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The Pythagorean fuzzy Einstein Choquet integral operators and their application in group decision making

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Abstract

Since Pythagorean fuzzy set is a powerful tool as it relaxing the condition that the sum of membership degrees is less than or equal to one with the square sum is less than or equal to one. Also Choquet integral is a very useful way of measuring the expected utility of an uncertain event. Therefore, in this paper we use the Choquet integral to develop Pythagorean fuzzy aggregation operators, namely Pythagorean fuzzy Einstein Choquet integral averaging operator and Pythagorean fuzzy Einstein Choquet integral geometric operator. The operators not only consider the importance of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions. It must be noted that several existing operators are the special cases of the developed operators. Further the properties such as boundedness, monotonicity and idempotency of the proposed operators have been studied in detail. Furthermore based on the developed operators a multi-criteria group decision-making method has been presented. Finally an illustrative example is presented to illustrate the validity and effectiveness of the proposed method.

Keywords Pythagorean fuzzy set (PFS) · Pythagorean fuzzy Einstein Choquet integral averaging (PFECIA) operator · Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) · Multi-criteria group decision making (MCGDM) problem

Mathematics Subject Classification 03E72 · 68T37 · 90B50 · 03B52

1 Introduction

Due to high complexity of socioeconomics it is difficult to acquire sufficient and accurate date for real-world decision making. To overcome this shortcoming fuzzy set introduced by Zadeh (1965) is one of the most effective tools. After the appearance of fuzzy set many extensions have been developed in both theoretical and practical fields. Among these extensions Atanassov intuitionistic fuzzy set (AIFS) introduced by Atanasov (1986) is one of the most

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powerful tool to deal with vagueness and fuzziness. Xu and Yager (2006) developed some intuitionistic fuzzy geometric operators to deal with MCDM problems. Xu (2007) established a series of aggregation operators to deal with MCGDM problems under IFS environment.

Since in some real world decision-making problems the decision makers (DMs) deals with the situation of particular attributes where the sum of membership's degrees is greater than 1. Therefore, Yager (2013) generalized the notion of IFS by initiating the idea of Pythagorean fuzzy set (PFS). In PFS, the square sum of its membership's degrees is greater than or equal to 1. To deal with multi-criteria decision-making (MCDM) problem a series of aggregation operators has been proposed by Yager (2014) under Pythagorean fuzzy environment. Using Einstein operation, a new generalization of Yager's operators have been developed by Garg (2016a, 2017a) and presented some new aggregation operators namely, the Pythagorean fuzzy Einstein weighted average (PFEWA) operator and Pythagorean fuzzy Einstein ordered weighted average (PFEOWA) operator, Pythagorean fuzzy Einstein weighted geometric (PFEWG) operator and Pythagorean fuzzy Einstein ordered weighted geometric (PFEOWG) operator for MCDM. Under the PFS environment Garg (2017b) developed the concept of confidence Pythagorean fuzzy weighted and ordered weighted operators to deal with MCDM problems. Garg (2018a) generalized the concept of PFEWG operator and developed Pythagorean fuzzy Einstein weighted interactive geometric (PFEWIG) operator by adding a pair of hesitation between the membership degrees. Garg (2018b) introduced a new decision-making model with probabilistic information and using the concept of immediate probabilities has been developed to aggregate the information under the Pythagorean fuzzy set environment. Khan et al. (2019a, b) presented Pythagorean fuzzy Prioritized aggregation operators for MADM problems. Khan et al. (2019a) developed Pythagorean fuzzy Einstein Prioritized aggregation operators.

Garg (2016c) initiated the notion of interval-valued Pythagorean fuzzy average (IVPFA) operator and interval-valued Pythagorean fuzzy geometric (IVPFG) operator. Khan and Abdullah (2018) developed interval-valued Pythagorean fuzzy Choquet integral averaging (IVPFCIA) operator for MADM problems. Khan et al. (2018) established interval-valued Pythagorean fuzzy Choquet integral geometric (IVPFCIG) operator. Garg (2018c) developed interval-valued Pythagorean fuzzy weighted exponential averaging (IVPFWEA) operator and the dual interval-valued Pythagorean fuzzy weighted exponential averaging (DIVPFWEA) operator.

Sugeno (1974) presented fuzzy measure to model interaction phenomena between the decision criteria (Kojadinovic 2002) and was utilized in numerous MCDM problems with inter-dependent decision criteria (Grabisch 1995, 1997). Based on the above discussion, we can see that in general the supposition of independence of criteria is too strong to be fulfilled in many MCDM and MCGDM problems. The aggregation operators proposed by Garg (2016a, 2017a) do not reflect the correlations and the importance between the elements or their ordered positions.

All of the existing Pythagorean fuzzy aggregation operators only consider situations where all the elements in PFS are independent, i.e., they only consider the addition of the importance of individual elements. However, in many real-world situations, the elements in PFS are usually interactive; for instance, Grabisch (1997) and Torra (2003) gave the following classical example: "Suppose we are to evaluate a set of students in relation to three subjects: {chemistry, mathematics, literature}, we want to give more importance to science-related subjects than to literature, but on the other hand we want to give some advantage to students that are good both in literature and in any of the science-related subjects". Therefore, we need to find some new ways to deal with these situations in which the decision data in question are correlative. The Choquet integral (1954) is a very useful way of measuring the expected utility of an uncertain event, and can be used to depict the interactions of the data under consideration.

Motivated by the interaction properties of Choquet integral in this paper two new aggregation operators have been studied. For this to aggregate the PFNs, two new operators namely Pythagorean fuzzy Einstein Choquet integral averaging (PFECIA) operator and Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) operator have been developed. Moreover, a method on the basis of presented operators has been developed for solving MADM problems. It also should be noted that most of the existing operators are the special cases of the presented operators. The rest of the paper is forming as:

In Sect. 2 a brief review of some basic definitions and operation has been given. In Sect. 3 based on Choquet integral and Einstein operational laws, Pythagorean fuzzy Einstein Choquet integral averaging (PFECIA) operator and Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) operator has been proposed. Then discuss some basic properties such as idempotency, boundary and monotonicity in detail. An algorithm for MCDM problem under PFS environment is given in Sect. 4. To show the viability and effectiveness of the presented method numerical example is given in Sect. 5. Concluding remarks is in Sect. 6.

2 Preliminaries

Atanassov (1986) initiated the notion of intuitionistic fuzzy set (IFS) as a generalization of fuzzy set initiated by Zadeh and is worth mentioned that IFS is a powerful tool to deal with imprecision and vagueness. The IFS can be defined as follows:

Definition 1 (Atanassov 1986) Suppose $Y = (y_1, y_2, \ldots y_n)$ is a fixed non-empty set, then an IFS, I in Y can be defined as follows:

$$I = \{(y, \mu_I(y_i), \upsilon_I(y_i))|y \in Y\}, \tag{1}$$

where $\mu_I(y_i)$ and $\upsilon_I(y_i)$ are mappings from Y to [0, 1], such that $0 \leqslant \mu_I(y_i) \leqslant 1$, $0 \leqslant \upsilon_I(y_i) \leqslant 1$ and $0 \leqslant \mu_I(y_i) + \upsilon_I(y_i) \leqslant 1$, for all $y \in Y$. Let $\pi_I(y_i) = 1 - \mu_I(y_i) - \upsilon_I(y_i)$, then it is commonly said to be an intuitionistic fuzzy index of element $y_i \in Y$ to set I, representing the degree of indeterminacy y_i to I. Also $0 \leqslant \pi_I(y_i) \leqslant 1$ for every $y_i \in Y$.

Since IFS fulfills the situations that the sum of its membership's degree is $\leqslant 1$. Though in real world decision making (DM) the DM may be deal with the situation of particular attributes such that the sum of membership's degree is > 1. Therefore, to overcome this situation Yager (2013) initiated the concept of Pythagorean fuzzy set (PFS) which full fill the situations that the square sum of its membership's degree is $\leqslant 1$. It can be defined as follows:

Definition 2 (Yager 2013) Suppose $Y = (y_1, y_2, \ldots y_n)$ is a fixed non-empty set. Then a PFS $\bar{B}$ in Y is a structure such as:

$$\bar{B} = \{(y_i, \mu_{\bar{B}}(y_i), \upsilon_{\bar{B}}(y_i))|y_i \in Y\}, \tag{2}$$

where $\mu_{\bar{B}}(y_i)$ and $\upsilon_{\bar{B}}(y_i)$ are mappings from Y to [0, 1], such that $0 \leqslant \mu_{\bar{B}}(y_i) \leqslant 1$, $0 \leqslant \upsilon_{\bar{B}}(y_i) \leqslant 1$ and also $0 \leqslant \mu^2_{\bar{B}}(y_i) + \upsilon^2_{\bar{B}}(y_i) \leqslant 1$, for all $y_i \in Y$, and they denote the membership degree and nonmembership degree of element $y_i \in Y$ to set $\bar{B}$, respectively. Let $\pi_{\bar{B}}(y_i) = \sqrt{1 - \mu^2_{\bar{B}}(y_i) - \upsilon^2_{\bar{B}}(y_i)}$, then it is commonly said to be the Pythagorean fuzzy index of element $y_i \in Y$ to set $\bar{B}$, representing the degree of indeterminacy of y_i to $\bar{B}$. Also $0 \leqslant \pi_{\bar{B}}(y_i) \leqslant 1$ for every $y_i \in Y$. Zhang and Xu (2014) called the pair $(\mu_{\bar{B}}, \upsilon_{\bar{B}})$, a

Pythagorean fuzzy number (PFN) indicated as $\bar{\beta}$ with the conditions that $0 \leqslant \mu_{\bar{\beta}}(y_i) \leqslant 1$, $0 \leqslant \upsilon_{\bar{\beta}}(y_i) \leqslant 1$ and also $0 \leqslant \mu^2_{\bar{\beta}}(y_i) + \upsilon^2_{\bar{\beta}}(y_i) \leqslant 1$, for all $y_i \in Y$, throughout in this paper a PFN is denoted by $\bar{\beta} = (\mu_{\bar{\beta}}, \upsilon_{\bar{\beta}})$.

Moreover Zhang and Xu (2014) defined some operational laws to aggregate the PFNs

Definition 3 (Zhang and Xu 2014) Suppose $\bar{\beta} = (\mu_{\bar{\beta}}, \upsilon_{\bar{\beta}})$, $\bar{\beta}_1 = (\mu_{\bar{\beta}_1}, \upsilon_{\bar{\beta}_1})$ and $\bar{\beta}_2 = (\mu_{\bar{\beta}_2}, \upsilon_{\bar{\beta}_2})$ are three PFNs, and $\lambda > 0$, then their operations are defined as follows:

1. $$\bar{\beta}_1 \cup \bar{\beta}_2 = (\max\{\mu_{\bar{\beta}_1}, \mu_{\bar{\beta}_2}\}, \min\{\upsilon_{\bar{\beta}_1}, \upsilon_{\bar{\beta}_2}\}), \tag{3}$$
2. $$\bar{\beta}_1 \cap \bar{\beta}_2 = (\min\{\mu_{\bar{\beta}_1}, \mu_{\bar{\beta}_2}\}, \max\{\upsilon_{\bar{\beta}_1}, \upsilon_{\bar{\beta}_2}\}), \tag{4}$$
3. $$\bar{\beta}^c = (\upsilon_{\bar{\beta}}, \mu_{\bar{\beta}}), \tag{5}$$
4. $$\bar{\beta}_1 \oplus \bar{\beta}_2 = \left(\sqrt{\mu^2_{\bar{\beta}_1} + \mu^2_{\bar{\beta}_2} - \mu^2_{\bar{\beta}_1}\mu^2_{\bar{\beta}_2}}, \upsilon_{\bar{\beta}_1}\upsilon_{\bar{\beta}_2}\right), \tag{6}$$
5. $$\bar{\beta}_1 \otimes \bar{\beta}_2 = \left(\mu_{\bar{\beta}_1}\mu_{\bar{\beta}_2}, \sqrt{\upsilon^2_{\bar{\beta}_1} + \upsilon^2_{\bar{\beta}_2} - \upsilon_{\bar{\beta}_1}\upsilon_{\bar{\beta}_2}}\right), \tag{7}$$
6. $$\lambda\bar{\beta} = \left(\sqrt{1 - (1 - \mu^2_{\bar{\beta}})^{\lambda}}, \upsilon^{\lambda}_{\bar{\beta}}\right), \ \lambda > \ 0, \tag{8}$$
7. $$\bar{\beta}^{\lambda} = \left(\mu^{\lambda}_{\bar{\beta}}, \sqrt{1 - (1 - \upsilon^2_{\bar{\beta}})^{\lambda}}\right), \ \lambda > \ 0. \tag{9}$$

To compare two PFNs, Zhang and Xu (2014) presented the score function and accuracy function and developed a total order relation for the comparison between PFNs as follows:

Definition 4 (Zhang and Xu 2014) Let $\bar{\beta}_1 = (\mu_{\bar{\beta}_1}, \upsilon_{\bar{\beta}_1})$ and $\bar{\beta}_2 = (\mu_{\bar{\beta}_2}, \upsilon_{\bar{\beta}_2})$ be the two PFNs. Then $S(\bar{\beta}_1) = \mu^2_{\bar{\beta}_1} - \upsilon^2_{\bar{\beta}_1}$ and $S(\bar{\beta}_2) = \mu^2_{\bar{\beta}_2} - \upsilon^2_{\bar{\beta}_2}$ are the scores of $\bar{\beta}_1$ and $\bar{\beta}_2$ respectively, and $H(\bar{\beta}_1) = \mu^2_{\bar{\beta}_1} + \upsilon^2_{\bar{\beta}_1}$, $H(\bar{\beta}_2) = \mu^2_{\bar{\beta}_2} + \upsilon^2_{\bar{\beta}_2}$ are the accuracy degrees of $\bar{\beta}_1$, $\bar{\beta}_2$ respectively. Then we have

(a) If $S(\bar{\beta}_1) < S(\bar{\beta}_2)$, then $\bar{\beta}_1$ is smaller than $\bar{\beta}_2$, denoted by $\bar{\beta}_1 < \bar{\beta}_2$,
(b) If $S(\bar{\beta}_1) = S(\bar{\beta}_2)$, then $\bar{\beta}_1 = \bar{\beta}_2$.
(c) If $H(\bar{\beta}_1) = H(\bar{\beta}_2)$, then $\bar{\beta}_1$ and $\bar{\beta}_2$ represent the same information, i.e., $\bar{\beta}_1 = \bar{\beta}_2$,
(d) If $H(\bar{\beta}_1) < H(\bar{\beta}_2)$ then $\bar{\beta}_1$ is smaller than $\bar{\beta}_2$ denoted by $\bar{\beta}_1 < \bar{\beta}_2$,
(e) If $H(\bar{\beta}_1) > H(\bar{\beta}_2)$ then $\bar{\beta}_1$ is greater than $\bar{\beta}_2$ denoted by $\bar{\beta}_1 > \bar{\beta}_2$.

