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An inverse problem for an inhomogeneous time-fractional diffusion equation: a regularization method and error estimate

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Abstract

In this paper, we study an inverse problem for an inhomogeneous time-fractional diffusion equation in the one-dimensional real-positive semiaxis domain. Such a problem is obtained from the classical diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative. After we show that the inverse problem is severely ill posed, we apply a modified regularization method based on the solution in the frequency domain to solve the inverse problem. A convergence estimate is also derived. We present two numerical examples to show the efficiency of the method.

Keywords Regularization method · Inverse problem · Caputo fractional derivative · Convergence estimate

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1 Introduction

In this paper, we consider the problem of recovering the temperature from the following inverse problem:

$$\begin{cases} -au_x(x,t) = {}_0D_t^{\gamma} u(x,t) + F(x,t,u(x,t)), & x > 0, t > 0, \\ u(1,t) = g(t), \quad t \geq 0, \\ u(x,0) = \lim_{x \to +\infty} u(x,t) = 0, \end{cases} \tag{1}$$

where a is the constant diffusivity coefficient, $F(x,t,u(x,t))$ is the nonlinear source term and ${}_0D_t^{\gamma} u(x,t)$ is the Caputo time fractional derivative of order $0 < \gamma < 1$ defined by

$${}_0D_t^{\gamma} u(x,t) := \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} \frac{\partial}{\partial \tau} u(x,\tau)\, \mathrm{d}\tau, \tag{2}$$

where Γ is the Gamma function. This was intended to properly handle initial values (Caputo 1967; Chen et al. 2012; Eidelman et al. 2004) since its Laplace transform(LT) $s^{\beta}\tilde{f}(s) - s^{\beta-1}f(0)$ incorporates the initial value in the same way as the first derivative. Here, $\tilde{f}(s)$ is the usual Laplace transform. It is well known that the Caputo derivative has a continuous spectrum (Chen et al. 2012), with eigenfunctions given in terms of the Mittag-Leffler function

$$E_{\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

In fact, it is easy to see that $f(t) = E_{\beta}(-\lambda t^{\beta})$ solves the eigenvalue equation

$$\frac{\partial^{\beta} f(t)}{\partial t^{\beta}} = -\lambda f(t), \quad f(0) = 1$$

for any $\lambda > 0$. This is easily verified by differentiating term-by-term and using the fact that t^p has Caputo derivative $t^{p-\beta}\frac{\Gamma(p+1)}{\Gamma(p+1-\beta)}$ for $p > 0$ and $0 < \beta \leq 1$. $0 < \beta < 1$ is taken for slow diffusion, and is related to the parameter specifying the large-time behavior of the waiting-time distribution function, see Podlubny (1999) and some of the references cited therein.

Recently, there has been a growing interest in inverse problems and regularization methods with fractional derivatives. For example, in Murio (2007), the authors studied the following inverse problem:

$$\begin{cases} u_x(x,t) = -a^{-\frac{1}{2}} \left({}_0^{\mathrm{RL}}D_t^{\frac{1}{2}} u(x,t) \right) + u_{\infty}(\pi a t)^{-\frac{1}{2}}, & x > 0,\ t \geq 0, \\ u(1,t) = f(t), & t > 0, \\ u(0,t) = g(t), & t > 0, \\ -u_x(0,t) = h(t), & t > 0, \end{cases} \tag{3}$$

where a is the diffusivity coefficient, $u_{\infty} = u(x,0) = \lim_{x\to\infty} u(x,t) = 0$, $f(t)$ is a measured data, $q(t)$ and $h(t)$ are unknown functions. After it is proved that inverse problem (3) is ill posed, they use the mollification method to stabilize the inverse problem. A simple algorithm based on space marching mollification techniques is introduced for the numerical solution of the discrete problem. Stability bounds, error estimates and some numerical examples are also presented. In Zheng and Wei (2011c), the authors studied the following inverse problem:

$$\begin{cases} -au_x(x,t) = {}_0D_t^{\gamma}u(x,t), & x>0,\ t>0,\\ u(1,t)=f(t), & t\ge 0,\\ \lim_{x\to+\infty}u(x,t)=u(x,0)=0, & t\ge 0. \end{cases} \tag{4}$$

In this inverse problem, the temperature and heat flux are sought from a measured temperature history at a fixed location inside the body. The difference of inverse problem (4) from (3) is that the order of the time fractional derivative is taken to be γ, not $\frac{1}{2}$ but no source term is considered. After they showed that inverse problem (4) is ill posed, they applied a spectral regularization method to solve the inverse problem using the solution given by the Fourier method. Convergence estimates are also presented under a priori bound assumptions for the exact solution. Inverse problem (4) is also studied in Cheng and Fu (2012) for $a = 1$. In this paper, the authors give a new iteration regularization method to deal with this problem, and error estimates are obtained for a priori and a posteriori parameter choice rules, respectively. In Li et al. (2014), the authors study the same problem as in Cheng and Fu (2012). They give a new dynamic method for choosing a regularization parameter. Using the spectral methods, some convergence rates on the temperature and heat flow are also given.

We now motivate the use of the governing equation in (4) for $\gamma = \frac{1}{2}$. We refer the readers to Oldham and Spanier (1972) for details. Consider the following diffusion equation

$$u_t(\xi,\eta,\zeta,t) = a\nabla^2 u(\xi,\eta,\zeta,t), \tag{5}$$

with

$$u(\xi,\eta,\zeta,0) = u_\infty, \tag{6}$$

where ∇^2 is the Laplacian operator. Three geometries of the boundary allow a reduction from three to one in the number of spatial coordinates for the diffusion equation. Three different values of a geometric factor β are used to characterize these geometries: $\beta = \frac{1}{2}$ for infinite planes, $\beta = 0$ for infinitely long cylinders and $\beta = -\frac{1}{2}$ for spheres. These geometries simplify the Laplacian operator so that (5) becomes

$$u_t(x,t) = au_{xx}(x,t) + \frac{a\big(1-2\beta\big)}{x+R}u_x(x,t), \tag{7}$$

and (6) becomes

$$u(x,0) = \lim_{x\to+\infty} u(x,t) = u_\infty, \tag{8}$$

where x is the spatial coordinate, R is the radius of the surface in the case of cylindrical and spherical coordinates and without significance in planar case. It is now proved that (7) and (8) are represented by

$$u_x(x,t) = -a^{-\frac{1}{2}}\left({}_0^{\mathrm{RL}}D_t^{\frac{1}{2}}u(x,t)\right) + u_\infty(\pi a t)^{-\frac{1}{2}} - \frac{\frac{1}{2}-\beta}{x+R}(u(x,t)-u_\infty), \tag{9}$$

where ${}_0^{\mathrm{RL}}D_t^{\gamma}u(x,t)$ is the Riemann–Liouville time fractional derivative of order $0<\gamma<1$ defined by

$${}_0^{\mathrm{RL}}D_t^{\gamma}u(x,t) = \frac{1}{\Gamma(1-\gamma)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\gamma}}\,\mathrm{d}\tau.$$

We note that the Riemann–Liouville and Caputo fractional derivatives agree when the initial condition is zero. Podlubny (1999) and Kilbas et al. (2006) can be referred for further properties of the Caputo and Riemann–Liouville fractional derivatives.

We note that a very similar inverse problem to (4) is studied in Xiong et al. (2012) for the two-dimensional case. Based on an a priori assumption, the authors derive a conditional stability result. Some new regularization methods are constructed for solving the inverse problem and the corresponding error estimates are also proved in this paper. In many phenomena, diffusion progresses occur in spatially inhomogeneous environments or the slow diffusion progresses require models containing an inhomogeneous or a nonlinear source term. As it is mentioned above, if the geometry of the body is an infinitely long cylinder or a sphere, the nonlinear term on the right-hand side of (9) must be taken into consideration. The inverse problem (1) is studied in Tuan et al. (2016) for just a linear source term. Motivated by this reason, in this paper, we study inverse problem (1). We note that the source term in (1) is a general source term. From this point of view, we generalize Eq. (9) in terms of both the order of the fractional time derivative and source function. As it is known, solving nonlinear problems require many difficult techniques and new ideas to deal the fractional and nonlinear terms. The techniques and methods used for homogeneous case cannot be applied directly to solve problem (1). The exact solution of a nonlinear problem can be represented by a nonlinear integral equation containing some instability terms, see Eq. (11). The leading idea of the method presented in this paper is to find a suitable integral equation for approximating the exact solution. Then we replace instability terms by regularization terms and show that the solution of regularized problem converges to the exact solution. In the homogeneous problem, we have many choices of stability term for regularization. However, in nonlinear problem, the solution u is expressed in complex terms and defined by an integral equation whose right-hand side depends on u. That is why studying with nonlinear problems is very difficult. In this paper, we develop some new techniques to overcome these difficulties.

