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An iterative method with residual vectors for solving the fixed point and the split inclusion problems in Banach spaces

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Abstract

In this paper, we propose an iterative technique with residual vectors for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of a split inclusion problem (SIP) with a way of selecting the stepsizes without prior knowledge of the operator norm in the framework of p-uniformly convex and uniformly smooth Banach spaces. Then strong convergence of the proposed algorithm to a common element of the above two sets is proved. As applications, we apply our result to find the set of common fixed points of a family of mappings which is also a solution of the SIP. We also give a numerical example and demonstrate the efficiency of the proposed algorithm. The results presented in this paper improve and generalize many recent important results in the literature.

Keywords Resolvent operator · Relatively nonexpansive mapping · Strong convergence · Iterative methods · Banach spaces

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1 Introduction

Let H_1 and H_2 be two Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be two set-valued maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Moudafi (2011) introduced the following so-called *split inclusion problem* (SIP):

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.1}$$

The set of solutions of problem (1.1) is denoted by Γ, *i.e.*, $\Gamma := \{x^* \in H_1 : x^* \in B_1^{-1}(0) \text{ and } Ax^* \in B_2^{-1}(0)\}$. In fact, we know that the split inclusion problem is a generalization of the inclusion problem and the split feasibility problem. Next, we provide some special cases of SIP (1.1).

- Let $f : H_1 \to \mathbb{R} \cup \{+\infty\}$ and $g : H_2 \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex functions. If we take $B_1 = \partial f$ and $B_2 = \partial g$, where ∂f and ∂g are the subdifferential of f and g, then the SIP (1.1) becomes the following so-called *proximal split feasibility problem*:

$$\text{Find } x^* \in \operatorname{argmin} f \text{ such that } Ax^* \in \operatorname{argmin} g, \tag{1.2}$$

  where $\operatorname{argmin} f = \{x \in H_1 : f(x) \le f(y),\ \forall y \in H_1\}$ and $\operatorname{argmin} g = \{x \in H_2 : g(x) \le g(y),\ \forall y \in H_2\}$. In particular, if we take $f(x) = \frac{1}{2}\|Mx - b\|^2$ and $g(x) = \frac{1}{2}\|Nx - c\|^2$, where M and N are matrices, and $b, c \in H_1$, then the SIP (1.2) becomes the least square problem. This problem has been intensively studied, especially, in Hilbert spaces; see for instance (Moudafi and Thakur 2014).
- Let C and Q be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. If $B_1 = N_C$, $B_2 = N_Q$, where N_C and N_Q are the normal cones of C and Q, respectively, then the SIP (1.2) becomes the following so-called *split feasibility problem*:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.3}$$

This problem was first introduced, in a finite dimensional Hilbert space, by Censor and Elfving (1994) for modeling inverse problems in radiation therapy treatment planning which arise from phase retrieval and in medical image reconstruction, especially intensity modulated therapy (Censor et al. 2006).

To solve the SIP (1.1), Byrne et al. (2011) gave the following convergence theorem in infinite dimensional Hilbert spaces:

Theorem 1.1 *Let H_1 and H_2 be real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings, $\lambda > 0$ and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. Suppose that $\Gamma \neq \emptyset$. For given $x_1 \in H_1$, let $\{x_n\}$ be the sequence defined by*

$$x_{n+1} = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \quad \forall n \ge 1. \tag{1.4}$$

Then $\{x_n\}$ converges weakly to an element $x^* \in \Gamma$.

In order to obtain strong convergence, Kazmi and Rizvi (2014) proposed an algorithm for solving SIP (1.1) with fixed points of a nonexpansive mapping T. They obtained the following result:

Theorem 1.2 *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and $f : H_1 \to H_1$ be a contraction mapping with a constant $\alpha \in (0, 1)$. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be set-valued maximal monotone mappings, $\lambda > 0$. Let $T : H_1 \to H_1$ be a nonexpansive mapping such that $F(T) \cap \Gamma \neq \emptyset$. For a given $x_1 \in H_1$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n - \gamma A^*(I - J_\lambda^{B_2})Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. Then the sequences $\{u_n\}$ and $\{x_n\}$ both converge strongly to $x^* \in F(T) \cap \Gamma$, where $x^* = P_{F(T)\cap\Gamma} f(x^*)$.

On the other hand, Takahashi and Takahashi (2016) first introduced the SIP outside Hilbert spaces. Let E_1 and E_2 be two Banach spaces. Let $B_1 : E_1 \multimap E_1$ and $B_2 : E_2 \multimap E_2$ be two set-valued maximal monotone operators and $A : E_1 \to E_2$ be a bounded linear operator. They proposed the SIP in Banach spaces as follows:

$$\text{Find } x^* \in E_1 \text{ such that } 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*). \tag{1.6}$$

In recent years, many authors have constructed several iterative methods for solving SIP (see, e.g., Sitthithakerngkiet et al. 2018; Takahashi and Takahashi 2016; Takahashi 2015, 2017; Takahashi and Yao 2015; Suantai et al. 2018; Jailoka and Suantai 2017; Ogbuisi and Mewomo 2017; Alofi et al. 2016).

Very recently, Alofi et al. (2016) introduced an algorithm based on Halpern's iteration for solving SIP (1.1) in a uniformly convex and smooth Banach space. They proved the following strong convergence theorem:

Theorem 1.3 *Let H be a Hilbert space and let E be a uniformly convex and smooth Banach space. Let J_E be the duality mapping on E. Let $B_1 : H \multimap H$ and $B_2 : E \multimap E^*$ be maximal monotone operators, respectively. Let $J_\lambda^{B_1}$ be the resolvent of B_1 for $\lambda > 0$ and let $J_\mu^{B_2}$ be the metric resolvent of B for $\mu > 0$. Let $A : H \to E$ be a bounded linear operator with its adjoint A^* such that $A \neq 0$. Suppose that $\Gamma \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. Let $x_1 \in H$ and let $\{x_n\} \subset H$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)J_{\lambda_n}^{B_1}(x_n - \lambda_n A^*(I - J_{\mu_n}^{B_2})Ax_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where $\{\lambda_n\}, \{\mu_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\} \subset (0, 1)$ satisfy the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$0 < a \leq \lambda_n \|A\|^2 \leq b < 2, \quad 0 < k \leq \mu_n, \quad 0 < c \leq \beta_n \leq d < 1,$$

for some $a, b, c, d \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $x^* \in \Gamma$, where $x^* = P_\Gamma u$.

However, it is observed that several iterative methods suggested require the computation of the norm of the bounded linear operator $\|A\|$, which may not be calculated easily in general. In this work, motivated by the previous works, we introduce an iterative technique with residual vectors for solving the fixed point problem of a relatively nonexpansive mapping and SIP with a way of selecting the step sizes without prior knowledge of the operator norm in the framework of p-uniformly convex and uniformly smooth Banach spaces. We prove its strong convergence of proposed algorithm to a common element of the set fixed points of a relatively nonexpansive mapping and the solutions of the SIP. As applications, we apply our result to finding the set of common fixed points of a family of mappings which is also a solution of the SIP. We also give some numerical examples and demonstrate the efficiency of the proposed algorithm. The results obtained in this paper improve and generalize many known results in the literature.

2 Preliminaries

Let E and E^* be real Banach spaces and the dual space of E, respectively. Let E_1 and E_2 be real Banach spaces and let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$ which is defined by

$$\langle A^*\bar{y}, x\rangle := \langle \bar{y}, Ax\rangle, \quad \forall x \in E_1, \quad \bar{y} \in E_2^*.$$

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon \right\}.$$

The *modulus of smoothness* of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \left\{ \frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : \|x\| = \|y\| = 1 \right\}.$$

Definition 2.1 A Banach space E is said to be

1. *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$;
2. *p-uniformly convex* (or to have a modulus of convexity of power type p) if there is a $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$;
3. *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$;
4. *q-uniformly smooth* if there exists a $c_q > 0$ such that $\rho_E(\tau) \leq c_q\tau^q$ for all $\tau > 0$.