Garg (2016a) initiated the Einstein operation for PFNs and analyze some desirable properties of these operations.

Definition 5 (Garg 2016a) Let $\bar{\beta} = \left(\mu_{\bar{\beta}}, \upsilon_{\bar{\beta}}\right)$, $\bar{\beta}_1 = \left(\mu_{\bar{\beta}_1}, \upsilon_{\bar{\beta}_1}\right)$ and $\bar{\beta}_2 = \left(\mu_{\bar{\beta}_2}, \upsilon_{\bar{\beta}_2}\right)$ be the three PFNs, and $\lambda > 0$ be any real number. Then,

1. $$\bar{\beta}_1 \oplus_{\varepsilon} \bar{\beta}_2 = \left(\frac{\sqrt{\mu^2_{\bar{\beta}_1} + \mu^2_{\bar{\beta}_2}}}{\sqrt{1 + \mu^2_{\bar{\beta}_1} \cdot_{\varepsilon} \mu^2_{\bar{\beta}_2}}}, \frac{\upsilon_{\bar{\beta}_1} \cdot_{\varepsilon} \upsilon_{\bar{\beta}_2}}{\sqrt{1 + \left(1 - \upsilon^2_{\bar{\beta}_1}\right) \cdot_{\varepsilon} \left(1 - \upsilon^2_{\bar{\beta}_2}\right)}}\right), \tag{10}$$
2. $$\bar{\beta}_1 \otimes_{\varepsilon} \bar{\beta}_2 = \left(\frac{\mu_{\bar{\beta}_1} \cdot_{\varepsilon} \mu_{\bar{\beta}_2}}{\sqrt{1 + \left(1 - \mu^2_{\bar{\beta}_1}\right) \cdot_{\varepsilon} \left(1 - \mu^2_{\bar{\beta}_2}\right)}}, \frac{\sqrt{\upsilon^2_{\bar{\beta}_1} + \upsilon^2_{\bar{\beta}_2}}}{\sqrt{1 + \upsilon^2_{\bar{\beta}_1} \cdot_{\varepsilon} \upsilon^2_{\bar{\beta}_2}}}\right), \tag{11}$$

3.
$$\bar{\beta}^{\wedge_\varepsilon \lambda} = \left(\frac{\sqrt{2\left(\mu_{\bar{\beta}}^2\right)^\lambda}}{\sqrt{\left(2-\mu_{\bar{\beta}}^2\right)^\lambda + \left(\mu_{\bar{\beta}}^2\right)^\lambda}}, \frac{\sqrt{\left(1+\upsilon_{\bar{\beta}}^2\right)^\lambda - \left(1-\upsilon_{\bar{\beta}}^2\right)^\lambda}}{\sqrt{\left(1+\upsilon_{\bar{\beta}}^2\right)^\lambda + \left(1-\upsilon_{\bar{\beta}}^2\right)^\lambda}} \right), \ \lambda > 0, \tag{12}$$

4.
$$\lambda \cdot_\varepsilon \bar{\beta} = \left(\frac{\sqrt{\left(1+\mu_{\bar{\beta}}^2\right)^\lambda - \left(1-\mu_{\bar{\beta}}^2\right)^\lambda}}{\sqrt{\left(1+\mu_{\bar{\beta}}^2\right)^\lambda + \left(1-\mu_{\bar{\beta}}^2\right)^\lambda}}, \frac{\sqrt{2\left(\upsilon_{\bar{\beta}}^2\right)^\lambda}}{\sqrt{\left(2-\upsilon_{\bar{\beta}}^2\right)^\delta + \left(\upsilon_{\bar{\beta}}^2\right)^\lambda}} \right), \ \lambda > 0. \tag{13}$$

Definition 6 (Garg 2016a) Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ is the collection of PFNs with $\leqslant_L$, then a PFEWA operator is a mapping PFFWA $: \ \Omega^n \to \Omega$ and

$$\begin{aligned} &\mathrm{PFEWA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = w_1 \cdot_\varepsilon \bar{\beta}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \bar{\beta}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon w_n \cdot_\varepsilon \bar{\beta}_n \\ &= \left(\frac{\sqrt{\prod_{i=1}^m \left(1+\mu_{\bar{\beta}_i}^2\right)^{w_i} - \prod_{i=1}^m \left(1-\mu_{\bar{\beta}_i}^2\right)^{w_i}}}{\sqrt{\prod_{i=1}^m \left(1+\mu_{\bar{\beta}_i}^2\right)^{w_i} + \prod_{i=1}^m \left(1-\mu_{\bar{\beta}_i}^2\right)^{w_i}}}, \frac{\sqrt{2\prod_{i=1}^m \left(\upsilon_{\bar{\beta}_i}^2\right)^{w_i}}}{\sqrt{\prod_{i=1}^m \left(2-\upsilon_{\bar{\beta}_i}^2\right)^{w_i} + \prod_{i=1}^m \left(\upsilon_{\bar{\beta}_i}^2\right)^{w_i}}} \right), \end{aligned} \tag{14}$$

where $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$ is the weighted vector of $\bar{\beta}_i (i = 1, 2, \ldots, m)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^m w_i = 1$.

Definition 7 (Garg 2017a) Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ is the collection of PFNs with $\leqslant_L$, then an PFEWG operator is a mapping PFEWG $: \ \Omega^n \to \Omega$, and

$$\begin{aligned} &\mathrm{PFEWG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = \bar{\beta}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{\beta}_2^{\wedge_\varepsilon w_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{\beta}_n^{\wedge_\varepsilon w_n} \\ &= \left(\frac{\sqrt{2\prod_{i=1}^m \left(\mu_{\bar{\beta}_i}^2\right)^{w_i}}}{\sqrt{\prod_{i=1}^m \left(2-\mu_{\bar{\beta}_i}^2\right)^{w_i} + \prod_{i=1}^m \left(\mu_{\bar{\beta}_i}^2\right)^{w_i}}}, \frac{\sqrt{\prod_{i=1}^m \left(1+\upsilon_{\bar{\beta}_i}^2\right)^{w_i} - \prod_{i=1}^m \left(1-\upsilon_{\bar{\beta}_i}^2\right)^{w_i}}}{\sqrt{\prod_{i=1}^m \left(1+\upsilon_{\bar{\beta}_i}^2\right)^{w_i} + \prod_{i=1}^m \left(1-\upsilon_{\bar{\beta}_i}^2\right)^{w_i}}} \right), \end{aligned} \tag{15}$$

where $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$ is the weighted vector of $\bar{\beta}_i (i = 1, 2, \ldots, m)$ such that $w_i \in [0, 1]$ and $\sum_{i=1}^m w_i = 1$.

2.1 Fuzzy measure and Choquet integral

In this subsection some basic definitions such as fuzzy measure, λ-fuzzy measure and discrete Choquet integral are given.

Definition 8 (Sugeno 1974) Let $Y = \{y_1, y_2, \ldots, y_n\}$ be a fixed set and $P(Y)$ be the power set of Y. Then a set function $\mu \ : \ P(Y) \to [0, 1]$ is said to be a fuzzy measure μ on Y if it fulfills the following properties:

1. $\mu(\phi) = 0, \mu(Y) = 1.$

2. If $U, V \in P(Y)$ and $U \subseteq V$ then $\mu(U) \leqslant \mu(V)$.

"However, it is essential to add the conditions of continuity whenever Y is infinite, it is sufficient to assume a finite fixed set in real exercise. $\mu(\{y_1, y_2, \ldots, y_n\})$ can be assumed as the rating of subjective status of decision criteria set $\{y_1, y_2, \ldots, y_n\}$. Therefore, with the discrete weights of criteria, weights of any arrangement of criteria can also be defined. Intuitively, we could say the following of pair of criteria sets $U, V \in P(X)$, $U \cap V = \phi$: U and V are assumed to be without interaction (or to be independent)" if

$$\mu(U \cup V) = \mu(U) + \mu(V), \tag{16}$$

μ is said to be an additive measure U and V show a positive synergetic interaction between them (or are complementary) if

$$\mu(U \cup V) > \mu(U) + \mu(V), \tag{17}$$

μ is said to be a superadditive measure U and V show a negative synergetic interaction between them (or are redundant or substitutive) if

$$\mu(U \cup V) < \mu(U) + \mu(V), \tag{18}$$

μ is said to be a sub-additive measure.

"Clearly it is hard to compute the fuzzy measure based on Definition (8). Thus, to confirm a fuzzy measure in MCGDM problems, Sugeno (1974) developed a λ-fuzzy measure which can be defined as:

$$\mu(U \cup V) = \mu(U) + \mu(V) + \lambda\mu(U)\mu(V), \tag{19}$$

$\lambda \in [-1, \infty)$, $U \cap V = \phi$. The parameter λ computes interaction among the criteria. In Eq. (19), if $\lambda = 0$, λ-fuzzy measure reduces to only an additive measure. And for negative and positive λ, the λ-fuzzy measure reduces to sub-additive and superadditive measures, respectively. However, when all the elements in Y are independent", then we get,

$$\mu(U) = \sum_{i=1}^{n} \mu(\{yi\}). \tag{20}$$

If Y is a finite set, then $\cup_{i=1}^{n}\{y_i\} = Y$. The λ-fuzzy measure μ fulfills the following Eq. (21)

$$\mu(Y) = \mu\left(\cup_{i=1}^{n}\{y_i\}\right) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{n}[1 + \lambda\mu(y_i)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{i=1}^{n} \mu(y_i) & \text{if } \lambda = 0 \end{cases}, \tag{21}$$

here $y_i \cap y_j = \phi \ \forall i, j = 1, 2, \ldots, n$ and $i \neq j$. It should be noted that for a subset with a single element y_i, $\mu(y_i)$ is said to be a fuzzy density and can be indicated as $\mu_i = \mu(y_i)$.

Particularly for each subset $U \in P(Y)$, we have

$$\mu(U) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{n}[1 + \lambda\mu(y_i)] - 1\right) & \text{if } \lambda \neq 0 \\ \sum_{i=1}^{n} \mu(y_i) & \text{if } \lambda = 0 \end{cases}, \tag{22}$$

On the basis of Eq. (2), λ can be uniquely computed from $\mu(Y)=1$, which is equivalent to solving

$$\lambda+1=\prod_{i=1}^{n}[1+\lambda\mu_i]. \tag{23}$$

Definition 9 (Choquet 1954) Let μ be a fuzzy measure on Y and g a positive real-valued function on Y. The discrete Choquet integral of g with respect to μ can be defined as:

$$C_{\mu}(g)=\sum_{i=1}^{n}g_{\sigma(i)}[\mu(\Im_{\sigma(i)})-\mu(\Im_{\sigma(i-1)})] \tag{24}$$

where $\sigma(i)$ shows a permutation on Y such that $g_{\sigma(1)}\geqslant g_{\sigma(2)}\geqslant\cdots\geqslant g_{\sigma(n)}$, $\Im_{\sigma(i)}=\{1,2,\ldots,i\}$, $\Im_{\sigma(0)}=\phi$.

Definition 10 (Peng and Yang 2016) Suppose $\bar{\beta}_i=(\mu_{\bar{\beta}_i},\upsilon_{\bar{\beta}_i})\,(i=1,2,\ldots,m)$ is a collection of PFNs and λ be a fuzzy measure on Y. Then a Pythagorean fuzzy Choquet integral average (PFECIA) operator based on fuzzy measure is a mapping PFCIA : $\Omega^m\rightarrow\Omega$ such that

$$\mathrm{PFCIA}\left(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\right)=\left\langle\sqrt{1-\prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}},\prod_{i=1}^{n}\left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}\right\rangle, \tag{25}$$

where $\{\sigma(1),\sigma(2),\ldots,\sigma(m)\}$ is a permutation of $\{1,2,\ldots,m\}$ such that $\bar{\beta}_{\sigma(1)}\geqslant\bar{\beta}_{\sigma(2)}\geqslant\cdots\geqslant\bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)}=\left\{y_{\sigma(k)}|i\leqslant k\right\}$ for $k\geqslant 1$, and $\Im_{\sigma(0)}=\phi$.

Definition 11 (Peng and Yang 2016) Suppose $\bar{\beta}_i=(\mu_{\bar{\beta}_i},\upsilon_{\bar{\beta}_i})\,(i=1,2,\ldots,m)$ is a collection of PFNs and λ be a fuzzy measure on Y. Then a Pythagorean fuzzy Choquet integral average (PFECIA) operator on fuzzy measure is a mapping PFCIG : $\Omega^m\rightarrow\Omega$ such that

$$\mathrm{PFCIG}\left(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\right)=\left\langle\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})},\sqrt{1-\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\right\rangle, \tag{26}$$

where $\{\sigma(1),\sigma(2),\ldots,\sigma(m)\}$ is a permutation of $\{1,2,\ldots,m\}$ such that $\bar{\beta}_{\sigma(1)}\geqslant\bar{\beta}_{\sigma(2)}\geqslant\cdots\geqslant\bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)}=\left\{y_{\sigma(k)}|i\leqslant k\right\}$ for $k\geqslant 1$, and $\Im_{\sigma(0)}=\phi$.