This paper is organized as follows: in the next section, it is proved that the considered inverse problem is severely ill posed. A regularization method and error estimates are also given. In Sect. 3, we present two numerical examples to show the efficiency of the method.

2 Regularization and error estimate

To use Fourier transform, we extend the functions $u(x,.)$ and $g(.)$ to the whole line $-\infty < t < +\infty$ by defining them to be zero for $t < 0$. The Fourier transform of a function $f(t) \in L^2(\mathbb{R})$ is defined by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{-i\omega t} \mathrm{d}t, \quad -\infty < \omega < +\infty.$$

Since the measurements usually contain an error, we assume that the measured data function $g_\alpha(t) \in L^2(\mathbb{R})$ satisfies $\|g_\alpha - g\|_{L^2(\mathbb{R})} \le \alpha$, where $\alpha > 0$ is a bound on the measurement error. If we take Fourier transformation in (1) with respect to t, by Podlubny (1999), we have

$$\begin{cases} \widehat{u_x}(x,\omega) + \frac{(i\omega)^\gamma}{a}\hat{u}(x,\omega) = \frac{1}{a}\widehat{F}(x,\omega,u(x,\omega)), & x > 0,\ \omega \in \mathbb{R}, \\ \hat{u}(1,\omega) = \hat{g}(\omega), & \omega \ge 0, \\ \lim_{x\to+\infty} \hat{u}(x,\omega) = 0, & \omega \ge 0, \end{cases} \tag{10}$$

where

$$(i\omega)^\gamma = |\omega|^\gamma \cos\frac{\gamma\pi}{2} + i|\omega|^\gamma \mathrm{sign}(\omega) \sin\frac{\gamma\pi}{2},$$

and $\widehat{F}(x,\omega,u(x,\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(x,t,u(x,t))\mathrm{e}^{-i\omega t}\mathrm{d}t$. By solving problem (10), the solution of problem (1) is obtained as follows, see Tuan et al. (2016):

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\Bigg[\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\widehat{g}(\omega) - \frac{1}{a}\int_x^1 \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,u(z,\omega))\mathrm{d}z\Bigg]\mathrm{e}^{i\omega t}\,\mathrm{d}\omega. \tag{11}$$

By (11) and noting that $(i\omega)^\gamma$ has the positive real part $|\omega|^\gamma\cos\frac{\gamma\pi}{2}$, we conclude that both $\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)$ and $\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)$ increases exponentially for $0\le z<x<1$ as $\omega\to+\infty$. So the small distribution for the data $g(t)$ will be amplified infinitely by this factor and lead to blow-up in (11). Therefore, recovering the temperature $u(x,t)$ from the measured data $g_\alpha(t)$ is severely ill posed. We must use some regularization methods to deal with this problem, for example, see Zheng and Wei (2010a, b, 2011a). To regularize the problem, we have to replace the terms $\exp\left(\frac{(i\omega)^\alpha(1-x)}{a}\right)$ and $\exp\left(\frac{(i\omega)^\alpha(z-x)}{a}\right)$ by other terms. For this purpose, we define a regularized solution $u_\varepsilon^\alpha(x,t)$ whose Fourier transform satisfies the following problem:

$$\begin{cases} \widehat{(u_\varepsilon^\alpha)_x}(x,\omega) + \frac{(i\omega)^\gamma}{a}\widehat{u_\varepsilon^\alpha}(x,\omega) = \frac{P(\varepsilon,\omega)}{a}\widehat{F}(x,\omega,u_\varepsilon^\alpha(x,\omega)), & (x,\omega)\in(0,+\infty)\times(0,+\infty),\\ \widehat{u_\varepsilon^\alpha}(1,\omega) = P(\varepsilon,\omega)\widehat{g}_\alpha(\omega), & \omega\in(0,+\infty),\\ \lim_{x\to+\infty}\widehat{u_\varepsilon^\alpha}(x,\omega)=0, & (x,\omega)\in(0,+\infty)\times(0,+\infty), \end{cases} \tag{12}$$

where $\varepsilon=\varepsilon(\alpha)$ is a regularization parameter satisfies $\lim_{\alpha\to 0}\varepsilon(\alpha)=0$ and $P(\varepsilon,\omega) = \frac{1}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}$. Following a similar procedure as above gives the formal solution of problem (12) as

$$u_\varepsilon^\alpha(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{g}_\alpha(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega - \frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1\frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{F}(z,\omega,u_\varepsilon^\alpha(z,\omega))\mathrm{e}^{i\omega t}\,\mathrm{d}z\,\mathrm{d}\omega. \tag{13}$$

Now, we state and prove the main result of the paper.

Theorem 1 *Let $F:\mathbb{R}\times[0,1]\times\mathbb{R}\to\mathbb{R}$. If F satisfies $F(x,t,0)=0$ and*

$$|F(x,t,u_1)-F(x,t,u_2)|\le K|u_1-u_2|, \tag{14}$$

for a constant $K>0$ independent of x,t,u_1,u_2, then problem (12) has a unique solution $u_\varepsilon^\alpha\in C([0,1];L^2(\mathbb{R}))$, where the space $C([0,1];L^2(\mathbb{R}))$ comprises all continuous functions $u:[0,1]\to L^2(\mathbb{R})$. Suppose that problem (1) has a unique solution $u\in C([0,1];L^2(\mathbb{R}))$ that satisfies

$$\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega<\infty. \tag{15}$$

Choose $\varepsilon := \varepsilon(\alpha)$ *such that*

$$\lim_{\alpha \to 0} \varepsilon = \lim_{\alpha \to 0} \varepsilon \alpha^{-1} = 0. \tag{16}$$

Then the following estimate holds for every $x \in (0, 1)$:

$$\|u(x,.) - u_\varepsilon^\alpha(x,.)\|_{L^2(\mathbb{R})} \le \sqrt{3} \exp\left(\frac{3(1-x)^2 K^2}{2a^2}\right) \times \left[\sqrt{\sup_{0 \le x \le 1} \int_{-\infty}^{+\infty} \exp\left(\frac{2}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2} x\right) |\widehat{u}(x,\omega)|^2 \mathrm{d}\omega + \varepsilon^{-1}\alpha}\right] \varepsilon^x. \tag{17}$$

Before we prove Theorem 1, we prove the following useful lemma.

Lemma 1 *Let* $\gamma \in (0, 1)$, $x \in [0, 1]$ *and* $\omega \in \mathbb{R}$. *If* $z \in [x, 1]$, *then we have*:

$$\left| \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1 + \varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \right| \le \varepsilon^{x-z}, \quad \left| \frac{\exp\left(\frac{(i\omega)^\gamma (1-x)}{a}\right)}{1 + \varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \right| \le \varepsilon^{x-1}. \tag{18}$$

Proof Since the modulus of the complex number $\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)$ is $\exp\left(\frac{z-x}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)$, by simple calculations, we obtain

$$\left| \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1 + \varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \right| = \frac{\exp\left(\frac{z-x-1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}{\varepsilon + \exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}$$
$$= \left(\frac{1}{\varepsilon + \exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}\right)^{z-x} \left(\frac{\exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}{\varepsilon + \exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}\right)^{1-z+x}.$$

Since $\frac{1}{\varepsilon + \exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \le \frac{1}{\varepsilon}$ and $\frac{\exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)}{\varepsilon + \exp\left(-\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \le 1$, we deduce that

$$\left| \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1 + \varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \right| \le \varepsilon^{x-z}.$$

By letting $z = 1$ in the above inequality, we obtain the second inequality. The proof is complete. □

Proof of Theorem 1 We divide the proof into two steps:

Step 1 First, we prove that problem (12) has a unique solution $u_\varepsilon^\alpha \in C([0, 1]; L^2(\mathbb{R}))$. For this purpose, we define the following function for $W \in C([0, 1]; L^2(\mathbb{R}))$:

$$G(W)(x,t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \frac{\exp\left(\frac{(i\omega)^\gamma (1-x)}{a}\right)}{1+\varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \widehat{g}_\alpha(\omega) \mathrm{e}^{i\omega t}\, \mathrm{d}\omega$$
$$- \frac{1}{a\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \int\limits_{x}^{1} \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1+\varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \widehat{F}(z,\omega, W(z,\omega)) \mathrm{e}^{i\omega t}\, \mathrm{d}z\, \mathrm{d}\omega. \tag{19}$$

