Nonlinear Schrödinger equation for envelope Rossby waves with complete Coriolis force and its solution

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Abstract

The physical features of the equatorial envelope Rossby waves including with complete Coriolis force and dissipation are investigated analytically. Staring with a potential vorticity equation, the wave amplitude evolution of equatorial envelope Rossby waves is described as a nonlinear Schrödinger equation by employing multiple scale analysis and perturbation expansions. The equation is more suitable for describing envelope Rossby solitary waves when the horizontal component of Coriolis force is stronger near the equator. Then, based on the Jacobi elliptic function expansion method and trial function method, the classical Rossby solitary wave solution and the corresponding stream function of the envelope Rossby solitary waves are obtained, respectively. With the help of these solutions, the effect of dissipation and the horizontal component of Coriolis parameter are discussed in detail by graphical presentations. The results reveal the effect of the horizontal component of Coriolis force and dissipation on the classical Rossby solitary waves.

Keywords Complete Coriolis force · Rossby solitary waves · Nonlinear Schrödinger equation · Jacobi elliptic function methods · The dissipation effect

Mathematics Subject Classification 76M45

1 Introduction

In recent years, equatorial Rossby wave theory has been an important research subject in the atmospheric and oceanic dynamics (Boyd 2018; Ching et al. 2015; Puy et al. 2016). In 1834, Russell first observed solitary wave. Since then, Rossby waves have attracted much more attention as a branch of solitary waves, and many mathematical models are constructed to study the generation and evolution of Rossby solitary waves, such as the Korteweg-de Vries (KdV), the modified KdV (mKdV) equation, the Zakharov–Kuznetsov

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equation, the nonlinear Schrödinger equation (NLS), and so on. Meanwhile, some natural phenomena related to Rossby waves were explained better with the help of these mathematical models.

The KdV equation was firstly obtained by Long (1964) in barotropic fluid, which is a famous soliton wave equation. Later, Redekopp (1977) discussed the general theory of solitary waves in zonal planetary shear flow and obtained mKdV equation. After these pioneer work, the classical Rossby solitary waves theory was formed step by step (Boyd 1980; Wadati 1973; Yang et al. 2014). However, the disadvantages of the KdV-type solitons must be satisfied the long-wave approximation. For the envelope Rossby solitary waves depicted by the NLS equation, it is not need to satisfy the long-wave approximation. Therefore, it is more appropriate to describe the Rossby waves contrast to the KdV or mKdV solitary wave models. The equation was firstly obtained by Benney (1979) in atmospheric dynamics for describing the envelope Rossby solitary waves. Then, Luo (1991) used the NLS equation to explain atmospheric vortex pair blocking in the mid-high latitudes. Tan and Boyd (2000) derived the dissipative NLS equation and studied the effects of forcing and dissipation on the collision interaction of two envelope Rossby solitary waves. Demiray (2003) sought a traveling wave solution to the dissipative NLS equation by using of the hyperbolic tangent method. Barletti et al. (2018) used the Hamiltonian Boundary Value Methods for solving the NLS equation. Fu et al. (2018) derived the time–space fractional $(2 + 1)$-dimensional NLS equation to describe the envelope gravity waves by using the multiple scale and obtained the solution of the equation. Recently, a new ZK-BO equation and ZK–ILW equation are derived to describe the evolution of Rossby solitary waves by Yang et al. (2018) and Guo et al. (2018). However, looking at the above results, none of them considered the effects of the horizontal component of Coriolis parameter. It is discarded in the usual shallow water equation. The simplifications are known as the 'traditional approximation' which involves the neglect of the locally horizontal components of the Earth's rotation vector (Hua et al. 1997).

For the traditional approximation, it is valid when the vertical length scales are small compared with the horizontal length scales. However, it becomes increasingly controversial questions from the dynamical perspective (Kasahara 2003; Philips 1968; Veronis 1968; Wangsness 1970) and many researchers indicated that the horizontal component of Coriolis parameter neglected has significant effects under the specific circumstances (Raymond 2000). Meanwhile, White and Bromley (1995) gave a set of "quasi-hydrostatic" models and examined the importance of the cosine Coriolis terms. They pointed out that the cosine Coriolis terms are not neglected in planetary-scale motion and in tropical synoptic-scale motion. Dellar and Salmon (2005) derived a conserved potential vorticity equation including with complete Coriolis force by variational method. In 2010, Stewart and Dellar (2010) extended this work and described the flow of multiple superposed layers of inviscid, incompressible fluids in a rotating frame. Stewart and Dellar (2012) investigated the behavior of linear plane waves with the complete Coriolis force in multilayer shallow water equations. Recently, Lu et al. (2018) constructed time-fractional Boussinesq equation to describe Rossby solitary waves from the quasi-geostrophic vorticity equation with dissipation and complete Coriolis force in stratified fluid.

On the other hand, the solutions of solitary wave models have been extensively investigated and many powerful methods have been proposed to solve nonlinear partial differential equations; for example, traveling wave solutions (Zayed and Gepreel 2009), Jacobi elliptic function expansion methods (Liu et al. 2001; Liu and Fan 2005), Bäcklund transformations (Quispel et al. 1984), trial function methods (Belytschko et al. 1994), homotopy perturbation methods (He 2003), new homotopy perturbation methods (Biazar and Eslami 2013; Eslami 2014;

Eslami and Mirzazadeh 2014), sine–cosine expansion methods (Triki and Wazwaz 2011, 2014), and so on. In the present paper, we give asymptotic solutions of the new NLS equation including with complete Coriolis force and dissipation by using Jacobi elliptic function expansion methods and trial function methods.

Considering all of the above discussions, this paper is organized as follows. From conserved potential vorticity equation including with complete Coriolis force and dissipation, we derive a NLS equation by multiple scale analysis and perturbation expansions in Sect. 2. In Sect. 3, the classical Rossby solitary wave solution and the stream function of the envelope Rossby solitary waves are obtained by using Jacobi elliptic function expansion methods and trial function methods, respectively. Then, graphical presentations are presented, and the effects of the dissipation and the horizontal component of Coriolis parameter are discussed. Finally, some conclusions are given in Sect. 4.