3 Pythagorean fuzzy Einstein Choquet integral aggregation operator

In this section based on Choquet integral and Einstein operations in we develop Pythagorean fuzzy Einstein Choquet integral average (PFECIA) Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) operator. We discuss some properties of the developed operators like idempotency, monotonicity and boundary detail.

Definition 12 Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ is a collection of PFNs and λ be a fuzzy measure on Y. Then a Pythagorean fuzzy Einstein Choquet integral average (PFECIA) operator based on fuzzy measure is a mapping PFECIA : $\Omega^m \to \Omega$ such that

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \begin{pmatrix} \lambda(\Im_{\sigma(1)}) - \lambda(\Im_{\sigma(0)}) \cdot_\varepsilon \bar{\beta}_1 \oplus_\varepsilon \lambda(\Im_{\sigma(2)}) - \\ \lambda(\Im_{\sigma(1)}) \cdot_\varepsilon \bar{\beta}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \lambda(\Im_{\sigma(m)}) - \lambda(\Im_{\sigma(m-1)}) \cdot_\varepsilon \bar{\beta}_m \end{pmatrix}, \tag{27}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ is a permutation of $\{1, 2, \ldots, m\}$ such that $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leqslant k\right\}$ for $k \geqslant 1$, and $\Im_{\sigma(0)} = \phi$.

Theorem 1 *Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ is a collection of PFNs and λ be a fuzzy measure on Y. Then,*

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \left(\begin{array}{c} \dfrac{\sqrt{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \\ \dfrac{\sqrt{2\prod_{i=1}^m \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^m \left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \end{array} \right), \tag{28}$$

where $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ is a permutation of $\{1, 2, \ldots, m\}$ such that $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leqslant k\right\}$ for $k \geqslant 1$, and $\Im_{\sigma(0)} = \phi$.

Proof Equation (28) can be shown based on mathematical induction. First to show that Eq. (28) holds for $m = 2$ we have,

$$\lambda(\Im_{\sigma(1)}) - \lambda(\Im_{\sigma(0)}) \cdot_\varepsilon \bar{\beta}_1 = \left(\begin{array}{c} \dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})} - \left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})} + \left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}, \\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})} + \left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}} \end{array} \right)$$

and

$$\lambda(\Im_{\sigma(2)}) - \lambda(\Im_{\sigma(1)}) \cdot_\varepsilon \bar{\beta}_2 = \left(\begin{array}{c} \dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})} - \left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})} + \left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}, \\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})} + \left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}} \end{array} \right)$$

$\square$

Now from (28) we have,

$$\mathrm{PFECIA}\big(\bar{\beta}_1,\bar{\beta}_2\big)=\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})\cdot_{\varepsilon}\bar{\beta}_1\oplus_{\varepsilon}\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})\cdot_{\varepsilon}\bar{\beta}_2$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}-\left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}+\left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}},\\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}+\left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}\end{array}\right)$$

$$\oplus_{\varepsilon}\left(\begin{array}{c}\dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}-\left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}+\left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}},\\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}+\left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}-\left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(1+\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}+\left(1-\mu^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(1-\mu^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}},\\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(2-\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(2)})-\lambda(\Im_{\sigma(1)})}+\left(\upsilon^2_{\bar{\beta}_{\sigma(1)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}\left(\upsilon^2_{\bar{\beta}_{\sigma(2)}}\right)^{\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{\prod_{i=1}^{2}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{2}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{2}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{2}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}},\\ \dfrac{\sqrt{2\prod_{i=1}^{2}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{2}\left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{2}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\end{array}\right)$$

This shows that Eq. (28) satisfies for $m=2$. Now suppose, Eq. (28) satisfies for $m=k$, that is

$$\mathrm{PFECIA}(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_k)=\left(\begin{array}{c}\lambda(\Im_{\sigma(1)})-\lambda(\Im_{\sigma(0)})\cdot_{\varepsilon}\bar{\beta}_1\oplus_{\varepsilon}\lambda(\Im_{\sigma(2)})-\\ \lambda(\Im_{\sigma(1)})\cdot_{\varepsilon}\bar{\beta}_2\oplus_{\varepsilon}\cdots\oplus_{\varepsilon}\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})\cdot_{\varepsilon}\bar{\beta}_k\end{array}\right)$$

$$=\left(\begin{array}{c}\dfrac{\sqrt{\prod_{i=1}^{k}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{k}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{k}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}},\\ \dfrac{\sqrt{2\prod_{i=1}^{k}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\hat{C}_{\sigma(i)})-\lambda(\hat{C}_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k}\left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{k}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\end{array}\right).$$

Next we show that Eq. (28) satisfies for $m = k + 1$, by the Einstein operational laws of the PFNs we have,

$$
\begin{aligned}
&\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{k+1})\\
&= \begin{pmatrix} \lambda(\Im_{\sigma(1)}) - \lambda(\Im_{\sigma(0)}) \cdot_{\varepsilon} \bar{\beta}_1 \oplus_{\varepsilon} \lambda(\Im_{\sigma(2)}) - \lambda(\Im_{\sigma(1)}) \cdot_{\varepsilon} \bar{\beta}_2 \oplus_{\varepsilon} \cdots \oplus_{\varepsilon} \\ \lambda(\Im_{\sigma(k)}) - \lambda(\Im_{\sigma(k-1)}) \cdot_{\varepsilon} \bar{\beta}_k \oplus_{\varepsilon} \lambda(\Im_{\sigma(k+1)}) - \lambda(\Im_{\sigma(k)}) \cdot_{\varepsilon} \bar{\beta}_{k+1} \end{pmatrix}\\
&= \left(\begin{array}{c} \dfrac{\sqrt{\prod_{i=1}^{k} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{k} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{k} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \\ \dfrac{\sqrt{2\prod_{i=1}^{k} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k} \left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{k} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \end{array} \right)\\
&\oplus_{\varepsilon} \left(\begin{array}{c} \dfrac{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} - \left(1-\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}{\sqrt{\left(1+\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} + \left(1-\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}, \\ \dfrac{\sqrt{2\left(\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}{\sqrt{\left(2-\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} + \left(\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}} \end{array} \right)\\
&= \left(\begin{array}{c} \dfrac{\sqrt{\prod_{i=1}^{k} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(1+\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} - \prod_{i=1}^{k} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(1-\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}{\sqrt{\prod_{i=1}^{k} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(1+\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} + \prod_{i=1}^{k} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(1-\mu^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}, \\ \dfrac{\sqrt{2\prod_{i=1}^{k} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}}{\sqrt{\prod_{i=1}^{k} \left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(2-\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})} + \prod_{i=1}^{k} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \left(\upsilon^2_{\bar{\beta}_{\sigma(k+1)}}\right)^{\lambda(\Im_{\sigma(k+1)})-\lambda(\Im_{\sigma(k)})}}} \end{array} \right)\\
&= \left(\begin{array}{c} \dfrac{\sqrt{\prod_{i=1}^{k+1} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{k} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k+1} \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{k+1} \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \\ \dfrac{\sqrt{2\prod_{i=1}^{k+1} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{k+1} \left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{k+1} \left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \end{array} \right).
\end{aligned}
$$

That is Eq. (28) satisfies for $m = k + 1$. Thus, Eq. (28) satisfies $\forall\ m$. This completes the proof.

Lemma 1 (Xu 2000) *Let* $\bar{\beta}_i > 0$, $w_i > 0$ $(i = 1, 2, \ldots m)$ *and* $\sum_{i=1}^{m} w_i = 1$, *then*

$$
\prod_{i=1}^{m} (\bar{\beta}_i)^{w_i} \leqslant \sum_{i=1}^{m} w_i \bar{\beta}_i,
$$

where the equality holds if and only if $\bar{\beta}_1 = \bar{\beta}_2 = \cdots = \bar{\beta}_m$.

Theorem 2 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})$ $(i = 1, 2, \ldots, m)$ *is a collection of PFNs. Then the aggregated value using PFECIA operator is also a PFN, i.e.*, $\mathrm{PFECIA}\left(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\right) \in$ PFN*s*, *where* $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leqslant k\right\}$ *for* $k \geqslant 1$, *and* $\Im_{\sigma(0)} = \phi$.

Proof As we know that $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i}) \in$ PFNs,$(i = 1, 2, \ldots, m)$, so by definition of PFNs we have $0 \leqslant \mu_{\bar{\beta}_i} \leqslant 1$, $0 \leqslant \upsilon_{\bar{\beta}_i} \leqslant 1$ and $\mu^2_{\bar{\beta}_i} + \upsilon^2_{\bar{\beta}_i} \leqslant 1$. Thus,

$$\begin{aligned}
&\frac{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\quad = 1 - \frac{2\prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}.\\
&\quad \leq 1 - \prod_{i=1}^m (1-\mu^2_{\bar{\beta}_{\sigma(i)}})^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}
\end{aligned} \tag{29}$$

□

Also,

$$\begin{aligned}
&1+\mu^2_{\bar{\beta}_{\sigma(i)}} \geq 1-\mu^2_{\bar{\beta}_{\sigma(i)}}\\
&\quad \Rightarrow \prod_{i=1}^m (1+\mu^2_{\bar{\beta}_{\sigma(i)}})^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \geq 0\\
&\quad \Rightarrow \frac{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(j)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \geq 0
\end{aligned}.$$

Hence, $0 \leq \mu_{\bar{\beta}_{\text{PFEPWA}}} \leq 1$.
Similarly,

$$\begin{aligned}
&\frac{2\prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^m \left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\quad \leq \prod_{i=1}^m \left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \leq 1
\end{aligned}.$$

$$\prod_{i=1}^m \left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \geq 0$$

Also,

$$\Leftrightarrow \frac{2\prod_{i=1}^m \left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^m \left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^m \left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \geq 0$$

Hence, $0 \leq \upsilon_{\bar{\beta}_{\text{PFEPWA}}} \leq 1$.

Furthermore,

$$\mu^2_{\bar{\beta}_{\text{PFECIA}}} + \upsilon^2_{\bar{\beta}_{\text{PFECIA}}}$$

$$= \left(\frac{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} + \frac{2\prod_{i=1}^{m}\left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \right).$$

$$\leq 1 - \frac{2\prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}_{\sigma(j)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$+ \frac{2\prod_{i=1}^{m}\left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \leq 1$$

Thus, PFECIA $\in [0, 1]$. Therefore, PFNs aggregated by PFECIA operator is again a PFN.

Theorem 3 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ is a collection of PFNs and λ be a fuzzy measure on Y such that $\sum_{i=1}^{m} \lambda(\Im_{\sigma(i)}) - \lambda(\Im_{\sigma(i-1)}) = 1$. *Then,*

$$\text{PFECIA}\big(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\big) \leqslant \text{PFCIA}\big(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\big), \tag{30}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leqslant k\}$ *for* $k \geqslant 1$, and $\Im_{\sigma(0)} = \phi$.

Proof Let PFECIA$(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = (\mu^{\varepsilon}_{\bar{\beta}}, \upsilon^{\varepsilon}_{\bar{\beta}}) = \bar{\beta}^{\varepsilon}$ and PFCIA$(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = (\mu_{\bar{\beta}}, \upsilon_{\bar{\beta}}) = \bar{\beta}$. As we know that

$$\sqrt{\prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\leqslant \sqrt{\sum_{i=1}^{m} \lambda(\Im_{\sigma(i)}) - \lambda(\Im_{\sigma(i-1)})\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right) + \sum_{i=1}^{m} \lambda(\Im_{\sigma(i)}) - \lambda(\Im_{\sigma(i-1)})\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)}.$$

$$= \sqrt{2}$$

□

Thus from Eq. (29) we have

$$\sqrt{\frac{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\geqslant \sqrt{1 - \prod_{i=1}^{m}\left(1-\mu_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}, \tag{31}$$
$$\Leftrightarrow \mu \leq \mu_{\bar{\beta}}$$

where the quality holds if and only if $\mu^2_{\bar{\beta}_i}(i=1,2,\ldots,m)$ are equal. Again as we know that

$$\sqrt{\prod_{i=1}^{m}(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}})^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}(\upsilon^2_{\bar{\beta}_{\sigma(i)}})^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\leqslant\sqrt{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}})+\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)}=\sqrt{2}.$$

Thus,

$$\frac{\sqrt{2\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)+\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)}}$$

$$\geq\frac{\sqrt{2\prod_{i=1}^{m}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\geqslant\prod_{i=1}^{m}\left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}$$

$$\Leftrightarrow\upsilon^{\varepsilon}_{\bar{\beta}}\geq\upsilon_{\bar{\beta}},\tag{32}$$

where the quality holds if and only if $\upsilon_{\bar{\beta}_i}(i=1,2,\ldots,m)$ are equal.