By $F(x,t,0) = 0$ and (14), we have $G(W) \in C([0,1]; L^2(\mathbb{R}))$ for every $W \in C([0,1]; L^2(\mathbb{R}))$. Now, we show that if $W_1, W_2 \in C([0,1]; L^2(\mathbb{R}))$, then the following estimate holds for $m \geq 1$:

$$\|G^m(W_1)(x,.) - G^m(W_2)(x,.)\|^2_{L^2(\mathbb{R})} \leq \left(\frac{K}{a\varepsilon}\right)^{2m} \frac{(1-x)^m}{m!} |||W_1 - W_2|||^2, \tag{20}$$

where $|||.|||$ is the sup norm in $C([0,1]; L^2(\mathbb{R}))$ defined by

$$\|w\|_{C([0,1];L^2(\mathbb{R}))} = \sup_{0\leq x\leq 1} \|w(x,.)\|_{L^2(\mathbb{R})}. \tag{21}$$

We prove (20) by induction. For $m = 1$, by (14) and Parseval equality, we obtain

$$\|G(W_1)(x,.) - G(W_2)(x,.)\|^2_{L^2(\mathbb{R})} = \|\widehat{G}(W_1)(x,.) - \widehat{G}(W_2)(x,.)\|^2_{L^2(\mathbb{R})}$$
$$= \frac{1}{a^2} \int\limits_{-\infty}^{+\infty} \left| \int\limits_{x}^{1} \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1+\varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \left(\widehat{F}(z,\omega,W_1) - \widehat{F}(z,\omega,W_2)\right) \mathrm{d}z \right|^2 \mathrm{d}\omega$$
$$\leq \frac{1}{a^2} \int\limits_{-\infty}^{+\infty} \left(\int\limits_{x}^{1} \left| \frac{\exp\left(\frac{(i\omega)^\gamma (z-x)}{a}\right)}{1+\varepsilon \exp\left(\frac{1}{a}|\omega|^\gamma \cos\frac{\gamma\pi}{2}\right)} \right|^2 \mathrm{d}z \int_x^1 \left|\widehat{F}(z,\omega,W_1) - \widehat{F}(z,\omega,W_2)\right|^2 \mathrm{d}z \right) \mathrm{d}\omega$$
$$\leq \frac{1-x}{(a\varepsilon)^2} \int\limits_{x}^{1} \|\widehat{F}(z,.,W_1(z,.)) - \widehat{F}(z,.,W_2(z,.))\|^2_{L^2(\mathbb{R})} \mathrm{d}z \leq \frac{K^2(1-x)}{(a\varepsilon)^2} |||W_1 - W_2|||^2. \tag{22}$$

Assuming that (20) holds for $m \in \mathbb{Z}^+$, it can easily be proven that (20) holds for $m+1$. Since this procedure is quite similar to the case $m = 1$, we omit the details. We consider $G : C([0,1]; L^2(\mathbb{R})) \to C[0,1]; L^2(\mathbb{R}))$. Since (20), we obtain

$$\|G^m(W_1)(x,.) - G^m(W_2)(x,.)\|^2_{L^2(\mathbb{R})} \leq \left(\frac{K}{a\varepsilon}\right)^{2m} \frac{1}{m!} |||W_1 - W_2|||^2. \tag{23}$$

Noting that the right-hand side of (23) does not depend on x, we deduce that

$$|||G^m(W_1) - G^m(W_2)|||^2 = \sup_{0\leq x\leq 1} \|G^m(W_1)(x,.) - G^m(W_2)(x,.)\|^2_{L^2(\mathbb{R})}$$
$$\leq \left(\frac{K}{a\varepsilon}\right)^{2m} \frac{1}{m!} |||W_1 - W_2|||^2. \tag{24}$$

Since $\lim_{m\to\infty} \sqrt{\left(\frac{K}{a\varepsilon}\right)^{2m} \frac{1}{m!}} = 0$, there exists a positive integer m_0 such that G^{m_0} is a contraction. It follows that $G^{m_0}(W) = W$ has a unique solution $u^\alpha_\varepsilon \in C([0,1]; L^2(\mathbb{R}))$. We claim that $G(u_\varepsilon^\alpha) = u_\varepsilon^\alpha$. In fact, one has $G(G^{m_0}(u_\varepsilon^\alpha)) = G(u_\varepsilon^\alpha)$. Hence, $G^{m_0}(G(u_\varepsilon^\alpha)) = G(u_\varepsilon^\alpha)$. By the uniqueness of the fixed point of G^{m_0}, one has $G(u_\varepsilon^\alpha) = u_\varepsilon^\alpha$, i.e, the equation $G(u_\varepsilon^\alpha) = u_\varepsilon^\alpha$ has a unique solution $u_\varepsilon^\alpha \in C([0,1]; L^2(\mathbb{R}))$.

Step 2 We estimate the error $\|u_\varepsilon^\alpha - u\|_{L^2(\mathbb{R})}$. By (11) and (13), we deduce that

$$
\begin{aligned}
&\widehat{u}(x,.) - \widehat{u_\varepsilon^\alpha}(x,.)\\
&= \left[\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right) - \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\right]\widehat{g}(\omega)\\
&\quad + \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}(\widehat{g}(\omega) - \widehat{g}_\alpha(\omega))\\
&\quad - \frac{1}{a}\int_x^1 \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,u(z,\omega))\,\mathrm{d}z\\
&\quad + \frac{1}{a}\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\widehat{F}(z,\omega,w_\varepsilon^\alpha(z,\omega))\,\mathrm{d}z\\
&= \frac{\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\Bigg[\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\widehat{g}(\omega)\\
&\quad - \frac{1}{a}\int_x^1 \exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,u(z,\omega))\,\mathrm{d}z\Bigg]\\
&\quad + \frac{1}{a}\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\Big(\widehat{F}(z,\omega,w_\varepsilon^\alpha(z,\omega)) - \widehat{F}(z,\omega,u(z,\omega))\Big)\mathrm{d}z\\
&\quad + \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}(\widehat{g}(\omega) - \widehat{g}_\alpha(\omega)).
\end{aligned}
\tag{25}
$$

By Parseval equality, we have

$$
\begin{aligned}
&\|u(x,.) - u_\varepsilon^\alpha(x,.)\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{+\infty} |\widehat{u}(x,.) - \widehat{u_\varepsilon^\alpha}(x,.)|^2\,\mathrm{d}\omega\\
&\le \underbrace{3\int_{-\infty}^{+\infty}\left|\frac{\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\right|^2\left[\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\widehat{g}(\omega) - \frac{1}{a}\int_x^1\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,u(z,\omega))\,\mathrm{d}z\right]^2\mathrm{d}\omega}_{:=\tilde{J}_1(x)}\\
&\quad + \underbrace{\frac{3}{a^2}\int_{-\infty}^{+\infty}\left|\int_x^1 \frac{\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\Big(\widehat{F}(z,\omega,w_\varepsilon^\alpha(z,\omega)) - \widehat{F}(z,\omega,u(z,\omega))\Big)\mathrm{d}z\right|^2\mathrm{d}\omega}_{:=\tilde{J}_2(x)}\\
&\quad + \underbrace{3\int_{-\infty}^{+\infty}\left|\frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}(\widehat{g}_\alpha(\omega) - \widehat{g}(\omega))\right|^2\mathrm{d}\omega}_{:=\tilde{J}_3(x)} = \tilde{J}_1(x) + \tilde{J}_2(x) + \tilde{J}_3(x).
\end{aligned}
\tag{26}
$$

We now estimate $\tilde{J}_1(x)$, $\tilde{J}_2(x)$ and $\tilde{J}_3(x)$. $\tilde{J}_1(x)$ is estimated as follows:

$$\begin{aligned}\tilde{J}_1(x) &= 3\int_{-\infty}^{+\infty}\left|\frac{\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}(1-x)\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\right|^2\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega\\ &\le 3\varepsilon^{2x}\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega, \qquad (27)\end{aligned}$$

where we have used the facts that

$$\widehat{u}(x,\omega)=\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\widehat{g}(\omega)-\frac{1}{a}\int_x^1\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\widehat{F}(z,\omega,u(z,\omega))\,\mathrm{d}z,$$

and

$$\left|\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)\right|=\left|\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}(1-x)\right)\right|\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right).$$

