Computational and Applied Mathematics (2019) 38:79
https://doi.org/10.1007/s40314-019-0848-y

An improved instability region for the extended Rayleigh problem of hydrodynamic stability

K. Reena Priya[1,2] · V. Ganesh[3]

Received: 22 April 2018 / Revised: 2 December 2018 / Accepted: 3 December 2018 /
Published online: 29 March 2019

Abstract
For the extended Rayleigh problem of hydrodynamics stability dealing with homogeneous shear flows with variable cross section, we have obtained a parabolic instability region. This improved parabolic instability region intersects with the semicircular instability region under certain condition. The validity of the result is illustrated with an example of basic flows. Furthermore, we have obtained a bound for the complex part of the phase velocity.

Keywords Hydrodynamic stability · Shear flows · Variable bottom · Sea straits

Mathematics Subject Classification 76E05

1 Introduction

The concept of inviscid incompressible shear flows in sea straits with variable cross section was first initiated by Pratt et al. (2000), a well-structured mathematical analysis of this problem was developed in Deng et al. (2003). An extended Taylor–Goldstein problem with respect to the variable topography that is applicable to sea straits was obtained in Deng et al. (2003). In the aforementioned studies, the density could be either a variable or a constant. In case of the density being the variable, it leads to the extended version of the Taylor–Goldstein problem of hydrodynamic stability. Whereas, the extended Rayleigh problem of hydrodynamic stability is attained, when the density is constant. It was observed that in many ways the general analytical results on the above said problems are varied. Specifically, the

Communicated by Corina Giurgea.

✉ V. Ganesh
profvganesh@gmail.com

K. Reena Priya
kreenapriya@gmail.com

1 Research and Development Centre, Bharathiyar University, Coimbatore 641046, India

2 Department of Mathematics, IFET College of Engineering, Villupuram, India

3 Department of Mathematics, Rajiv Gandhi College of Engineering and Technology, Kirumampakkam, Pondicherry 607402, India

role of the curvature of the basis flow velocity plays a vital place in the homogeneous case but not so in the stratified case.

In this paper, the extended Rayleigh problem is taken for the study. Several analytical results associated with this problem have already been obtained in Pratt et al. (2000), Ganesh and Subbiah (2013), Subbiah and Ganesh (2010), Dou and Ganesh (2014), Jafer and Godo (2017) and Deng et al. (2003). Specifically, Subbiah and Ganesh (2007) derived a parabolic instability and it intersects with the semicircular instability region of Deng et al. (2003) for a class of basic flows. The functions $f(z)$ and $g(z)$ were defined in Subbiah and Ganesh (2007). The semicircular instability region of Deng et al. (2003) is reduced in Subbiah and Ganesh (2007) for the flows satisfying the conditions like $f(z) < 0$ or $g(z) > 0$. In this paper, we obtained an improved parabolic instability region.

Banerjee et al. (1988) has proved the parabolic instability region for the Rayleigh problem. This was taken deeper to extended Rayleigh problem by Subbiah and Ganesh (2007). Reenapriya and Ganesh (2015) extended the work and derived a parabolic instability region with the condition that $U_{0\min} > 0$. The parabolic instability region referred in Subbiah and Ganesh (2007) is dependent on conditions like $f(z) < 0$ or $g(z) > 0$. Reenapriya and Ganesh (2015) have obtained a parabolic instability region which is valid for $U_{0\min} > 0$. Therefore, it is essential to improve the parabolic instability region for the extended Rayleigh problem that has none of the conditions as given in Subbiah and Ganesh (2007), Reenapriya and Ganesh (2015). The new parabolic instability region is a generalized instability region for both the Rayleigh problem and the extended Rayleigh problem. Also, we derived a bounds for complex part of the phase velocity.