Then Eqs. (31), (32) can be transformed into the following forms: $\mu_{\bar{\beta}}\leqslant\mu_{\bar{\beta}^{\varepsilon}}$ and $\upsilon_{\bar{\beta}}\geqslant\upsilon_{\bar{\beta}^{\varepsilon}}$ respectively. Hence, $S(\bar{\beta})=\mu^2_{\bar{\beta}}-\upsilon^2_{\bar{\beta}}\succcurlyeq\mu^2_{\bar{\beta}^{\varepsilon}}-\upsilon^2_{\bar{\beta}^{\varepsilon}}=S(\bar{\beta}^{\varepsilon})$. So $S(\bar{\beta})\geqslant S(\bar{\beta}^{\varepsilon})$. If $S(\bar{\beta})>S(\bar{\beta}^{\varepsilon})$. Then we have

$$\mathrm{PFCIA}\big(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\big)>\mathrm{PFECIA}\big(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\big).\tag{33}$$

If $S(\bar{\beta})=S(\bar{\beta}^{\varepsilon})$, i.e., $\mu^2_{\bar{\beta}}-\upsilon^2_{\bar{\beta}}=\mu^2_{\bar{\beta}^{\varepsilon}}-\upsilon^2_{\bar{\beta}^{\varepsilon}}$, then we have $H(\bar{\beta})=\mu^2_{\bar{\beta}}+\upsilon^2_{\bar{\beta}}=\mu^2_{\bar{\beta}^{\varepsilon}}+\upsilon^2_{\bar{\beta}^{\varepsilon}}=H(\bar{\beta}^{\varepsilon})$. Thus we have

$$\mathrm{PFCIA}\big(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\big)=\mathrm{PFECIA}\big(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m\big).\tag{34}$$

From Eqs. (33) to (34) we have Eq. (30) always holds, where the equality holds if and only if $\bar{\beta}_i(i=1,2,\ldots,m)$ are equal. Thus,

$$\mathrm{PFCIA}(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m)\geq\mathrm{PFECIA}(\bar{\beta}_1,\bar{\beta}_2,\ldots,\bar{\beta}_m)$$

Example 1 Let $\bar{\beta}_1=(0.6,0.7)$, $\bar{\beta}_2=(0.8,0.5)$ and $\bar{\beta}_3=(0.9,0.4)$ be the three PFNs. Let $\mu(\Im_1)=0.35$, $\mu(\Im_2)=0.3$, $\mu(\Im_3)=0.25$, then λ of $\bar{\beta}_1,\bar{\beta}_2,\bar{\beta}_3$ can be determine as $\lambda=0.36$. Thus by Eq. (7) we calculate $\mu(\Im_1,\Im_2)=0.69$, $\mu(\Im_1,\Im_3)=0.63$, $\mu(\Im_2,\Im_3)=0.58$, $\mu(\Im_1,\Im_2,\Im_3)=1$. By Definition 4, $\bar{\beta}_i$ $(i=1,2,3)$ is reordered such that $\bar{\beta}_i>\bar{\beta}_{(i-1)}$. Hence

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = \left(\frac{\sqrt{\prod_{i=1}^{3}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{3}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{3}\left(1+\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})} + \prod_{i=1}^{3}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \frac{\sqrt{2\prod_{i=1}^{3}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{3}\left(2-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{3}\left(\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \right).$$

$$= (0.7734, 0.5499)$$

$S(\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)) = 0.2235$

Now using PFCIA operator, we have

$$\mathrm{PFCIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = \left(\sqrt{1 - \prod_{i=1}^{3}\left(1-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}, \prod_{i=1}^{3}\left(\upsilon_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} \right).$$

$$= (0.7806, 0.5447)$$

$S(\mathrm{PFCIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)) = 0.2359$

Thus, from the above example, we have

$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) \leqslant \mathrm{PFCIA}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)$.

The PFECIA operator has the following properties:

Theorem 4 (Idempotency) *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *be a fuzzy measure on Y:*

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \bar{\beta}. \tag{35}$$

Proof We have,

$$\begin{aligned} \mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) &= \mathrm{PFECIA}(\bar{\beta}, \bar{\beta}, \ldots, \bar{\beta}) \\ &= \begin{pmatrix} \lambda(\Im_{\sigma(1)}) - \lambda(\Im_{\sigma(0)}) \cdot_{\varepsilon} \bar{\beta} \oplus_{\varepsilon} \lambda(\Im_{\sigma(2)}) - \lambda(\Im_{\sigma(1)}) \\ \cdot_{\varepsilon} \bar{\beta} \oplus_{\varepsilon} \cdots \oplus_{\varepsilon} \lambda(\Im_{\sigma(m)}) - \lambda(\Im_{\sigma(m-1)}) \cdot_{\varepsilon} \bar{\beta} \end{pmatrix} \\ &= \left(\frac{\sqrt{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} - \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\mu^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(1-\mu^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \frac{\sqrt{2\prod_{i=1}^{m}\left(\upsilon^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\upsilon^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})} + \prod_{i=1}^{m}\left(\upsilon^2_{\bar{\beta}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \right). , \\ &= \left(\frac{1+\mu_{\bar{\beta}}-1-\mu_{\bar{\beta}}}{1+\mu_{\bar{\beta}}+1-\mu_{\bar{\beta}}}, \frac{2\upsilon_{\bar{\beta}}}{2-\upsilon_{\bar{\beta}}+\upsilon_{\bar{\beta}}} \right) = (\mu_{\bar{\beta}}, \upsilon_{\bar{\beta}}) = \bar{\beta} \end{aligned} \tag{36}$$

□

This completes the proof.

Theorem 5 (Boundary) *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})\,(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *a fuzzy measure on Y. Where* $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$

such that $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leqslant k\right\}$ *for* $k \geqslant 1$, *and* $\Im_{\sigma(0)} = \phi$. *If* $\bar{\beta}_{\min} = \min\left(\bar{\beta}_i\right)$, $\bar{\beta}_{\max} = \max\left(\bar{\beta}_i\right)$. *Then*

$$\bar{\beta}_{\min} \leqslant \text{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) \leqslant \bar{\beta}_{\max} \tag{37}$$

Proof Let, $g(a) = \sqrt{\frac{2-a^2}{a^2}}$, $a \in (0, 1]$. Then, $g'(x) = \frac{-2}{a^3}\sqrt{\frac{a^2}{2-a^2}} < 0$. So $g(x)$ is decreasing function on $(0, 1]$. Since, $\mu_{\bar{\beta}_{\min}} \leqslant \mu_{\bar{\beta}_i} \leqslant \mu_{\bar{\beta}_{\max}}$, for all i. Then $g\left(\mu_{\bar{\beta}_{\max}}\right) \leqslant g\left(\mu_{\bar{\beta}_i}\right) \leqslant g\left(\mu_{\bar{\beta}_{\min}}\right)$ $(i = 1, 2, \ldots, m)$ i.e., $\sqrt{\frac{2-\mu^2_{\bar{\beta}_{\max}}}{\mu^2_{\bar{\beta}_{\max}}}} \leqslant \sqrt{\frac{2-\mu^2_{\bar{\beta}_i}}{\mu^2_{\bar{\beta}_i}}} \leqslant \sqrt{\frac{2-\mu^2_{\bar{\beta}_{\min}}}{\mu^2_{\bar{\beta}_{\min}}}}$ and since $\Im_{\sigma(i)} \supseteq \Im_{\sigma(i-1)}$, then $\lambda(\Im_{\sigma(i)}) - \lambda(\Im_{\sigma(i-1)}) \geqslant 0$. Let $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ is a permutation of $\{1, 2, \ldots, m\}$ such that $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$, we have,

$$\begin{aligned}
&\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\max}}}{\mu^2_{\bar{\beta}_{\max}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\min}}}{\mu^2_{\bar{\beta}_{\min}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\max}}}{\mu^2_{\bar{\beta}_{\max}}}\right)^{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\min}}}{\mu^2_{\bar{\beta}_{\min}}}\right)^{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\max}}}{\mu^2_{\bar{\beta}_{\max}}}\right)^{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_i}}{\mu^2_{\bar{\beta}_i}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\\
&\leqslant \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\min}}}{\mu^2_{\bar{\beta}_{\min}}}\right)}\\
&\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\max}}}{\mu^2_{\bar{\beta}_{\max}}}\right)+1}\\
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}\\
&\leqslant \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\min}}}{\mu^2_{\bar{\beta}_{\min}}}\right)+1}\\
&\Leftrightarrow \frac{\sqrt{2}}{\sqrt{\mu^2_{\bar{\beta}_{\max}}}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu_{\bar{\beta}_i}^2}{\mu_{\bar{\beta}_{\sigma(i)}}^2}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}\\
&\leqslant \frac{\sqrt{2}}{\sqrt{\mu_{\bar{\beta}_{\min}}^2}}\\
&\Leftrightarrow \frac{\sqrt{\mu_{\bar{\beta}_{\min}}^2}}{\sqrt{2}}\\
&\leqslant \frac{1}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu_{\bar{\beta}_{\sigma(i)}}^2}{\mu_{\bar{\beta}_{\sigma(i)}}^2}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}}\\
&\leqslant \frac{\sqrt{\mu_{\bar{\beta}_{\max}}^2}}{\sqrt{2}}\\
&\Leftrightarrow \sqrt{\mu_{\bar{\beta}_{\min}}^2}\\
&\leqslant \frac{\sqrt{2}}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu_{\bar{\beta}_{\sigma(i)}}^2}{\mu_{\bar{\beta}_{\sigma(i)}}^2}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}}\\
&\leqslant \sqrt{\mu_{\bar{\beta}_{\max}}^2}\\
&\Leftrightarrow \mu_{\bar{\beta}_{\min}}\\
&\leqslant \frac{\sqrt{2}}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu_{\bar{\beta}_{\sigma(i)}}^2}{\mu_{\bar{\beta}_{\sigma(i)}}^2}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}}\\
&\leqslant \mu_{\bar{\beta}_{\max}}\\
&\Leftrightarrow \mu_{\bar{\beta}_{\min}}\\
&\leqslant \frac{\sqrt{2}}{\sqrt{\frac{\prod_{i=1}^{m}\left(2-\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}\\
&\leqslant \mu_{\bar{\beta}_{\max}}\\
&\Leftrightarrow \mu_{\bar{\beta}_{\min}}\\
&\leqslant \frac{\sqrt{2}}{\frac{\sqrt{\prod_{i=1}^{m}\left(2-\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}\\
&\leqslant \mu_{\bar{\beta}_{\max}}\\
&\Leftrightarrow \mu_{\bar{\beta}_{\min}}\\
&\leqslant \frac{\sqrt{2\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\\
&\leqslant \mu_{\bar{\beta}_{\max}}.
\end{aligned}
\tag{38}
$$

□

Again let, $h(b)=\sqrt{\frac{1-b^2}{1+b^2}}$, $b\in[0,1]$. Then, $h'(b)=\frac{-2b}{(1+b^2)^2}\sqrt{\frac{1+b^2}{1-b^2}}<0$. Since $h(b)$ is a decreasing function on [0, 1]. Thus, $\upsilon_{\bar{\beta}_{\max}}\leqslant\upsilon_{\bar{\beta}_j}\leqslant\upsilon_{\bar{\beta}_{\min}}$ for all i. Then $h(\upsilon_{\bar{\beta}_{\min}})\leqslant h(\upsilon_{\bar{\beta}_i})\leqslant h(\upsilon_{\bar{\beta}_{\max}})$ for all i. i.e., $\sqrt{\frac{1-\upsilon^2_{\bar{\beta}_{\min}}}{1+\upsilon^2_{\bar{\beta}_{\min}}}}\leqslant\sqrt{\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}}\leqslant\sqrt{\frac{1-\upsilon^2_{\bar{\beta}_{\max}}}{1+\upsilon^2_{\bar{\beta}_{\max}}}}$, $(i=1,2,\ldots,m)$, and let $\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})>0(i=1,2,\ldots,m)$ and $\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})=1$, we have

$$\Leftrightarrow\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\min}}}{1+\upsilon^2_{\bar{\beta}_{\min}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\leqslant\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\leqslant\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\max}}}{1+\upsilon^2_{\bar{\beta}_{\max}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\Leftrightarrow\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\min}}}{1+\upsilon^2_{\bar{\beta}_{\min}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\leqslant\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\leqslant\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\max}}}{1+\upsilon^2_{\bar{\beta}_{\max}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\Leftrightarrow\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\min}}}{1+\upsilon^2_{\bar{\beta}_{\min}}}\right)^{\sum_{i=1}^{m}\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}\leqslant\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\leqslant\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\max}}}{1+\upsilon^2_{\bar{\beta}_{\max}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$
$$\Leftrightarrow\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\min}}}{1+\upsilon^2_{\bar{\beta}_{\min}}}\right)+1}\leqslant\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}$$
$$\leqslant\sqrt{\left(\frac{1-\upsilon^2_{\bar{\beta}_{\max}}}{1+\upsilon^2_{\bar{\beta}_{\max}}}\right)+1}$$
$$\Leftrightarrow\frac{\sqrt{2}}{\sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}}\leqslant\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_i}}{1+\upsilon^2_{\bar{\beta}_i}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}$$
$$\leqslant\frac{\sqrt{2}}{\sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}}}$$
$$\Leftrightarrow\sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}}$$
$$\leqslant\frac{\sqrt{2}}{\sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\hat{C}_{\sigma(i)})-\lambda(\hat{C}_{\sigma(i-1)})}+1}}\leqslant\sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}$$
$$\Leftrightarrow\sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}}$$
$$\leqslant\frac{\sqrt{2}}{\sqrt{\frac{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}\leqslant\sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}$$

$$\Leftrightarrow \sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}} \leqslant \frac{\sqrt{2}\sqrt{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\leqslant \sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}$$
$$\Leftrightarrow \sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}} \leqslant \frac{\sqrt{2\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\leqslant \sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}$$
$$\Leftrightarrow \sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}} \leqslant \sqrt{\frac{2\prod_{j=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\leqslant \sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}}$$
$$\Leftrightarrow \sqrt{1+\upsilon^2_{\bar{\beta}_{\max}}-1} \leqslant \sqrt{\frac{2\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}-1}$$
$$\leqslant \sqrt{1+\upsilon^2_{\bar{\beta}_{\min}}-1}$$
$$\Leftrightarrow \sqrt{\upsilon^2_{\bar{\beta}_{\max}}} \leqslant \sqrt{\frac{2\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\leqslant \sqrt{\upsilon^2_{\bar{\beta}_{\min}}}$$
$$\Leftrightarrow \upsilon_{\bar{\beta}_{\max}} \leqslant \frac{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$
$$\leqslant \upsilon_{\bar{\beta}_{\min}}. \tag{39}$$

Let PFECIA$(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = \bar{\beta}$. Then Eqs. (38) and (39) can be written as: $\mu_{\bar{\beta}_{\min}} \leqslant \mu_{\bar{\beta}} \leqslant \mu_{\bar{\beta}_{\max}}$ and $\upsilon_{\bar{\beta}_{\max}} \leqslant \upsilon_{\bar{\beta}} \leqslant \upsilon_{\bar{\beta}_{\min}}$, respectively. Thus $S(\bar{\beta}) = \mu^2_{\bar{\beta}} - \upsilon^2_{\bar{\beta}} \leqslant \mu^2_{\bar{\beta}_{\max}} - \upsilon^2_{\bar{\beta}_{\max}} = S(\bar{\beta}_{\max})$ and $S(\bar{\beta}) = \mu^2_{\bar{\beta}} - \upsilon^2_{\bar{\beta}} \geqslant \mu^2_{\bar{\beta}_{\min}} - \upsilon^2_{\bar{\beta}_{\min}} = S(\bar{\beta}_{\min})$. If, $S(\bar{\beta}) < S(\bar{\beta}_{\max})$ and $S(\bar{\beta}) > S(\bar{\beta}_{\min})$. Then we have

$$\bar{\beta}_{\min} < \text{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) < \bar{\beta}_{\max}. \tag{40}$$