By Holder's inequality and noting that $\left|\exp\left(\frac{(i\omega)^\gamma(z-x)}{a}\right)\right| = \exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}(z-x)\right)$, we estimate $\tilde{J}_2(x)$ as follows:

$$\begin{aligned}\tilde{J}_2(x) \le&\frac{3(1-x)}{a^2}\int_{-\infty}^{+\infty}\int_x^1\left|\frac{\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}(z-x)\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}\right.\\ &\left.\times\left(\widehat{F}(z,\omega,w_\varepsilon^\alpha(z,\omega))-\widehat{F}(z,\omega,u(z,\omega))\right)\mathrm{d}z\right|^2\mathrm{d}\omega\\ \le&\frac{3(1-x)K^2}{a^2}\int_{-\infty}^{+\infty}\int_x^1\varepsilon^{2x-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2\,\mathrm{d}z\,\mathrm{d}w. \qquad (28)\end{aligned}$$

By Lemma 1, $\tilde{J}_3(x)$ is estimated as follows:

$$\tilde{J}_3(x)\le 3\varepsilon^{2(x-1)}\int_{-\infty}^{+\infty}|\widehat{g_\alpha}(\omega)-\widehat{g}(\omega)|^2\,\mathrm{d}\omega\le 3\varepsilon^{2(x-1)}\|g_\alpha-g\|^2_{L^2(\mathbb{R})}\le 3\varepsilon^{2(x-1)}\alpha^2. \qquad (29)$$

Combining (26)–(29), we conclude that

$$\begin{aligned}\|u(x,.)-u_\varepsilon^\alpha(x,.)\|^2_{L^2(\mathbb{R})}\le&\ 3\varepsilon^{2(x-1)}\alpha^2\\ &+3\varepsilon^{2x}\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega\\ &+\frac{3(1-x)K^2}{a^2}\int_{-\infty}^{+\infty}\int_x^1\varepsilon^{2x-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2\,\mathrm{d}z\,\mathrm{d}w. \qquad (30)\end{aligned}$$

The last inequality implies that

$$
\begin{aligned}
&\varepsilon^{-2x}\|u(x,.)-u_\varepsilon^\alpha(x,.)\|^2_{L^2(\mathbb{R})}\\
&\le 3\varepsilon^{-2}\alpha^2+3\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega\\
&+\frac{3(1-x)K^2}{a^2}\int_{-\infty}^{+\infty}\int_x^1\varepsilon^{-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2\,\mathrm{d}z\,\mathrm{d}w\\
&=3\varepsilon^{-2}\alpha^2+3\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega\\
&+\frac{3(1-x)K^2}{a^2}\int_x^1\varepsilon^{-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}z,
\end{aligned}
\tag{31}
$$

where we applied Fubini's theorem to get

$$
\int_{-\infty}^{+\infty}\int_x^1\varepsilon^{-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2\,\mathrm{d}z\,dw=\int_x^1\varepsilon^{-2z}\|u_\varepsilon^\alpha(z,.)-u(z,.)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}z.
$$

Using the Gronwall's inequality in (31), we obtain that

$$
\begin{aligned}
&\varepsilon^{-2x}\|u(x,.)-u_\varepsilon^\alpha(x,.)\|^2_{L^2(\mathbb{R})}\\
&\le 3\exp\left(\frac{3(1-x)^2K^2}{a^2}\right)\left[\varepsilon^{-2}\alpha^2+\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega\right].
\end{aligned}
\tag{32}
$$

Multiplying both sides of (32) by ε^{2x} and using the inequality $\sqrt{a_1+a_2}\le\sqrt{a_1}+\sqrt{a_2}$ for any nonnegative real numbers a_1, a_2, we complete the proof. □

Remark 1 The solution to Problem 1 for the homogeneous source, i.e $F(x,t,u)=0$ is given by

$$
u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\widehat{g}(\omega)\right]\mathrm{e}^{i\omega t}\,\mathrm{d}\omega\cdot \tag{33}
$$

To regularize solution (33), we give a general regularized solution by finding the term $\mathbf{R}_\alpha(\epsilon,\omega,x)$ such that

- there exists a function $\mathbf{\Gamma}(\alpha)>\mathbf{0}$ such that $\lim_{\alpha\to 0}\mathbf{\Gamma}(\alpha)=+\infty$ and $\mathbf{R}_\alpha(\epsilon,\omega,x)\le\mathbf{\Gamma}(\alpha)$;
- $\lim_{\varepsilon\to 0}\mathbf{R}_\alpha(\epsilon,\omega,x)=\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)$.

Then define the following regularized solution:

$$
W_\varepsilon^\alpha(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}[\mathbf{R}_\alpha(\epsilon,\omega,x)\widehat{g}_\alpha(\omega)]\mathrm{e}^{i\omega t}\,\mathrm{d}\omega\cdot \tag{34}
$$

- We obtain the regularized solution found in this paper for $\mathbf{R}_\alpha^{(1)}(\epsilon,\omega,x) = \frac{\exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)}{1+\varepsilon\exp\left(\frac{1}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}\right)}$.
- We obtain another regularized solution found in Zheng and Wei (2011b) for $\mathbf{R}_\alpha^{(2)}(\epsilon,\omega,x) = \exp\left(\frac{1}{a}\frac{(i\omega)^\gamma(1-x)}{1+\varepsilon\omega^2}\right)$.
- We obtain another regularized solution found in Zheng and Wei (2011c) for $\mathbf{R}_\alpha^{(3)}(\epsilon,\omega,x) = \exp\left(\frac{(i\omega)^\gamma(1-x)}{a}\right)\chi_{\max}$, where $\chi_{\max}$ is the characteristic function of the interval $[-\omega_{\max},\omega_{\max}]$.

We obtain the regularized solution to problem (1) using $\mathbf{R}_\alpha(\epsilon,\omega,x)$ as follows:

$$\begin{aligned} U_\varepsilon^\alpha(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathbf{R}_\alpha(\epsilon,\omega,x)\widehat{g}_\alpha(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega \\ &\quad-\frac{1}{a\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_x^1\mathbf{R}_\alpha(\epsilon,\omega,z)\widehat{F}(z,\omega,U_\varepsilon^\alpha(z,\omega))\mathrm{e}^{i\omega t}\,\mathrm{d}z\,\mathrm{d}\omega\cdot \end{aligned}$$

We note that the analysis presented in this paper for the inhomogeneous problem is based on Lemma 1. However, some terms in $\mathbf{R}_\alpha(\epsilon,\omega,x)$ may not generally satisfy Lemma 1. This implies that the method for the homogeneous problem cannot be applied for the inhomogeneous problem.

Remark 2 Convergence estimate (17) is a Hölder-type estimate and it is proved for $x\in(0,1)$. We note that we have a faster convergence rate for $x>1$. If $x=0$, we also need a convergence estimate. For this purpose, we assume that

$$\sup_{0\le z\le 1}\|u_z(z,.)\|_{L^2(\mathbb{R})} = P<\infty,\quad P>0.$$

Using (17), we have

$$\begin{aligned} \|u(0,.)-u_\varepsilon^\alpha(x,.)\|_{L^2(\mathbb{R})} &\le \|u(0,.)-u(x,.)\|_{L^2(\mathbb{R})}+\|u(x,.)-u_\varepsilon^\alpha(x,.)\|_{L^2(\mathbb{R})} \\ &\le x\sup_{0\le z\le 1}\|u_z(z,.)\|_{L^2(\mathbb{R})}+Q\varepsilon^x\le Px+Q\varepsilon^x, \end{aligned}$$

where

$$\begin{aligned} Q &= \sqrt{3}\exp\left(\frac{3(1-x)^2K^2}{2a^2}\right) \\ &\quad\times\left[\sqrt{\sup_{0\le x\le 1}\int_{-\infty}^{+\infty}\exp\left(\frac{2}{a}|\omega|^\gamma\cos\frac{\gamma\pi}{2}x\right)|\widehat{u}(x,\omega)|^2\,\mathrm{d}\omega+\varepsilon^{-1}\alpha}\,\right]. \end{aligned}$$

Following exactly the method given in Trong et al. (2007), we conclude that

$$\|u(0,.)-u_\varepsilon^\alpha(x_\varepsilon,.)\|_{L^2(\mathbb{R})}\le(P+Q)\sqrt{\frac{1}{\ln\left(\frac{1}{\varepsilon}\right)}}.$$

