of integer orders. An interpretation of the Caputo—Fabrizio operators has been proposed. It can be interpreted as operator of continuously (exponentially) distributed lag between input and output. As a result, the Caputo–Fabrizio operators can be applied for modeling processes with continuously distributed lag in physics and economics, but it cannot be used to describe processes with memory. However, the advantage of using equations with the Caputo–Fabrizio operators to describe processes with distributed lag is questionable. The use of corresponding differential equations with derivatives of integer orders has obvious advantages.

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