If $S(\bar{\beta}) = S(\bar{\beta}_{\max})$ i.e., $\mu^2_{\bar{\beta}} - \upsilon^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\max}} - \upsilon^2_{\bar{\beta}_{\max}}$. Then we have $\mu^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\max}}$ and $\upsilon^2_{\bar{\beta}} = \upsilon^2_{\bar{\beta}_{\max}}$. Thus $H(\bar{\beta}) = \mu^2_{\bar{\beta}} + \upsilon^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\max}} + \upsilon^2_{\bar{\beta}_{\max}} = H(\bar{\beta}_{\max})$. Hence

$$\text{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = \bar{\beta}_{\max}. \tag{41}$$

If $S(\bar{\beta}) = S(\bar{\beta}_{\min})$ i.e., $\mu^2_{\bar{\beta}} - \upsilon^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\min}} - \upsilon^2_{\bar{\beta}_{\min}}$. Then we have $\mu^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\min}}$ and $\upsilon^2_{\bar{\beta}} = \upsilon^2_{\bar{\beta}_{\min}}$. Thus, $H(\bar{\beta}) = \mu^2_{\bar{\beta}} + \upsilon^2_{\bar{\beta}} = \mu^2_{\bar{\beta}_{\min}} + \upsilon^2_{\bar{\beta}_{\min}} = H(\bar{\beta}_{\min})$. Thus

$$\text{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_n) = \bar{\beta}_{\min}. \tag{42}$$

Thus from Eqs. (40) to (44) we have Eq. (37) always holds. Thus

$$\bar{\beta}_{\min} \leqslant \mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leq \bar{\beta}_{\max}.$$

Theorem 6 (Monotonicity) *Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})$ $(i = 1, 2, \ldots, m)$ is a collection of PFNs and λ a fuzzy measure on Y. Let $\bar{\beta}_i^* = (\mu^*_{\bar{\beta}_i}, \upsilon^*_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ be a collection of PFNs. Then*

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leqslant \mathrm{PFECIA}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*) \tag{43}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ is a permutation of $\{1, 2, \ldots, m\}$ such that $\bar{\beta}_{\sigma(1)} \geqslant \bar{\beta}_{\sigma(2)} \geqslant \cdots \geqslant \bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leqslant k\}$ for $k \geqslant 1$, and $\Im_{\sigma(0)} = \phi$.

Proof Let, $g(x) = \sqrt{\frac{2-a^2}{a^2}}$, $a \in (0, 1]$. Then, $g'(a) = \frac{-2}{a^3}\sqrt{\frac{a^2}{2-a^2}} < 0$. Since $g(a)$ is decreasing function on (0, 1]. If $\mu_{\bar{\beta}_i} \leqslant \mu_{\bar{\beta}_i^*}$ for all i. Then, $g(\mu_{\bar{\beta}_i^*}) \leqslant g(\mu_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$. i.e., $\sqrt{\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}} \leqslant \sqrt{\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}}$, $(i = 1, 2, \ldots, m)$.

we have,

$$\Leftrightarrow \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \leqslant \sqrt{\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\Leftrightarrow \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\Leftrightarrow \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1} \leqslant \sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}$$

$$\Leftrightarrow \frac{1}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}} \leqslant \frac{1}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}}$$

$$\Leftrightarrow \frac{\sqrt{2}}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}_{\sigma(i)}}}{\mu^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}} \leqslant \frac{\sqrt{2}}{\sqrt{\prod_{i=1}^{m}\left(\frac{2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}}{\mu^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+1}}$$

$$\Leftrightarrow \frac{\sqrt{2}}{\frac{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}$$

$$\leqslant \frac{\sqrt{2}}{\frac{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(I)})-\lambda(\Im_{\sigma(I-1)})}}}}$$

$$\Leftrightarrow \frac{\sqrt{2}\sqrt{\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\leqslant \frac{\sqrt{2}\sqrt{\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\Leftrightarrow \frac{\sqrt{2\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\leqslant \frac{\sqrt{2\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(j)}}\right)^{\lambda(\aleph_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}^*_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}. \tag{44}$$

$\square$

Again let, $h(b)=\sqrt{\frac{1-b^2}{1+b^2}}, b\in[0,1]$, then $h'(b)=\frac{-2b}{(1+b^2)^2}\sqrt{\frac{1+b^2}{1-b^2}}<0$. Since $h(b)$ is a decreasing function on $(0,1]$. If $\upsilon_{\bar{\beta}_j}\geqslant\upsilon_{\bar{\beta}^*_j}$ for all i. Then $h(\upsilon_{\bar{\beta}^*_j})\geqslant h(\upsilon_{\bar{\beta}_j})$ for all i. i.e., $\sqrt{\frac{1-\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}}\geqslant\sqrt{\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}}$, $(i=1,2,\ldots,m)$. Now,

$$\Leftrightarrow \sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \geqslant \sqrt{\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\Leftrightarrow \sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \geqslant \sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}$$

$$\Leftrightarrow \frac{1}{\sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \geqslant \frac{1}{\sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\Leftrightarrow \frac{\sqrt{2}}{\sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \geqslant \frac{\sqrt{2}}{\sqrt{1+\prod_{i=1}^{m}\left(\frac{1-\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}{1+\upsilon^2_{\bar{\beta}^*_{\sigma(i)}}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}$$

$$\Leftrightarrow \frac{\sqrt{2}}{\sqrt{\frac{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}$$

$$
\begin{aligned}
&\geqslant \frac{\sqrt{2}}{\sqrt{\dfrac{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}}\\
&\Leftrightarrow \frac{\sqrt{2\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\\
&\geqslant \frac{\sqrt{2\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\\
&\Leftrightarrow \sqrt{\frac{2\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}-1}\\
&\geqslant \sqrt{\frac{2\prod_{j=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(i-1)})}}{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}-1}\\
&\Leftrightarrow \frac{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}\\
&\geqslant \frac{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon_{\bar{\beta}^*_{\sigma(i)}}^2\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}.
\end{aligned}
\tag{45}
$$

Let, $\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \bar{\beta}$ and $\mathrm{PFECIA}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*) = \bar{\beta}^*$.

Then (44) and (45) can be transformed into the following forms: $\mu_{\bar{\beta}} \leqslant \mu_{\bar{\beta}^*}$ and $\upsilon_{\bar{\beta}}////\upsilon_{\bar{\beta}^*}$ respectively. Since $S(\bar{\beta}) = \mu_{\bar{\beta}}^2 - \upsilon_{\bar{\beta}}^2 \leqslant \mu_{\bar{\beta}^*}^2 - \upsilon_{\bar{\beta}^*}^2 = S(\bar{\beta}^*)$. Thus, $S(\bar{\beta}) \leqslant S(\bar{\beta}^*)$. If $S(\bar{\beta}) < S(\bar{\beta}^*)$, Then we have

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) < \mathrm{PFECIA}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*). \tag{46}$$

If $S(\bar{\beta}) = S(\bar{\beta}^*)$. i.e., $\mu_{\bar{\beta}}^2 - \upsilon_{\bar{\beta}}^2 = \mu_{\bar{\beta}^*}^2 - \upsilon_{\bar{\beta}^*}^2$, then we have $\mu_{\bar{\beta}}^2 = \mu_{\bar{\beta}^*}^2$ and $\upsilon_{\bar{\beta}}^2 = \upsilon_{\bar{\beta}^*}^2$. Thus, $H(\bar{\beta}) = \mu_{\bar{\beta}}^2 + \upsilon_{\bar{\beta}}^2 = \mu_{\bar{\beta}^*}^2 + \upsilon_{\bar{\beta}^*}^2 = H(\bar{\beta}^*)$. Then we have

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \mathrm{PFECIA}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*). \tag{47}$$

From Eqs. (46) to (47), we have

$$\mathrm{PFECIA}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leqslant \mathrm{PFECIA}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*)$$

This completes the proof. □

Theorem 7 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m), \bar{\rho}_i = (\mu_{\bar{\rho}_i}, \upsilon_{\bar{\rho}_i})(i = 1, 2, \ldots, m)$ *are two collections of PFNs on Y and* λ *be a fuzzy measure on Y. Where* $\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \left\{y_{\sigma(k)}|i \leq k\right\}$ *for* $k \geq 1$, *and* $\aleph_{\sigma(0)} = \phi$. *Then the following statements are equivalent:*

1. *For* $\aleph, \Im \in P(Y)$ *such that* $|\aleph| = |\Im|$, *we have* $\lambda(\aleph) = \lambda(\Im)$.
2. *There is an exponential weighted vector* $w = (w_1, w_2, \ldots, w_m)$ *such that*

$$\text{PFECIA}\left(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\right) = \text{PFEOWA}_w\left(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\right)$$

Proof Proof of the Theorem is same as Proposition 1 (Sect. 4.2) in (Marichal 2002). □

Remark Assume that PFEWA operator has an exponential weighted vector $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$. Then by Theorem 1 it is clear that λ is an additive fuzzy measure $\lambda(\Im) = \sum_{i \in \aleph} w_i$, PFECIA operator will be reduce to a PFEWA operator, where $w_i = \lambda(i)$. Conversely it is clear that PFEWA is a PFECIA operator with additive fuzzy measure λ: $\lambda(\Im) = \sum_{i \in \aleph} w_i$, $\aleph \subseteq Y$.

From Theorem 1 and above Remark, "it is clear that PFECIA operator is a generalization of both PFEOWA and PFEWA operators. Thus, PFEOWA and PFEWA operators are two special cases of PFECIA operator".

Definition 13 Suppose $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ is the collection of PFNs and λ a fuzzy measure on Y. Then a Pythagorean fuzzy Einstein Choquet integral geometric PFECIG operator based on fuzzy measure is a mapping PFECIG : $\Omega^n \to \Omega$, and

$$\text{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \begin{pmatrix} \bar{\beta}_1^{\wedge_\varepsilon \lambda(\Im_{\sigma(1)}) - \lambda(\Im_{\sigma(0)})} \otimes_\varepsilon \bar{\beta}_2^{\wedge_\varepsilon \lambda(\Im_{\sigma(2)}) - \lambda(\Im_{\sigma(1)})} \\ \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{\beta}_m^{\wedge_\varepsilon \lambda(\Im_{\sigma(m)}) - \lambda(\Im_{\sigma(m-1)})} \end{pmatrix}, \tag{48}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ is a permutation of $\{1, 2, \ldots, m\}$ such that $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ and $\Im_{\sigma(k)} = \{y_{\sigma(k)}|i \leq k\}$ for $k \geq 1$, and $\Im_{\sigma(0)} = \phi$. Based on the Einstein operational law we have the following result.

Theorem 8 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})$ $(i = 1, 2, \ldots, m)$ *is a collection of PFNs with* $\wedge_\varepsilon$ *and* λ *be a fuzzy measure on Y*

$$\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \left(\frac{\sqrt{2\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(2-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \frac{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{j=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{m}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{m}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \right), \tag{49}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leq k\}$ *for* $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$.

Proof Proof of the Theorem follows from Theorem 1. □

Theorem 9 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *be a fuzzy measure on Y. Then the aggregated value by utilizing PFECIG operator is also a PFN, i.e.,* $\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \in \mathrm{PFN}$, *where* $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leq k\}$ for $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$.

Proof Proof of the Theorem is same as Theorem 2. □

Theorem 10 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ *is a collection of PFNs with* $\leq_L$, *and* λ *be a fuzzy measure on* Y, *then*

$$\mathrm{PFCIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leqslant \mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m), \tag{50}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leq k\}$ *for* $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$.

Proof Proof of the Theorem is same as Theorem 3. □

Example 2 If we apply the PFECIG operator to Example (1), then we have

$$\begin{aligned}\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) &= \left(\frac{\sqrt{2\prod_{i=1}^{3}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{3}\left(2-\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{3}\left(\mu^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}, \frac{\sqrt{\prod_{i=1}^{2}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}-\prod_{i=1}^{3}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}}{\sqrt{\prod_{i=1}^{2}\left(1+\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}+\prod_{i=1}^{3}\left(1-\upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}}} \right)\\ &=(0.7383, 0.5798)\end{aligned}$$

$$S\big(\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)\big) = 0.1585$$

Now by utilizing PFCIG operator, we have

$$\mathrm{PFCIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = \left(\prod_{i=1}^{3} (\mu_{\bar{\beta}_{\sigma(i)}})^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}, \sqrt{1 - \prod_{i=1}^{3}\left(1 - \upsilon^2_{\bar{\beta}_{\sigma(i)}}\right)^{\lambda(\Im_{\sigma(i)})-\lambda(\Im_{\sigma(i-1)})}} \right)$$
$$= (0.7301, 0.5864)$$
$$S(\mathrm{PFCIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)) = 0.1437.$$

This shows that $\mathrm{PFCIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) \leqslant \mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3)$

Theorem 11 (Idempotency) *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *a fuzzy measure on* Y. *Where* $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leq k\right\}$ *for* $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$, *then*

$$\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) = \bar{\beta}. \tag{51}$$

Proof Proof of the Theorem follows from Theorem 4. □

Theorem 12 (Boundary) *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *a fuzzy measure on* Y. *Where* $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leq k\right\}$ *for* $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$. *If* $\bar{\beta}_{\min} = \min(\bar{\beta}_i)$, $\bar{\beta}_{\max} = \max(\bar{\beta}_i)$. *Then*

$$\bar{\beta}_{\min} \leqslant \mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leqslant \bar{\beta}_{\max} \tag{52}$$

Proof Proof of the Theorem follows from Theorem 5. □

Theorem 13 (Monotonicity) *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})$ $(i = 1, 2, \ldots, m)$ *is a collection of PFNs and* λ *a fuzzy measure on* Y. *Suppose* $\bar{\beta}_i^* = (\mu^*_{\bar{\beta}_i}, \upsilon^*_{\bar{\beta}_i})$ $(i = 1, 2, \ldots, m)$ *is a collection of PFNs. Then*

$$\mathrm{PFECIG}(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m) \leqslant \mathrm{PFECIG}(\bar{\beta}_1^*, \bar{\beta}_2^*, \ldots, \bar{\beta}_m^*) \tag{53}$$

where $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\Im_{\sigma(k)} = \{y_{\sigma(k)} | i \leq k\}$ *for* $k \geq 1$, *and* $\Im_{\sigma(0)} = \phi$.