Table 1 The relative errors for $\gamma = 0.1$

x, α	$\alpha = 0.1086$	$\alpha = 0.05$	$\alpha = 0.01$
$x = 0.15$	1.16E+00	9.20E−01	7.55E−01
$x = 0.25$	1.04E+00	8.14E−01	6.60E−01
$x = 0.35$	9.20E−01	7.08E−01	5.63E−01
$x = 0.45$	8.08E−01	6.09E−01	4.74E−01
$x = 0.55$	7.01E−01	5.15E−01	3.89E−01
$x = 0.65$	6.12E−01	4.37E−01	3.18E−01
$x = 0.75$	5.41E−01	3.75E−01	2.62E−01
$x = 0.85$	4.99E−01	3.38E−01	2.28E−01
$x = 0.95$	4.89E−01	3.29E−01	2.20E−01

3 Numerical experiments

In this section, we examine the regularization method given in this paper with two problems. Both examples are considered in the following form:

$$\begin{cases} -au_x(x,t) = {}_0D_t^\gamma u(x,t) + u + G(x,t), & x > 0,\ t > 0, \\ u(1,t) = g(t), & t \geq 0, \\ u(x,0) = \lim_{x\to+\infty} u(x,t) = 0. \end{cases} \tag{35}$$

The examples are solved for $x \in (0,1)$, $t \in (0, 2\pi)$ and $a = 1.2$. We form the noisy data in the following way:

$$g_\alpha(t) = g(t) + \alpha\,(2\,\mathrm{rand}(\mathrm{size}(.) - 1),$$

where α is the error level and rand(size(.)) is a random number in $[-1, 1]$. We generate spatial and temporal discretizations by $x_i = i\Delta x$, $\Delta x = \frac{1}{M}$, $i = \overline{0, M}$ and $t_j = j\Delta t$, $\Delta t = \frac{2\pi}{N}$, $j = \overline{0, N}$, respectively. We take $M = N = 256$ and $a = 1.2$ in the numerical examples below. The discrete Fourier method is used to find the approximation of the regularized solution. We also note that the nonlinear term in the regularized solution is controlled by using Gauss–Legendre quadrature method given in Press et al. (1996). Errors between the exact and its regularized solutions are estimated by the relative error estimation defined by

$$E(x) = \frac{\left(\sum_{j=0}^N |U^\epsilon(:, t_j) - u(:, t_j)|^2\right)^{1/2}}{\left(\sum_{j=0}^N |u(:, t_j)|^2\right)^{1/2}}.$$

Example 1 Consider inverse problem (35) for

$$\begin{cases} G(x,t) = ax^2 \exp\left(-\frac{x^3}{3}\right) t^4 - \exp\left(-\frac{x^3}{3}\right) \frac{24t^{4-\gamma}}{\Gamma(5-\gamma)} - \exp\left(-\frac{x^3}{3}\right) t^4, \\ g(t) = \exp\left(-\frac{1}{3}\right) t^4. \end{cases} \tag{36}$$

By noting $G(x,t) = -au_x(x,t) -{}_0 D_t^\gamma u(x,t) - u$, ${}_0D_t^\gamma t^4 = \frac{24t^{4-\gamma}}{\Gamma(5-\gamma)}$ and $u_x(x,t) = -x^2t^4 \exp(-\frac{x^3}{3})$, it is easy to check that $u(x,t) = t^4 \exp(-\frac{x^3}{3})$ is the solution of (35) for $G(x,t)$ and $g(t)$ given in (36). Tables 1, 2 and 3 show the relative errors for $\gamma = 0.1$, $\gamma = 0.45$ and $\gamma = 0.9$, respectively. It is very clear that the regularized solution converges to the exact solution with different values of γ. The numerical results are also shown in Figs. 1, 2, 3, 4,

Table 2 The relative errors for $\gamma = 0.45$

x, α	$\alpha = 0.1086$	$\alpha = 0.05$	$\alpha = 0.01$
$x = 0.15$	1.29E+00	1.01E+00	8.18E−01
$x = 0.25$	1.13E+00	8.77E−01	7.05E−01
$x = 0.35$	9.77E−01	7.49E−01	5.93E−01
$x = 0.45$	8.37E−01	6.31E−01	4.91E−01
$x = 0.55$	7.06E−01	5.22E−01	3.96E−01
$x = 0.65$	6.00E−01	4.33E−01	3.19E−01
$x = 0.75$	5.17E−01	3.63E−01	2.59E−01
$x = 0.85$	4.67E−01	3.22E−01	2.24E−01
$x = 0.95$	4.54E−01	3.13E−01	2.16E−01

Table 3 The relative error for $\gamma = 0.9$

x, α	$\alpha = 0.1086$	$\alpha = 0.05$	$\alpha = 0.01$
$x = 0.15$	9.71E−01	8.34E−01	7.41E−01
$x = 0.25$	8.76E−01	7.47E−01	6.59E−01
$x = 0.35$	7.70E−01	6.49E−01	5.67E−01
$x = 0.45$	6.63E−01	5.52E−01	4.76E−01
$x = 0.55$	5.57E−01	4.55E−01	3.86E−01
$x = 0.65$	4.68E−01	3.74E−01	3.10E−01
$x = 0.75$	3.97E−01	3.09E−01	2.49E−01
$x = 0.85$	3.55E−01	2.71E−01	2.14E−01
$x = 0.95$	3.46E−01	2.63E−01	2.06E−01

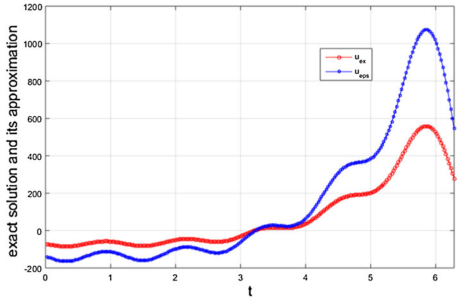
5 and 6 for different values of γ, α and x. From these Figures, it can be easily observed that the numerical results near the boundary $x = 1$ are better than the ones around $x = 0$. That is why the boundary condition is given analytically at $x = 1$ with only small measurement errors. From Tables 1, 2 and 3, we can see that the error for $\alpha = 0.01$ is less than the errors for $\alpha = 0.05$ and $\alpha = 0.1086$. Moreover, we cannot see any relation between numerical accuracy and the order γ of Caputo fractional derivative as in Zheng and Wei (2011c). The reason is that the right-hand side of (17) may be a decreasing or an increasing function of γ in some intervals.

Example 2 Consider inverse problem (35) for

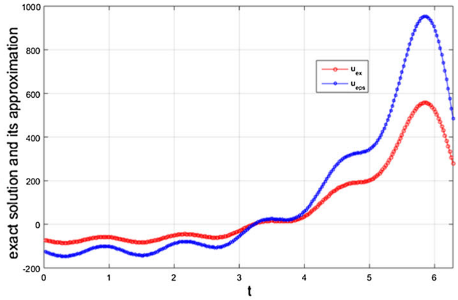
$$\begin{cases} G(x,t) = \dfrac{\gamma t^{\gamma+1}}{\Gamma(1-\gamma)}\left(\dfrac{a}{(x+1)^2} - \dfrac{1}{(x+1)}\right) - \dfrac{\gamma(\gamma+1)!t}{(x+1)\Gamma(2)\Gamma(1-\gamma)}, \\ g(t) = \dfrac{\gamma t^{\gamma+1}}{2\Gamma(1-\gamma)}. \end{cases} \tag{37}$$

By noting that ${}_0D_t^{\gamma} u = \dfrac{\gamma t(\gamma+1)!}{(x+1)\Gamma(1-\gamma)\Gamma(2)}$, $u_x(x,t) = -\dfrac{\gamma t^{\gamma+1}}{\Gamma(1-\gamma)(x+1)^2}$, $u(x,0) = 0$ and

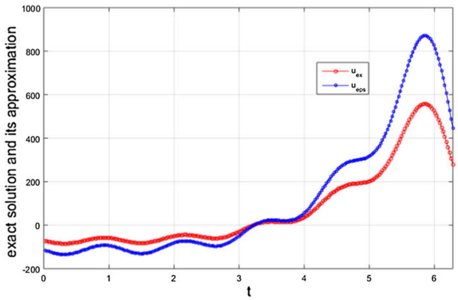
$$\lim_{x\to+\infty} u(x,t) = \lim_{x\to+\infty} \frac{\gamma t^{\gamma+1}}{(x+1)\Gamma(1-\gamma)} = 0,$$