2 Extended Rayleigh problem (Fig. 1)

We consider a channel with variable topography aligned in the x-direction and $b(z) = y_L - y_R$ be the width of the channel, where $y = y_L$ and $y = y_R$ are the lateral positions of the side walls. The channel contains an incompressible, inviscid, homogeneous fluid flow for which the density ρ_0(constant), Pressure P and velocity $\overrightarrow{u} = (u, v, w)$ are governed by Euler's equations such as

$$\rho\left[\frac{\mathrm{d}u}{\mathrm{d}t}\right] = -\frac{\partial p}{\partial x}, 0 = -\frac{\partial p}{\partial y},$$
$$\rho\left[\frac{\mathrm{d}w}{\mathrm{d}t}\right] = -\frac{\partial p}{\partial z}$$

and

$$\nabla.\overrightarrow{u} = 0,$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial p}{\partial x} + w\frac{\partial}{\partial z}$ is the Eulerian derivative.

Integrating $\nabla.\overrightarrow{u} = 0$, across the channel and applying the boundary conditions at the walls, we have

$$\frac{\partial u}{\partial x} + wT(z) + \frac{\partial w}{\partial z} = 0,$$

where $T(z) = \frac{1}{b(z)}\frac{\mathrm{d}b(z)}{\mathrm{d}z} = \frac{b'}{b} = \frac{\mathrm{d}(logz)}{\mathrm{d}z}$ is the topography.

Let the basic flow be given by the velocity $(U(z), 0, 0)$ and Pressure $P = P_0$ (constant)and the perturbed state is given by velocity field $(U_0(z) + u, v, w)$ and the Pressure $P = P_0 + P$.

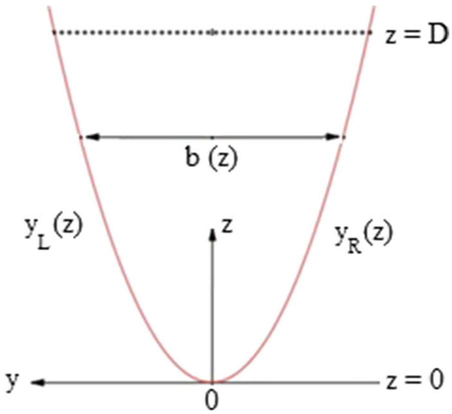

Fig. 1 The observer faces the x-direction

After linearizing the equation of motions, we have

$$\rho_0\left(\frac{\partial u}{\partial t}+U_0\frac{\partial u}{\partial t}+w\frac{\mathrm{d}U_0}{\mathrm{d}z}\right)=-\frac{\partial p}{\partial x},$$
$$0=-\frac{\partial p}{\partial y},$$
$$\rho_0\left(\frac{\partial w}{\partial t}+U_0\frac{\partial w}{\partial x}\right)=-\frac{\partial p}{\partial z},$$
$$\frac{\partial u}{\partial x}+wT(z)+\frac{\partial w}{\partial z}=0.$$

By applying the normal mode disturbances $(u, w, P, \rho)=(\hat{u}, \hat{w}, \hat{P}, \hat{\rho})e^{ik(x-ct)}$ in the above equations and omitting the hat sign we have

$$\rho_0\left[(U_0-c)iku+w\frac{\partial U_0}{\partial z}\right]=-ikP,$$
$$\rho_0\left[(U_0-c)ikw\right]=-\frac{\partial P}{\partial z},$$
$$iku+wT(z)+\frac{\partial w}{\partial z}=0.$$

Eliminating all variables except w from the above equations we obtained the extended Rayleigh problem (cf. Deng et al. (2003)), given by

$$\left[\frac{(bW)'}{b}\right]'-\left[k^2+\frac{b\left[\frac{U_0'}{b}\right]'}{U_0-c}\right]W=0, \tag{1}$$

with boundary conditions

$$W(0)=0=W(D). \tag{2}$$

Here W is the complex eigen function(note we have used $W(z)$ instead of $w(z)$ for convention), $k>0$ is the wave number, $c=c_r+ic_i$ is the complex phase velocity, U_0 is the basic velocity profile, $b(z)$ is the breadth function.

For unstable models $c_i>0$ and so $(U_0-c)^{\frac{1}{2}}$ is well defined.