Proof Proof of the Theorem follows from Theorem 6. □

Theorem 14 *Suppose* $\bar{\beta}_i = (\mu_{\bar{\beta}_i}, \upsilon_{\bar{\beta}_i})(i = 1, 2, \ldots, m)$, $\bar{\rho}_i = (\mu_{\bar{\rho}_i}, \upsilon_{\bar{\rho}_i})$ $(i = 1, 2, \ldots, m)$ *are two collections of PFNs on* Y *and* λ *be a fuzzy measure on* Y. *Where* $\{\sigma(1), \sigma(1), \ldots, \sigma(m)\}$ *is a permutation of* $\{1, 2, \ldots, m\}$ *such that* $\bar{\beta}_{\sigma(1)} \geq \bar{\beta}_{\sigma(2)} \geq \cdots \geq \bar{\beta}_{\sigma(m)}$ *and* $\aleph_{\sigma(k)} = \left\{y_{\sigma(k)} | i \leq k\right\}$ *for* $k \geq 1$, and $\aleph_{\sigma(0)} = \phi$. *Then the following statements are equivalent*:

1. *For* $\aleph, \Im \in P(Y)$ *such that* $|\aleph| = |\Im|$, *we have* $\lambda(\aleph) = \lambda(\Im)$.
2. *There is an exponential weighted vector* $w = (w_1, w_2, \ldots, w_m)$ *such that*

$$\mathrm{PFECIG}\big(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\big) = \mathrm{PFEOWG}_w\big(\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m\big)$$

Proof Proof of the Theorem is same as Proposition 1 (Sect. 4.2) in (Marichal 2002). □

Remark Assume that PFEWA operator has an exponential weighted vector $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}}$. Then by Theorem 8 it is clear that λ is an additive fuzzy measure $\lambda(\aleph) = \sum_{i \in \aleph} w_i$, PFECIG operator will be reduced to a PFEWG operator, where $w_i = \lambda(i)$. Conversely it is clear that PFEWG is a PFECIG operator with additive fuzzy measure λ: $\lambda(\aleph) = \sum_{i \in \aleph} w_i$, $\aleph \subseteq Y$.

From Theorem 8 and above Remark, it is clear that PFECIG operator is a generalization of both PFEOWG and PFEWG operators. Thus, PFEOWG and PFEWG operators are two special cases of PFECIG operator.

Both the PFECIA operator and PFECIG operator address not only the importance of the elements or their ordered positions, but also the correlations of the elements or their ordered positions. However, the difference between these two operators is that the PFECIG operator is much more sensitive to the given arguments. Especially in the case where there is an argument taking the value of zero, the aggregated value of these arguments using the PFECIG operator must be zero no matter what the other given arguments are. For instance if we take $\bar{\beta}_1 = (0.0, 0.0)$, $\bar{\beta}_2 = (1.0, 0.0)$ and $\bar{\beta}_3 = (1.0, 0.0)$. Consider the fuzzy density and its λ parameter as presented in Example 1, then by applying PFECIG operator we get $\mathrm{PFCIG}(\bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_3) = (0.0, 0.0)$.

4 Multi-attribute decision making based on Pythagorean fuzzy Einstein Choquet integral aggregation operators

In this section, we utilize the proposed aggregation operators namely Pythagorean fuzzy Einstein Choquet integral average (PFECIA) operator and Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) operator to multiple attribute decision-making problems under Pythagorean fuzzy environment.

MCDM problems usually comprise the following two Phases: "(1) Aggregation phase: to get the overall value for each alternative the aggregation phase combines the individual criteria value of each alternative given by experts. (2) Exploitation phase: to get the best alternatives exploitation phase orders the overall values. In practical decision making the experts or DMs may not have a particular or adequate knowledge about the DM problem, or powerless to classify clearly the point to which one alternative is better than the other. Usually, MCGDM problem contains ambiguous and vague information. We propose MCGDM problems under interval-valued Pythagorean fuzzy environment using the concept of Choquet integral and Einstein operations. The main advantage of the Choquet integral is that it coincides with the Lebesgue integral when the measure is additive. An additive measure may be openly tied to the notions of additive expected utility (Schmeidler 1989) and common preferential independence (Marichal 2002). The Choquet integral is able to perform aggregation of criteria even when mutual preferential independence is violated".

Suppose $D = \{d_1, d_2, \ldots, d_r\}$ is the set of the DMs involved in the decision procedure, $Y = (Y_1, Y_2, \ldots, Y_m)$ the set of the alternatives and $C = (C_1, C_2, \ldots, C_n)$ be the set of the criteria used for evaluating the alternatives.

In the following, we shall utilize the PFECIA/PFECIG operator to propose an approach to MCGDM under Pythagorean fuzzy setting, which involves the following steps:

Suppose $M^{(k)} = [\bar{\beta}_{ij}^{(k)}]_{m\times n} = [(\mu_{\bar{\beta}_{ij}}^{(k)}, \upsilon_{\bar{\beta}_{ij}}^{(k)})]_{m\times n}$ is a Pythagorean fuzzy decision matrix providing by the DMs $D_k \in D$ and is expressed as a PFNs where $\mu_{\bar{\beta}_{ij}}^{(k)}$ indicates the degree that the alternatives $Y_i \in Y$ fulfills the criteria $C_j \in C$ expressed by the DMs D_k, and $\upsilon_{\bar{\beta}_{ij}}^{(k)}$ indicates the degree that the alternatives $Y_i \in Y$ does not fulfills the criteria $C_j \in C$ expressed by the DMs D_k, such that $\mu_{\bar{\beta}_{ij}}^{(k)} \in [0, 1]$, $\upsilon_{\bar{\beta}_{ij}}^{(k)} \in [0, 1]$, $(\mu_{\bar{\beta}_{ij}}^{(k)})^2 + (\upsilon_{\bar{\beta}_{ij}}^{(k)})^2 \leq 1$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

To synchronize the data, "first step is to look at the criteria. If all the criteria C_j $(j = 1, 2, \ldots, n)$ are of same type, then there is no need for normalization. Conversely if these contain different scales and/or units then there is needed to transform them all to the same scale and/or unit. Let us consider two types of criteria, namely, (1) cost type and the (2) benefit type. Considering their natures, a benefit criteria (the bigger the values better is it) and cost criteria (the smaller the values the better is it) are of rather opposite type. In such cases, we need to first transform the criteria values of cost type into the criteria values of benefit type". So transform the Pythagorean fuzzy decision matrix $M^{(k)} = [\bar{\beta}_{ij}^{(k)}]_{m\times n}$ into normalized decision matrix $M^{*(k)} = [\bar{\gamma}_{ij}^{(k)}]_{m\times n}$, where $\bar{\gamma}_{ij}^{(k)} = (\mu_{\bar{\gamma}_{ij}}^{(k)}, \upsilon_{\bar{\gamma}_{ij}}^{(k)})$ and

$$\bar{\gamma}_{ij}^{(k)} = \begin{cases} \bar{\beta}_{ij}^{(k)} & \text{for benefit criteria, } i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n. \\ (\bar{\beta}_{ij}^{(k)})^c & \text{for cost criteria, } i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n. \end{cases}$$

where $(\bar{\beta}_{ij}^{(k)})^c$ is the complement of $\bar{\beta}_{ij}^{(k)}$, such that $(\bar{\beta}_{ij}^{(k)})^c = (\upsilon_{\bar{\beta}_{ij}}^{(k)}, \mu_{\bar{\beta}_{ij}}^{(k)})$

With attributes normalized and using the PFECIA/PFECIG operator, we now develop an algorithm to solve multiple attribute group decision-making problems under Pythagorean fuzzy environment:

Step 1: For each alternative $Y_i (i = 1, \ldots, m)$ all DMs $d_k (k = 1, \ldots, r)$ is requested to definite their individual estimation or preference based on each criteria $C_j (j = 1, \ldots, n)$ by an PFN $\bar{\beta}_j^k = (\mu_{\bar{\beta}_j^k}, \upsilon_{\bar{\beta}_j^k})$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$; $k = 1,2,\ldots,r$). Then we can get a decision-making matrix as:

$$M^{(k)} = \begin{array}{c} \\ Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \ldots & C_n \end{array} \\ \begin{bmatrix} \bar{\beta}_{11}^{(k)} & \bar{\beta}_{12}^{(k)} & \ldots & \bar{\beta}_{1n}^{(k)} \\ \bar{\beta}_{21}^{(k)} & \bar{\beta}_{22}^{(k)} & \ldots & \bar{\beta}_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\beta}_{m1}^{(k)} & \bar{\beta}_{m2}^{(k)} & \ldots & \bar{\beta}_{mn}^{(k)} \end{bmatrix} \end{array}.$$

Step 2: First confirm the fuzzy density $\mu_i = \mu(\Im_i)$ of each criterion and $\mu_i = \mu(\Im_i)$ of each decision maker. According to Eq. (8) λ_1 parameter of criteria and λ_2 parameter of decision maker can be computed, respectively

Step 3: By Definition 5, $\bar{\beta}_{ij}^{(k)}$ in i-th line of $M^{(k)}$ is reordered such that $\bar{\beta}_{i(j)}^{(k)} > \bar{\beta}_{i(j-1)}^{(k)}$: utilizing the PFECIA\PFECIG operator:

$$\bar{\gamma}_{ij} = (\mu_{\bar{\gamma}_{ij}}, \upsilon_{\bar{\gamma}_{ij}}) = \text{PFECIA}(\bar{\beta}_{ij}^{(1)}, \bar{\beta}_{ij}^{(2)}, \ldots, \bar{\beta}_{ij}^{(l)})$$
$$= \left(\sqrt{\frac{\prod_{k=1}^{l}\left(1+\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} - \prod_{k=1}^{l}\left(1-\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}{\prod_{k=1}^{l}\left(1+\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} + \prod_{j=1}^{l}\left(1-\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}, \frac{\sqrt{2\prod_{k=1}^{l}\left(\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{l}\left(2-\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} + \prod_{k=1}^{l}\left(\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}} \right), \tag{54}$$

or

$$\bar{\gamma}_{ij} = (\mu_{\bar{\gamma}_{ij}}, \upsilon_{\bar{\gamma}_{ij}})$$
$$= \text{PFECIG}(\bar{\beta}_{ij}^{(1)}, \bar{\beta}_{ij}^{(2)}, \ldots, \bar{\beta}_{ij}^{(l)})$$

$$= \left(\frac{\sqrt{2\prod_{k=1}^{l}\left(\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{l}\left(2-\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}+\prod_{k=1}^{l}\left(\left(\mu_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}, \frac{\sqrt{\prod_{k=1}^{l}\left(1+\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}-\prod_{k=1}^{l}\left(1-\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{l}\left(1+\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}+\prod_{k=1}^{l}\left(1-\left(\upsilon_{\bar{\beta}_{ij}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}} \right), \tag{55}$$

to get the collective Pythagorean fuzzy decision $M = [\bar{\gamma}_{ij}]_{m\times n}$ $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Step 4: Same to step 3, all $\bar{\beta}_i^{(k)}(k = 1, 2, \ldots, r)$ is reordered such that $\bar{\beta}_i^{(k)} > \bar{\beta}_i^{(k-1)}$

Step 5: Aggregate all the Pythagorean fuzzy numbers $\bar{\gamma}_{ij}$ for each alternative Y_i by utilizing PFECIA/PFECIG operator

$$\bar{\gamma}_i = (\mu_{\bar{\gamma}_i}, \upsilon_{\bar{\gamma}_i}) = \text{PFECIA}(\bar{\gamma}_{i1}, \bar{\gamma}_{i2}, \ldots, \bar{\gamma}_{in})$$
$$= \left(\frac{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}-\prod_{j=1}^{n}\left(1-\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{n}\left(1+\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}+\prod_{j=1}^{n}\left(1-\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}, \frac{\sqrt{2\prod_{j=1}^{n}\left(\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{n}\left(2-\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}+\prod_{j=1}^{n}\left(\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}} \right), \tag{56}$$

or

$$\bar{\gamma}_i = (\mu_{\bar{\gamma}_i}, \upsilon_{\bar{\gamma}_i}) = \text{PFECIG}(\bar{\gamma}_{i1}, \bar{\gamma}_{i2}, \ldots, \bar{\gamma}_{in})$$
$$= \left(\frac{\sqrt{2\prod_{j=1}^{n}\left(\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{n}\left(2-\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}+\prod_{j=1}^{n}\left(\mu_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}, \frac{\sqrt{\prod_{j=1}^{n}\left(1+\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}-\prod_{j=1}^{n}\left(1-\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{n}\left(1+\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}+\prod_{j=1}^{n}\left(1-\upsilon_{\bar{\gamma}_{ij}}^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}} \right), \tag{57}$$

to derive the overall Pythagorean fuzzy preference numbers $\bar{\gamma}_i$, $i = 1, 2, \ldots, m$, of the alternatives Y_i, $i = 1, 2, \ldots, m$.