2 Basic mathematical model

Based on the potential vorticity equation near the equator with complete Coriolis force and turbulent dissipation, it can be written as follows:

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + f - f_H \frac{\partial B}{\partial y} \right] = -\mu \nabla^2 \psi + Q, \tag{1}$$

where $\psi(x, y)$ is the total stream function; $f = \beta(y)y$ is the vertical component of Coriolis parameter; f_H is the horizontal component of Coriolis parameter, it is constant; $B(x, y)$ is the bottom topography; μ is the turbulent dissipation parameter; Q denotes the external heating source due to the tropical ocean; ∇^2 is the two-dimensional Laplace operator.

The boundary conditions are defined by the following:

$$\frac{\partial \psi}{\partial x} = 0, \ y = y_1, \ y = y_2. \tag{2}$$

Here, $y = y_1, \ y = y_2$ denote the southern and northern edges of the zonal flow.

Introducing the dimensionless as

$$(x, y) = L_0(x^*, y^*), t = \frac{L_0}{U_0} t^*, \psi = L_0 U_0 \psi^*, B = \frac{U_0 H}{f_H L_0} B^*, \beta = \frac{U_0}{L_0^2} \beta^{\bullet}, \mu_0 = \frac{U_0}{L_0} \mu^*, Q = \frac{U_0^2}{L_0^2} Q^*, \tag{3}$$

where dimensionless variables are marked by an asterisk. L_0 is the zonal characteristic length of the mean zonal flow, and H is a vertical characteristic length, respectively; U_0 is the characteristic velocity.

Substituting of (3) into (1) and (2) gets

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \beta(y)y - \delta' \frac{\partial B}{\partial y} \right] = -\mu \nabla^2 \psi + Q(x, y, t), \tag{4}$$

and the boundary conditions of nondimensional form is as follows:

$$\frac{\partial \psi}{\partial x} = 0, \ y = 0, \ y = 1. \tag{5}$$

Here, the subscript asterisks have been dropped for simplicity. $\delta' = H/L_0$ represents factors of the aspect ratio. To balance the external source and nonlinearity, we introduce a new small parameter δ,

$$\delta' = \varepsilon\delta. \tag{6}$$

Here, ε is a small parameter characterizing the weakness of the nonlinearity.

Assuming that the total stream function can be decomposed as follows:

$$\psi(x, y, t) = -\int_0^y [U(s) - c_0]ds + \varepsilon\psi'(x, y, t), \tag{7}$$

where c_0 is a constant, $U(s)$ is the basic flow, and the external source balances the diffusion of the basic flow (Caillol and Grimshaw 2008). Substituting of (6) and (7) into (4) yields:

$$\left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\nabla^2\psi' + p(y)\frac{\partial\psi'}{\partial x} + \varepsilon J(\psi', \nabla^2\psi') - \varepsilon\delta J\left(\psi', \frac{\partial B}{\partial y}\right)$$

$$- \delta(U - c_0)\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial y}\right) = -\mu\nabla^2\psi', \tag{8}$$

where $p(y) = \frac{d(\beta(y)y)}{dy} - U''$, $J(A, B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$, $Q = -\mu\frac{dU}{dy}$.

Introducing the slow time and space variables:

$$T_1 = \varepsilon t, T_2 = \varepsilon^2 t; X_1 = \varepsilon x, X_2 = \varepsilon^2 x. \tag{9}$$

The derivative transformations are as follows:

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2}; \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}, \tag{10}$$

By adopting the transformation, the Eq. (8) becomes

$$\left[\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2}\right) + (U - c_0)\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\right]\left[\nabla^2\psi' + 2\varepsilon\frac{\partial^2\psi'}{\partial x\partial X_1} + \varepsilon^2\left(\frac{\partial^2\psi'}{\partial X_1^2} + 2\frac{\partial^2\psi'}{\partial x\partial X_2}\right)\right.$$

$$\left. + 2\varepsilon^3\frac{\partial^2\psi'}{\partial X_1\partial X_2} + \varepsilon^4\frac{\partial^2\psi'}{\partial X_2^2}\right] + \varepsilon\left[\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\psi'\left(\frac{\partial}{\partial y}\right) - \frac{\partial\psi'}{\partial y}\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\right]$$

$$\times\left[\nabla^2\psi' + 2\varepsilon\frac{\partial^2\psi'}{\partial x\partial X_1} + \varepsilon^2\left(\frac{\partial^2\psi'}{\partial X_1^2} + 2\frac{\partial^2\psi'}{\partial x\partial X_2}\right) + 2\varepsilon^3\frac{\partial^2\psi'}{\partial X_1\partial X_2} + \varepsilon^4\frac{\partial^2\psi'}{\partial X_2^2}\right] - \varepsilon\delta\left[\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1}\right.\right.$$

$$\left.\left. + \varepsilon^2\frac{\partial}{\partial X_2}\right)\psi'\frac{\partial}{\partial y}\left(\frac{\partial B}{\partial y}\right) - \frac{\partial\psi'}{\partial y}\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\left(\frac{\partial B}{\partial y}\right)\right] + p(y)\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\psi' \tag{11}$$

$$- \delta(U - c_0)\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X_1} + \varepsilon^2\frac{\partial}{\partial X_2}\right)\frac{\partial B}{\partial y} = -\varepsilon^2\lambda\left[\nabla^2\psi' + 2\varepsilon\frac{\partial^2\psi'}{\partial x\partial X_1} + \varepsilon^2\left(\frac{\partial^2\psi'}{\partial X_1^2} + 2\frac{\partial^2\psi'}{\partial x\partial X_2}\right)\right.$$

$$\left. + 2\varepsilon^3\frac{\partial^2\psi'}{\partial X_1\partial X_2} + \varepsilon^4\frac{\partial^2\psi'}{\partial X_2^2}\right].$$

In Eq. (11), further assumptions are as follows:

$$\frac{\partial B}{\partial y} = \varepsilon D(y), \mu = \varepsilon^2\lambda. \tag{12}$$

Assuming the disturbance stream function can be expanded as follows:

$$\psi' = \psi_0(x, y, t; X_1, X_2; T_1, T_2) + \varepsilon\psi_1(x, y, t; X_1, X_2; T_1, T_2) + \varepsilon^2\psi_2(x, y, t; X_1, X_2; T_1, T_2) + \cdots \tag{13}$$

Substituting of (13) into (11), the lowest order equation is as follows:

$$o(\varepsilon^0): \left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\nabla^2\psi_0 + p(y)\frac{\partial\psi_0}{\partial x} = 0. \tag{14}$$

Assuming that the solution of Eq. (14) is as follows:

$$\psi_0 = A(X_1, X_2; T_1, T_2)\Phi_0(y)e^{i(kx-wt)} + \text{c.c.}, \tag{15}$$

where A is a slowly varying envelope amplitude, k is the zonal wave number, ω is the frequency of Rossby waves, and c.c. denotes the complex conjugate of the preceding term.