(a) $\alpha = 0.1086$

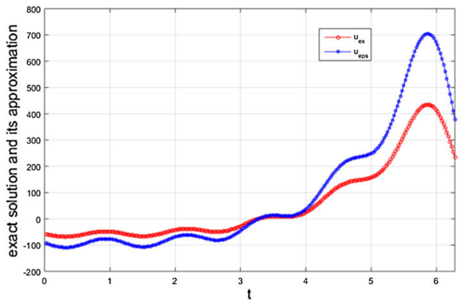


(b) $\alpha = 0.05$

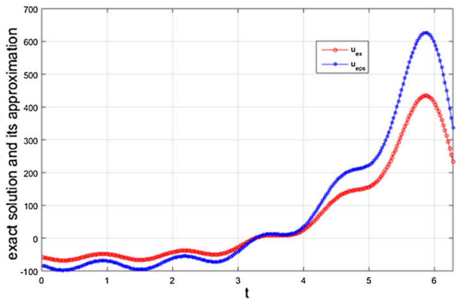


(c) $\alpha = 0.01$

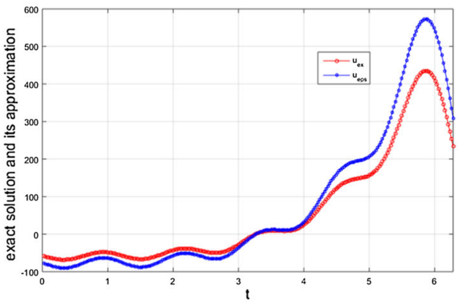
Fig. 1 The exact and regularized solutions for $\gamma = 0.1$ and $x = 0.35$



(a) $\alpha = 0.1086$

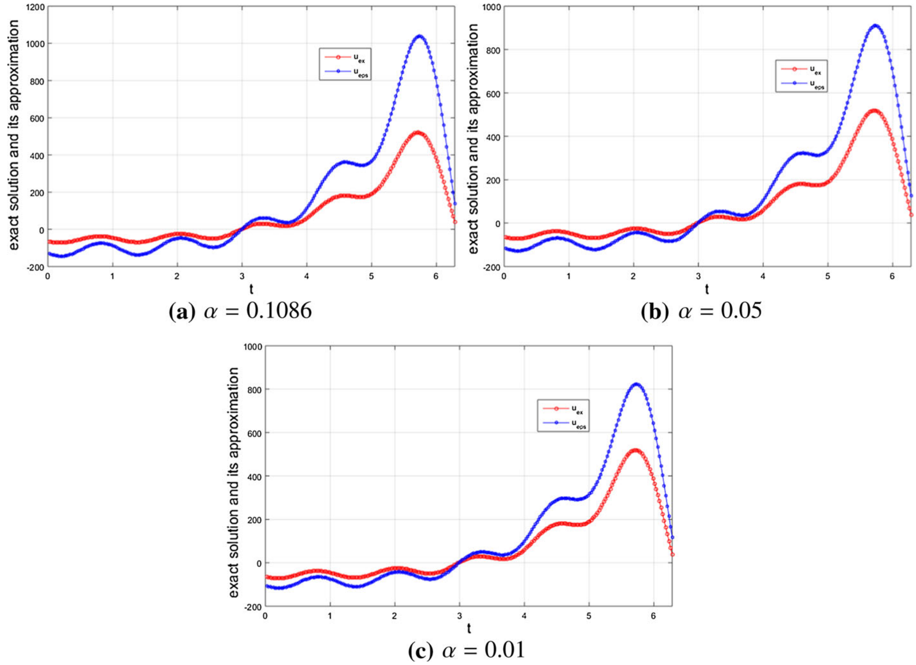


(b) $\alpha = 0.05$



(c) $\alpha = 0.01$

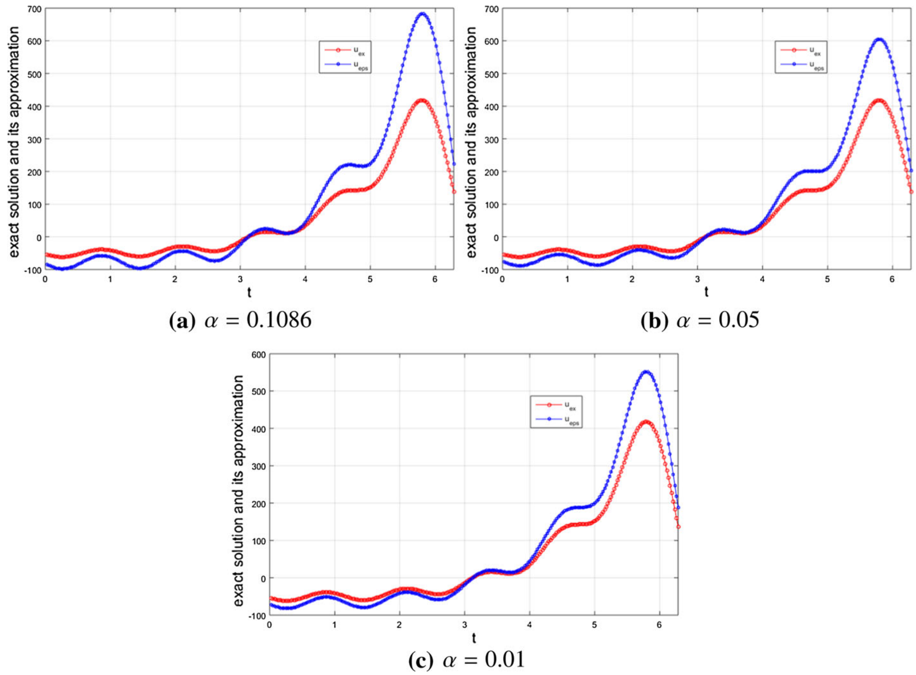
Fig. 2 The exact and regularized solutions for $\gamma = 0.1$ and $x = 0.65$



(a) $\alpha = 0.1086$ **(b)** $\alpha = 0.05$

(c) $\alpha = 0.01$

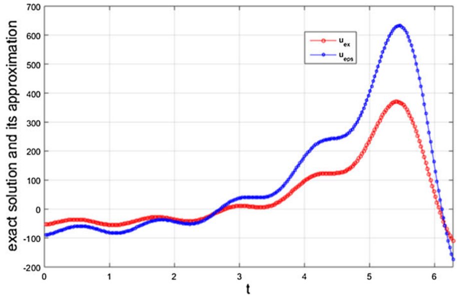
Fig. 3 The exact and regularized solutions for $\gamma = 0.45$ and $x = 0.35$



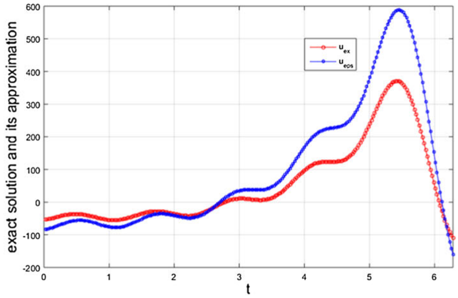
(a) $\alpha = 0.1086$ **(b)** $\alpha = 0.05$

(c) $\alpha = 0.01$

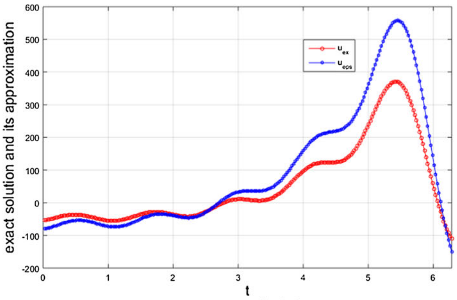
Fig. 4 The exact and regularized solutions for $\gamma = 0.45$ and $x = 0.65$



(a) $\alpha = 0.1086$

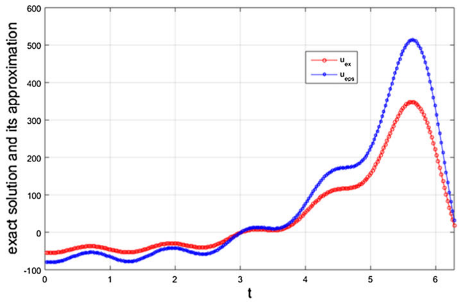


(b) $\alpha = 0.05$

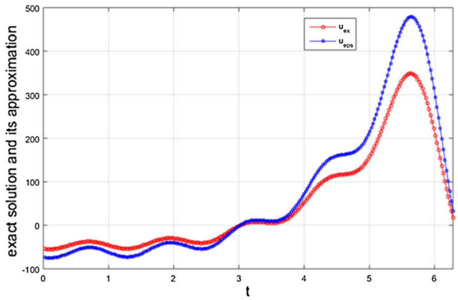


(c) $\alpha = 0.01$

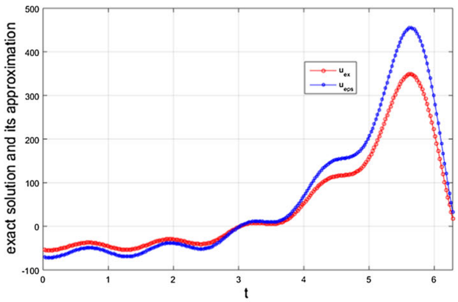
Fig. 5 The exact and regularized solutions for $\gamma = 0.9$ and $x = 0.35$