Introducing the transformation $W = (U_0 - c)^{1/2} G$, we can get the equation satisfied by G to be

$$\left[(U_0 - c)\frac{(bG)'}{b}\right]' - \frac{1}{2} b \left(\frac{U_0'}{b}\right)' G - k^2 (U_0 - c) G - \frac{\left[\frac{(U_0')^2}{4}\right]}{(U_0 - c)} G = 0, \tag{3}$$

with boundary conditions

$$G(0) = 0 = G(D). \tag{4}$$

3 Instability regions

Theorem 3.1 *The following integral relations are valid when the imaginary part of the complex phase velocity c_i is strictly greater than zero and by considering U_{0s} as an arbitrary real number.*

(i)

$$\int (U_0 - c_r) Q \mathrm{d}z + \frac{1}{2} \int b \left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z + \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} (U_0 - U_{0s}) b|G|^2 \mathrm{d}z$$
$$- (c_r - U_{0s}) \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 \mathrm{d}z = 0,$$

(ii)

$$-c_i \int Q \mathrm{d}z + c_i \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 \mathrm{d}z = 0.$$

Proof Multiplying (3) by (bG^*), where * stands for complex conjugation; integrating over [0,D] and using (4), we get

$$\int (U_0 - c) \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] \mathrm{d}z + \frac{1}{2} \int b \left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{(U_0 - c)} b|G|^2 \mathrm{d}z = 0. \tag{5}$$

Let $Q = \frac{|(bG)'|^2}{b} + k^2 b|G|^2$ then the aforementioned equation becomes

$$\int (U_0 - c) Q \mathrm{d}z + \frac{1}{2} \int b \left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z + \int \frac{\frac{(U_0')^2}{4}}{(U_0 - c)} b|G|^2 \mathrm{d}z = 0. \tag{6}$$

The real part of the Eq. (6) is given by

$$\int (U_0 - c_r) Q \mathrm{d}z + \frac{1}{2} \int b \left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{|(U_0 - c)|^2} (U_0 - c_r) b|G|^2 \mathrm{d}z = 0.$$

Rewriting $(U_0 - c_r)$ as $(U_0 - U_{0s}) - (c_r - U_{0s})$, where U_{0s} is an arbitrary real number, we get

$$\int (U_0 - c_r) Q \mathrm{d}z + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z + \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}(U_0 - U_{0s}) b|G|^2 \mathrm{d}z$$
$$- (c_r - U_{0s}) \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 \mathrm{d}z = 0. \tag{7}$$

The imaginary part of the Eq. (6) is given by

$$- c_i \int Q \mathrm{d}z + c_i \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 \mathrm{d}z = 0. \tag{8}$$

Thus we arrive with the results of Theorem 3.1.

Theorem 3.2 *For the existence of unstable mode, the following integral relation is true*

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z \geq \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}\left[U_0 - c_r + \frac{U_{0\max}}{2} - \frac{U_{0\min}}{2}\right] b|G|^2 \mathrm{d}z.$$

Proof The real part of Eq. (5) is given by

$$\int (U_0 - c_r)\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] \mathrm{d}z + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{|(U_0 - c)|^2}(U_0 - c_r) b|G|^2 \mathrm{d}z = 0. \tag{9}$$

The imaginary part of Eq. (5) is given by

$$- c_i \int \left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] \mathrm{d}z + c_i \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2} b|G|^2 \mathrm{d}z = 0. \tag{10}$$

Multiplying the Eq. (10) by $\left(\frac{c_r + U_{0s}}{c_i}\right)$ and subtracting from (9), we get

$$\int (U_0 + U_{0s})\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}[U_0 - U_{0s} - 2c_r] b|G|^2 \mathrm{d}z = 0. \tag{11}$$

Multiplying the Eq. (10) by $\left(\frac{U_{0\max} - U_{0\min}}{2c_i}\right)$ and adding the resultant with (9), we get

$$\int \left(U_0 - c_r - \frac{U_{0\max}}{2} + \frac{U_{0\min}}{2}\right)\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right] + \frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2 \mathrm{d}z$$
$$+ \int \frac{\frac{(U_0')^2}{4}}{|U_0 - c|^2}\left[U_0 - c_r + \frac{U_{0\max}}{2} - \frac{U_{0\min}}{2}\right] b|G|^2 \mathrm{d}z = 0.$$

Since

$$\left(U_0 - c_r - \frac{U_{0\max}}{2} + \frac{U_{0\min}}{2}\right) \leq 0,$$

dropping the term, we get

$$\frac{1}{2}\int b\left(\frac{U_0'}{b}\right)' b|G|^2\mathrm{d}z \geq \int \frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}\left[U_0 - c_r + \frac{U_{0\max}}{2} - \frac{U_{0\min}}{2}\right]]b|G|^2\mathrm{d}z. \quad (12)$$

Hence the Theorem 3.2 is proved.