When $U - c_0 - c \neq 0$, Φ_0 satisfies the following equation:

$$\Phi_0'' + \left(\frac{p(y)}{U - c_0 - c} - k^2\right)\Phi_0 = 0, \tag{16}$$

$$\Phi_0(0) = \Phi_0(1) = 0, \tag{17}$$

where c is the really constant. For the $o(\varepsilon)$, we get the following:

$$\left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\nabla^2\psi_1 + p(y)\frac{\partial\psi_1}{\partial x} \equiv F_1, \tag{18}$$

$$\Phi_1(0) = \Phi_1(1) = 0, \tag{19}$$

where F_1 being defined as follows:

$$F_1 = -\left\{\left[\frac{\partial}{\partial T_1} + (U - c_0)\frac{\partial}{\partial X_1}\right]\nabla^2\psi_0 + p(y)\frac{\partial\psi_0}{\partial X_1} + J(\psi_0, \nabla^2\psi_0) + 2\left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\frac{\partial^2\psi_0}{\partial x\partial X_1}\right\}. \tag{20}$$

Equation (20) becomes

$$F_1 = \frac{p(y)\Phi_0}{U - c_0 - c}\left(\frac{\partial A}{\partial T_1} + c_1\frac{\partial A}{\partial X_1}\right)e^{i(kx-\omega t)} + ik\left(\frac{p(y)}{U - c_0 - c}\right)_y\Phi_0^2A^2e^{2i(kx-\omega t)} + \text{c.c.}, \tag{21}$$

with $c_1 = c + \frac{2k^2(U-c_0-c)^2}{p(y)}$. By eliminating these secular terms, we get the following:

$$\frac{\partial A}{\partial T_1} + c_g\frac{\partial A}{\partial X_1} = 0, \tag{22}$$

where

$$c_g = c + \frac{I_1}{I}, I_1 = \int_0^1 2k^2\Phi_0^2 dy, I = \int_0^1 \frac{p(y)\Phi_0^2}{(U - c_0 - c)^2}dy, \tag{23}$$

where c_g is the group velocity of Rossby wave. Equation (18) then reduces to

$$\left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\nabla^2\psi_1 + p(y)\frac{\partial\psi_1}{\partial x} \equiv ikA^2Q(y)e^{2i(kx-\omega t)} + \text{c.c.} \tag{24}$$

Here, $Q(y) = \left(\frac{p(y)}{U - c_0 - c}\right)_y \Phi_0^2$.

Assuming that the solution of Eq. (24) is as follows:

$$\psi_1 = B(X_1, X_2; T_1, T_2)\Phi_1 e^{2i(kx - \omega t)} + \text{c.c.} \tag{25}$$

By combining (24) with (25), get

$$\left[\Phi_1'' - 4k^2\Phi_1 + \frac{p(y)}{U - c - c_0}\Phi_1\right]B = A^2 Q(y)\frac{1}{2(U - c - c_0)}. \tag{26}$$

In Eq. (26), A and B are not two independent variables. For simplification, B will be taken into the following forms, namely

$$\psi_1 = A^2\Phi_1 e^{2i(kx - \omega t)} + \text{c.c.} \tag{27}$$

For

$$o(\varepsilon^2) : \left[\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right]\nabla^2\psi_2 + p(y)\frac{\partial\psi_2}{\partial x} = F_2, \tag{28}$$

With

$$
\begin{aligned}
F_2 = -\Bigg\{ &\left(\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi_0}{\partial X_1^2} + 2\frac{\partial^2\psi_0}{\partial x \partial X_2}\right) + \left(\frac{\partial}{\partial T_1} + (U - c_0)\frac{\partial}{\partial X_1}\right)\nabla^2\psi_1 + \left(\frac{\partial}{\partial T_1} + (U - c_0)\frac{\partial}{\partial X_1}\right)2\frac{\partial^2\psi_0}{\partial x \partial X_1} \\
&+ \left(\frac{\partial}{\partial t} + (U - c_0)\frac{\partial}{\partial x}\right)\left(2\frac{\partial^2\psi_1}{\partial x \partial X_1}\right) + \left(\frac{\partial}{\partial T_2} + (U - c_0)\frac{\partial}{\partial X_2}\right)\nabla^2\psi_0 + \frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi_1 + \frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}\left(2\frac{\partial^2\psi_0}{\partial x \partial X_1}\right) \\
&+ \frac{\partial\psi_0}{\partial X_1}\frac{\partial}{\partial y}\nabla^2\psi_0 + \frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi_0 - \frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi_1 - \frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial x}\left(2\varepsilon\frac{\partial^2\psi_0}{\partial x \partial X_1}\right) - \frac{\partial\psi_0}{\partial y}\frac{\partial}{\partial X_1}\nabla^2\psi_0 - \frac{\partial\psi_1}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi_0 \\
&+ p(y)\frac{\partial\psi_1}{\partial X_1} + p(y)\frac{\partial\psi_0}{\partial X_2} - \delta\frac{\partial\psi_0}{\partial x}\frac{\partial}{\partial y}D(y) + \lambda\left(\frac{\partial^2\psi_0}{\partial x^2} + \frac{\partial^2\psi_0}{\partial y^2}\right)\Bigg\}.
\end{aligned}
\tag{29}
$$

Next, Eq. (29) becomes

$$
\begin{aligned}
F_2 = \frac{p(y)\Phi_0}{U - c - c_0}\Bigg[&\frac{\partial A}{\partial T_2} + c_1\frac{\partial A}{\partial X_2} + ik\frac{U - c - c_0}{p(y)}(c + 2c_g - 3(U - c_0))\frac{\partial^2 A}{\partial X_1^2} + ik\delta\frac{U - c - c_0}{p(y)}D'(y)A - \lambda A\Bigg]e^{i(kx - wt)} \\
&- ik|A|^2 A\Big[\Phi_0(\Phi_1'' - 4k^2\Phi_1)' + 2\Phi_0'(\Phi_1'' - 4k^2\Phi_1) + 2\Phi_1(\Phi_0'' - k^2\Phi_0)' \\
&+ \Phi_1(\Phi_0'' - k^2\Phi_0)'\Big]e^{i(kx - \omega t)} + \text{c.c.} + \square,
\end{aligned}
\tag{30}
$$

where $\square$ for other items related to the $e^{\pm 2i(kx - \omega t)}$, and $e^{\pm 3i(kx - \omega t)}$.