From the Definition 2.1, we observe that every p-uniformly convex space is uniformly convex and if E is q-uniformly smooth, then E is also uniformly smooth. It is known that (Agarwal et al. 2009)

$$\begin{cases} E \text{ is } p\text{-uniformly convex if and only if } E^* \text{ is } q\text{-uniformly smooth,} \\ E \text{ is } q\text{-uniformly smooth if and only if } E^* \text{ is } p\text{-uniformly convex,} \end{cases} \tag{2.1}$$

where $p \geq 2$ and $1 < q \leq 2$ are conjugate exponents, *i.e.*, p, q satisfy $\frac{1}{p} + \frac{1}{q} = 1$ (see Xu and Roach 1991). For the sequence spaces ℓ_p, Lebesgue spaces L_p and Sobolev spaces W_p^m, we also know that (Agarwal et al. 2009; Hanner 1956; Xu and Roach 1991)

$$\begin{cases} \ell_p,\ L_p \text{ and } W_p^m \text{ are 2-uniformly convex and } p\text{-uniformly smooth with } 1 < p \leq 2, \\ \ell_p,\ L_p \text{ and } W_p^m \text{ are } q\text{-uniformly convex and 2-uniformly smooth with } 2 \leq q < \infty. \end{cases}$$

Definition 2.2 A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *gauge* if $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.3 The mapping $J_\varphi^E : E \multimap E^*$ associated with a gauge function φ defined by

$$J_\varphi^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\varphi(\|x\|),\ \|f\| = \varphi(\|x\|),\ \ \forall x \in E\},$$

is called the *duality mapping with gauge* φ, where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between E and E^*.

If $\varphi(t) = t$, then $J_\varphi^E = J_2^E = J$ is the *normalized duality mapping*. In particular, $\varphi(t) = t^{p-1}$, where $p > 1$, the duality mapping $J_\varphi^E = J_p^E$ is called the *generalized duality mapping* which is defined by

$$J_p^E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}.$$

It is well known that if E is uniformly smooth, the generalized duality mapping J_p^E is norm to norm uniformly continuous on bounded subsets of E (see Reich 1981). Furthermore, J_p^E is one-to-one, single-valued and satisfies $J_p^E = (J_q^{E^*})^{-1}$, where $J_q^{E^*}$ is the generalized duality mapping of E^* (see Reich 1992; Cioranescu 1990 for more details).

For a gauge φ, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s$$

is a continuous convex strictly increasing differentiable function on $\mathbb{R}^+$ with $\Phi'(t) = \varphi(t)$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. Therefore, Φ has a continuous inverse function Φ^{-1}.

We next recall the Bregman distance, which was introduced and studied in Bregman (1967).

Definition 2.4 Let E be a real smooth Banach space. The Bregman distance $D_\varphi(x, y)$ between x and y in E is defined by

$$D_\varphi(x, y) = \Phi(\|y\|) - \Phi(\|x\|) - \langle J_\varphi(x), y - x\rangle.$$

We note that the Bregman distance D_φ does not satisfy the well-known properties of a metric because D_φ is not symmetric and does not satisfy the triangle inequality. Moreover, the Bregman distance has the following important properties:

$$D_\varphi(x, y) = D_\varphi(x, z) + D_\varphi(z, y) + \langle J_\varphi^E x - J_\varphi^E z, z - y\rangle, \tag{2.2}$$

$$D_\varphi(x, y) + D_\varphi(y, x) = \langle J_\varphi^E x - J_\varphi^E y, x - y\rangle, \tag{2.3}$$

for all $x, y, z \in E$.

In the case $\varphi(t) = t^{p-1}$, where $p > 1$, the distance $D_\varphi = D_p$ is called the *p-Lyapunov function* which was studied in Bonesky et al. (2008) and it is given by

$$D_p(x, y) = \frac{1}{q}\|x\|^p - \langle J_p^E x, y\rangle + \frac{1}{p}\|y\|^p,$$

where p, q are conjugate exponents. For the p-uniformly convex space, the Bregman distance has the following relation (see Schöpfer et al. 2008):

$$\tau\|x - y\|^p \le D_p(x, y) \le \langle J_p^E x - J_p^E y, x - y\rangle, \tag{2.4}$$

where $\tau > 0$ is some fixed number. If $p = 2$, we get

$$D_2(x, y) := \phi(x, y) = \|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2,$$

where ϕ is called the *Lyapunov function* which was introduced by Alber (1993, 1996).

The following Lemma can be obtained from Theorem 2.8.17 of Agarwal et al. (2009) (see also Lemma 5 of Kuo and Sahu 2013).

Lemma 2.5 *Let $p > 1$, $r > 0$ and E be a Banach space. Then the following statements are equivalent:*

(i) *E is uniformly convex;*
(ii) *There exists a strictly increasing convex function $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_r^*(0) = 0$ such that*

$$\Big\| \sum_{k=1}^{N} \alpha_k x_k \Big\|^p \leq \sum_{k=1}^{N} \alpha_k \|x_k\|^p - \alpha_i \alpha_j g_r^*(\|x_i - x_j\|),$$

for all $i, j \in \{1, 2, \ldots, N\}$, $x_k \in B_r := \{x \in E : \|x\| \leq r\}$, $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$, where $k \in \{1, 2, \ldots, N\}$.

Lemma 2.6 (Xu 1991) *Let $1 < q \leq 2$ and E be a Banach space. Then the following statements are equivalent:*

(i) *E is q-uniformly smooth;*
(ii) *there is a constant $\kappa_q > 0$ which is called the q-uniform smoothness coefficient of E such that for all $x, y \in E$*

$$\|x - y\|^q \leq \|x\|^q - q\langle y, J_q^E(x)\rangle + \kappa_q \|y\|^q. \tag{2.5}$$

In what follows, we shall use the following notations: $x_n \to x$ means that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. Let C be a closed and convex subset of E and let T be a mapping from C into itself. We denote $F(T)$ by the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : x = Tx\}$. A point $z \in C$ is called an *asymptotic fixed point* (Reich 1996) of T, if there exists a sequence $\{x_n\}$ in C which converges weakly to z and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote $\widehat{F}(T)$ by the set of asymptotic fixed points of T. A mapping $T : C \to C$ is called *Bregman relatively nonexpansive* (Butnariu et al. 2001, 2003; Censor and Reich 1996; Matsushita and Takahashi 2005), if the following conditions are satisfied:

(R1) $F(T) = \widehat{F}(T) \neq \emptyset$;
(R2) $D_p(Tx, z) \leq D_p(x, z), \quad \forall z \in F(T), \ \forall x \in C$.

Let E be a p-uniformly convex Banach space which is also uniformly smooth. Following Censor and Lent (1981) and Alber (1993), we make use of the function $V_p : E^* \times E \to \mathbb{R}^+$ which is defined by

$$V_p(x^*, x) = \frac{1}{q}\|x^*\|^q - \langle x^*, x\rangle + \frac{1}{p}\|x\|^p \tag{2.6}$$

for all $x \in E$ and $x^* \in E^*$, where p, q are conjugate exponents. Then V_p is nonnegative and convex in the first variable. It is observed that

$$V_p(x^*, x) = D_p(J_q^{E^*}(x^*), x) \tag{2.7}$$

for all $x \in E$ and $x^* \in E^*$. In addition,

$$V_p(x^*, x) \leq V_p(x^* - y^*, x) + \langle J_q^{E^*}(x^*) - x, y^*\rangle \tag{2.8}$$

for all $x \in E$ and $x^* \in E^*$.

Lemma 2.7 (Bonesky et al. 2008) *Let $p > 1$ and E be a real p-uniformly convex and uniformly smooth Banach space. For $x \in E$ and a sequence $\{x_n\}$ in E. Then,* $\lim_{n\to\infty} D_p(x_n, x) = 0 \Longleftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0$.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that

$$D_p(x, z) = \min_{y\in C} D_p(x, y). \tag{2.9}$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the *generalized projection* of E onto C.

