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An inertial forward–backward splitting method for solving combination of equilibrium problems and inclusion problems

Suhel Ahmad Khan[1] · **Watcharaporn Cholamjiak**[2] · **K. R. Kazmi**[3]

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Abstract
In this paper, we prove a weak convergence theorem for finding a common solution of combination of equilibrium problems, infinite family of nonexpansive mappings, and the modified inclusion problems using inertial forward–backward algorithm. Further, we discuss some applications of our obtained results. Furthermore, we provide some numerical results to illustrate the convergence behavior of some of our results, and compare the convergence rate between the existing projection method and the proposed inertial forward–backward algorithm.

Keywords Equilibrium problem · Inertial method · Inclusion problems · Nonexpansive mapping · α-inverse strongly monotone mapping · Fixed point problem

Mathematics Subject Classification 47H10 · 49J40 · 49J52 · 90C30

1 Introduction

Throughout the paper, unless otherwise stated, let H be a real Hilbert space. Inner product and induced norm are, respectively, denoted by the notations $\langle ., . \rangle$ and $\|.\|$. Weak convergence and strong convergence are denoted by "$\rightharpoonup$" and "$\rightarrow$", respectively. Let C be a nonempty, closed, and convex subset of H. The fixed point problem for the mapping $T : C \rightarrow H$ is to find $x \in C$, such that $x = Tx$. We denote the fixed point set of a mapping T by $\mathrm{Fix}(T)$.

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✉ Suhel Ahmad Khan
khan.math@gmail.com

Watcharaporn Cholamjiak
c-wchp007@hotmail.com

K. R. Kazmi
krkazmi@gmail.com

1 Department of Mathematics, BITS-Pilani, Dubai Campus, P.O. Box 345055, Dubai, UAE

2 School of Science, University of Phayao, Phayao 56000, Thailand

3 Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is called α-inverse strongly monotone if there exists a positive real number $\alpha > 0$, such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction; then, the classical equilibrium problem (for short, EP) is to find $x \in C$, such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of all solutions of the equilibrium problem EP (1.1) is denoted by EP(F), that is

$$\mathrm{EP}(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}. \tag{1.2}$$

Equilibrium problem EP (1.1) introduced by Blum and Oettli (1994) in 1994 is the most intensively studied class of problems. This theory has helped in many ways of developing several thrust areas in physics, optimization, economics, and transportation problems. In recent past, various classes and forms of equilibrium problems and their applications have been studied, and as a result, various techniques and iterative schemes have been developed over the year to solve equilibrium problems; see (Blum and Oettli 1994; Combettes and Hirstoaga 2005; Farid et al. 2017; Khan and Chen 2015; Suwannaut and Kangtunyakarn 2014) and references therein.

Recently, Suwannaut and Kangtunyakarn (2014) proposed the following combination of equilibrium problems: for each $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction and $a_i \in (0, 1)$ with $\sum_{i=1}^{N} a_i = 1$. The combination of equilibrium problems (for short, CEP) is to find $x \in C$, such that

$$\sum_{i=1}^{N} a_i F_i(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of all solutions of the combination of equilibrium problem CEP (1.3) is denoted by $\mathrm{EP}\big(\sum_{i=1}^{N} a_i F_i\big)$, that is

$$\mathrm{EP}\left(\sum_{i=1}^{N} a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(x, y) \geq 0, \quad \forall y \in C\right\}. \tag{1.4}$$

If $F_i = F$, $\forall i = 1, 2, \ldots, N$, then CEP (1.3) reduces to EP (1.1).

Let $A : H \to H$ is an operator and $B : H \to 2^H$ is a multi-valued operator. The variational inclusion problem (for short, VIP) is to find $x \in H$, such that

$$0 \in Ax + Bx. \tag{1.5}$$

The set of the solution of VIP (1.5) is denoted by $(A + B)^{-1}(0)$. Variational inclusion problems are investigated and studied in minimization problem, complementarity problems, optimal control, convex programming, split feasibility problem, and variational inequalities.

An important method for solving problem VIP (1.5) is the forward–backward splitting method given by

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \geq 1, \tag{1.6}$$

where $J_r^B = (I + rB)^{-1}$ with $r > 0$. Forward–backward splitting methods are versatile in offering ways of exploiting the special structure of variational inequality problems. In this algorithm, $I - rA$ gives a forward step with step size r, whereas $(I + rB)^{-1}$ gives a backward step. Forward–backward splitting method is very useful and feasible, because computation of resolvent of $(I + rA)^{-1}$ and $(I + rB)^{-1}$ is much easier than computation of sum of resolvent the two operators $A + B$. This method provides a range of approaches to solve large-scale optimization problems and variational inequality problems; see (Bauschke and Combettes 2011; Cholamjiak 1994; Combettes and Wajs 2005; Lions and Mercier 1979; Lopez et al. 2012; Passty 1979; Tseng 2000 and reference therein. Forward–backward splitting method includes the proximal point algorithm and the gradient method as special cases; see (Alvarez 2004; Douglas and Rachford 1956; Lions and Mercier 1979; Peaceman and Rachford 1955) and references therein.
If $A = \bigtriangledown h$ and $B = \partial k$, where $\bigtriangledown h$ is the gradient of h and ∂g is the subdifferential of k, then VIP (1.5) problem reduces to the following minimization problem:

$$\min_{x \in H} h(x) + k(x), \tag{1.7}$$

and solution (1.6) reduces to

$$x_{n+1} = \text{prox}_{rk}(x_n - r \bigtriangledown h(x_n)), \quad n \geq 1, \tag{1.8}$$

where $\text{prox}_{rk} = (I + r\partial k)^{-1}$ is the proximity operator of k.

In 1964, Polyak (1964) introduced a two-step iterative method known as the heavy-ball method involving minimizing a smooth convex function h given by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = y_n - r \bigtriangledown h(x_n), \ n \geq 1, \end{cases} \tag{1.9}$$

where $\theta_n \in [0, 1)$ is an extrapolation factor with step size r that has to be chosen sufficiently small. Inspired by work of Polyak, in 2001, Alvarez and Attouch (2001) introduced an inertial forward–backward algorithm which was modification of the forward–backward splitting algorithm (1.9), and is given by

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + rB)^{-1} y_n, \ n \geq 1. \end{cases} \tag{1.10}$$

They proved the general convergence for monotone inclusion problems under the condition $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$ with $\{\theta_n\} \subset [0, 1)$ in a Hilbert space setting. The term $\theta_n(x_n - x_{n-1})$ is known as inertia with an extrapolation factor θ_n which leads to faster convergence while keeping nature of each iteration basically unchanged; see (Alvarez 2004; Dang et al. 2017; Dong et al. 2017, 2018; Khan et al. 2018; Lorenz and Pock 2015; Nesterov 1983).