Eliminating these secular terms $\int_0^1 \frac{\Phi_0}{U - c - c_0}F_2 dy = 0$, we get the following:

$$
\begin{aligned}
&\int_0^1 \frac{p(y)}{(U - c - c_0)^2}\Phi_0^2 dy\left(\frac{\partial A}{\partial T_2} + c_1\frac{\partial A}{\partial X_2}\right) + ik\int_0^1 \frac{c + 2c_g - 3(U - c_0)}{U - c - c_0}\Phi_0^2 dy\frac{\partial^2 A}{\partial X_1^2} + ik\delta\int_0^1 \frac{\Phi_0^2}{U - c - c_0}D'(y)dyA - \lambda A \\
&\int_0^1 \frac{p(y)\Phi_0^2}{(U - c - c_0)^2}dy + ik|A|^2 A\int_0^1 \frac{\Phi_0}{U - c - c_0}\left[\frac{\Phi_0}{2}\left(\frac{Q(y)}{U - c - c_0}\right)_y + \Phi_0\Phi_1\left(\frac{p(y)}{U - c - c_0}\right)_y + \frac{Q(y)}{U - c - c_0}\frac{d\Phi_0}{dy}\right]dy = 0.
\end{aligned}
\tag{31}
$$

Set

$$\alpha = \frac{I_1}{I}, \lambda = \frac{I_2}{I}, \gamma = \frac{I_3}{I}, \eta = \frac{I_4}{I},$$ (32)

where

$$I = \int_0^1 \frac{p(y)}{(U - c - c_0)^2} \Phi_0^2 \mathrm{d}y, I_1 = k \int_0^1 \frac{c + 2c_g - 3(U - c_0)}{U - c - c_0} \Phi_0 \mathrm{d}y,$$

$$I_2 = \delta k \int_0^1 \frac{\Phi_0^2}{U - c - c_0} D'(y) \mathrm{d}y, \ I_4 = -\lambda \int_0^1 \frac{p(y)\Phi_0^2}{(U - c - c_0)^2} \mathrm{d}y.$$

$$I_3 = k \int_0^1 \frac{\Phi_0}{U - c - c_0} \left[\frac{\Phi_0}{2} \left(\frac{Q(y)}{U - c - c_0} \right)_y + \Phi_0 \Phi_1 \left(\frac{p(y)}{U - c - c_0} \right)_y + \frac{Q(y)}{U - c - c_0} \frac{d\Phi_0}{dy} \right] \mathrm{d}y.$$

Equation (32) can be rewritten as follows;

$$i\left(\frac{\partial A}{\partial T_2} + c_1 \frac{\partial A}{\partial X_2} \right) + \alpha \frac{\partial^2 A}{\partial X_1^2} + \lambda A + \gamma |A|^2 A + i\eta A = 0.$$ (33)

Equation (33) is a nonlinear Schrödinger equation with the horizontal component of Coriolis parameter and dissipation. The coefficients α, γ are the dispersion coefficient and the Landau number which are related to the shear basic flow and generalized beta effect, respectively. The term with coefficient η represents the dissipation effect. The term with coefficient λ denotes that the horizontal components of Coriolis parameter and bottom topography are interacting with the equatorial envelop Rossby waves. When the horizontal component of the Coriolis parameter is neglected, namely $\lambda = 0$, the equation deduces to the results of Demiray (2003).

3 Asymptotic solution

3.1 Solutions of NLS equation with the horizontal components of Coriolis parameter

In this section, we will consider the effects of the horizontal component of the Coriolis parameter. For Eq. (33), by adopting the axis transform by Jeffrey and Kawahara (1982) as follows:

$$T = T_2, X = \frac{1}{\varepsilon}(X_2 - c_g T_2) = X_1 - c_g T_1.$$ (34)

Equation (33) can be rewritten as follows:

$$i\frac{\partial A}{\partial T} + \alpha \frac{\partial^2 A}{\partial X^2} + \lambda A + \gamma |A|^2 A + i\eta A = 0.$$ (35)

In Eq. (35), when the coefficient $\lambda = 0, \eta = 0$, the equation becomes a standard NLS equation obtained by Benney and Newell (1967):

$$i\frac{\partial A}{\partial T} + \alpha \frac{\partial^2 A}{\partial X^2} + \gamma |A|^2 A = 0.$$ (36)

The single envelope Rossby solitary wave solution of Eq. (36) is as follows:

$$A_0(X, T) = \sqrt{\frac{2\alpha}{\gamma}} M \operatorname{sech} M(X - 2\alpha k_0 T) \exp\left\{i[k_0 X - \alpha(k_0^2 - M^2)T]\right\}, \qquad (37)$$

which the parameter M is the amplitude and k_0 is the moving speed of the envelope Rossby solitary waves, whose value is determined by the original state of $A_0(X, T)$.