Lemma 2.8 (Kuo and Sahu 2013) *Let C be a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and let $x \in E$. Then the following assertions hold:*

(i) $z = \Pi_C x$ *if and only if* $\langle J_p^E(x) - J_p^E(z), y - z\rangle \le 0$, $\forall y \in C$.
(ii) $D_p(\Pi_C x, y) + D_p(x, \Pi_C x) \le D_p(x, y)$, $\forall y \in C$.

Let $B : E \multimap E^*$ be a mapping. The effective domain of B is denoted by $D(B)$, *i.e.*, $D(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be *monotone* if

$$\langle u - v, x - y\rangle \ge 0, \quad \forall x, y \in D(B),\ u \in Bx \text{ and } v \in By. \tag{2.10}$$

A monotone operator B on E is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on E.

Let E be a p-uniformly convex and uniformly smooth Banach space and let $B : E \multimap E^*$ be a maximal monotone operator. Then, for $x \in E$ and $\lambda > 0$, we define a mapping $Q_\lambda^B : E \to D(B)$ by

$$Q_\lambda^B(x) := (I + \lambda(J_p^E)^{-1}B)^{-1}(x) \text{ for all } x \in E. \tag{2.11}$$

This mapping is called the *metric resolvent* of B for $\lambda > 0$. The set of null points of B is defined by $B^{-1}(0) = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}(0)$ is closed and convex (see Takahashi 2000). We see that

$$0 \in J_p^E(Q_\lambda^B(x) - x) + \lambda B Q_\lambda^B(x). \tag{2.12}$$

Further, $F(Q_\lambda^B) = B^{-1}(0)$ for $\lambda > 0$ (see Zeidler 1984). From Kuo and Sahu (2013), we also know that

$$\langle Q_\lambda^B(x) - Q_\lambda^B(y), J_p^E(x - Q_\lambda^B(x)) - J_p^E(y - Q_\lambda^B(y))\rangle \ge 0, \tag{2.13}$$

for all $x, y \in E$ and if $B^{-1}(0) \neq \emptyset$, then

$$\langle J_p^E(x - Q_\lambda^B(x)), Q_\lambda^B(x) - z\rangle \ge 0, \tag{2.14}$$

for all $x \in E$ and $z \in B^{-1}(0)$.

In addition, we can define a single-valued mapping $R_\lambda^B : E \to D(B)$ so-called the *resolvent* of B by (Kohsaka and Takahashi 2005)

$$R_\lambda^B(x) := (J_p^E + \lambda B)^{-1} J_p^E(x) \quad \text{for all } x \in E.$$

It is known that R_λ^B is a relatively nonexpansive mapping and $F(R_\lambda^B) = B^{-1}(0)$ for $\lambda > 0$ (see Kuo and Sahu 2013).

Lemma 2.9 (Kohsaka and Takahashi 2005) *Let $B : E \multimap E^*$ be a maximal monotone operator with $B^{-1}(0) \neq \emptyset$ and let R_λ^B be a resolvent operator of B for $\lambda > 0$. Then*

$$D_p(R_\lambda^B(x), z) + D_p(R_\lambda^B(x), x) \leq D_p(x, z),$$

for all $x \in E$ and $z \in B^{-1}(0)$.

The following Theorem is proved by Kohsaka and Takahashi (see Kohsaka and Takahashi 2005, Lemma 7.2).

Lemma 2.10 (Kohsaka and Takahashi 2005) *Let $B : E \multimap E^*$ be a monotone operator. Then B is maximal if and only if for each $\lambda > 0$,*

$$R(J_p^E + \lambda B) = E^*,$$

where $R(J_p^E + \lambda B)$ is the range of $J_p^E + \lambda B$.

The following lemma was proved by Suantai et al. (2018).

Lemma 2.11 *Let E_1 and E_2 be uniformly convex and smooth Banach spaces. Let $A : E_1 \to E_2$ be a bounded linear operator with the adjoint operator A^*. Let $R_\lambda^{B_1}$ be the resolvent operator of a maximal monotone operator B_1 for $\lambda_1 > 0$ and $Q_{\lambda_2}^{B_2}$ be a metric resolvent of a maximal monotone operator B_2 for $\lambda_2 > 0$. Suppose that $\Gamma \neq \emptyset$. Let $r > 0$ and $x^* \in E_1$. Then x^* is a solution of problem (1.6) if and only if*

$$x^* = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(x^*) - rA^*J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Ax^*)).$$

Lemma 2.12 *Let E be a real p-uniformly convex and uniformly smooth Banach spaces. Suppose that $x \in E$ and $\{x_n\}$ is a sequence in E. Then the following statements are equivalent:*

(a) *$\{D_p(x_n, x)\}$ is bounded;*
(b) *$\{x_n\}$ is bounded.*

Proof For the implication $(a) \implies (b)$ was proved in Reich and Sabach (2010). For the converse implication $(b) \implies (a)$, we assume that $x \in E$ and $\{x_n\}$ are bounded. From (2.4), we observe that

$$\begin{aligned} D_p(x_n, x) &\leq \langle J_p^E x_n - J_p^E x, x_n - x \rangle \\ &\leq \|J_p^E x_n - J_p^E x\| \|x_n - x\| \\ &\leq M, \end{aligned}$$

for all $n \in \mathbb{N}$, where $M = \sup_{n \geq 1}\{\|x_n\|, \|x_n\|^{p-1}, \|x\|, \|x\|^{p-1}\}$. This implies that $\{D_p(x_n, x)\}$ is bounded. □

Lemma 2.13 (Reich 1979) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14 (Maingé 2008) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \ldots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau_n} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

Lemma 2.15 *Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, \ldots, N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Then, we have*

$$D_p\left(J_q^{E^*}\left(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)\right), z\right) \leq \sum_{k=1}^{N} \alpha_k D_p(x_k, z) - \alpha_i \alpha_j g_r^*\left(\|J_p^E(x_i) - J_p^E(x_j)\|\right),$$

for all $i, j \in \{1, 2, \ldots, N\}$.

Proof Since p-uniformly convex, hence it is uniformly convex. From Lemma 2.5, we have

$$\begin{aligned}
&D_p\left(J_q^{E^*}\left(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)\right), z\right)\\
&= V_p\left(\sum_{k=1}^{N} \alpha_k J_p^E(x_k), z\right)\\
&= \frac{1}{q}\Big\| \sum_{k=1}^{N} \alpha_k J_p^E(x_k)\Big\|^q - \left\langle \sum_{k=1}^{N} \alpha_k J_p^E(x_k), z \right\rangle + \frac{1}{p}\|z\|^p\\
&\leq \frac{1}{q} \sum_{k=1}^{N} \alpha_k \|J_p^E(x_k)\|^q - \alpha_i \alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|) - \left\langle \sum_{k=1}^{N} \alpha_k J_p^E(x_k), z \right\rangle + \frac{1}{p}\|z\|^p\\
&= \frac{1}{q} \sum_{k=1}^{N} \alpha_k \|J_p^E(x_k)\|^q - \sum_{k=1}^{N} \alpha_k \langle J_p^E(x_k), z\rangle + \frac{1}{p}\|z\|^p - \alpha_i \alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|)\\
&= \sum_{k=1}^{N} \alpha_k D_p(x_k, z) - \alpha_i \alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),
\end{aligned}$$

for all $i, j \in \{1, 2, \ldots, N\}$. This completes the proof. □

3 Algorithm and strong convergence theorem

In this section, we introduce an iterative algorithm for finding a common element of the set of solutions of split inclusion problem (1.6) and the set of fixed points of a Bregman relatively nonexpansive mapping. More specifically, we assume the following assumptions:

- E_1 and E_2 are p-uniformly convex and uniformly smooth Banach spaces;
- $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ are maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively;

- $R_{\lambda_1}^{B_1}$ is the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and $Q_{\lambda_2}^{B_2}$ is the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$;
- $A : E_1 \to E_2$ is a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$;
- $T : E_1 \to E_1$ is a Bregman relatively nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$;
- The set of solution of SIP is consistent, *i.e.*, $\Gamma \neq \emptyset$;
- $\Omega := F(T) \cap \Gamma \neq \emptyset$;
- ϵ_n denotes the residual vector in E_1 such that $\lim_{n\to\infty} \epsilon_n = u \in E_1$.