Step 6: Calculate the score value as follows

$$S(\bar{\gamma}_i) = \mu_{\bar{\gamma}_i}^2 - \upsilon_{\bar{\gamma}_i}^2, i = 1, 2, \ldots, m. \tag{58}$$

Step 7: Rank all the alternatives $Y_i (i = 1, 2, \ldots, m)$, according to the score values $S(\bar{\gamma}_i)(i = 1, 2, \ldots, m)$, in descending order. The greater Y_i, with the highest value of $S(\bar{\gamma}_i)$, is the best alternative

Table 1 Pythagorean fuzzy decision matrix $R^{(1)}$ with respect to d_1

	A_1	A_2	A_3	A_4
Y_1	(0.6, 0.7)	(0.8, 0.4)	(0.8, 0.6)	(0.2, 0.9)
Y_2	(0.6, 0.8)	(0.6, 0.7)	(0.8, 0.4)	(0.7, 0.6)
Y_3	(0.7, 0.5)	(0.3, 0.9)	(0.6, 0.7)	(0.6, 0.5)
Y_4	(0.8, 0.4)	(0.6, 0.8)	(0.7, 0.6)	(0.4, 0.8)

Table 2 Pythagorean fuzzy decision matrix $R^{(1)}$ with respect to d_2

	A_1	A_2	A_3	A_4
Y_1	(0.8, 0.6)	(0.5, 0.7)	(0.9, 0.2)	(0.5, 0.8)
Y_2	(0.6, 0.7)	(0.8, 0.3)	(0.6, 0.5)	(0.7, 0.6)
Y_3	(0.7, 0.5)	(0.5, 0.8)	(0.8, 0.6)	(0.3, 0.9)
Y_4	(0.5, 0.6)	(0.7, 0.6)	(0.4, 0.8)	(0.7, 0.5)

Table 3 Pythagorean fuzzy decision matrix $R^{(1)}$ with respect to d_3

	A_1	A_2	A_3	A_4
Y_1	(0.4, 0.8)	(0.9, 0.3)	(0.5, 0.6)	(0.8, 0.5)
Y_2	(0.6, 0.7)	(0.7, 0.6)	(0.8, 0.5)	(0.6, 0.8)
Y_3	(0.8, 0.6)	(0.4, 0.8)	(0.7, 0.6)	(0.5, 0.6)
Y_4	(0.9, 0.3)	(0.5, 0.7)	(0.5, 0.8)	(0.7, 0.5)

5 Illustrative example

In this section we present two numerical examples to illustrate the proposed method. The first example is about supplier selection and the second one is adapted from Garg (2017a)

Example 3 Suppose that there are four suppliers Y_1, Y_2, Y_3 and Y_4 whose core competencies are evaluated by means of the following four attributes:

A_1: the level of technology innovation as how much they have developed their skill.

A_2: the control ability of flow that shows their supply ways and mechanism.

A_3: the ability of management as how they manage their supplying activities.

A_4: the level of service to fulfil their duties supplies.

Now there are three decision makers $D = \{d_1, d_2, d_3\}$ who are invited to evaluate the core competencies of four candidates under these four attributes. For decision maker $d_k (k = 1, 2, 3)$, the evaluated value of supplier $Y_i (i = 1, 2, 3, 4)$ with respect to $A_j (j = 1, 2, 3, 4)$ can be expressed by PFN $R^{(k)} = [\bar{\beta}_{ij}^{(k)}]_{m\times n}$: the Pythagorean fuzzy decision matrix $R^{(k)} = [\bar{\beta}_{ij}^{(k)}]_{4\times 4} (k = 1, 2, 3)$ can be gotten as listed in Tables 1, 2, and 3.

To get the best supplier(s), the following steps are involved:

Step 1: We first determine fuzzy density of the attributes and decision maker, and their λ parameter, respectively

Suppose $\mu(\Im_1) = 0.4$, $\mu(\Im_2) = 0.35$, $\mu(\Im_3) = 0.25$, $\mu(\Im_4) = 0.3$. Then by Eq. (8), λ of attributes can be determined; $\lambda_1 = -0.54$, and $\mu(\Im_1, \Im_2) = 0.67$, $\mu(\Im_1, \Im_3) = 0.6$, $\mu(\Im_1, \Im_4) = 0.64$, $\mu(\Im_2, \Im_3) = 0.55$, $\mu(\Im_2, \Im_4) = 0.59$, $\mu(\Im_3, \Im_4) = 0.51$, $\mu(\Im_1, \Im_2, \Im_3) = 0.83$, $\mu(\Im_1, \Im_2, \Im_4) = 0.87$, $\mu(\Im_2, \Im_3, \Im_4) = 0.76$, $\mu(\Im_1, \Im_3, \Im_4) = 0.80$, $\mu(\Im_1, \Im_2, \Im_3, \Im_4) = 1$

Suppose that $\mu(d_1) = 0.4$, $\mu(d_2) = 0.4$, $\mu(d_3) = 0.4$. Then λ of decision maker can be determined: $\lambda_2 = -0.44$ and $\mu(d_1, d_2) = 0.73$, $\mu(d_1, d_3) = 0.73$, $\mu(d_2, d_3) = 0.73$, $\mu(d_1, d_2, d_3) = 1$.

For PFECIA Operator

Step 2: For Pythagorean fuzzy decision matrix $R^{(1)}$, according to Definition 4, the evaluated value $\bar{\beta}_{ij}^{(1)}$ of supplier $Y_i (i = 1, 2, 3, 4)$ is reordered such that $\bar{\beta}^1_{i(j)} \geq \bar{\beta}^1_{i(j-1)}$, then utilize the PFCIA operator

$$\bar{a}_i^{(1)} = \text{PFECIA}\left(\bar{\beta}_{i1}^{(1)}, \bar{\beta}_{i2}^{(1)}, \bar{\beta}_{i2}^{(1)}, \bar{\beta}_{i4}^{(1)}\right)$$
$$= \left(\frac{\sqrt{\prod_{j=1}^{4}\left(1+\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})} - \prod_{j=1}^{4}\left(1-\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{4}\left(1+\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})} + \prod_{j=1}^{4}\left(1-\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}, \frac{\sqrt{2\prod_{j=1}^{4}\left(\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{4}\left(2-\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})} + \prod_{j=1}^{4}\left(\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}} \right),$$

to aggregate $\bar{a}_{ij}^{(1)} (j = 1, 2, 3, 4)$ corresponding to supplier $Y_i (i = 1, 2, 3, 4)$: $\bar{a}_1^{(1)} = (0.6926, 0.5642)$, $\bar{a}_2^{(1)} = (0.6321, 0.6122)$, $\bar{a}_3^{(1)} = (0.5801, 0.6108)$, $\bar{a}_4^{(1)} = (0.6782, 0.5840)$. Similarly, for Tables 2 and 3, we have, respectively,

$$\bar{a}_1^{(2)} = (0.7236, 0.4961), \bar{a}_2^{(2)} = (0.6546, 0.4810), \bar{a}_3^{(2)} = (0.5937, 0.7214), \bar{a}_4^{(2)} = (0.5949, 0.6058).$$

and

$$\bar{a}_1^{(3)} = (0.7423, 0.4937), \bar{a}_2^{(3)} = (0.6532, 0.6329), \bar{a}_3^{(3)} = (0.6568, 0.6382), \bar{a}_4^{(3)} = (0.7425, 0.4840).$$

Step 3: For supplier X_1, we reorder $\bar{a}_1^{(k)} (k = 1, 2, 3)$ such that $\bar{a}_1^{(k)} \geq \bar{a}_1^{(k-1)}$: using the PFECIG operator aggregates $\bar{a}_1^{(k)} (k = 1, 2, 3)$ into collective overall values $\bar{a}_1$:

$$\bar{a}_1^{(1)} = \text{PFECIA}\left(\bar{a}_1^{(1)}, \bar{a}_1^{(2)}, \bar{a}_1^{(3)}\right)$$
$$= \left(\frac{\sqrt{\prod_{k=1}^{3}\left(1+\left(\mu^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} - \prod_{k=1}^{3}\left(1-\left(\mu^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{3}\left(1+\left(\mu^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} + \prod_{k=1}^{3}\left(1-\left(\mu^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}, \frac{\sqrt{2\prod_{k=1}^{3}\left(\left(\upsilon^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{3}\left(2-\left(\upsilon^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})} + \prod_{k=1}^{3}\left(\left(\upsilon^{(k)}_{\bar{a}_{\sigma(1)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}} \right).$$

$$\bar{a}_1 = (0.7234, 0.5129)$$

Similar to supplier Y_1; for Y_2, Y_3, Y_4, we have, respectively
$\bar{a}_2 = (0.6484, 0.5714)$, $\bar{a}_3 = (0.6161, 0.6510)$, $\bar{a}_4 = (0.6867, 0.5481)$

Step 4: Calculate the score value we have:
$S(\bar{a}_1) = 0.2602$, $S(\bar{a}_2) = 0.0939$, $S(\bar{a}_3) = -0.0442$, $S(\bar{a}_4) = 0.1710$.

Step 5: According to the score values of the collective PFN's $\bar{a}_i$ of supplier $Y_i (i = 1, 2, 3, 4)$, we can obtain that $\bar{a}_1 > \bar{a}_4 > \bar{a}_2 > \bar{a}_3$.
Ranking the suppliers according to the score values we get, $Y_1 > Y_4 > Y_2 > Y_3$. Hence the best supplier is Y_1.

For PFECIG Operator

Step 2′ For Pythagorean fuzzy decision matrix $R^{(1)}$, according to Definition 4, the evaluated value $\bar{\beta}_{ij}^{(1)}$ of supplier $Y_i (i = 1, 2, 3, 4)$ is reordered such that $\bar{\beta}_{i(j)}^{1} \geq \bar{\beta}_{i(j-1)}^{1}$, then utilize the PFCIG operator

$$\bar{g}_i^{(1)} = \text{PFECIG}\left(\bar{\beta}_{i1}^{(1)}, \bar{\beta}_{i2}^{(1)}, \bar{\beta}_{i2}^{(1)}, \bar{\beta}_{i4}^{(l)}\right)$$

$$= \left(\frac{\sqrt{2\prod_{j=1}^{4}\left(\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{4}\left(2-\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}+\prod_{j=1}^{4}\left(\left(\mu_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}, \frac{\sqrt{\prod_{j=1}^{4}\left(1+\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(k-1)})}-\prod_{j=1}^{4}\left(1-\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}}{\sqrt{\prod_{j=1}^{4}\left(1+\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}+\prod_{j=1}^{4}\left(1-\left(\upsilon_{\bar{\beta}_{\sigma(j)}}\right)^2\right)^{\lambda(\Im_{\sigma(j)})-\lambda(\Im_{\sigma(j-1)})}}} \right),$$

to aggregate $\bar{\beta}_{ij}^{(1)} (j = 1, 2, 3, 4)$ corresponding to supplier Y_i:

$$\bar{g}_1^{(1)} = (0.6328, 0.5241), \bar{g}_2^{(1)} = (0.6741, 0.5156), \bar{g}_3^{(1)} = (0.5490, 0.4732), \bar{g}_4^{(1)} = (0.6569, 0.5517).$$

Similarly, for Tables 2 and 3, we have, respectively,

$$\bar{g}_1^{(2)} = (0.6858, 0.4641), \bar{g}_2^{(2)} = (0.6920, 0.4049), \bar{g}_3^{(2)} = (0.6095, 0.5857), \bar{g}_4^{(2)} = (0.5890, 0.5010).$$

and

$$\bar{g}_1^{(3)} = (0.6719, 0.3946), \bar{g}_2^{(3)} = (0.6782, 0.5620), \bar{g}_3^{(3)} = (0.6279, 0.5410), \bar{g}_4^{(3)} = (0.7003, 0.4256).$$

Step 3′ For supplier Y_1, we reorder $\bar{g}_1^{(k)} (k = 1, 2, 3)$ such that $\bar{g}_1^{(k)} \geq \bar{g}_1^{(k-1)}$: using the PFECIG operator aggregates $\bar{g}_1^{(k)} (k = 1, 2, 3)$ into collective overall values $\bar{g}_1$:

$$\bar{g}_1 = \text{PFECIG}\left(\bar{g}_1^{(1)}, \bar{g}_1^{(2)}, \bar{g}_1^{(3)}\right)$$

$$= \left(\frac{\sqrt{2\prod_{k=1}^{3}\left(\left(\mu_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{3}\left(2-\left(\mu_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}+\prod_{k=1}^{3}\left(\left(\mu_{\bar{\gamma}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}, \frac{\sqrt{\prod_{k=1}^{3}\left(1+\left(\upsilon_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}-\prod_{k=1}^{3}\left(1-\left(\upsilon_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}}{\sqrt{\prod_{k=1}^{3}\left(1+\left(\upsilon_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}+\prod_{k=1}^{3}\left(1-\left(\upsilon_{\bar{g}_{\sigma(1)}}^{(k)}\right)^2\right)^{\lambda(\Im_{\sigma(k)})-\lambda(\Im_{\sigma(k-1)})}}} \right).$$

$$\bar{g}_1 = (0.6658, 0.4561)$$

Similar to supplier Y_1; for Y_2, Y_3, Y_4, we have, respectively

Table 4 Pythagorean decision matrix provided by DM (d_1)

	C_1	C_2	C_3	C_4	C_5
Y_1	(0.9, 0.4)	(0.7, 0.6)	(0.8, 0.4)	(0.5, 0.8)	(0.7, 0.8)
Y_2	(0.6, 0.8)	(0.9, 0.4)	(0.7, 0.5)	(0.8, 0.6)	(0.9, 0.3)
Y_3	(0.6, 0.7)	(0.8, 0.5)	(0.5, 0.8)	(0.6, 0.8)	(0.5, 0.8)
Y_4	(0.5, 0.6)	(0.4, 0.9)	(0.9, 0.3)	(0.8, 0.4)	(0.7, 0.6)

Table 5 Pythagorean decision matrix provided by DM (d_2)

	C_1	C_2	C_3	C_4	C_5
Y_1	(0.7, 0.6)	(0.9, 0.4)	(0.5, 0.6)	(0.6, 0.8)	(0.3, 0.9)
Y_2	(0.8, 0.5)	(0.7, 0.5)	(0.9, 0.3)	(0.4, 0.9)	(0.8, 0.6)
Y_3	(0.3, 0.9)	(0.8, 0.6)	(0.7, 0.6)	(0.5, 0.6)	(0.9, 0.4)
Y_4	(0.5, 0.8)	(0.6, 0.7)	(0.8, 0.4)	(0.9, 0.4)	(0.6, 0.8)

$$\bar{g}_2 = (0.6824, 0.4895), \bar{g}_3 = (0.5805, 0.5332), \bar{g}_4 = (0.6491, 0.4935)$$

Step 4$'$ Calculate the score value we have:

$$S(\bar{g}_1) = 0.2353, \quad S(\bar{g}_2) = 0.2261, \quad S(\bar{g}_3) = 0.0527, \quad S(\bar{g}_4) = 0.1778.$$

Step 5$'$ According to the score values of the collective PFN's $\bar{g}_i$ of supplier $Y_i (i = 1, 2, 3, 4)$, we can obtain that: $\bar{g}_1 > \bar{g}_2 > \bar{g}_4 > \bar{g}_3$.