Recently, Moudafi and Oliny (2003) proposed the following inertial proximal point algorithm for solving the zero-finding problem of the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + rB)^{-1}(y_n - r_n Ax), \ n \geq 1. \end{cases} \tag{1.11}$$

They proved the weak convergence and computed the operator B as the inertial extrapolate y_n under the condition $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$.

Very recently, Khan et al. (2018) proposed inertial forward–backward splitting algorithm for solving the inclusion problems as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n u + \beta_n y_n + \gamma_n(I + rB)^{-1}(y_n - s_n Ax), \ n \geq 1, \end{cases} \tag{1.12}$$

and proved a strong convergence theorem of the sequence $\{x_n\}$ under suitable conditions of the parameters $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\theta_n\}$ in the setting of Hilbert space.

In 2014, Khuangsatung and Kangtunyakarn (2014) generalized variational inclusion problem (1.5) as follows: for $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be a single-valued mapping and let $B : H \to H$ be a multi-valued mapping. The combination of variational inclusion problem (for short, CVIP) is to find $x \in H$, such that

$$0 \in \sum_{i=1}^{N} b_i A_i x + Bx, \tag{1.13}$$

for all $b_i \in (0, 1)$ with $\sum_{i=1}^{N} b_i = 1$. The set of all solutions of the combination of variational inclusion problem CVIP (1.13) is denoted by $\left(\sum_{i=1}^{N} b_i A_i + B\right)^{-1}(0)$. If $A_i = A, \ \forall i = 1, 2, \ldots, N$, then CVIP (1.13) reduces to VIP (1.5).

Motivated by the recent research works (Cholamjiak 1994; Dang et al. 2017; Dong et al. 2017, 2018; Khan et al. 2018; Khuangsatung and Kangtunyakarn 2014) going in this direction, we propose an iterative method of modified forward–backward algorithm involving the inertial technique for solving the combination of equilibrium problems, modified inclusion problems, and fixed point problems. Furthermore, we prove a weak convergence theorem for finding a common element of the combination of inclusion problems, fixed point sets of a infinite family of nonexpansive mappings, and the solution sets of a combination of equilibrium problems in the setting of Hilbert space. Furthermore, we utilize our main theorem to provide some applications in finding a common element of the set of fixed points of a finite family of k-strictly pseudo-contractive mappings and the set of solution of equilibrium problem in Hilbert space. Finally, we give some numerical examples to support and justify our results, which shows that our proposed inertial projection method has a better convergence rate than the standard projection method.

2 Preliminaries

To prove our main result, we recall some basic definitions and lemmas, which will be needed in the sequel.

Lemma 2.1 Takahashi (2000) *Let H be a real Hilbert space. Then, the following holds:*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for all $x, y \in H$;*
(ii) $\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|-\beta\gamma\|y-z\|-\gamma\alpha\|z-x\|$, *for all $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha+\beta+\gamma = 1$ and $x, y, z \in H$.*

A mapping $P_C : H \to C$ is said to be metric projection if, for every point $x \in H$, there exists a unique nearest point in C denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C.$$

It is well known that P_C is nonexpansive and firmly nonexpansive, that is

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \quad \forall x, y \in H.$$

We also recall the following basic result in the setting of a real Hilbert space.

Lemma 2.2 Lopez et al. (2012) *Let H be a Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone and $B : H \to 2^H$ a maximal monotone operator. If $T_r^{A,B} := J_r^B(I - rA) = (I + rB)^{-1}(I - rA),\ r > 0$, then the following holds:*

(i) *for* $r > 0$, $\text{Fix}(T_r^{A,B}) = (A+B)^{-1}(0)$. *Further, if* $r \in (0, 2\alpha]$, *then* $(A+B)^{-1}(0)$ *is a closed convex subset in* H*;*
(ii) *for* $0 < s \leq r$ *and* $x \in H$, $\|x - T_s^{A,B}x\| \leq 2\|x - T_r^{A,B}x\|$.

Lemma 2.3 Lopez et al. (2012) *Let* H *be a Hilbert space. Let* A *is* α*-inverse strongly monotone operator. Then, for given* $r > 0$

$$\|T_r^{A,B}x - T_r^{A,B}y\|^2 \leq \|x-y\|^2 - r(2\alpha - r)\|Ax - Ay\|^2$$
$$-\|(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y\|,$$

for all $x, y \in H$.

Lemma 2.4 Goebel and Kirk (1990) *Let* C *be a nonempty closed convex subset of a uniformly convex space* X *and* T *a nonexpansive mapping with* $\text{Fix}(T) \neq \emptyset$*. If* $\{x_n\}$ *is a sequence in* C*, such that* $x_n \rightharpoonup x$ *and* $(I-T)x_n \to y$*, then* $(I-T)x = y$*. In particular, if* $y = 0$*, then* $x \in \text{Fix}(T)$.

Lemma 2.5 Alvarez and Attouch (2001) *Let* $\{\psi_n\}$, $\{\delta_n\}$ *and* $\{\alpha_n\}$ *be the sequences in* $[0, +\infty)$*, such that* $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ *for all* $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$*, and there exists a real number* α *with* $0 \leq \alpha_n \leq \alpha < 1$ *for all* $n \geq 1$*. Then, the following holds:*

(i) $\sum_{n\geq 1}[\psi_n - \psi_{n-1}]_+ < +\infty$*, where* $[t]_+ = \max\{t, 0\}$*;*
(ii) *there exists* $\psi^* \in [0, +\infty)$*, such that* $\lim_{n\to+\infty} \psi_n = \psi^*$.

Lemma 2.6 Opial (1967) *Each Hilbert space* H *satisfies the Opial's condition that is, for any sequence* $\{x_n\}$ *with* $x_n \rightharpoonup x$*, the inequality*

$$\lim\inf_{n\to\infty} \|x_n - x\| < \lim\inf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ *with* $y \neq x$.