Algorithm 3.1 *Choose an initial guess $u_1 \in E_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), \quad \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \quad n \in N, \tag{3.2}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Note that the choice in (3.2) of the stepsize λ_n is independent of the norms $\|A\|$.

Lemma 3.2 *Let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by Algorithm 3.1. Then, $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded.*

Proof. By the choice of λ_n, we observe that

$$\begin{aligned} &\lambda_n^{q-1} \leq \frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \\ \iff &\kappa_q \lambda_n^{q-1} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \leq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p - \epsilon\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \\ \iff &\frac{\epsilon\kappa_q}{q} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q \leq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p - \frac{\kappa_q \lambda_n^{q-1}}{q} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q. \end{aligned} \tag{3.3}$$

Let $z \in \Omega$. From (2.14), we observe that

$$\begin{aligned} &\langle J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n, Au_n - Az\rangle \\ &= \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p + \langle J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n, Q_{\lambda_2}^{B_2}(Au_n) - Az\rangle \\ &\geq \|(I - Q_{\lambda_2}^{B_2})Au_n\|^p. \end{aligned} \tag{3.4}$$

Set $v_n := J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)$ for all $n \geq 1$. By (3.4) and Lemma 2.6, we have

$$\begin{aligned} &D_p(x_n, z) \leq D_p(v_n, z) \\ &= D_p\big(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n), z\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q}\|J_q^{E_1^*}(J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n)\|^p - \langle J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
&= \frac{1}{q}\|J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n)\|^q - \langle J_p^{E_1}(u_n)-\lambda_n A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, z\rangle + \frac{1}{p}\|z\|^p \\
&\le \frac{1}{q}\|J_p^{E_1}(u_n)\|^q - \lambda_n\langle Au_n, J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q - \langle J_p^{E_1}(u_n), z\rangle \\
&\quad +\lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az\rangle + \frac{1}{p}\|z\|^p \\
&= \frac{1}{q}\|u_n\|^p - \langle J_p^{E_1}(u_n), z\rangle \\
&\quad +\frac{1}{p}\|z\|^p + \lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az-Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \\
&= D_p(u_n,z) + \lambda_n\langle J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n, Az-Au_n\rangle + \frac{\kappa_q\lambda_n^q}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \\
&\le D_p(u_n,z) - \lambda_n\left(\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^* J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\right),
\end{aligned} \tag{3.5}
$$

which implies by (3.3) that

$$D_p(x_n, z) \le D_p(u_n, z).$$

Since $\lim_{n\to\infty}\epsilon_n = u \in E_1$, which implies that $\{\epsilon_n\}$ is bounded, then from Lemma 2.12, we have $\{D_p(\epsilon_n, z)\}$ is bounded. So there exists a constant $K > 0$ such that $D_p(\epsilon_n, z) \le K$ for all $n \ge 1$. From Lemma 2.15, we have

$$
\begin{aligned}
D_p(x_{n+1}, z) &\le D_p(u_{n+1}, z) \\
&= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), z) \\
&\le \alpha_n D_p(\epsilon_n, z) + \beta_n D_p(x_n, z) + \gamma_n D_p(Tx_n, z) \\
&\quad -\beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) \\
&\le \alpha_n D_p(\epsilon_n, z) + (1-\alpha_n)D_p(x_n, z) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) \\
&\le \alpha_n K + (1-\alpha_n)D_p(x_n, z) \\
&\le \max\{K, D_p(x_n, z)\} \\
&\ \ \vdots \\
&\le \max\{K, D_p(x_1, z)\}.
\end{aligned} \tag{3.6}
$$

By induction, we have $\{D_p(x_n, z)\}$ is bounded. Hence, $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{Au_n\}$.

Theorem 3.3 *Let $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ be sequences generated by Algorithm 3.1. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) $0 < k \le \beta_n\gamma_n \le 1$ *for some* $k \in (0, 1)$.

Then $\{x_n\}_{n=1}^\infty$ and $\{u_n\}_{n=1}^\infty$ converge strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the generalized projection from E_1 onto Ω.

Proof Let $x^* = \Pi_{F(T)\cap\Gamma}u$. From (2.7) and (3.6), we have

$$
\begin{aligned}
&D_p(x_{n+1}, x^*)\\
&\quad\le D_p(u_{n+1}, x^*)\\
&\quad= V_p\big(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n), x^*\big)\\
&\quad\le V_p(\alpha_n J_p^{E_1}(\epsilon_n)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n) - \alpha_n(J_p^{E_1}(\epsilon_n)-J_p^{E_1}(x^*), x^*))\\
&\qquad+\alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\quad= V_p(\alpha_n J_p^{E_1}(x^*)+\beta_n J_p^{E_1}(x_n)+\gamma_n J_p^{E_1}(Tx_n), x^*)+\alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\quad= D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(x^*) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), x^*)\\
&\qquad+\alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\quad\le \alpha_n D_p(x^*, x^*) + \beta_n D_p(x_n, x^*) + \gamma_n D_p(Tx_n, x^*) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\qquad+\alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle\\
&\quad\le (1-\alpha_n)D_p(x_n, x^*) - \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\qquad+\alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle. \qquad (3.7)
\end{aligned}
$$

We now divide the proof into two cases:

Case 1 Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_p(x_n, x^*)\}_{n=n_0}^{\infty}$ is non-increasing. So we have $\{D_p(x_n, x^*)\}_{n=1}^{\infty}$ converges and it is bounded. From (3.7), we have

$$
\begin{aligned}
0 &\le k g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\le \beta_n\gamma_n g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|)\\
&\le D_p(x_n, x^*) - D_p(x_{n+1}, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle. \qquad (3.8)
\end{aligned}
$$

This implies that

$$\lim_{n\to\infty} g_r^*(\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\|) = 0.$$

By the property of g_r^*, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| = 0. \qquad (3.9)$$

Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \qquad (3.10)$$

By Lemma 2.7, we also have

$$\lim_{n\to\infty} D_p(x_n, Tx_n) = 0. \qquad (3.11)$$

By the boundedness of $\{x_n\}$ and the reflexivity of E_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x} \in E_1$. From (3.10), we obtain $\hat{x} \in \widehat{F}(T) = F(T)$. From (3.3), (3.5) and (3.6), we see that

$$\frac{\epsilon^2\kappa_q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \le \lambda_n\left(\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p - \frac{\kappa_q\lambda_n^{q-1}}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q\right)$$
$$\le D_p(u_n,\hat{x}) - D_p(x_n,\hat{x})$$
$$\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x}) + D_p(x_{n-1},\hat{x}) - D_p(x_n,\hat{x}) \to 0 \text{ as } n\to\infty,$$

which implies that

$$\lim_{n\to\infty}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\| = 0. \tag{3.12}$$

From (3.5) and (3.6), we have

$$\epsilon\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p \le \lambda_n\|(I-Q_{\lambda_2}^{B_2})Au_n\|^p$$
$$\le D_p(u_n,\hat{x}) - D_p(x_n,\hat{x}) + \frac{\kappa_q\lambda_n^q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q$$
$$\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x}) + D_p(x_{n-1},\hat{x}) - D_p(x_n,\hat{x})$$
$$+\frac{\kappa_q\lambda_n^q}{q}\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|^q \to 0 \quad \text{as} \quad n\to\infty.$$