Theorem 3.3 *An essential condition for the existence of unstable mode is*

$$c_i^2 \leq \lambda\,[c_r + U_{0\max}]\,,$$

$$where\,\lambda = \frac{\left|\frac{(U_0')^2}{4}\right|_{\max}}{\left|\frac{(3U_{0\min}+U_{0\max})}{2}\right|\left[\frac{b_{\min}\pi^2}{b_{\max}D^2}+k^2\right]}.$$

Proof Substituting the Eq. (12) in (11), we get

$$\int (U_0 + U_{0s})\left[\frac{|(bG)'|^2}{b} + k^2 b|G|^2\right]\mathrm{d}z$$
$$+\int \frac{\frac{(U_0')^2}{4}}{|U_0-c|^2}\left[\frac{U_{0\min}}{2} - \frac{U_{0\max}}{2} - c_r - U_{0s}\right] b|G|^2\mathrm{d}z \leq 0;$$

Since

$$\frac{1}{|U_0-c|^2} \leq \frac{1}{c_i^2}$$

and using Rayleigh–Ritz inequality,

$$(U_{0\min} + U_{0s})\left[\frac{b_{\min}\pi^2}{b_{\max}D^2} + k^2\right]\int b|G|^2\mathrm{d}z$$
$$\leq \frac{[\frac{(U_0')^2}{4}]_{\max}}{c_i^2}\left[\frac{U_{0\max}}{2} - \frac{U_{0\min}}{2} + c_r + U_{0s}\right]\int b|G|^2\mathrm{d}z.$$

Substituting $U_{0s} = \frac{U_{0\max}+U_{0\min}}{2}$ in the above equation we get

$$c_i^2 \leq \frac{\left|\frac{(U_0')^2}{4}\right|_{\max}}{\left[\frac{3U_{0\min}+U_{0\max}}{2}\right]\left[\frac{b_{\min}\pi^2}{b_{\max}D^2}+k^2\right]}(c_r + U_{0\max})\,;$$
$$c_i^2 \leq \lambda\,[c_r + U_{0\max}]\,, \quad (13)$$

where

$$\lambda = \frac{\left|\frac{(U_0')^2}{4}\right|_{\max}}{\left|\frac{(3U_{0\min}+U_{0\max})}{2}\right|\left[\frac{b_{\min}\pi^2}{b_{\max}D^2}+k^2\right]}.$$

Theorem 3.4 *If* $\lambda < \lambda_c$
where

$$\lambda_c = (U_{0\min} + 3U_{0\max}) - 2\sqrt{2U_{0\max}(U_{0\max} + U_{0\min})}$$

then the parabola

$$c_i^2 \leq \lambda \left[c_r + U_{0\max}\right],$$

intersects the semicircle

$$\left[c_r - \left(\frac{U_{0\min} + U_{0\max}}{2}\right)\right]^2 + c_i^2 \leq \left(\frac{U_{0\max} - U_{0\min}}{2}\right)^2.$$

Proof The semi-circle given in Deng et al. (2003) is

$$\left[c_r - \left(\frac{U_{0\max} + U_{0\min}}{2}\right)\right]^2 + c_i^2 \leq \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2. \tag{14}$$

Substituting (13) in (14), we get

$$\left[c_r - \left(\frac{U_{0\max} + U_{0\min}}{2}\right)\right]^2 + \lambda\left[c_r + U_{0\max}\right] \leq \left[\frac{U_{0\max} - U_{0\min}}{2}\right]^2;$$
$$c_r^2 + c_r[\lambda - U_{0\max} - U_{0\min}] + [\lambda U_{0\max} + U_{0\max} U_{0\min}] \leq 0.$$