Definition 2.1 Kangtunyakarn (2011) Let C be a nonempty convex subset of a real Banach space X. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself, and let $\lambda_1, \lambda_2, \ldots,$ be real numbers in $[0, 1]$. Define the mapping $K_n : C \to C$ as follows:

$$\begin{aligned}
U_0 &= I,\\
U_1 &= \lambda_1 T_1 U_0 + (1-\lambda_1)U_0,\\
U_2 &= \lambda_2 T_2 U_1 + (1-\lambda_2)U_1,\\
&\vdots\\
U_k &= \lambda_k T_k U_{k-1} + (1-\lambda_k)U_{k-1},\\
U_{k+1} &= \lambda_{k+1} T_{k+1} U_k + (1-\lambda_{k+1})U_k,\\
U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})U_{N-2},\\
K_n &= U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)U_{N-1}.
\end{aligned}$$

Such a mapping K_n is called the K*-mapping* generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

Lemma 2.7 Kangtunyakarn (2011) *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^{\infty} \text{Fix}(T_i) \neq \emptyset$*, and let* $\lambda_1, \lambda_2, \ldots,$ *be real numbers, such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots,$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$*. For every* $n \in N$*, let* K_n *be the* K*-mapping generated* *by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_N$. *Then, for every* $x \in C$ *and* $k \in N$, $\lim_{n\to\infty} K_n x$ *exists.*

For every $k \in N$ *and* $x \in C$*, a mapping* $K : C \to C$ *is defined by* $Kx = \lim_{n\to\infty} K_n x$ *is called* K*-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$.

Remark 2.1 Kangtunyakarn (2011) For every $n \in N$, K_n is a nonexpansive mapping and $\lim_{n\to\infty} \sup_{x\in D} \|K_n x - Kx\| = 0$, for every bounded subset D of C.

Lemma 2.8 Kangtunyakarn (2011) *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* C *into itself with* $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$*, and let* $\lambda_1, \lambda_2, \ldots,$ *be real numbers, such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \ldots,$ *with* $\sum_{i=1}^{\infty} \lambda_i < \infty$. *For every* $n \in N$*, let* K_n *be the* K*-mapping generated by* $T_1, T_2, \ldots, T_N$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_N$*, and let* K *be the* K*-mapping generated by* $T_1, T_2, \ldots$ *and* $\lambda_1, \lambda_2, \ldots$. *Then,* $\mathrm{Fix}(K) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$.

Assumption 2.1 Blum and Oettli (1994) We assume that $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0, \ \forall x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \ \forall x, y \in C$;
(A3) F is upper hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) For each $x \in C$ fixed, the function $y \to F(x, y)$ is convex and lower semicontinuous;
(A5) For fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H and $x \in C \cap K$, such that

$$F(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C \backslash K.$$

Lemma 2.9 Combettes and Hirstoaga (2005) *Assume that the bifunction* $F : C \times C \to \mathbb{R}$ *satisfies Assumption* 2.1. *For* $r > 0$ *and for all* $x \in H$*, define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{z \in C : \ F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\right\},$$

for all $x \in H$. *Then, the following holds:*

(i) T_r *is nonempty and single-valued.*
(ii) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle.$$

(iii) $\mathrm{Fix}(T_r) = \mathrm{EP}(F)$.
(iv) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.10 Suwannaut and Kangtunyakarn (2014) *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *For each* $i = 1, 2, \ldots, N$*, let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption* 2.1 *with* $\bigcap_{i=1}^{N} \mathrm{EP}(F_i) \neq \emptyset$. *Then*

$$\mathrm{EP}\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} \mathrm{EP}(F_i),$$

where $a_i \in (0, 1)$ *for* $i = 1, 2, \ldots, N$ *and* $\sum_{i=1}^{N} a_i = 1$.

Remark 2.2 Suwannaut and Kangtunyakarn (2014) From Lemma 2.10, it is easy to see that $\sum_{i=1}^{N} a_i F_i$ satisfies Assumption 2.1. Using Lemma 2.9, we obtain

$$\mathrm{Fix}(T_r^{\sum}) = \mathrm{EP}\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} \mathrm{EP}(F_i),$$

where

$$T_r^{\sum}(x) = \left\{z \in C : \left(\sum_{i=1}^{N} a_i F_i\right)(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\},$$

and $a_i \in (0, 1)$, for each $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} a_i = 1$.

Theorem 2.1 Khuangsatung and Kangtunyakarn (2014) *Let H be a real Hilbert space and let $B : H \to 2^H$ be a maximal monotone mapping. For every $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,\ldots,N}\{\alpha_i\}$ and $\bigcap_{i=1}^{N}(A_i + B)^{-1}(0) \neq \emptyset$. Then*

$$\left(\sum_{i=1}^{N} b_i A_i + B\right)^{-1}(0) = \bigcap_{i=1}^{N}(A_i + B)^{-1}(0),$$

where $\sum_{i=1}^{N} b_i = 1$ and $b_i \in (0, 1)$ for every $i = 1, 2, \ldots, N$. Moreover, $J_s^B(I - s\sum_{i=1}^{N} b_i A_i)$ is a nonexpansive mapping for all $0 < s < 2\eta$.

Remark 2.3 From Lemma 2.2 and Theorem 2.1, we obtain

$$\mathrm{Fix}(T_r^{\sum A,B}) = \left(\sum_{i=1}^{N} b_i A_i + B\right)^{-1}(0) = \bigcap_{i=1}^{N}(A_i + B)^{-1}(0),$$

where $T_r^{\sum A,B} := J_r^B(I - r\sum_{i=1}^{N} b_i A_i) = (I + rB)^{-1}(I - r\sum_{i=1}^{N} b_i A_i),\ r > 0.$

Lemma 2.11 Xu (2003) *Assume that $\{s\}$ is a sequence of nonnegative real numbers, such that*

$$s_{n+1} \leq (1 - \alpha_n)s + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence, such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s = 0$.

3 Main result

In this section, we prove a weak convergence theorem for finding a common element of the fixed point sets of a infinite family of nonexpansive mappings, the solution sets of a combination of equilibrium problems, and combination of inclusion problems

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. For each $i = 1, 2, \dots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \dots,$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \dots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in N$, let K_n be the K-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$, and let K be the K-mapping generated by $T_1, T_2, \dots$ and $\lambda_1, \lambda_2, \dots$ for every $x \in C$. For every $i = 1, 2, \dots, N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,\dots,N}\{\alpha_i\}$ and $B : H \to 2^H$ be a maximal monotone mapping. Assume that $\Omega := \bigcap_{i=1}^{N}(A_i + B)^{-1}(0) \bigcap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^{N} \mathrm{EP}(F_i) \neq \emptyset$. For given initial points $x_0, x_1 \in H$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^B \left(I - s\sum_{i=1}^{N} b_i A_i\right) u_n, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$ and $\{\theta_n\} \subset [0, \theta]$, $\theta \in [0, 1]$, $\liminf_{n\to\infty} r_n > 0$ and $0 < s < 2\eta$, where $\eta = \min_{i=1,\dots,N}\{\alpha_i\}$. Suppose the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, sequence $\{x_n\}$ converges weakly to $q \in \Omega$.