Hence

$$\lim_{n\to\infty}\|Au_n - Q_{\lambda_2}^{B_2}Au_n\| = 0. \tag{3.13}$$

Then, we have

$$\|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| \le \lambda_n\|A^*J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|$$
$$\le \lambda_n\|A^*\|\|J_p^{E_2}(I-Q_{\lambda_2}^{B_2})Au_n\|$$
$$\le \lambda_n\|A^*\|\|Au_n - Q_{\lambda_2}^{B_2}Au_n\|^{p-1} \to 0 \quad \text{as} \quad n\to\infty,$$

which implies that

$$\lim_{n\to\infty}\|J_p^{E_1}(v_n) - J_p^{E_1}(u_n)\| = 0. \tag{3.14}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty}\|v_n - u_n\| = 0. \tag{3.15}$$

By Lemma 2.9 and (3.6), we have

$$D_p(x_n,v_n) = D_p(R_{\lambda_1}^{B_1}v_n, v_n)$$
$$\le D_p(v_n,\hat{x}) - D_p(x_n,\hat{x})$$
$$\le D_p(u_n,\hat{x}) - D_p(x_n,\hat{x})$$
$$\le \alpha_{n-1}D_p(\epsilon_{n-1},\hat{x}) + D_p(x_{n-1},\hat{x}) - D_p(x_n,\hat{x}) \to 0 \quad \text{as} \quad n\to\infty.$$

Thus, we have

$$\lim_{n\to\infty}\|R_{\lambda_1}^{B_1}v_n - v_n\| = \lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.16}$$

Since $x_{n_i}\rightharpoonup\hat{x}\in E_1$, we also have $v_{n_i}\rightharpoonup\hat{x}\in E_1$. From (3.16), we get $\hat{x}\in F(R_{\lambda_1}^{B_1})\in B_1^{-1}(0)$.

From (3.15) and (3.16), we obtain

$$\|x_n - u_n\| \le \|x_n - v_n\| + \|v_n - u_n\| \to 0 \quad \text{as} \quad n\to\infty. \tag{3.17}$$

Since $x_{n_i} \rightharpoonup \hat{x} \in E_1$ and from (3.17), we also get $u_{n_i} \rightharpoonup \hat{x} \in E_1$. From (2.14), we have

$$\begin{aligned}\|(I - Q_{\lambda_2}^{B_2})A\hat{x}\|^p &= \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}\rangle \\ &= \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Au_{n_i}\rangle \\ &\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Au_{n_i} - Q_{\lambda_2}^{B_2}Au_{n_i}\rangle \\ &\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Q_{\lambda_2}^{B_2}Au_{n_i} - Q_{\lambda_2}^{B_2}A\hat{x}\rangle \\ &\le \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), A\hat{x} - Au_{n_i}\rangle \\ &\quad + \langle J_p^{E_2}(A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}), Au_{n_i} - Q_{\lambda_2}^{B_2}Au_{n_i}\rangle. \end{aligned} \tag{3.18}$$

Since A is continuous, we have $Au_{n_i} \rightharpoonup A\hat{x}$ as $i \to \infty$. Then, we have

$$\|A\hat{x} - Q_{\lambda_2}^{B_2}A\hat{x}\| = 0,$$

that is, $A\hat{x} = Q_{\lambda_2}^{B_2}A\hat{x}$. This shows that $A\hat{x} \in F(Q_{\lambda_2}^{B_2}) = B_2^{-1}(0)$. So $\hat{x} \in \Gamma$. Therefore, we conclude that $\hat{x} \in \Omega := F(T) \cap \Gamma$.

Now, we see that

$$\begin{aligned}D_p(u_{n+1}, x_n) &\le D_p(J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(Tx_n)), x_n) \\ &\le \alpha_n D_p(\epsilon_n, x_n) + \beta_n D_p(x_n, x_n) + \gamma_n D_p(Tx_n, x_n) \to 0 \ \text{ as } \ n \to \infty,\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|u_{n+1} - x_n\| = 0. \tag{3.19}$$

So, we have

$$\|u_{n+1} - u_n\| \le \|u_{n+1} - x_n\| + \|x_n - u_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.20}$$

We now choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle = \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle,$$

where $x^* = \Pi_\Omega u$. From (3.17) and Lemma 2.8, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_n - x^*\rangle &= \limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_n - x^*\rangle \\ &= \lim_{i\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), x_{n_i} - x^*\rangle \\ &= \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), \hat{x} - x^*\rangle \le 0.\end{aligned}$$

From (3.20), we also have

$$\limsup_{n\to\infty}\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle \le 0. \tag{3.21}$$

By (3.7), we note that

$$\begin{aligned}D_p(x_{n+1}, x^*) &\le (1-\alpha_n)D_p(x_n, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle \\ &= (1-\alpha_n)D_p(x_n, x^*) + \alpha_n\langle J_p^{E_1}(\epsilon_n) - J_p^{E_1}(u), u_{n+1} - x^*\rangle \\ &\quad + \alpha_n\langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{n+1} - x^*\rangle.\end{aligned}$$

Since $\epsilon_n \to u$ implies $J_p^{E_1}(\epsilon_n) \to J_p^{E_1}(u)$. Considering this together with (3.21), we conclude by Lemma 2.13 that $D_p(x_n, x^*) \to 0$ as $n \to \infty$. Therefore,S $x_n \to x^* \in \Omega$.

Case 2 Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Let us define a mapping $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then, by Lemma 2.14, we obtain

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

Put $\Gamma_n = D_p(x_n, x^*)$ for all $n \in \mathbb{N}$. Then, we have from (3.6) that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &\leq \lim_{n\to\infty} (D_p(\epsilon_{\tau(n)}, x^*) + (1-\alpha_{\tau(n)})D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)) \\ &= \lim_{n\to\infty} \alpha_{\tau(n)}\big(D_p(\epsilon_{\tau(n)}, x^*) - D_p(x_{\tau(n)}, x^*)\big) = 0, \end{aligned}$$

which implies that

$$\lim_{n\to\infty} (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) = 0. \tag{3.22}$$

Following the proof line in *Case 1*, we can show that

$$\begin{aligned} &\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0, \\ &\lim_{n\to\infty} \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2} Au_{\tau(n)}\| = 0, \\ &\lim_{n\to\infty} \|Au_{\tau(n)} - Q_{\lambda_2}^{B_2} Au_{\tau(n)}\| = 0, \\ &\lim_{n\to\infty} \|x_{\tau(n)} - v_{\tau(n)}\| = \lim_{n\to\infty} \|x_{\tau(n)} - u_{\tau(n)}\| = 0 \end{aligned}$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)+1} - u_{\tau(n)}\| = 0.$$

Furthermore, we can show that

$$\limsup_{n\to\infty} \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle \leq 0.$$

From (3.7), we have

$$D_p(x_{\tau(n)+1}, x^*) \leq (1-\alpha_{\tau(n)})D_p(x_{\tau(n)}, x^*) + \alpha_{\tau(n)}\langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle,$$

which implies that

$$\begin{aligned} \alpha_{\tau(n)} D_p(x_{\tau(n)}, x^*) \leq\ & D_p(x_{\tau(n)}, x^*) - D_p(x_{\tau(n)+1}, x^*) \\ & + \alpha_{\tau(n)}\langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle. \end{aligned}$$

Since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\alpha_{\tau(n)} > 0$, we get

$$\begin{aligned} D_p(x_{\tau(n)}, x^*) &\leq \langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle \\ &= \langle J_p^{E_1}(\epsilon_{\tau(n)}) - J_p^{E_1}(u), u_{\tau(n)+1} - x^*\rangle + \langle J_p^{E_1}(u) - J_p^{E_1}(x^*), u_{\tau(n)+1} - x^*\rangle. \end{aligned}$$