(a) $\alpha = 0.1086$



(b) $\alpha = 0.05$



(c) $\alpha = 0.01$

Fig. 6 The exact and regularized solutions for $\gamma = 0.9$ and $x = 0.65$

Table 4 The relative errors between exact solution and the regularized solution for $\gamma = 0.1$

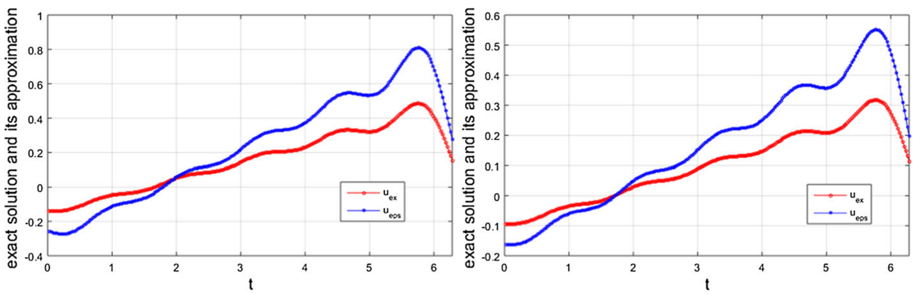
x, α	$\alpha = 0.2086$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
$x = 0.11$	0.7635317	0.252512587	0.183867595	0.152466191
$x = 0.33$	0.788320251	0.281986478	0.184129443	0.148419919
$x = 0.55$	0.806174486	0.240119711	0.185342757	0.1419990451
$x = 0.77$	0.574524471	0.210412969	0.156846767	0.157654904
$x = 0.99$	1.354317	0.195685175	0.229835052	0.13844801

Table 5 The relative errors between exact solution and the regularized solution for $\gamma = 0.45$

x, α	$\alpha = 0.2086$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
$x = 0.11$	0.608790934	0.235966663	0.168524039	0.143631154
$x = 0.33$	0.631142913	0.250958095	0.170172446	0.144866765
$x = 0.55$	0.687332575	0.237741415	0.171604018	0.145714397
$x = 0.77$	0.593704729	0.240012391	0.161175947	0.140351604
$x = 0.99$	0.584578057	0.243626184	0.172139829	0.138123937

Table 6 The relative errors between exact solution and the regularized solution for $\gamma = 0.9$

x, α	$\alpha = 0.2086$	$\alpha = 0.05$	$\alpha = 0.02$	$\alpha = 0.01$
$x = 0.11$	0.608790934	0.235966663	0.168524039	0.143631154
$x = 0.33$	0.631142913	0.250958095	0.170172446	0.144866765
$x = 0.55$	0.687332575	0.237741415	0.171604018	0.145714397
$x = 0.77$	0.593704729	0.240012391	0.161175947	0.140351604
$x = 0.99$	0.584578057	0.243626184	0.172139829	0.138123937



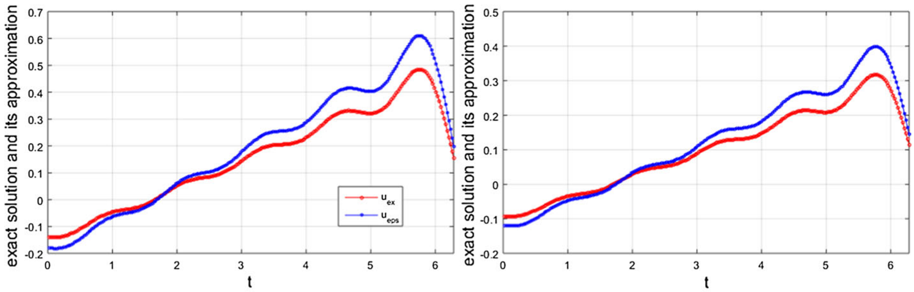
(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 7 The exact and regularized solutions for $\gamma = 0.1$ and $\alpha = 0.2086$

it is easy to check that the following function is the solution to (35):

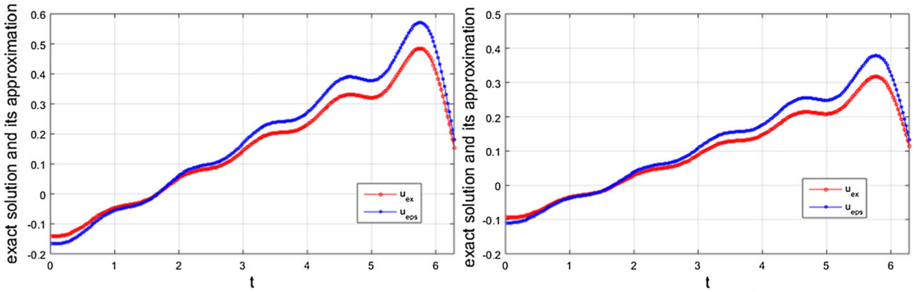
$$u(x,t) = \frac{\gamma t^{\gamma+1}}{(x+1)\Gamma(1-\gamma)}. \tag{38}$$

The difference of this example from the first example is that the solution $u(x,t)$ depends on γ as it is seen in (38). Similar to Example 1, it is clear that the regularized solution



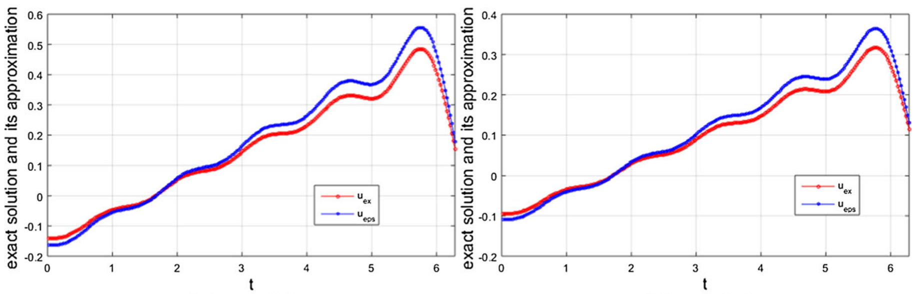
(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 8 The exact and regularized solutions for $\gamma = 0.1$ and $\alpha = 0.05$



(a) $x = 0.3$ **(b)** $x = 0.6$

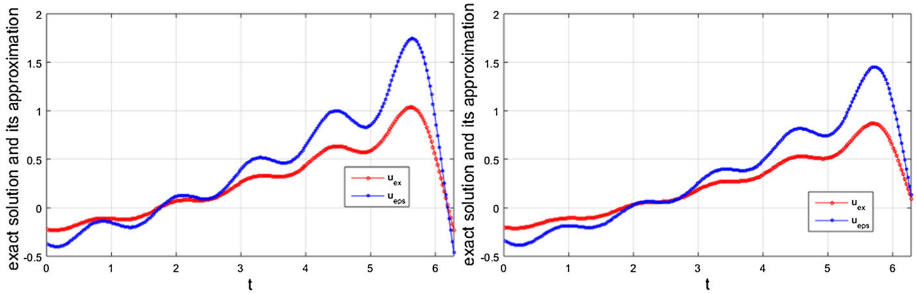
Fig. 9 The exact and regularized solutions for $\gamma = 0.1$ and $\alpha = 0.02$



(a) $x = 0.3$ **(b)** $x = 0.6$

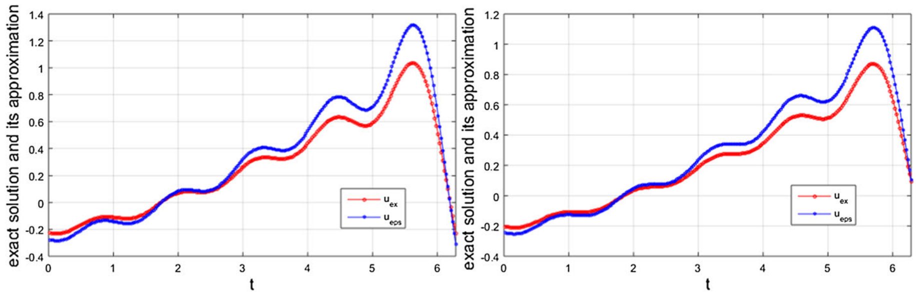
Fig. 10 The exact and regularized solutions for $\gamma = 0.1$ and $\alpha = 0.01$

converges to the exact solution with different values of γ. We also observe that the errors are generally bigger than the ones in Example 1. That is why the solution depends on γ. In contrast to Example 1, it can be easily seen that if γ increases, the numerical accuracy decreases. Tables 4, 5 and 6 show the relative errors for $\gamma = 0.1$, $\gamma = 0.45$ and $\gamma = 0.9$, with parameter regularization $\alpha = 0.2086$, $\alpha = 0.05$, $\alpha = 0.02$ and $\alpha = 0.01$, respectively. The numerical results are also shown in Figs. 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17 and 18 for different values of γ, α and x.