Ranking the suppliers according to the score values we get, $Y_1 > Y_2 > Y_4 > Y_3$. Hence the best supplier is Y_1.

Example 4 Consider an investor wants to invest his/her money in a certain company. After the careful analysis of the market, he/she considers the five possible alternatives denoted by $Y_i (i = 1, 2, 3, 4)$: Y_1: is a computer company, Y_2: is a furniture company, Y_3: is a car company, Y_4: is a chemical company.

In evaluating these alternatives, the investor has summarized the ability of these companies with six attributes denoted by $C_j (j = 1, 2, 3, 4, 5)$ where

C_1: technical ability, C_2: expected benefit, C_3: competitive power on the market, C_4: ability to bear risk, C_5: management capability

Suppose four companies $Y_i (i = 1, 2, 3, 4)$ are selected as the possible alternatives, which are evaluated by three decision makers (DMs) under the above five criteria $C_j (j = 1, 2, 3, 4, 5)$ and construct the following three Pythagorean fuzzy decision matrix $M^{(k)} = [\bar{\beta}_{ij}^{(k)}]_{m\times n}$ (see Tables 4, 5, 6). Since all the criteria are of benefit types, therefore, no need for normalization and $M^{(k)} = M^{*(k)} = [\bar{\beta}_{ij}^{(k)}]_{m\times n} = [\bar{\gamma}_{ij}^{(k)}]_{m\times n}$.

Based on the PFECIA operator the main steps are as follows:

Step 1: Suppose that $\mu(d_1) = 0.3$, $\mu(d_2) = 0.5$, $\mu(d_3) = 0.45$. Then λ of experts can be determined: $\lambda_2 = -0.53$, and $\mu(d_1, d_2) = 0.72$, $\mu(d_1, d_3) = 0.68$, $\mu(d_2, d_3) = 0.83$, $\mu(d_1, d_2, d_3) = 1$

Step 2: For Pythagorean fuzzy decision matrix $M^{(k)} = [\bar{\gamma}_{ij}^{(k)}]_{4\times 5} (k = 1, 2, 3)$, based on Definition (5), the calculated value $\bar{\gamma}_{ij}^{(k)} (k = 1, 2, 3)$ of alternative X_i is reordered

Table 6 Pythagorean decision matrix provided by DM (d_3)

	C_1	C_2	C_3	C_4	C_5
Y_1	(0.8, 0.5)	(0.4, 0.8)	(0.6, 0.5)	(0.8, 0.6)	(0.6, 0.8)
Y_2	(0.3, 0.9)	(0.7, 0.6)	(0.8, 0.4)	(0.9, 0.4)	(0.7, 0.5)
Y_3	(0.6, 0.8)	(0.9, 0.3)	(0.7, 0.5)	(0.8, 0.5)	(0.5, 0.6)
Y_4	(0.7, 0.5)	(0.8, 0.5)	(0.4, 0.9)	(0.7, 0.6)	(0.6, 0.7)

Table 7 Collective Pythagorean fuzzy decision matrix M

	C_1	C_2	C_3	C_4	C_5
Y_1	(0.8253, 0.4836)	(0.7880, 0.5405)	(0.6511, 0.4975)	(0.6853, 0.7076)	(0.5766, 0.8321)
$\boldsymbol{Y_2}$	(0.6723, 0.6682)	(0.7828, 0.4938)	(0.8477, 0.3609)	(0.7906, 0.5867)	(0.8172, 0.4675)
$\boldsymbol{Y_3}$	(0.5629, 0.7868)	(0.8532, 0.4287)	(0.6730, 0.5845)	(0.6811, 0.5845)	(0.7738, 0.5214)
$\boldsymbol{Y_4}$	(0.6035, 0.6126)	(0.6866, 0.6369)	(0.7781, 0.4776)	(0.8406, 0.4501)	(0.6895, 0.5675)

such that $\bar{\gamma}_{i(j)}^{(k)} \geq \bar{\gamma}_{i(j-1)}^{(k)}$, then utilize the PFECIA operator (Eq. 55) to aggregate all the individual matrices $M^{(k)} = [\bar{\gamma}_{ij}^{(k)}]_{4\times 5}$ $(k = 1, 2, 3)$, into collective Pythagorean fuzzy decision matrix $M = [\bar{\gamma}_{ij}]_{4\times 5}$ (see Table 7)

Step 3: Suppose $\mu(\Im_1) = 0.4$, $\mu(\Im_2) = 0.35$, $\mu(\Im_3) = 0.25$, $\mu(\Im_4) = 0.3$, $\mu(\Im_5) = 0.45$, so by Eq. (8), λ of criteria can be computed; $\lambda_1 = -0.79$ and $\mu(\Im_1, \Im_2) = 0.68$, $\mu(\Im_1, \Im_3) = 0.60$, $\mu(\Im_1, \Im_4) = 0.64$, $\mu(\Im_1, \Im_5) = 0.71$, $\mu(\Im_2, \Im_3) = 0.56$, $\mu(\Im_2, \Im_4) = 0.6$, $\mu(\Im_2, \Im_5) = 0.68$, $\mu(\Im_3, \Im_4) = 0.51$, $\mu(\Im_3, \Im_5) = 0.61$, $\mu(\Im_4, \Im_5) = 0.64$, $\mu(\Im_1, \Im_2, \Im_3) = 0.84$, $\mu(\Im_1, \Im_2, \Im_4) = 0.88$, $\mu(\Im_1, \Im_2, \Im_5) = 0.86$, $\mu(\Im_1, \Im_3, \Im_4) = 0.81$, $\mu(\Im_1, \Im_3, \Im_5) = 0.82$, $\mu(\Im_2, \Im_3, \Im_4) = 0.77$, $\mu(\Im_2, \Im_3, \Im_5) = 0.88$, $\mu(\Im_2, \Im_4, \Im_5) = 0.82$, $\mu(\Im_3, \Im_4, \Im_5) = 0.77$, $\mu(\Im_1, \Im_2, \Im_3, \Im_4) = 0.88$, $\mu(\Im_1, \Im_2, \Im_3, \Im_5) = 0.94$, $\mu(\Im_1, \Im_3, \Im_4, \Im_5) = 0.92$, $\mu(\Im_2, \Im_3, \Im_4, \Im_5) = 0.90$, $\mu(\Im_1, \Im_2, \Im_3, \Im_4, \Im_5) = 1$

Step 4: Utilize the PFECIA operator (Eq. 57) to aggregate all the preference values $\bar{\gamma}_{ij}$ (j = 1, 2, 3, 4,5) in the ith line of M and get the overall preference values $\bar{\gamma}_i$
$\bar{\gamma}_1 = (0.7649, 0.5472)$ $\bar{\gamma}_2 = (0.8093, 0.4504)$ $\bar{\gamma}_3 = (0.7870, 0.5025)$ $\bar{\gamma}_4 = (0.7694, 0.5006)$

Step 5: Compute the score of $\bar{\gamma}_i (i = 1, 2, 3, 4)$, respectively:
$S(\bar{\gamma}_1) = 0.2856$, $S(\bar{\gamma}_2) = 0.4521$, $S(\bar{\gamma}_3) = 0.3669$, $S(\bar{\gamma}_4) = 0.3414$.

Step 6: Ranking the alternatives according to the score we have $Y_2 > Y_3 > Y_4 > Y_1$
Thus, the most desirable alternative is Y_2.

Based on the PFECIG operator, the main steps are as follows:

Step 1′: See step 1

Step 2′: Utilize the PFECIG operator (Eq. 56) to aggregate all the individual matrices $M^{(k)} = [\bar{\gamma}_{ij}^{(k)}]_{4\times 5}$ $(k = 1, 2, 3)$, into collective Pythagorean fuzzy decision matrix $M = [\bar{\gamma}_{ij}]_{4\times 5}$ (see Table 8)

Step 3′: Utilize the PFECIG operator (Eq. 59) to aggregate all the preference values $\bar{\gamma}_{ij}$ (j = 1, 2, 3, 4, 5) in the ith line of M and get the overall preference values $\bar{\gamma}_i$
$\bar{\gamma}_1 = (0.7078, 0.5991)$, $\bar{\gamma}_2 = (0.7768, 0.5131)$, $\bar{\gamma}_3 = (0.7206, 0.5783)$, $\bar{\gamma}_4 = (0.7027, 0.6250)$

Table 8 Collective Pythagorean fuzzy decision matrix M

	C_1	C_2	C_3	C_4	C_5
Y_1	(0.8129, 0.4929)	(0.7004, 0.5992)	(0.6227, 0.5099)	(0.6861, 0.7262)	(0.5190, 0.8394)
Y_2	(0.5879, 0.7377)	(0.7595, 0.5054)	(0.8330, 0.3753)	(0.6993, 0.6911)	(0.8018, 0.5014)
Y_3	(0.5377, 0.7986)	(0.8452, 0.4653)	(0.6635, 0.6073)	(0.6451, 0.6073)	(0.6895, 0.5675)
Y_4	(0.5853, 0.6459)	(0.6466, 0.6857)	(0.7025, 0.6236)	(0.8219, 0.4671)	(0.6661, 0.7004)

Table 9 Comparison analysis with existing methods

Method	Score values				Order of alternatives
	Y_1	Y_2	Y_3	Y_4	
PFEOWA Garg (2016a)	0.2404	0.3434	0.1595	0.2457	$Y_2 > Y_4 > Y_3 > Y_1$
PFEOWG Garg (2017a)	0.0397	0.2197	0.0753	0.1076	$Y_2 > Y_4 > Y_3 > Y_1$
PFCIA (Peng and Yang 2016)	0.3308	0.4480	0.3834	0.3552	$Y_2 > Y_3 > Y_4 > Y_1$
PFCIG (Peng and Yang 2016)	0.1057	0.1893	0.1516	0.0481	$Y_2 > Y_3 > Y_1 > Y_4$
PFECIA	0.2856	0.4521	0.3669	0.3414	$Y_2 > Y_3 > Y_4 > Y_1$
PFECIG	0.1421	0.3402	0.1848	0.1032	$Y_2 > Y_3 > Y_1 > Y_4$

Step 4′: Compute the score of $\bar{\gamma}_i (i = 1, 2, 3, 4)$, respectively:
$S(\bar{\gamma}_1) = 0.1421$, $S(\bar{\gamma}_2) = 0.3402$, $S(\bar{\gamma}_3) = 0.1848$, $S(\bar{\gamma}_4) = 0.1032$.
Step 5′: Ranking the alternatives according to the score we have $Y_2 > Y_3 > Y_1 > Y_4$

Thus the most desirable alternative is Y_2.

From the above analysis, "the main advantages of our developed Pythagorean fuzzy Einstein Choquet integral aggregation operators is that, it not only accommodate the Pythagorean fuzzy environment and not only consider the importance of the elements or their ordered positions, but also reflect the correlations among the elements or their ordered positions. It must be noted that many existing operators are the special cases of the proposed operators. Therefore, our proposed Pythagorean fuzzy Einstein Choquet integral aggregation operators are more flexible than the intuitionistic fuzzy Einstein Choquet integral aggregation operators as in the Pythagorean fuzzy environment the decision makers deals with the situations where the degree of membership and nonmembership of particular criteria are such that its sum is greater than 1".

On the other hand if the comparison analyses based on the different study are conducted, which was proposed by various authors such as Garg (2017a, b) and Peng and Yang (2016), then their subsequently results are summarized as shown in Table 9.

From this comparison table we conclude that the best alternative obtained using the developed operators coincided with these existing methods. Therefore, the considered approach can be taken as an alternative way to solve these types of problem in a more profitable way. According to the above comparison analysis, the proposed method for addressing the decision-making problems has the following advantages with respect to the existing ones.

(a) Additionally, it has observed from Table 9 that the results computed by the various existing approaches are under the environment where all the elements in PFS are independent, i.e., they only consider the addition of the importance of individual elements. While the proposed operators not only consider the importance of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions.

(b) The existing operators for PFS are a special case of the proposed operators. Therefore, it has been concluded that the proposed aggregation operators are more generalized and suitable to solve the real-life problems more accurately than the existing ones.

6 Conclusion

Garg (2016a, 2017a) proposed Pythagorean fuzzy Einstein operators and proposed a MCDM approach based on the developed operators. However, these operators only consider situations where all the elements in a PFS are independent, i.e., they only consider the addition of the importance of individual elements. Therefore, to address this situation in this paper due to the consideration of the inter-dependent phenomena among the evaluated criteria and Einstein operations we developed Pythagorean fuzzy Einstein Choquet integral average (PFECIA) operator and Pythagorean fuzzy Einstein Choquet integral geometric (PFECIG) operator. The important achievement of the developed Pythagorean fuzzy Choquet integral aggregation operators is that, it considers inter-dependent phenomena among the evaluated criteria. We discussed some basic properties of the developed operators namely idempotency, boundary and monotonicity. Moreover, we applied the proposed aggregation operators to multi-attribute decision making under Pythagorean fuzzy environment. Furthermore, to verify and demonstrate the practicality and effectiveness of the developed operators two illustrative examples were given. Finally we have compared the proposed approach to existing methods.

In the future, we will extend the developed approach to linguistic Pythagorean fuzzy set environment (Garg 2018d), Pythagorean hesitant fuzzy set environment (Khan et al. 2017), Hesitant Pythagorean fuzzy set environment (Garg 2018e) and correlation coefficient environment (Garg 2016b)

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