Since $\epsilon_{\tau(n)} \to u$ implies $J_p^{E_1}(\epsilon_{\tau(n)}) \to J_p^{E_1}(u)$. Hence, $\lim_{n\to\infty} D_p(x_{\tau(n)}, x^*) = 0$. From (3.22), we obtain

$$D_p(x_n, x^*) \le D_p(x_{\tau(n)+1}, x^*) = D_p(x_{\tau(n)}, x^*) + (D_p(x_{\tau(n)+1}, x^*) - D_p(x_{\tau(n)}, x^*)) \to 0 \text{ as } n \to \infty,$$

which implies that $D_p(x_n, x^*) \to 0$. That is $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

We consequently obtain the following result in Hilbert spaces:

Corollary 3.4 *Let H_1 and H_2 be Hilbert spaces. Let $B_1 : H_1 \multimap H_1$ and $B_2 : H_2 \multimap H_2$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator $A^* : H_2 \to H_1$. Let $T : H_1 \to H_1$ be a relatively nonexpansive mapping such that $F(T) = \widehat{F}(T) \neq \emptyset$. Assume that $\Omega := F(T) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in H_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(u_n - \lambda_n A^*(I - Q_{\lambda_2}^{B_2})Au_n) \\ u_{n+1} = \alpha_n \epsilon_n + \beta_n x_n + \gamma_n T x_n, \quad \forall n \ge 1, \end{cases} \tag{3.23}$$

where $\{\epsilon_n\} \subset H_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \le \lambda_n \le \frac{2\|(I - Q_{\lambda_2}^{B_2})Au_n\|^2}{\|A^*(I - Q_{\lambda_2}^{B_2})Au_n\|^2} - \epsilon, \quad n \in N, \tag{3.24}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < k \le \beta_n \gamma_n \le 1$ *for some* $k \in (0, 1)$.

Then $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to $x^* = \Pi_\Omega u$.

4 Convergence theorems for a family of mappings

In this section, we apply our result to the common fixed point problems of a family of mappings.

4.1 A countable family of relatively nonexpansive mappings

Definition 4.1 (Aoyama et al. 2007) Let C be a subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C in to E such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the *AKTT-condition* if, for any bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{z\in B}\{\|J_p^E(T_{n+1}z) - J_p^E(T_n z)\|\} < \infty.$$

As in Suantai et al. (2012), we can prove the following Proposition:

Proposition 4.2 *Let C be a nonempty, closed and convex subset of a real p-uniformly convex and uniformly smooth Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x, \quad \forall x \in B \tag{4.1}$$

and

$$\lim_{n\to\infty} \sup_{z\in B} \|J_p^E(Tz) - J_p^E(T_n z)\| = 0.$$

In the sequel, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and T is defined by (4.1) with $\bigcap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 4.3 *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of Bregman relatively nonexpansive mappings on E_1 such that $F(T_n) = \widehat{F}(T_n)$ for all $n \geq 1$. Assume that $\Omega := \bigcap_{n=1}^{\infty} F(T_n) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in E_1$, and let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(T_n x_n)), \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where $\{\epsilon_n\} \subset E_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \quad n \in N, \tag{4.3}$$

for some $\epsilon > 0$, where the index set $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ and $\lambda_n = \lambda$ (λ being any nonnegative value), otherwise. Suppose that the following conditions hold:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < k \leq \beta_n \gamma_n \leq 1$ *for some* $k \in (0, 1)$.

Suppose in addition that $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition and $F(T) = \widehat{F}(T)$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ converge strongly to $x^* = \Pi_\Omega u$, where Π_Ω is the generalized projection from E_1 onto Ω.

Proof To this end, it suffices to show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. By following the method of proof in Theorem 3.3, we can show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $J_p^{E_1}$ is uniformly continuous on bounded subsets of E_1, we have

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_n x_n)\| = 0.$$

By Proposition 4.2, we see that

$$\begin{aligned}\|J_p^{E_1}(x_n) - J_p^{E_1}(Tx_n)\| &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| + \|J_p^{E_1}(T_nx_n) - J_p^{E_1}(Tx_n)\| \\ &\le \|J_p^{E_1}(x_n) - J_p^{E_1}(T_nx_n)\| \\ &\quad + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_nx) - J_p^{E_1}(Tx)\| \to 0 \text{ as } n\to\infty.\end{aligned}$$

Since $J_q^{E_1^*}$ is norm-to-norm uniformly continuous on bounded subsets of E_1^*,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

4.2 A semigroup of relatively nonexpansive mappings

Definition 4.4 A one-parameter family $\mathcal{S} = \{T_t\}_{t\ge 0}$ from E into E is said to be a *nonexpansive semigroup* if it satisfies the following conditions:

(S1) $T_0x = x$ for all $x \in E$;
(S2) $T_{s+t} = T_sT_t$ for all $s, t \ge 0$;
(S3) for each $x \in C$ the mapping $t \mapsto T_tx$ is continuous;
(S4) for each $t \ge 0$, T_t is nonexpansive, *i.e.*,

$$\|T_tx - T_ty\| \le \|x - y\|, \ \forall x, y \in E.$$

Remark 4.5 We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, *i.e.*, $F(\mathcal{S}) = \bigcap_{t\ge 0} F(T_t)$.

We now give some examples of semigroup operator. The following classical examples were one of the main sources for the development of semigroup theory (see Engel and Nagel 2000):

Example 4.6 Let E be a real Banach space and let $\mathcal{L}(E)$ be the space of all bounded linear operators on E. For $A \in \mathcal{L}(E)$ and define a bounded linear operator T_t by

$$T_t := e^{tA} = \sum_{n=0}^{\infty} \frac{t^nA^n}{n!},$$

for $t \ge 0$. Then, the operator T_t is a semigroup on E.

Example 4.7 Let $E := L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Consider the initial value problem for the heat equation:

$$\begin{aligned}\tfrac{\partial u}{\partial t} &= \Delta u, \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ u(x, 0) &= f(x), \quad \text{for } x \in \mathbb{R}^n,\end{aligned} \tag{4.4}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator on E. We can solve the heat equation using Fourier transform and the solution (4.4) can be written as follows:

$$u(x, t) = \frac{1}{\sqrt{(4\pi t)^n}} \int_{\mathbb{R}^n} e^{\frac{-\|s-\xi\|^2}{4t}} f(\xi)d\xi,$$

where $t > 0$, $s \in \mathbb{R}^n$ and $f \in E$. Then, we can write the solution $u(x,t)$ in the form of convolution integral as follows:

$$u(x,t) = (K_t * f)(x),$$

where K_t is heat kernel given by $K_t(x) = \frac{1}{\sqrt{(4\pi t)^n}} e^{\frac{-\|x\|^2}{4t}}$. Then the solution of (4.4) can be written as follows:

$$T_t f(x) := u(x,t) = (K_t * f)(x).$$

We can check that the operator $T_t f(x)$ is a semigroup on E.

Definition 4.8 A one-parameter family $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be a *family of uniformly Lipschitzian mappings* if there exists a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that

$$\|T_t x - T_t y\| \leq L_t \|x - y\|, \ \forall x, y \in E.$$

We now first give the following definition:

Definition 4.9 A one-parameter family $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be a *Bregman relatively nonexpansive semigroup* if it satisfies $(S1)$, $(S2)$, $(S3)$ and the following conditions:

(a) $F(\mathcal{S}) = \widehat{F}(\mathcal{S}) \neq \emptyset$;
(b) $D_p(T_t x, z) \leq D_p(x, z), \ \ \forall x \in E, \ z \in F(\mathcal{S})$ and $t \geq 0$.