Proof We divide the proof in the following steps.

Step 1. First, we show that $\{x_n\}$ is bounded.
Let $p \in \Omega$, and then, from Lemma 2.9, we have $u_n = T_{r_n}^{\sum} y_n$. We estimate that

$$\begin{aligned} \|u_n - p\| &= \left\| T_{r_n}^{\sum} y_n - T_{r_n}^{\sum} p \right\| \\ &\leq \|y_n - p\| \\ &\leq \|x_n - p\| + \theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.2}$$

From (3.2) and nonexpansiveness of $J_s^B(I - s\sum_{i=1}^{N} b_i A_i)$, we arrive that

$$\begin{aligned} \|x_{n+1} - p\| &= \left\| \alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^B \left(I - s\sum_{i=1}^{N} b_i A_i\right) u_n - p \right\| \\ &\leq \alpha_n \|x_n - p\| + \beta_n \|K_n u_n - p\| + \gamma_n \left\| J_s^B \left(I - s\sum_{i=1}^{N} b_i A_i\right) u_n - p \right\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|u_n - p\| \\ &\leq \|x_n - p\| + (1 - \alpha_n)\theta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.3}$$

From Lemma 2.5 and condition (i), we obtain $\lim_{n\to\infty} \|x_n - p\|$ exists and it follows that $\{x_n\}$ is bounded and also $\{y_n\}$ and $\{u_n\}$ are bounded.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
Let us take $J_s^{\sum A,B} = J_s^B(I - s\sum_{i=1}^N b_i A_i)$. Then, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^{\sum A,B} u_n - \alpha_{n-1}x_{n-1} - \beta_{n-1}K_{n-1}u_{n-1} - \gamma_{n-1}J_s^{\sum A,B} u_{n-1}\\
&\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + \beta_n\|K_n u_n - K_n u_{n-1}\| + \beta_n\|K_n u_{n-1} - K_{n-1}u_{n-1}\|\\
&\quad +|\beta_n - \beta_{n-1}|\|K_{n-1}u_{n-1}\| + \gamma_n\|J_s^{\sum A,B} u_n - J_s^{\sum A,B} u_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|J_s^{\sum A,B} u_{n-1}\|\\
&\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + \beta_n\|u_n - u_{n-1}\| + \beta_n\|K_n u_{n-1} - K_{n-1}u_{n-1}\|\\
&\quad +|\beta_n - \beta_{n-1}|\|K_{n-1}u_{n-1}\| + \gamma_n\|u_n - u_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|J_s^{\sum A,B} u_{n-1}\|\\
&\le \alpha_n\|x_n - x_{n-1}\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|x_{n-1}\| + \beta_n\|K_n u_{n-1} - K_{n-1}u_{n-1}\|\\
&\quad +|\beta_n - \beta_{n-1}|\|K_{n-1}u_{n-1}\| + |\gamma_n - \gamma_{n-1}|\|J_s^{\sum A,B} u_{n-1}\|.
\end{aligned} \tag{3.4}$$

Since $u_n = T_{r_n}^{\sum} y_n$, therefore, using the definition of $T_{r_n}^{\sum}$, we have

$$\sum_{i=1}^N a_i F_i\left(T_{r_n}^{\sum} y_n, y\right) + \frac{1}{r_n}\left\langle y - T_{r_n}^{\sum} y_n, T_{r_n}^{\sum} y_n - y_n\right\rangle \ge 0, \quad \forall y \in C, \tag{3.5}$$

and

$$\sum_{i=1}^N a_i F_i\left(T_{r_{n+1}}^{\sum} y_{n+1}, y\right) + \frac{1}{r_{n+1}}\left\langle y - T_{r_{n+1}}^{\sum} y_{n+1}, T_{r_{n+1}}^{\sum} y_{n+1} - y_{n+1}\right\rangle \ge 0, \quad \forall y \in C. \tag{3.6}$$

From (3.5) and (3.6), it follows that:

$$\sum_{i=1}^N a_i F_i\left(T_{r_n}^{\sum} y_n, T_{r_{n+1}}^{\sum} y_{n+1}\right) + \frac{1}{r_n}\langle T_{r_{n+1}}^{\sum} y_{n+1} - T_{r_n}^{\sum} y_n, T_{r_n}^{\sum} y_n - y_n\rangle \ge 0, \quad \forall y \in C, \tag{3.7}$$

and

$$\begin{aligned}
&\sum_{i=1}^N a_i F_i\left(T_{r_{n+1}}^{\sum} y_{n+1}, T_{r_n}^{\sum} y_n\right)\\
&\quad + \frac{1}{r_{n+1}}\left\langle T_{r_n}^{\sum} y_n - T_{r_{n+1}}^{\sum} y_{n+1}, T_{r_{n+1}}^{\sum} y_{n+1} - y_{n+1}\right\rangle \ge 0, \quad \forall y \in C.
\end{aligned} \tag{3.8}$$

From (3.7), (3.8), and monotonicity of $\sum_{i=1}^N a_i F_i$, we have

$$\frac{1}{r_n}\left\langle T_{r_{n+1}}^{\sum} y_{n+1} - T_{r_n}^{\sum} y_n, T_{r_n}^{\sum} y_n - y_n\right\rangle + \frac{1}{r_{n+1}}\left\langle T_{r_n}^{\sum} y_n - T_{r_{n+1}}^{\sum} y_{n+1}, T_{r_{n+1}}^{\sum} y_{n+1} - y_{n+1}\right\rangle \ge 0,$$

which follows that

$$\left\langle T_{r_n}^{\sum} y_n - T_{r_{n+1}}^{\sum} y_{n+1}, \ \frac{T_{r_{n+1}}^{\sum} y_{n+1} - y_{n+1}}{r_{n+1}} - \frac{T_{r_n}^{\sum} y_n - y_n}{r_n}\right\rangle \ge 0.$$

It follows that

$$\left\langle T_{r_{n+1}}^{\sum} y_{n+1} - T_{r_n}^{\sum} y_n, \ T_{r_n}^{\sum} y_n - T_{r_{n+1}}^{\sum} y_{n+1} + T_{r_{n+1}}^{\sum} y_{n+1} - y_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}^{\sum} y_{n+1} - y_{n+1}\right)\right\rangle \ge 0.$$