Substituting of (37) and (13) into (7) yields

$$\psi_0 = -\int^y [U(s) - c_0]ds + \varepsilon \sqrt{\frac{2\alpha}{\gamma}} M \operatorname{sech} \varepsilon M(x - V_0 t) \Phi_0(y) \exp\{i[K_0 x - \Omega_0]t\}, \qquad (38)$$

where

$$V_0 = c_g + 2\varepsilon\alpha k_0, K_0 = k + \varepsilon k_0, \Omega_0 = \omega + \varepsilon k_0 c_g + \varepsilon^2 \alpha(k_0^2 - M^2). \qquad (39)$$

In the following discussion, we suppose $\lambda \neq 0, \eta = 0$. Equation (35) becomes

$$i\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \lambda A + \gamma|A|^2 A = 0; \qquad (40)$$

taking the following transformation:

$$A = B \exp\{i\lambda T\}. \qquad (41)$$

Substituting of (41) into (40), we get the following:

$$i\frac{\partial B}{\partial T} + \alpha\frac{\partial^2 B}{\partial X^2} + \gamma|B|^2 B = 0. \qquad (42)$$

Based on the above discussion, the envelope Rossby solitary wave solution of Eq. (40) is as follows:

$$A_1(X, T) = \sqrt{\frac{2\alpha}{\gamma}} M \operatorname{sech} M(X - 2\alpha k_0 T) \exp\left\{i[k_0 X - [\alpha(k_0^2 - M^2)]T + \lambda T]\right\}. \qquad (43)$$

The behavior of function $A(X,T)$ defined by Eq. (43) is given in Figs. 1 and 2. Figure 1 describes the evolution of Rossby solitary wave amplitude $A(X, T)$ changing with time and spatial when the dissipation is absent. The amplitude of Rossby solitary waves is not changed
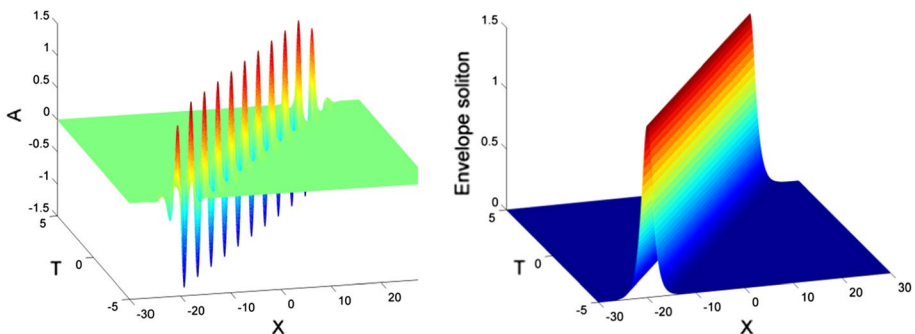
Fig. 1 The real part and envelope of the solution (43) with parameters chosen as: $\alpha = 1$, $\gamma = 1, M = 1, k_0 = 2$, and $\lambda = 1$

(a) : $\lambda = 0$ **(b)** : $\lambda = 2$
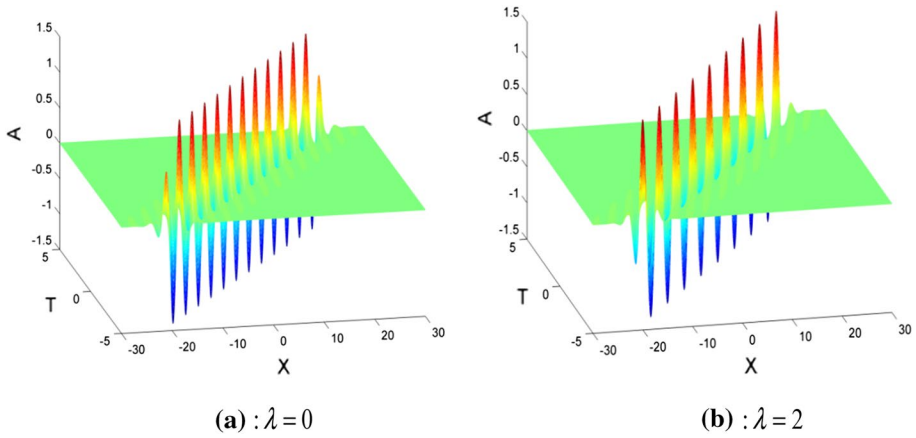
Fig. 2 The real part of the solution (43) with different parameters chosen as: **a** $\alpha = 1, \gamma = 1, M = 1, k_0 = 2$, and $\lambda = 0$; **b** $\alpha = 1, \gamma = 1, M = 1, k_0 = 2$, and $\lambda = 2$

with time; it is the idealized solitary wave. Then, we discuss the effects of the horizontal component of Coriolis parameter λ on the amplitude of solitary wave in the absent of dissipation in Figs. 2 and 3; we find that with increasing of the Coriolis parameter λ, the amplitude of solitary wave decreases when the latitude $X = 0.5$ and $X = 1$, and it means that it is more beneficial to form the large amplitude solitary wave when the Coriolis parameter λ is smaller.

The stream function of envelope Rossby solitary waves in Eq. (40) is as follows:

$$\psi_1 = -\int^y [U(s) - c_0]ds + \varepsilon\sqrt{\frac{2\alpha}{\gamma}}M \operatorname{sech} M\varepsilon(x - V_1 t)\Phi_0(y)\exp\{i[K_1 x - \Omega_1 t]\},$$

(44)

where

$$V_1 = c_g + 2\varepsilon\alpha k_0, \; K_1 = k + \varepsilon k_0, \; \Omega_1 = \omega + \varepsilon k_0 c_g + \varepsilon^2\alpha(k_0^2 - M^2) - \varepsilon^2\lambda.$$

(45)

(a) : $X = 1$ **(b)** : $X = 0.5$
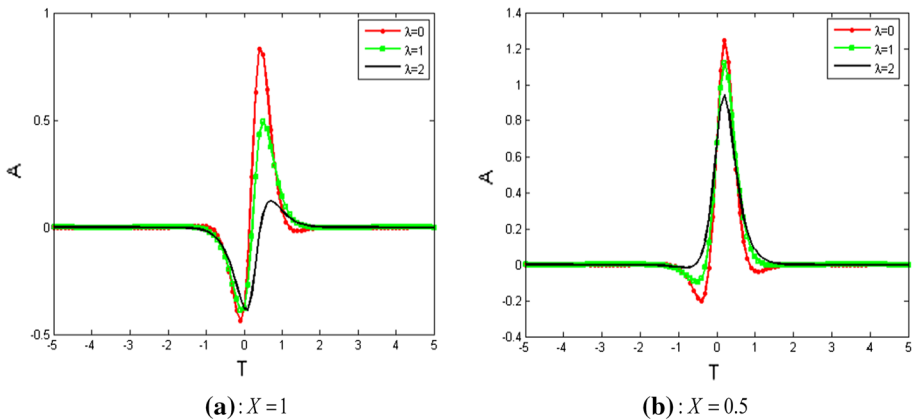
Fig. 3 The effects of λ on the amplitude of Rossby solitary waves when $X = 1$ and $X = 0.5$

Comparing with Eqs. (39) and (45), we find that the carrier frequency Ω_1 is based on Ω_0 plus a small correction $\varepsilon^2\lambda$, but the propagation velocity and the carrier wave number have nothing to do with the Coriolis parameter. It means that the horizontal component of the Coriolis parameter can amend the carrier frequency.