Using idea in Aleyner and Censor (2005), Aleyner and Reich (2005) and Benavides et al. (2002), we define the following concept:

Definition 4.10 A continuous operator semigroup $\mathcal{S} = \{T_t\}_{t\geq 0} : E \to E$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) if for all $s \geq 0$ and any bounded subset B of E such that

$$\lim_{t\to\infty} \sup_{x\in B} \|J_p^E(T_t x) - J_p^E\big(T_s T_s x\big)\| = 0.$$

Theorem 4.11 *Let E_1 and E_2 be p-uniformly convex and uniformly smooth Banach spaces. Let $B_1 : E_1 \multimap E_1^*$ and $B_2 : E_2 \multimap E_2^*$ be maximal monotone operators such that $B_1^{-1}(0) \neq \emptyset$ and $B_2^{-1}(0) \neq \emptyset$, respectively. Let $R_{\lambda_1}^{B_1}$ be the resolvent operator of a maximal monotone B_1 for $\lambda_1 > 0$ and let $Q_{\lambda_2}^{B_2}$ be the metric resolvent operator of a maximal monotone B_2 for $\lambda_2 > 0$. Let $A : E_1 \to E_2$ be a bounded linear operator with its adjoint operator $A^* : E_2^* \to E_1^*$. Let $\mathcal{S} = \{T_t\}_{t\geq 0}$ be a u.a.r. Bregman relatively nonexpansive semigroup of uniformly Lipschitzian mappings on E_1 into E_1 with a bounded measurable function $L_t : (0,\infty) \to [0,\infty)$ such that $F(\mathcal{S}) := \bigcap_{h\geq 0} F(T_h) \neq \emptyset$. Assume that $\Omega := F(\mathcal{S}) \cap \Gamma \neq \emptyset$. Choose an initial guess $u_1 \in E_1$; let $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ be sequences generated by*

$$\begin{cases} x_n = R_{\lambda_1}^{B_1}(J_q^{E_1^*}(J_p^{E_1}(u_n) - \lambda_n A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n)) \\ u_{n+1} = J_q^{E_1^*}(\alpha_n J_p^{E_1}(\epsilon_n) + \beta_n J_p^{E_1}(x_n) + \gamma_n J_p^{E_1}(T_{t_n} x_n)), \ \ \forall n \geq 1, \end{cases} \tag{4.5}$$

where $\{\epsilon_n\} \subset E_1$ is a residual vector such that $\epsilon_n \to u$, and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Suppose that stepsize λ_n is a bounded sequence chosen in such a way that

$$0 < \epsilon \leq \lambda_n \leq \left(\frac{q\|(I - Q_{\lambda_2}^{B_2})Au_n\|^p}{\kappa_q \|A^* J_p^{E_2}(I - Q_{\lambda_2}^{B_2})Au_n\|^q} - \epsilon \right)^{\frac{1}{q-1}}, \ \ n \in N, \tag{4.6}$$

for some $\epsilon > 0$, *where the index set* $N := \{n \in \mathbb{N} : (I - Q_{\lambda_2}^{B_2})Au_n \neq 0\}$ *and* $\lambda_n = \lambda$ *(*λ *being any nonnegative value), otherwise. Suppose that the following conditions hold:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < k \leq \beta_n \gamma_n \leq 1$ *for some* $k \in (0, 1)$;
(C3) $\{t_n\} \subset (0, \infty)$ *with* $\lim_{n\to\infty} t_n = \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ *and* $\{u_n\}_{n=1}^{\infty}$ *converge strongly to* $x^* = \Pi_\Omega u$, *where* Π_Ω *is the generalized projection from* E_1 *onto* Ω.

Proof We only have to show that $\lim_{n\to\infty} \|x_n - T_t x_n\| = 0$ for all $t \geq 0$. By following the method of proof in Theorem 3.3, we can show that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_n - T_{t_n} x_n\| = 0. \tag{4.7}$$

Since $\{T_t\}_{t\geq 0}$ is a uniformly of Lipschitzian mappings with a bounded measurable function L_t. Then, we have

$$\begin{aligned} \|T_t T_{t_n} x_n - T_t x_n\| &\leq L_t \|T_{t_n} x_n - x_n\| \\ &\leq \sup_{t\geq 0}\{L_t\} \|T_{t_n} x_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $J_p^{E_1}$ is uniformly norm-to-norm continuous on bounded subsets of E_1, then we also have

$$\lim_{n\to\infty} \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| = 0. \tag{4.8}$$

For each $t \geq 0$, we note that

$$\begin{aligned} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_t x_n)\| &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_{t_n} x_n)\| + \|J_p^{E_1}(T_{t_n} x_n) - J_p^{E_1}(T_t T_{t_n} x_n)\| \\ &\quad + \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| \\ &\leq \|J_p^{E_1}(x_n) - J_p^{E_1}(T_{t_n} x_n)\| + \|J_p^{E_1}(T_t T_{t_n} x_n) - J_p^{E_1}(T_t x_n)\| \\ &\quad + \sup_{x\in\{x_n\}} \|J_p^{E_1}(T_{t_n} x) - J_p^{E_1}(T_t T_{t_n}) x\|. \end{aligned}$$

Since $\{T_t\}_{t\geq 0}$ is a u.a.r. Bregman relatively nonexpansive semigroup with $\lim_{n\to\infty} t_n = \infty$, then from (4.7) and (4.8), we get

$$\lim_{n\to\infty} \|J_p^{E_1}(x_n) - J_p^{E_1}(T_t x_n)\| = 0$$

for all $t \geq 0$. Since $J_q^{E_1^*}$ is uniformly norm-to-norm continuous on bounded subsets of E_1^*, we get

$$\lim_{n\to\infty} \|x_n - T_t x_n\| = 0.$$

This completes the proof. □

5 Numerical experiments

In this section, we give some numerical examples to support our main theorem.

Table 1 Numerical results of Algorithm 5.2 with different choices of N and M

The choices of N and M	No. of iterations	cpu (time)
$N = 50$, $M = 50$	250	0.007260
$N = 100$, $M = 50$	290	0.010884
$N = 200$, $M = 200$	357	0.031999
$N = 150$, $M = 300$	347	0.024889
$N = 500$, $M = 1000$	460	0.260444

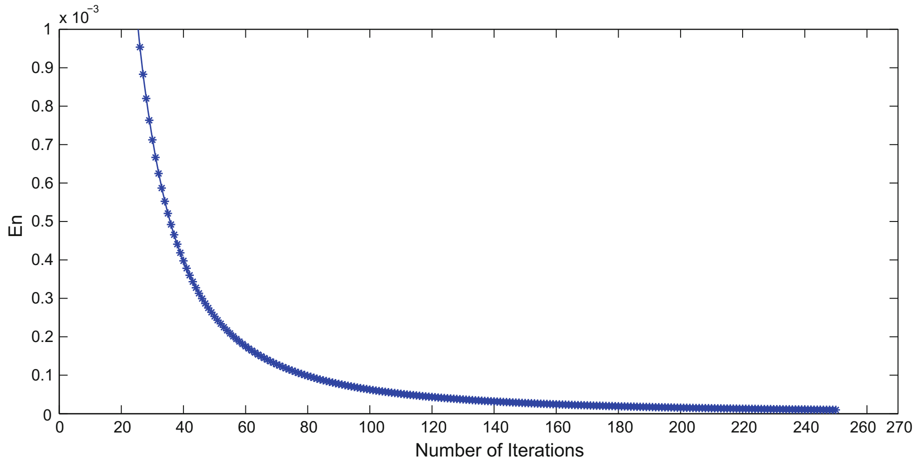

Fig. 1 The convergence behavior of E_n for $N = 50$ and $M = 50$

Example 5.1 For each $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(\mathbf{x}) := \|\mathbf{x}\|_2 \text{ and } g(\mathbf{x}) = -\sum_{i=1}^{N} \log x_i.$$

Then, we have

$$\text{prox}_{\lambda f}(\mathbf{x}) \begin{cases} \left(1 - \frac{\lambda}{\|\mathbf{x}\|_2}\right)x; & \|\mathbf{x}\|_2 \geq \lambda \\ 0; & \|\mathbf{x}\|_2 < \lambda \end{cases} \tag{5.1}$$

and

$$\text{prox}_{\lambda g}(\mathbf{x})_i = \frac{x_i + \sqrt{x_i^2 + 4\lambda}}{2}$$

for $i = 1, 2, 3, \ldots, N$. Let a mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$T\mathbf{x} = (2 - x_1, 2 - x_2, 2 - x_3, \ldots, 2 - x_N).$$