It follows that

$$\begin{aligned}\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\|^2 &\le \left\langle T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n,\ T^{\sum}_{r_{n+1}}y_{n+1}-y_n-\frac{r_n}{r_{n+1}}\big(T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\big)\right\rangle\\
&\le \left\langle T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n,\ y_{n+1}-y_n+\big(1-\frac{r_n}{r_{n+1}}\big)\big(T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\big)\right\rangle\\
&\le \left\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\right\|\left\|y_{n+1}-y_n+\big(1-\frac{r_n}{r_{n+1}}\big)\big(T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\big)\right\|\\
&\le \left\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\right\|\left\{\|y_{n+1}-y_n\|+\left|1-\frac{r_n}{r_{n+1}}\right|\|T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\|\right\}\\
&\le \left\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\right\|\left\{\|y_{n+1}-y_n\|+\frac{1}{r_{n+1}}|r_{n+1}-r_n|\|T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\|\right\}\\
&\le \left\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\right\|\left\{\|y_{n+1}-y_n\|+\frac{1}{d}|r_{n+1}-r_n|\|T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\|\right\},\end{aligned}$$

which implies

$$\left\|T^{\sum}_{r_{n+1}}y_{n+1}-T^{\sum}_{r_n}y_n\right\| \le \|y_{n+1}-y_n\|+\frac{1}{d}|r_{n+1}-r_n|\|T^{\sum}_{r_{n+1}}y_{n+1}-y_{n+1}\|,$$

which follows that

$$\|u_{n+1}-u_n\| \le \|y_{n+1}-y_n\|+\frac{1}{d}|r_{n+1}-r_n|\|u_{n+1}-y_{n+1}\|,$$

Which implies that

$$\|u_n-u_{n-1}\| \le \|y_n-y_{n-1}\|+\frac{1}{d}|r_n-r_{n-1}|\|u_n-y_n\|. \tag{3.9}$$

From (3.1) and (3.9), we have

$$\begin{aligned}\|u_n-u_{n-1}\| \le\ &\|x_n-x_{n-1}\|+\theta_n\|x_n-x_{n-1}\|-\theta_{n-1}\|x_{n-1}-x_{n-2}\|\\
&+\frac{1}{d}|r_n-r_{n-1}|\|u_n-y_n\|. \le (1+\theta_n)\|x_n-x_{n-1}\|+\frac{1}{d}|r_n-r_{n-1}|\|u_n-y_n\|.\end{aligned} \tag{3.10}$$

Now, from (3.4) and (3.10), we have

$$\begin{aligned}\|x_{n+1}-x_n\| \le\ &(1-\alpha_n\theta_n)\|x_n-x_{n-1}\|+\theta_n\|x_n-x_{n-1}\|\\
&+\frac{1-\alpha_n}{d}|r_n-r_{n-1}|\|u_n-y_n\|+|\alpha_n-\alpha_{n-1}|\|x_{n-1}\|\\
&+\beta_n\|K_nu_{n-1}-K_{n-1}u_{n-1}\|+|\beta_n-\beta_{n-1}|\|K_{n-1}u_{n-1}\|\\
&+|\gamma_n-\gamma_{n-1}|\|J_s^{\sum A,B}u_{n-1}\|.\end{aligned} \tag{3.11}$$

Following the lines of Lemma 2.11 in Kangtunyakarn (2011), we have

$$K_nu_{n-1}-K_{n-1}u_{n-1}=\lambda_N\big(T_NK_{n-1}u_{n-1}-K_{n-1}u_{n-1}\big).$$

Since $\lambda_N\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|K_nu_{n-1}-K_{n-1}u_{n-1}\|=0. \tag{3.12}$$

From (3.11), (3.12), Lemma 2.11, and conditions (i), (iii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.13}$$

Step 3. We will show that $q\in\bigcap_{i=1}^N(A_i+B)^{-1}(0)$.

From Lemma 2.3, we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^{\sum A,B} u_n - p\|^2 \\
&= \|\alpha_n (x_n - p) + \beta_n (K_n u_n - p) + \gamma_n (J_s^{\sum A,B} u_n - p)\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|K_n u_n - p\|^2 + \gamma_n \|J_s^{\sum A,B} u_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + \beta_n \|K_n u_n - p\|^2 \\
&\quad + \gamma_n \Bigg(\|u_n - p\|^2 - s \sum_{i=1}^{N} b_i (2\eta - s) \|A_i u_n - A_i p\|^2 \\
&\quad - \|u_n - J_s^{\sum A,B} u_n + s \sum_{i=1}^{N} b_i A_i p - s \sum_{i=1}^{N} b_i A_i u_n\| \Bigg) \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 - \gamma_n s \sum_{i=1}^{N} b_i (2\eta - s) \|A_i u_n - A_i p\|^2 \\
&\quad - \gamma_n \|u_n - J_s^{\sum A,B} u_n + s \sum_{i=1}^{N} b_i A_i p - s \sum_{i=1}^{N} b_i A_i u_n\| \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\| + (1-\alpha_n)\theta_n \|x_n - x_{n-1}\|)^2 \\
&\quad - \gamma_n s \sum_{i=1}^{N} b_i (2\eta - s) \, \|A_i u_n - A_i p\|^2 \\
&\quad - \gamma_n \left\| u_n - J_s^{\sum A,B} u_n + s \sum_{i=1}^{N} b_i A_i p - s \sum_{i=1}^{N} b_i A_i u_n \right\| \\
&\le \|x_n - p\|^2 + 2(1-\alpha_n)^2 \theta_n \langle x_n - x_{n-1}, y_n - p \rangle \\
&\quad - \gamma_n s \sum_{i=1}^{N} b_i (2\eta - s) \|A_i u_n - A_i p\|^2 \\
&\quad - \gamma_n \left\| u_n - J_s^{\sum A,B} u_n + s \sum_{i=1}^{N} b_i A_i p - s \sum_{i=1}^{N} b_i A_i u_n \right\|. \qquad (3.14)
\end{aligned}
$$

Now, from (3.14), we obtain

$$
\begin{aligned}
&\gamma_n s \sum_{i=1}^{N} b_i (2\eta - s) \|A_i u_n - A_i p\|^2 \le \|x_{n+1} - x_n\| (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad + 2(1-\alpha_n)^2 \theta_n \langle x_n - x_{n-1}, y_n - p \rangle \\
&\quad - \gamma_n \left\| u_n - J_s^{\sum A,B} u_n + s \sum_{i=1}^{N} b_i A_i p - s \sum_{i=1}^{N} b_i A_i u_n \right\|. \\
&\le \|x_{n+1} - x_n\| (\|x_n - p\| + \|x_{n+1} - p\|) + 2(1-\alpha_n)^2 \theta_n \langle x_n - x_{n-1}, y_n - p \rangle.
\end{aligned}
$$