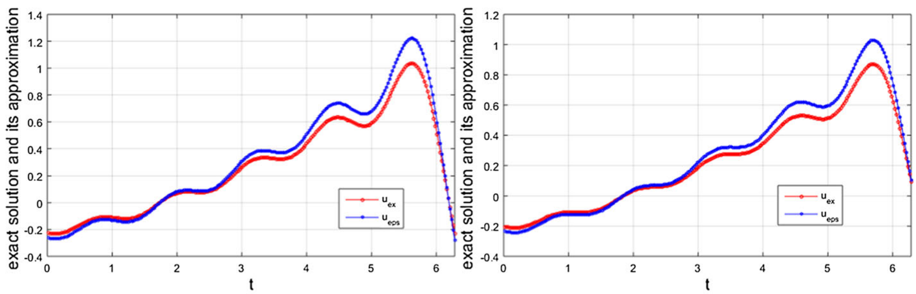
(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 11 The exact and regularized solutions for $\gamma = 0.45$ and $\alpha = 0.2086$



(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 12 The exact and regularized solutions for $\gamma = 0.45$ and $\alpha = 0.05$

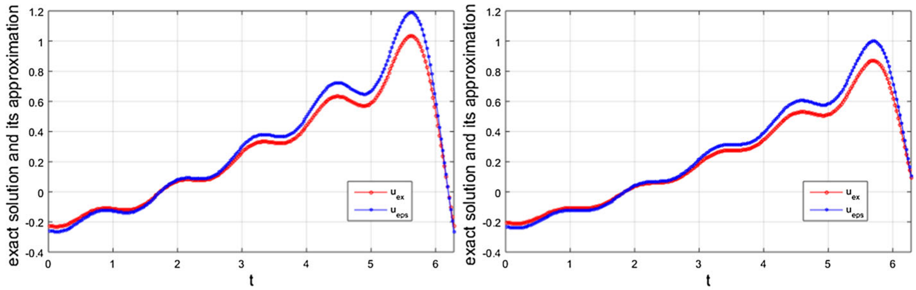


(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 13 The exact and regularized solutions for $\gamma = 0.45$ and $\alpha = 0.02$

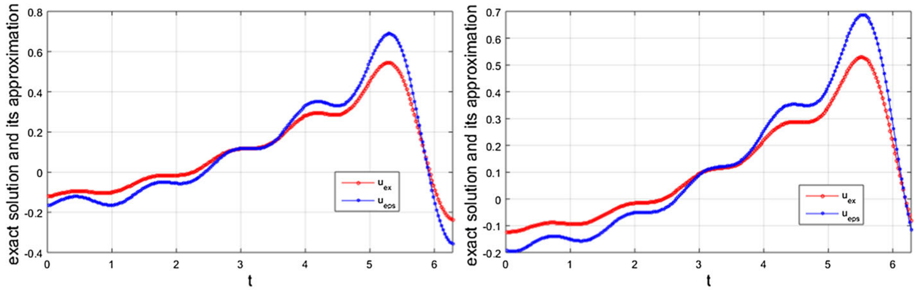
4 Conclusion

In this paper, we consider an inverse problem for a time fractional diffusion equation with a nonlinear source. The inverse problem here is studied for the linear source function $F(x, t, u) = b(x, t)u(x, t) + H(x, t)$, where $L^\infty([0, T]; L^2(\mathbb{R}))$ and $L^2([0, 1]; L^2(\mathbb{R}))$ in Tuan et al. (2016). We note that the proof in this paper is based on Lemma 1. We present a modified method to regularize the solution and investigate the convergence rate between the regularized solution and the sought solution. We have many possible stability terms in the homogeneous case. However, in the nonlinear case, the solution u is given an integral equation whose right-hand side depends on the solution u that makes the nonlinear case difficult. In this



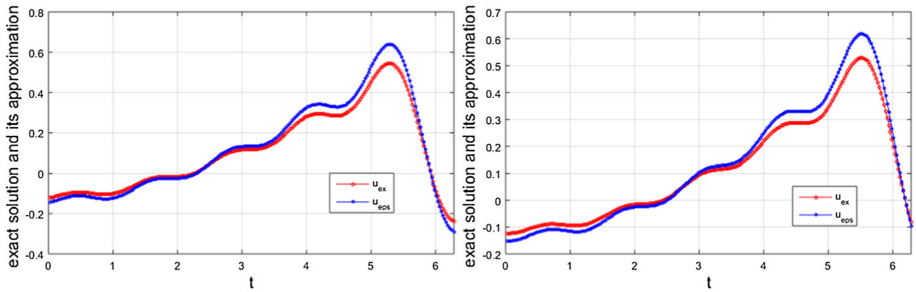
(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 14 The exact and regularized solutions for $\gamma = 0.45$ and $\alpha = 0.01$



(a) $x = 0.3$ **(b)** $x = 0.6$

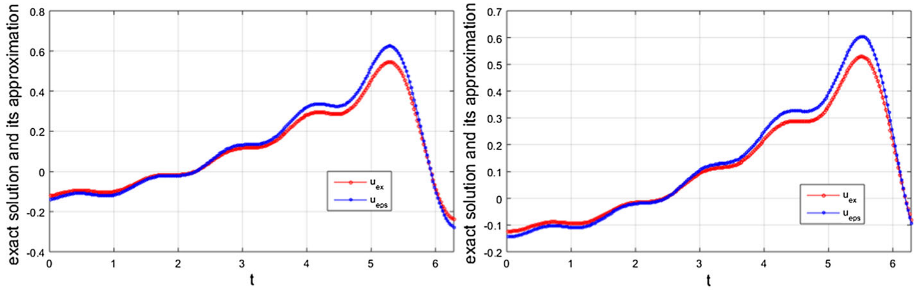
Fig. 15 The exact and regularized solutions for $\gamma = 0.9$ and $\alpha = 0.2086$



(a) $x = 0.3$ **(b)** $x = 0.6$

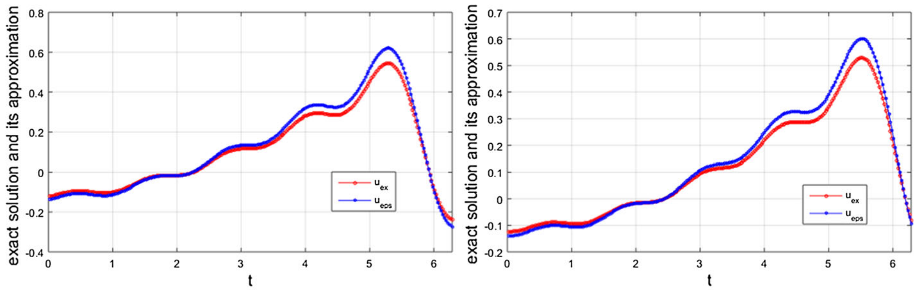
Fig. 16 The exact and regularized solutions for $\gamma = 0.9$ and $\alpha = 0.05$

paper, we develop some new techniques to overcome these difficulties. Numerical examples are given to illustrate our results. In the future, we extend the results in this paper for more general source term $F(x, t, u(x, t), u_x(x, t))$. We also plan to determine the source term numerically in the time-fractional equation $-au_x(x, t) =_0 D_t^{\gamma} u(x, t) + F(x, t, u(x, t))$. In this context, we prove the existence and uniqueness to the solution. Then after we show that the inverse problem is ill posed, we find a regularized solution. Later, we will study simultaneous determination problem of the diffusivity coefficient a and the order of the Caputo time fractional derivative γ for both numerically and theoretically. These are subjects of the future studies by the authors of this paper.



(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 17 The exact and regularized solutions for $\gamma = 0.9$ and $\alpha = 0.02$



(a) $x = 0.3$ **(b)** $x = 0.6$

Fig. 18 The exact and regularized solutions for $\gamma = 0.9$ and $\alpha = 0.01$

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