3.2 Solutions of NLS equation with the horizontal components and dissipation

In this section, we mainly find asymptotic solution of Eq. (35) and analyze the dissipation effects on the evolution of Rossby solitary waves. We assume the coefficient $\lambda \neq 0, \eta \neq 0$, and $\eta << 1$, taking the following transformation:

$$A = \left[a_0(T) + a_1(T)cn\xi\right]e^{i(k(T)-\omega(T))}, \xi = f(T)(X - g(T)). \tag{46}$$

Substituting of (46) into (35) get

$$(-\gamma a_1^3 + 2\alpha a_1 f^2 m^2)sn^2\xi cn\xi + \left[i\left(\eta a_1 + \frac{da_1}{dT}\right) + \left(-\frac{dk}{dT}a_1X - \alpha k^2 a_1 + 3\gamma a_0^2 a_1 + \frac{d\omega}{dT}a_1 + \gamma a_1^3 - \alpha a_1 f^2 + \lambda a_1\right)\right]$$
$$\times cn\xi + i\left(-a_1\frac{df}{dT}X + a_1\frac{df}{dT}g - 2\alpha k a_1 f + a_1 f\frac{dg}{dT}\right)sn\xi dn\xi - 3\gamma a_0 a_1^2 sn^2\xi + i\left(\frac{da_0}{dT} + \eta a_0\right) + \left(-\frac{dk}{dT}a_0X\right)$$
$$+3\gamma a_0 a_1^2 + \gamma a_0^3 + \frac{d\omega}{dT}a_0 - \alpha k^2 a_0 + \lambda a_0\right) = 0. \tag{47}$$

Therefore

$$-\gamma a_1^3 + 2\alpha a_1 f^2 m^2 = 0, \eta a_1 + \frac{da_1}{dT} = 0, -\frac{dk}{dT}a_1X - \alpha k^2 a_1 + 3\gamma a_0^2 a_1 + \frac{d\omega}{dT}a_1 + \gamma a_1^3 - \alpha a_1 f^2 + \lambda a_1 = 0$$
$$-a_1\frac{df}{dT}X + a_1\frac{df}{dT}g - 2\alpha k a_1 f + a_1 f\frac{dg}{dT} = 0, -3\gamma a_0 a_1^2 = 0, \frac{da_0}{dT} + \eta a_0 = 0,$$
$$-\frac{dk}{dT}a_0X + 3\gamma a_0 a_1^2 + \gamma a_0^3 + \frac{d\omega}{dT}a_0 - \alpha k^2 a_0 + \lambda a_0 = 0. \tag{48}$$

By solving Eq. (48), we obtain the following:

$$a_0 = 0, a_1 = \sqrt{\frac{2\alpha}{\gamma}}he^{-\eta T}, f = \frac{h}{m}e^{-\eta T}, k(T) = k_0, \omega = \alpha\left[\left(k_0^2 - \frac{\lambda}{\alpha}\right)T + \frac{h^2(2m^2-1)}{2\eta m^2}e^{-2\eta T}\right]; \tag{49}$$
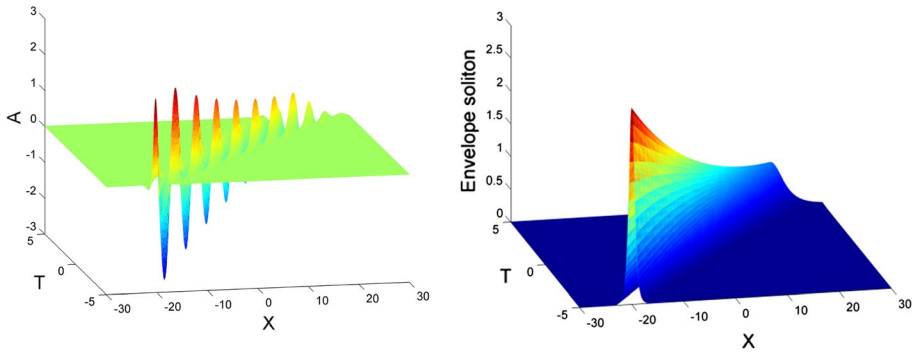
substituting of (49) into (48), we get

$$\eta(X - g(T)) + \left(\frac{dg}{dT} - 2\alpha k(T)\right) = 0. \tag{50}$$

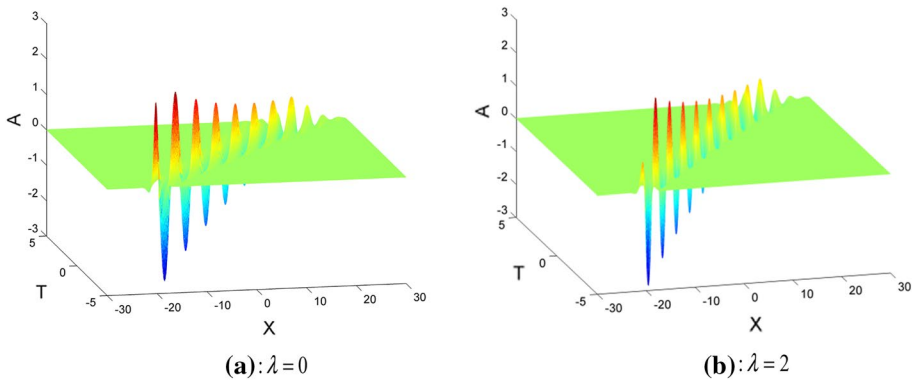Because of $\eta << 1$, the item of $\eta(X - g(T))$ can be omitted in Eq. (50); therefore

$$g(T) = 2\alpha k_0 T. \tag{51}$$

The solution of the Eq. (35) is given by the following:

$$A(X, T) = \sqrt{\frac{2\alpha}{\gamma}}he^{-\eta T}cn\left(\frac{h}{m}e^{-\eta T}(X - 2\alpha k_0 T)\right)\exp\left\{i\left[k_0X - \alpha\left(\left(k_0^2 - \frac{\lambda}{\alpha}\right)T + \frac{(2m^2-1)h^2}{2\eta m^2}e^{-2\eta T}\right)\right]\right\}. \tag{52}$$