From condition (i) and (3.13), it follows that

$$
\lim_{n \to \infty} \|A_i u_n - A_i p\| = 0. \qquad (3.15)
$$

By the following same line as above and using (3.15), we have

$$\lim_{n\to\infty} \|u_n - J_s^{\sum A,B} u_n\| = 0. \tag{3.16}$$

Since $p \in \Omega$ and $T_r^{\sum}$ is firmly nonexpansive, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{\sum} y_n - T_{r_n}^{\sum} p\|^2 \le \left\langle T_{r_n}^{\sum} y_n - T_{r_n}^{\sum} p, y_n - p\right\rangle \\ &= \langle u_n - p, y_n - p\rangle \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|y_n - u_n\|^2\}.\end{aligned}$$

Hence, it follows that

$$\|u_n - p\|^2 \le \|y_n - p\|^2 - \|y_n - u_n\|^2. \tag{3.17}$$

Now, from (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + \beta_n(K_n u_n - p) + \gamma_n(J_s^{\sum A,B} u_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + \beta_n\|K_n u_n - p\|^2 + \gamma_n\|J_s^{\sum A,B} u_n - p)\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2.\end{aligned}$$

From (3.17) and (3.2), above inequality can be written as

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - (1-\alpha_n)\|y_n - u_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\|x_n - p\| + \theta_n\|x_n - x_{n-1}\|)^2 - (1-\alpha_n)\|y_n - u_n\|^2 \\ &\le \|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\langle x_n - x_{n-1}, y_n - p\rangle - (1-\alpha_n)\|y_n - u_n\|^2.\end{aligned} \tag{3.18}$$

From (3.13), (3.18), and condition (i), it follows that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0. \tag{3.19}$$

From the definition of y_n and condition (i), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = \lim_{n\to\infty} \theta_n\|x_n - x_{n-1}\| = 0. \tag{3.20}$$

From (3.19), we obtain

$$\|u_n - x_n\| \le \|u_n - y_n\| + \|y_n - x_n\| \to 0, \tag{3.21}$$

as $n \to \infty$. From (3.13) and (3.21), it follows that

$$\|x_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \|x_n - u_n\| \to 0, \tag{3.22}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded and H is reflexive, $w_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x,\ \{x_{n_i}\} \subset \{x_n\}\}$ is nonempty. Let $q \in w_w(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to q. Let $p \in w_w(x_n)$ and $\{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_m} \rightharpoonup p$. From (3.21), we also have $u_{n_i} \rightharpoonup q$ and $u_{n_m} \rightharpoonup p$. Since $J_s^{\sum A,B}$ is nonexpansive, by Lemma 2.4, we have $p, q \in \bigcap_{i=1}^N (A_i + B)^{-1}(0)$. Applying Lemma 2.6, we obtain $p = q$.

Step 4. We will show that $q \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) = \mathrm{Fix}(K)$.
Now, from Lemma 2.1 and (3.18), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + \beta_n(K_n u_n - p) + \gamma_n(J_s^{\sum A,B} u_n - p)\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + \beta_n\|K_n u_n - p\|^2 \\ &\quad + \gamma_n\|J_s^{\sum A,B} u_n - p\|^2 - \alpha_n\beta_n\|x_n - K_n u_n\| \\ &\quad - \beta_n\gamma_n\|K_n u_n - J_s^{\sum A,B} u_n\| - \gamma_n\alpha_n\|J_s^{\sum A,B} u_n - x_n\|^2 \\ &\leq \|x_n - p\|^2 + 2(1-\alpha_n)\theta_n\langle x_n - x_{n-1}, y_n - p\rangle - \alpha_n\beta_n\|x_n - K_n u_n\| \\ &\quad - \beta_n\gamma_n\|K_n u_n - J_s^{\sum A,B} u_n\| - \gamma_n\alpha_n\|J_s^{\sum A,B} u_n - x_n\|^2. \end{aligned} \tag{3.23}$$

From (3.13), and conditions (i), (ii), we obtain

$$\lim_{n\to\infty} \|K_n u_n - J_s^{\sum A,B} u_n\| = 0. \tag{3.24}$$

From (3.11), we have

$$\begin{aligned}\|x_{n+1} - K_n u_n\| &= \left\|\alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^{\sum A,B} u_n - K_n u_n\right\| \\ &= \left\|\alpha_n(x_n - K_n u_n) + \gamma_n(J_s^{\sum A,B} u_n - K_n u_n)\right\|.\end{aligned}$$

In addition, we can estimate

$$\begin{aligned}\|K_n u_n - u_n\| &\leq \|K_n u_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq \alpha_n\|x_n - K_n u_n\| + \gamma_n\|J_s^{\sum A,B} u_n - K_n u_n\| + \|x_{n+1} - x_n\|. \end{aligned} \tag{3.25}$$

From (3.13), (3.24), (3.25), and condition (ii), we obtain

$$\lim_{n\to\infty} \|K_n u_n - u_n\| = 0. \tag{3.26}$$

Now, suppose to the contrary that $q \notin \mathrm{Fix}(K)$, i.e., $Kq \neq q$ and by Lemma 2.6, we see that

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - q\| &< \liminf_{i\to\infty} \|u_{n_i} - Kq\| \\ &\leq \liminf_{i\to\infty} \{\|u_{n_i} - Ku_{n_i}\| + \|Ku_{n_i} - Kq\|\} \\ &\leq \liminf_{i\to\infty} \{\|u_{n_i} - Ku_{n_i}\| + \|u_{n_i} - q\|\}. \end{aligned} \tag{3.27}$$

On the other hand, we have

$$\|Ku_n - u_n\| \leq \|Ku_n - K_n u_n\| + \|K_n u_n - u_n\| \leq \sup_{y\in C} \|Ky - K_n y\| + \|K_n u_n - u_n\|. \tag{3.28}$$

Using Remark 2.1 and (3.26), we obtain that $\lim_{n\to\infty} \|Ku_n - u_n\| = 0$. From (3.27), we obtain

$$\liminf_{i\to\infty} \|u_{n_i} - q\| < \liminf_{i\to\infty} \|u_{n_i} - q\|,$$

which is a contradiction, so we have $q \in \mathrm{Fix}(K) = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$.