We aim to solve the following SIP and the fixed point problem: find $x^* \in \Gamma \cap F(T)$, *i.e.*, find $x^* \in \text{argmin } f$ such that $Ax^* \in \text{argmin } g$ and x^* is a fixed point of T, where A is a real

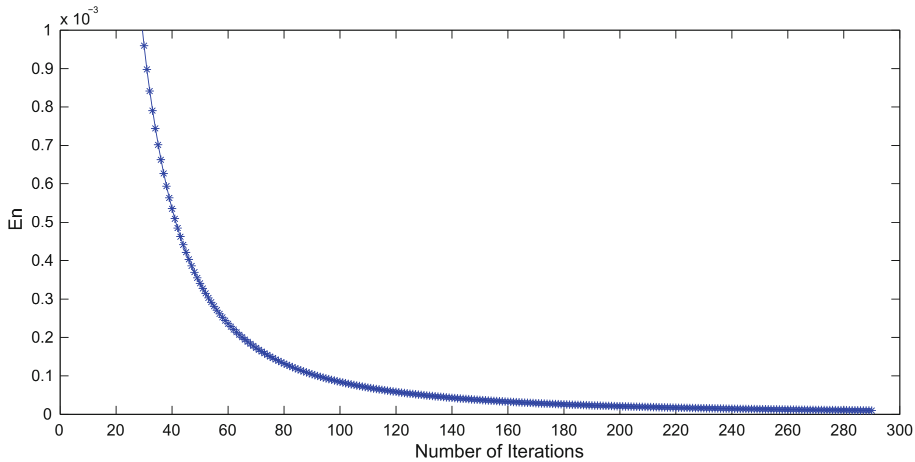

Fig. 2 The convergence behavior of E_n for $N = 100$ and $M = 50$

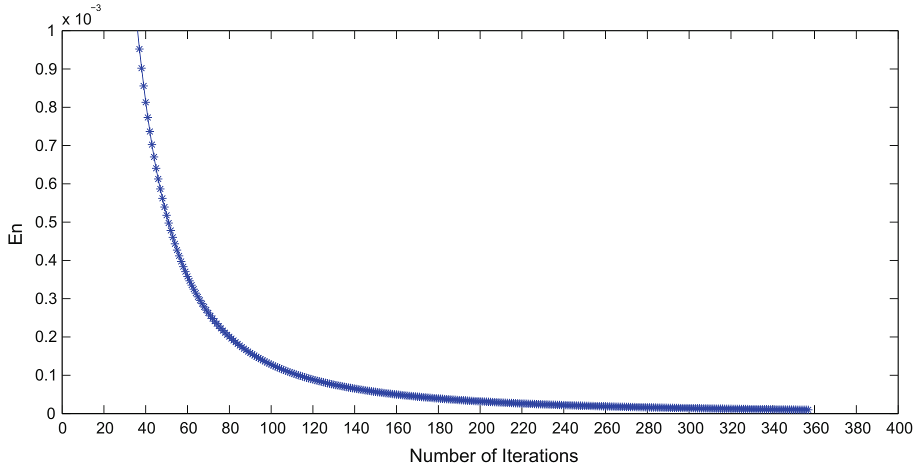

Fig. 3 The convergence behavior of E_n for $N = 200$ and $M = 200$

$N \times M$ matrix. So our iterative scheme (3.1) becomes

$$\begin{cases} \mathbf{x}_n = \text{prox}_{\lambda_1}^{f}\left[\mathbf{u}_n - \lambda_n A^t(A\mathbf{u}_n - \text{prox}_{\lambda_2}^{g}(A\mathbf{u}_n))\right] \\ \mathbf{u}_{n+1} = \alpha_n \epsilon_n + \beta_n \mathbf{x}_n + \gamma_n T\mathbf{x}_n, \quad \forall n \geq 1. \end{cases} \tag{5.2}$$

Let $\lambda_1 = \lambda_2 = 1$, $\alpha_n = \frac{1}{20n+1}$, $\beta_n = 0.5$, $\gamma_n = \frac{10n-0.5}{20n+1}$ and $\lambda_n = \frac{\|A\mathbf{u}_n - \text{prox}_{\lambda_2}^{g}(A\mathbf{u}_n)\|^2}{\|A^t(A\mathbf{u}_n - \text{prox}_{\lambda_2}^{g}(A\mathbf{u}_n))\|^2}$. The stopping criterion is defined by $E_n = \|u_{n+1} - u_n\| < 10^{-6}$. The matrix A is generated from a normal distribution with mean zero and one variance. For an initial guess $\mathbf{x}_1 \in \mathbb{R}^N$ and residual vector $\epsilon_n \in \mathbb{R}^N$ randomly, we obtain the following numerical results, given in Table 1 and Figs. 1, 2, 3, 4 and 5:

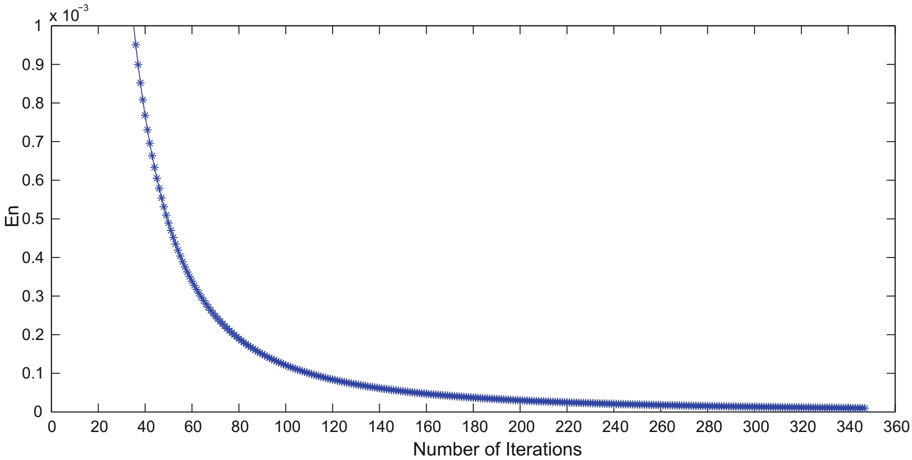

Fig. 4 The convergence behavior of E_n for $N = 150$ and $M = 300$

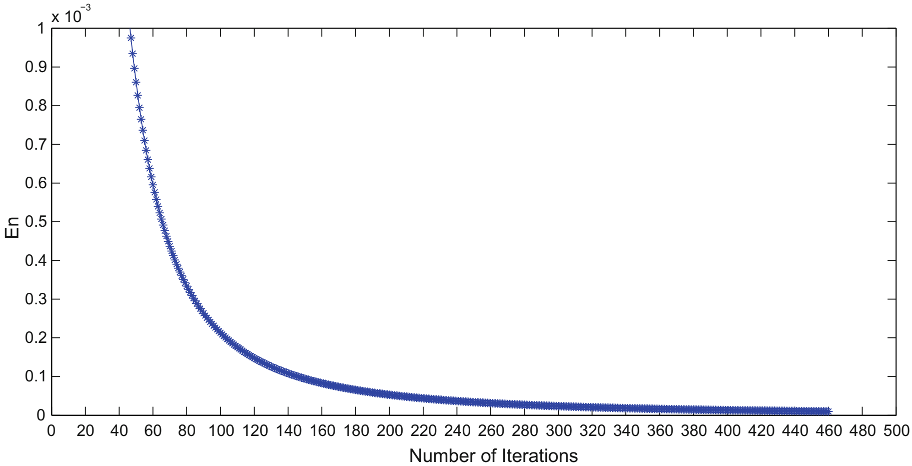

Fig. 5 The convergence behavior of E_n for $N = 500$ and $M = 1000$

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