Fig. 4 The real part and envelope of the solution (54) with parameters chosen as: $\alpha = 1$ $\gamma = 1$ $h = 1$ $\eta = 0.15$ $k_0 = 2$ $\lambda = 1$

(a): $\lambda = 0$ **(b)**: $\lambda = 2$

Fig. 5 The real part of the solution (54) with parameters chosen as: $\alpha = 1$ $\gamma = 1$ $h = 1$ $\eta = 0.15$ and $k_0 = 2$

For $m \to 1$, the solution is degenerated as follows:

$$A = \sqrt{\frac{2\alpha}{\gamma}} h e^{-\eta T} \operatorname{sech}\left[h e^{-\eta T}(X - 2\alpha k_0 T) \right] \exp\left\{ i\left[k_0 X - \alpha\left(\left(k_0^2 - \frac{\lambda}{\alpha} \right)T + \frac{h^2}{2\eta}e^{-2\eta T} \right) \right] \right\}.$$
(53)

The behavior of function $A(X, T)$ defined by (53) is given in Figs. 4 and 5.

Figures 4 and 5 show the effects of the horizontal component of Coriolis parameter λ and the dissipation on the amplitude of solitary waves, and we can obtain the following conclusions. When the dissipation is exist, the wave amplitude is decreasing with time as $he^{-\eta T}$, where h is the initial amplitude. This is different from the previous results. From Figs. 4 and 5, with the increasing the Coriolis parameter λ,, the amplitude of the solitary waves will accelerate decay. There is obvious difference in the absence of the dissipation (Fig. 6).

Corresponding, the stream function of equatorial envelope Rossby solitary waves is as follows:

$$\psi_0 = -\int^y [U(s) - c_0]ds + \varepsilon\sqrt{\frac{2\alpha}{\gamma}} h e^{-\eta\varepsilon^2 t} \operatorname{sech}[\varepsilon h e^{-\eta\varepsilon^2 t}(x - V_2 t)]\Phi_0(y)\exp\{i[K_2 x - \Omega_2]\};$$
(54)

(a): $X = 1$ **(b)**: $X = 0.5$
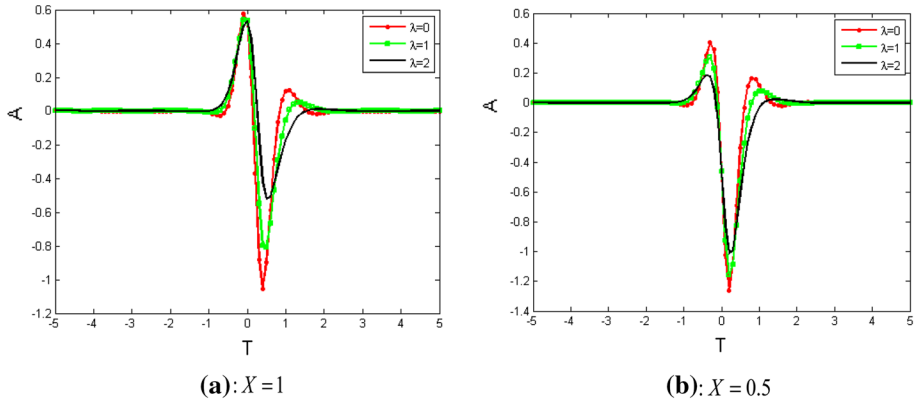
Fig. 6 The effects of λ on the amplitude of Rossby waves when $X = 1$ and $X = 0.5$

here

$$V_2 = c_g + 2\varepsilon\alpha k_0, K_2 = k + \varepsilon k_0, \Omega_2 = (\omega + \varepsilon k_0 c_g + \varepsilon^2 \alpha k_0^2 - \varepsilon^2 \lambda)t + \frac{h^2}{2\eta}e^{-2\eta\varepsilon t^2}. \quad (55)$$

Comparing with Eqs. (45) and (55), it shows that the dissipation has no effect on the propagation velocity and the carrier wave number, but it has certain influence on the carrier frequency of equatorial envelope Rossby solitary waves. This conclusion was also obtained by Yun-Long et al. (2015)but they have not considered the horizontal component of Coriolis parameter. In addition, Eq. (55) reveals that the carrier frequency is equal to frequency of linear Rossby waves plus three small corrections, which are related to dispersion and the Coriolis horizontal component.

4 Conclusion

In this paper, we investigate the evolution of equatorial envelope Rossby solitary waves with complete Coriolis force and dissipation. The evolution of Rossby solitary waves is described by a nonlinear Schrödinger equation by means of multiple scale analysis and perturbation methods, it is more suitable for describing the equatorial envelope solitary Rossby waves. Then, based on Jacobi elliptic function expansion methods and trial function methods, the asymptotic solutions of the dissipative NLS equation are obtained. With the help of these solutions, the effect of dissipation and the horizontal component of Coriolis parameter on the evolution of envelope Rossby solitary waves are investigated, respectively. The results indicate that the dissipations influence on the carrier frequency of equatorial envelope Rossby solitary waves and accelerate the fragmentation of wave amplitude, while it has no effect on the propagation speed.

On the other hand, the wave amplitude is also affected by the horizontal component of Coriolis parameter. With the increasing the Coriolis parameter λ, the amplitude of Rossby solitary waves will accelerate decay. The Coriolis parameter also makes a correction to the wave frequency. However, the propagation velocity and the carrier wave number have nothing to do with the Coriolis parameter.