It's discriminant part is given by

$$\begin{aligned}
&[\lambda - (U_{0\max} + U_{0\min})]^2 - 4(1)\,[\lambda U_{0\max} + U_{0\max} U_{0\min}] \geq 0;\\
&\quad \lambda^2 - 2\lambda(U_{0\min} + U_{0\max}) + U_{0\min}^2 + U_{0\max}^2 + 2U_{0\min} U_{0\max}\\
&\quad - 4\lambda U_{0\max} - 4U_{0\min} U_{0\max} \geq 0;\\
&\quad [\lambda^2 + \lambda(-2U_{0\min} - 6U_{0\max}) + [U_{0\max} - U_{0\min}]^2 \geq 0.
\end{aligned}$$

Solving for λ, we get

$$\lambda = (U_{0\min} + 3U_{0\max}) \pm 2\sqrt{2U_{0\max}(U_{0\max} + U_{0\min})}.$$

If $\lambda < \lambda_c$, $\lambda_c = (U_{0\min} + 3U_{0\max}) - 2\sqrt{2U_{0\max}(U_{0\max} + U_{0\min})}$,

then the parabola will intersects the semicircle.

Remark If $U_0 = constant$ then by semicircle theorem, the flow is stable. If $U_0 \neq constant$ and $U_{0\min} = 0$, then the above instability region is valid. Moreover, if U_0 changes its sign then $\left|\frac{(3U_0\min + U_{0\max})}{2}\right| > 0$ and the region is valid. Now we shall illustrate the applicability of our results for an example of basic flow.

Example $U_0 = z - \frac{1}{2}$, $0 \leq z \leq 1$. In this case $U_{0\min} = -\frac{1}{2}$, $U_{0\max} = \frac{1}{2}$. Our result is valid for this case (Fig. 2).

Theorem 3.5 *If* $c_i > 0$ *then we have the integral relation*

$$\int \left[\left|W'\right|^2 + k^2 \left|W\right|^2\right] \mathrm{d}z - \frac{1}{2}\int T' \left|W\right|^2 \mathrm{d}z$$
$$+ \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2}(U_0 - U_{0s})\left|W\right|^2 \mathrm{d}z - \frac{(c_r - U_{0s})}{c_i} Im. \int T W' W^* \mathrm{d}z = 0.$$

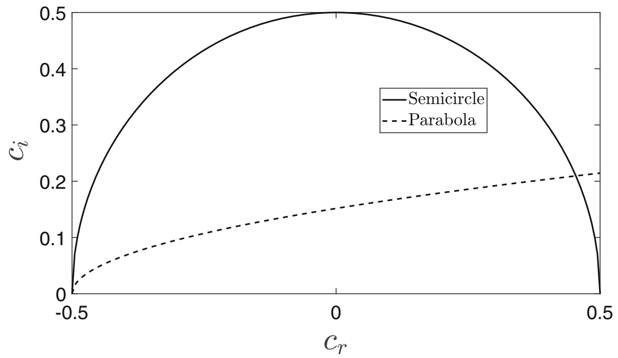

Fig. 2 Instability Region

Proof The extended Rayleigh problem is given by

$$W'' + TW' + T'W - \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c}\right] W = 0, \tag{15}$$

with boundary conditions

$$W(0) = 0 = W(D). \tag{16}$$

Multiplying (15) by W^*, integrating over [0, D] and using (16), we get

$$\int |W'|^2 \, dz + \int k^2 |W|^2 \, dz - \int TW'W^* dz - \int T' |W|^2 \, dz$$
$$+ \int \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c} |W|^2 \, dz = 0,$$

since Re $\int TW'W^* dz = -\frac{1}{2}\int T' |W|^2 \, dz$ and taking real and imaginary part, we get

$$\int \left[|W'|^2 + k^2 |W|^2\right] dz + \frac{1}{2}\int T' |W|^2 \, dz - \int T' |W|^2 \, d$$
$$+ \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} (U_0 - c_r) |W|^2 \, dz = 0;$$
$$\int \left[|W'|^2 + k^2 |W|^2\right] dz - \frac{1}{2}\int T' |W|^2 \, dz + \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} (U_0 - c_r) |W|^2 \, dz = 0 \tag{17}$$

and

$$- Im. \int TW'W^* dz + c_i \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} |W|^2 \, dz = 0. \tag{18}$$