Step 5. Show that $q \in \bigcap_{i=1}^N \mathrm{EP}(F_i)$.
Since $u_n = T_{r_n}^{\sum} y_n$, we have

$$\sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C.$$

Since $\sum_{i=1}^N a_i F_i$ satisfies Assumption 2.1, so from monotonicity of $\sum_{i=1}^N a_i F_i$, we get

$$\frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge \sum_{i=1}^N a_i F_i(y, u_n), \quad \forall y \in C. \tag{3.29}$$

Since $\liminf_{n\to\infty} r_n > 0$ and from (3.19), it follows that

$$\lim_{n\to\infty} \frac{\|u_n - y_n\|}{r_n} = 0. \tag{3.30}$$

It follows from (3.29), (3.30), and (A4) that

$$\sum_{i=1}^N a_i F_i(y, q) \le 0, \quad \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, let $y_t := ty + (1-t)q$. Since $y \in C$, we have $y_t \in C$, and hence, $\sum_{i=1}^N a_i F_i(y_t, q) \le 0$. Therefore, we have

$$\begin{aligned}
0 &= \sum_{i=1}^N a_i F_i(y_t, y_t)\\
&= \sum_{i=1}^N a_i F_i(y_t, ty + (1-t)q)\\
&\le t\sum_{i=1}^N a_i F_i(y_t, y) + (1-t)\sum_{i=1}^N a_i F_i(y_t, q))\\
&\le t\sum_{i=1}^N a_i F_i(y_t, y).
\end{aligned}$$

Dividing by t, we get

$$\sum_{i=1}^N a_i F_i(ty + (1-t)q, y) \ge 0 \quad \forall y \in C.$$

Letting $t \downarrow 0$ and from (A3), we get

$$\sum_{i=1}^N a_i F_i(q, y) \ge 0 \quad \forall y \in C.$$

Therefore, $q \in \mathrm{EP}(\sum_{i=1}^N a_i F_i)$. Hence, by Lemma 2.10, we obtain $q \in \bigcap_{i=1}^N \mathrm{EP}(F_i)$. Therefore, $q \in \Omega$. This completes the proof. □

As direct consequences of Theorem 3.1, we have the following corollaries.

Corollary 3.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots,$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in N$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$, and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ for every $x \in C$. For every $i = 1, 2, \ldots, N$, let $A : H \to H$ be α-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone mapping. Assume that $\Omega := (A + B)^{-1}(0) \bigcap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \bigcap \mathrm{EP}(F) \neq \emptyset$. For given initial points $x_0, x_1 \in H$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^B(I - sA)u_n, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$ and $\{\theta_n\} \subset [0, \theta], \theta \in [0, 1]$, $\liminf_{n\to\infty} r_n > 0$ and $0 < s < 2\alpha$. Suppose that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, sequence $\{x_n\}$ converges weakly to $q \in \Omega$.

Proof By taking $F_i = F$ and $A_i = A$, $\forall i = 1, 2, \ldots N$, in Theorem 3.1, the conclusion of Corollary 3.1 is followed. □

Corollary 3.2 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself with $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots,$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in N$, let K_n be the K-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$, and let K be the K-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ for every $x \in C$. For every $i = 1, 2, \ldots, N$, let $A : H \to H$ be α-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone mapping. Assume that $\Omega := (A + B)^{-1}(0) \bigcap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. For given initial points $x_0, x_1 \in H$, let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = \alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^B(I - sA)u_n, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$ and $\{\theta_n\} \subset [0, \theta], \theta \in [0, 1]$, $0 < s < 2\alpha$. Suppose that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, sequence $\{x_n\}$ converges weakly to $q \in \Omega$.

Proof By taking $F_i \equiv 0$ and $A_i = A$, $\forall i = 1, 2, \ldots N$, in Theorem 3.1, the conclusion of Corollary 3.2 is followed. □

4 Applications

In this section, we discuss various applications of inertial forward–backward method to establish weak convergence result for finding a common element of the fixed point set of infinite family of nonexpansive mappings, solution sets of a combination of equilibrium problem, and k-strict pseudo-contraction mapping in the setting of Hilbert space. To prove these results, we need the following results.

Definition 4.1 A mapping $T : C \to C$ is said to be a k-strict pseudo-contraction mapping, if there exists $k \in [0, 1)$, such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \;\; \forall x, y \in C.$$

Lemma 4.1 Zhou (2008) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a k-strict pseudo-contraction. Define $S : C \to C$ by $Sx = ax + (1-a)Tx$, for each $x \in C$. Then, S is nonexpansive, such that* $\text{Fix}(S) = \text{Fix}(T)$, *for $a \in [k, 1)$.*

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. For each $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of k_i-strictly pseudo-contractive mappings of C into itself. Define a mapping T_{k_i} by $T_{k_i} = k_i x + (1 - k_i)T_i x$, $\forall x \in C, i \in \mathbb{N}$ with $\bigcap_{i=1}^{\infty} \text{Fix}(T_{k_i}) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots,$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots$, with $\sum_{i=1}^{\infty} \lambda_i < \infty$. For every $n \in \mathbb{N}$, let K_n be the K-mapping generated by $T_{k_1}, T_{k_2}, \ldots, T_{k_n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$, and let K be the K-mapping generated by $T_{k_1}, T_{k_2}, \ldots$ and $\lambda_1, \lambda_2, \ldots$ for every $x \in C$. For every $i = 1, 2, \ldots, N$, let $A_i : H \to H$ be α_i-inverse strongly monotone mapping with $\eta = \min_{i=1,\ldots,N}\{\alpha_i\}$ and $B : H \to 2^H$ be a maximal monotone mapping. Assume that $\Omega := \bigcap_{i=1}^{N}(A_i + B)^{-1}(0) \bigcap \bigcap_{i=1}^{\infty} \text{Fix}(T_i) \bigcap \bigcap_{i=1}^{N} \text{EP}(F_i) \neq \emptyset$. For given initial points $x_0, x_1 \in H$, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n K_n u_n + \gamma_n J_s^B \left(I - s\sum_{i=1}^{N} b_i A_i\right) u_n, \end{cases} \tag{4.1}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\} \subset [0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, for all $n \geq 1$ and $\{\theta_n\} \subset [0, \theta], \theta \in [0, 1]$, $\liminf_{n\to\infty} r_n > 0$ and $0 < s < 2\eta$, where $\eta = \min_{i=1,\ldots,N}\{\alpha_i\}$. Suppose that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then, sequence $\{x_n\}$ converges weakly to $q \in \Omega$.

Proof For every $i \in \mathbb{N}$, by Lemma 4.1, we have that T_{k_i} is a nonexpansive mapping and $\bigcap_{i=1}^{\infty} \text{Fix}(T_{k_i}) = \bigcap_{i=1}^{\infty} \text{Fix}(T_i)$. From Theorem 3.1 and Lemma 2.8, the conclusion of Theorem 4.1 is followed.