References

Barletti L, Brugnano L, Caccia GF et al (2018) Energy-conserving methods for the nonlinear Schrödinger equation. Appl Math Comput 318:3–18

Belytschko T, Lu YY, Gu L (1994) Element-free Galerkin methods. Int J Numer Methods Eng 37:229–256

Benney DJ (1979) Large amplitude Rossby waves. Stud Appl Math 6:01–10

Benney DJ, Newell AC (1967) The propagation of nonlinear wave envelopes. J Math Phys 46:133–139

Biazar J, Eslami M (2013) A new technique for non-linear two-dimensional wave equations. Sci Iran 20:359–363

Boyd JP (1980) Equatorial solitary waves. Part I: Rossby solitons. J Phys Oceanogr 10:1699–1717

Boyd JP (2018) Nonlinear equatorial waves. Berlin, Heidelberg

Caillol P, Grimshaw RH (2008) Rossby elevation waves in the presence of a critical layer. Stud Appl Math 120:35–64

Ching L, Sui CH, Yang MJ et al (2015) A modeling study on the effects of MJO and equatorial Rossby waves on tropical cyclone genesis over the western North Pacific in June 2004. Dyn Atmos Oceans 2:70–87

Dellar PJ, Salmon R (2005) Shallow water equations with a complete Coriolis force and topography. Phys Fluids 17:106601

Demiray H (2003) An analytical solution to the dissipative nonlinear Schrödinger equation. Appl Math Comput 145:179–184

Eslami M (2014) An efficient method for solving fractional partial differential equations. Thai J Math 12:601–611

Eslami M, Mirzazadeh M (2014) Study of convergence of Homotopy perturbation method for two-dimensional linear Volterra integral equations of the first kind. Int J Comput Sci Math 5:72–80

Fu C, Lu CN, Yang HW (2018) Time–space fractional (2+1) dimensional nonlinear Schrödinger equation for envelope gravity waves in baroclinic atmosphere and conservation laws as well as exact solutions. Adv Differ Equ 1:56

Guo M, Zhang Y, Wang M, Chen Y, Yang H (2018) A new ZK-ILW equation for algebraic gravity solitary waves in finite depth stratified atmosphere and the research of squall lines formation mechanism. Comput Math Appl. https://doi.org/10.1016/j.camwa.2018.02.019

He JH (2003) Homotopy perturbation method: a new nonlinear analytical technique. Appl Math Comput 135:73–79

Hua BL, Moore DW, Le Gentil S (1997) Inertial nonlinear equilibration of equatorial flows. J Fluid Mech 331:345–371

Jeffrey A, Kawahara T (1982) Asymptotic methods in nonlinear wave theory. Applicable mathematics series. Pitman, Boston

Kasahara A (2003) On the nonhydrostatic atmospheric models with inclusion of the horizontal component of the Earth's angular velocity. J Meteorol Soc Jpn 81:935–950

Liu GT, Fan TY (2005) New applications of developed Jacobi elliptic function expansion methods. Phys Lett A 345:161–166

Liu S, Fu Z, Liu S et al (2001) Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys Lett A 289:69–74

Long RR (1964) Solitary waves in the Westerlies. J Atmos Sci 21:197–200

Lu C, Fu C, Yang H (2018) Time-fractional generalized Boussinesq Equation for Rossby solitary waves with dissipation effect in stratified fluid and conservation laws as well as exact solutions. Appl Math Comput 327:104–116

Luo DH (1991) Nonlinear Schrödinger equation in the rotational barotropic atmosphere and atmospheric blocking. Acta Meteorol Sin 5:587–590

Philips NA (1968) Reply to G. Veronis's comments on Phillips (1966). J Atmos Sci 25:1155–1157

Puy M, Vialard J, Lengaigne M et al (2016) Modulation of equatorial Pacific westerly/easterly wind events by the Madden–Julian oscillation and convectively-coupled Rossby waves. Clim Dyn 46:2155–2178

Quispel GRW, Nijhoff FW, Capel HW et al (1984) Bäklund transformations and singular integral equations. Phys A 123:319–359

Raymond WH (2000) Equatorial meridional flows: rotationally induced circulations. Pure Appl Geophys 157:1767–1779

Redekopp LG (1977) On the theory of solitary Rossby waves. J Fluid Mech 82:725–745

Stewart AL, Dellar PJ (2010) Multilayer shallow water equations with complete Coriolis force. Part 1. Derivation on a non-traditional beta-plane. J Fluid Mech 651:387–413

Stewart AL, Dellar PJ (2012) Multilayer shallow water equations with complete Coriolis force. Part 2. Linear plane waves. J Fluid Mech 690:16–50

Tan B, Boyd JP (2000) Coupled-mode envelope solitary waves in a pair of cubic Schrödinger equations with cross modulation: analytical solution and collisions with application to Rossby waves. Chaos Solitons Fractals 11:1113–1129

Triki H, Wazwaz AM (2011) Solitary wave solutions for a K (m, n, p, q + r) equation with generalized evolution. Int J Nonlinear Sci 11:387–395

Triki H, Wazwaz AM (2014) Traveling wave solutions for fifth-order KdV type equations with time-dependent coefficients. Commun Nonlinear Sci Numer Simul 19:404–408

Veronis G (1968) Comments on Phillips's (1966) proposed simplification of the equations of motiofor shallow rotating atmosphere. J Atmos Sci 25:1154–1155

Wadati M (1973) The modified Korteweg-deVries equation. J Phys Soc Jpn 34:1289–1296

Wangsness RK (1970) Comments on "The equations of motion for a shallow rotating atmosphere and the 'traditional approximation'". J Atmos Sci 27:504–506

White AA, Bromley RA (1995) Dynamically consistent, quasi-hydrostatic equations for global models with a complete representation of the Coriolis force. Q J R Meteorol Soc 121:399–418

Yang HW, Wang XR, Yin BS (2014) A kind of new algebraic Rossby solitary waves generated by periodic external source. Nonlinear Dyn 76:1725–1735

Yang HW, Chen X, Guo M et al (2018) A new ZK–BO equation for three-dimensional algebraic Rossby solitary waves and its solution as well as fission property. Nonlinear Dyn 91:2019–2032

Yun-Long S, Hong-Wei Y, Bao-Shu Y et al (2015) Dissipative nonlinear Schrödinger equation with external forcing in rotational stratified fluids and its solution. Commun Theor Phys 64:464–472

Zayed EME, Gepreel KA (2009) The (G'/G)-expansion method for finding traveling wave solutions of nonlinear partial differential equations in mathematical physics. J Math Phys 50:013502