Multiplying (18) by $\frac{c_r - U_{0s}}{c_i}$ and adding to (17),we get

$$\int \left[\left|W'\right|^2 + k^2 \left|W\right|^2 \right] \mathrm{d}z - \frac{1}{2} \int T' \left|W\right|^2 \mathrm{d}z$$
$$+ \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} (U_0 - U_{0s}) \left|W\right|^2 \mathrm{d}z - \frac{(c_r - U_{0s})}{c_i} Im. \int T W' W^* \mathrm{d}z = 0. \quad (19)$$

Theorem 3.6 *If* $c_i > 0$ *then we have the following integral relation*

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z - k^4 \int |W|^2 \mathrm{d}z - 2k^2 \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} (U_0 - c_r) |W|^2 \mathrm{d}z$$
$$- \int \frac{\left[b\left[\frac{U_0'}{b}\right]' \right]^2}{|U_0 - c|^2} |W|^2 \mathrm{d}z = 0.$$

Proof Multiplying (1) by $[\frac{(bW^*)'}{b}]'$ and integrating over [0,D] and using (2), we get

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z - \int \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c} \right] W \left[\frac{(bW^*)'}{b} \right]' \mathrm{d}z = 0. \quad (20)$$

From (1),

$$\left[\frac{(bW)'}{b} \right]' = \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c} \right] W,$$
$$\left[\frac{(bW^*)'}{b} \right]' = \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c^*} \right] W^*. \quad (21)$$

Substituting (21) in (20), we get

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z - \int \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c} \right] \left[k^2 + \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c^*} \right] W W^* \mathrm{d}z = 0,$$
$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z - k^4 \int |W|^2 \mathrm{d}z - k^2 \int \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c^*} |W|^2 \mathrm{d}z$$
$$- k^2 \int \frac{b\left[\frac{U_0'}{b}\right]'}{U_0 - c} |W|^2 \mathrm{d}z - \int \frac{\left[b\left[\frac{U_0'}{b}\right]' \right]^2}{|U_0 - c|^2} |W|^2 \mathrm{d}z = 0.$$

Equating the real part, we get

$$\int \left| \left[\frac{(bW)'}{b} \right]' \right|^2 \mathrm{d}z - k^4 \int |W|^2 \mathrm{d}z - 2k^2 \int \frac{b\left[\frac{U_0'}{b}\right]'}{|U_0 - c|^2} (U_0 - c_r) |W|^2 \mathrm{d}z$$

$$-\int \frac{\left[b\left[\frac{U_0'}{b}\right]'\right]^2}{|U_0-c|^2}|W|^2\,\mathrm{d}z=0. \tag{22}$$

Theorem 3.7 *If $T' \leq 0$ and $c_r = U_{0s}$ then bounds for c_i is given by*

$$c_i \leq \sqrt{\frac{k^2 b\left(\frac{U_0'}{b}\right)'(U_0-U_{0s})+\left[b\left(\frac{U_0'}{b}\right)'\right]^2}{\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}+k^2\frac{\pi^2}{D^2}\right]}}.$$

Proof Multiplying (18) by $\frac{(c_r-U_{0s})}{c_i}2k^2$ and subtracting from (22), we get

$$\begin{aligned}&\int\left|\left[\frac{(bW)'}{b}\right]'\right|^2\mathrm{d}z-k^4\int|W|^2\,\mathrm{d}z-2k^2\int\frac{b\left[\frac{U_0'}{b}\right]'}{|U_0-c|^2}(U_0-U_{0s})|W|^2\,\mathrm{d}z\\&+2k^2\frac{(c_r-U_{0s})}{c_i}Im.\int TW'W^*\mathrm{d}z-\int\frac{\left[b\left[\frac{U_0'}{b}\right]'\right]^2}{|U_0-c|^2}|W|^2\,\mathrm{d}z=0.\end{aligned} \tag{23}$$

Multiplying (19) by k^2 and adding to (23), we get

$$\begin{aligned}&\int\left|\left[\frac{(bW)'}{b}\right]'\right|^2\mathrm{d}z+k^2\int\left|W'\right|^2\mathrm{d}z-\frac{k^2}{2}\int T'|W|^2\,\mathrm{d}z\\&-k^2\int\frac{b\left[\frac{U_0'}{b}\right]'}{|U_0-c|^2}(U_0-U_{0s})|W|^2\,\mathrm{d}z\\&+k^2\frac{(c_r-U_{0s})}{c_i}Im.\int TW'W^*\mathrm{d}z-\int\frac{\left[b\left[\frac{U_0'}{b}\right]'\right]^2}{|U_0-c|^2}|W|^2\,\mathrm{d}z=0,\end{aligned}$$

If $T' \leq 0$, $c_r = U_{0S}$ and using Rayleigh -Ritz inequality, we get

$$\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}+k^2\frac{\pi^2}{D^2}\right]\int|W|^2\,\mathrm{d}z-\int\frac{\left[k^2 b\left[\frac{U_0'}{b}\right]'(U_0-U_{0s})+\left[b\left[\frac{U_0'}{b}\right]'\right]^2\right]}{|U_0-c|^2}|W|^2\,\mathrm{d}z\leq 0.$$

Since

$$\begin{aligned}&\frac{1}{|U_0-c|^2}\leq\frac{1}{c_i^2},\\&\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}+k^2\frac{\pi^2}{D^2}\right]c_i^2-\left[k^2 b\left[\frac{U_0'}{b}\right]'(U_0-U_{0s})+\left[b\left[\frac{U_0'}{b}\right]'\right]^2\right]\leq 0,\\&\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2}+k^2\frac{\pi^2}{D^2}\right]c_i^2\leq\left[k^2 b\left[\frac{U_0'}{b}\right]'(U_0-U_{0s})+\left[b\left[\frac{U_0'}{b}\right]'\right]^2\right],\end{aligned}$$

$$c_i \leq \sqrt{\frac{k^2 b \left(\frac{U_0'}{b}\right)' (U_0 - U_{0s}) + \left[b\left(\frac{U_0'}{b}\right)'\right]^2}{\left[\frac{\pi^4 b_{\min}^2}{D^4 b_{\max}^2} + k^2 \frac{\pi^2}{D^2}\right]}}.$$

4 Concluding remarks

For the extended Rayleigh problem of hydrodynamic stability, we have obtained a parabolic instability region. The parabolic instability region is an unbounded region it will be important only if the parabolic instability region intersects the semicircular instability region. Hence, we have proved that the parabolic instability region which intersects the semicircular instability region under some conditions. We have illustrated the validity of our results for a basic flow. Unlike the previous parabolic instability regions, our new parabolic instability region does not depend on any conditions. When b=constant or $T = 0$, we can get the instability region for standard Rayleigh problem. Hence, our parabolic instability region is also true for standard Rayleigh problem. Also, we have derived a bound for the complex part of phase velocity.

References

Banerjee MB, Gupta JR, Subbiah M (1988) On reducing Howards semicircle for homogeneous shear flows. J Math Anal Appl 130:398–402

Deng J, Pratt L, Howard L, Jones C (2003) On stratified Shear flow in sea straits of arbitrary cross section. Stud Appl Math 111:409–434

Dou HS, Ganesh V (2014) Short wave stability of homogenous shear flows with variable topography. Appl Math Mech 35(5):541–548

Ganesh V, Subbiah M (2013) Series solutions and a perturbation formula for the extended Rayleigh problem of hydrodynamic stability. Proc Indian Acad-Sci (Math Sci) 123(2):293–302

Jafer AB, Godo AA (2017) On the stability of shear flows. Int J Sci Eng Appl Sci 3(11):1–6

Pratt L, Deese HE, Murray SP, Johns W (2000) Continuous dynamical modes in straits having arbitrary cross sections with applications to the Bab al Mandab. J Phys Oceanogo 303:2515–2534

Reenapriya K, Ganesh V (2015) On the instability region for the extended rayleigh problem of hydrodynamic stability. Appl Math Sci 9(46):265–272

Subbiah M, Ganesh V (2007) On the stability of homogeneous shear flows in sea straits of arbitrary cross section. Indian J Pure Appl Math 38(1):43–50

Subbiah M, Ganesh V (2010) On short wave stability and sufficient conditions for stability in the extended Rayleigh problem of hydrodynamic stability. Proc Indian Acad Sci (Math Sci) 120(3):387–394