Now, we consider a property of finite family of strictly pseudo-contractive mappings in Hilbert space as follows: □

Proposition 4.1 Fan et al. (2009) *Let C be a nonempty closed convex subset of a real Hilbert space H.*

(i) *For any integer* $N \geq 1$, *let, for each* $1 \leq i \leq N$, $S_i : C \to H$ *is* k_i*-strict pseudo-contraction for some* $0 \leq k_i < 1$. *Let* $\{b_i\}_i^N$ *is a positive sequence, such that* $\sum_{i=1}^N b_i = 1$. *Then,* $\sum_{i=1}^N b_i S_i$ *is a k-strict pseudo-contraction, with* $k = \max_{i=1,\dots,N}\{k_i\}$;
(ii) *Let* $\{S_i\}_i^N$ *and* $\{b_i\}_i^N$ *be given as in (i) above. Suppose that* $\{S_i\}_i^N$ *has a common fixed point. Then*

$$\mathrm{Fix}\left(\sum_{i=1}^N b_i S_i\right) = \bigcap_{i=1}^N \mathrm{Fix}(S_i).$$

Theorem 4.2 *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H. *For each* $i = 1, 2, \dots, N$, *let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying Assumption 2.1. Let* $\{S_i\}_{i=1}^N$ *be an finite family of* k_i*-strictly pseudo-contractive mappings of* C *into itself with* $k = \max_{i=1,\dots,\mathbb{N}}\{k_i\}$. *Let* $\{T_i\}_{i=1}^\infty$ *be an infinite family of nonexpansive mappings with* $\bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$, *and let* $\lambda_1, \lambda_2, \dots,$ *be real numbers, such that* $0 < \lambda_i < 1$ *for every* $i = 1, 2, \dots,$ *with* $\sum_{i=1}^\infty \lambda_i < \infty$. *For every* $n \in \mathbb{N}$, *let* K_n *be the* K*-mapping generated by* $T_1, T_2, \dots, T_n$ *and* $\lambda_1, \lambda_2, \dots, \lambda_N$, *and let* K *be the* K*-mapping generated by* $T_1, T_2, \dots$ *and* $\lambda_1, \lambda_2, \dots$ *for every* $x \in C$. *Assume that* $\Omega := \bigcap_{i=1}^N \mathrm{Fix}(S_i) \bigcap \bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^N \mathrm{EP}(F_i) \neq \emptyset$. *For given initial points* $x_0, x_1 \in H$, *let the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be generated by*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n K_n u_n + \gamma_n \left((1-s)u_n + s\sum_{i=1}^N b_i S_i u_n\right), \end{cases} \tag{4.2}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\} \subset [0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 1$ *and* $\{\theta_n\} \subset [0, \theta], \theta \in [0,1]$, $\lim\inf_{n\to\infty} r_n > 0$*and* $0 < s < 1 - k$. *Suppose that the following conditions hold:*

(i) $\sum_{n=1}^\infty \theta_n \|x_n - x_{n-1}\| < \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n < \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(iii) $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^\infty |\gamma_{n+1} - \gamma_n| < \infty$.

Then, sequence $\{x_n\}$ *converges weakly to* $q \in \Omega$.

Proof Let $A_i = I - S_i$ and $B = 0$ in Theorem 3.1, and then, we have that A_i is α_i-inverse strongly monotone with $\frac{1-k}{2}$. Now, we show that $\bigcap_{i=1}^N (A_i + B)^{-1}(0) = \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. Since $A_i = I - S_i$ and $B = 0$, therefore, using Theorem 2.1 and Proposition 4.1, we have

$$\begin{aligned} x \in \bigcap_{i=1}^N (A_i + B)^{-1}(0) &\Leftrightarrow x \in \left(\sum_{i=1}^N b_i A_i + B\right)^{-1}(0) \Leftrightarrow 0 \in \sum_{i=1}^N b_i A_i x + Bx \\ &\Leftrightarrow 0 \in \sum_{i=1}^N b_i A_i x \Leftrightarrow 0 \in \sum_{i=1}^N b_i (I - S_i)x \\ &\Leftrightarrow x = \sum_{i=1}^N b_i S_i x \Leftrightarrow x \in \mathrm{Fix}\left(\sum_{i=1}^N b_i S_i x\right) \Leftrightarrow x \in \bigcap_{i=1}^N \mathrm{Fix}(S_i). \end{aligned}$$

It follows that

$$\bigcap_{i=1}^N (A_i + B)^{-1}(0) = \bigcap_{i=1}^N \mathrm{Fix}(S_i).$$

We know that $J_s^B(I - s\sum_{i=1}^N b_iA_i)u_n = (I + sB)^{-1}(I - s\sum_{i=1}^N b_iA_i)u_n$.
Since $B = 0$, we have $J_s^B\left(I - s\sum_{i=1}^N b_iA_i\right)u_n = u_n - s\sum_{i=1}^N b_iA_iu_n$

$$= u_n - s\sum_{i=1}^N b_i(I - S_i)u_n$$

$$= (1 - s)u_n + s\sum_{i=1}^N b_iS_iu_n.$$

Since $s \in (0, 1-k) \subset (0, 1)$, then $(1-s)u_n + s\sum_{i=1}^N b_iS_iu_n \in H$. Therefore, from Theorem 3.1, we obtain the desired result. □

5 Example and numerical results

Finally, we give the following numerical example to illustrate Theorems 3.1 and 4.2.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers. For each $i = 1, 2, \ldots, N$, let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_i(x, y) = i(y^2 - 2x^2 + xy + 3x - 3y).$$

Furthermore, let $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, such that $\sum_{i=1}^N a_i = 1$, for every $i = 1, 2, \ldots, N$. Then, we have

$$\begin{aligned}\sum_{i=1}^N a_iF_i(x, y) &= \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N}\right)i(y^2 - 2x^2 + xy + 3x - 3y)\\ &= \Psi(y^2 - 2x^2 + xy + 3x - 3y),\end{aligned}$$

where $\Psi = \sum_{i=1}^N \left(\frac{4}{5^i} + \frac{1}{N5^N}\right)i$.

It is easy to check that $\sum_{i=1}^N a_iF_i$ satisfies all the conditions of Theorem 3.1 and $\mathrm{EP}\big(\sum_{i=1}^N a_iF_i\big) = \bigcap_{i=1}^N \mathrm{EP}(F_i) = \{1\}$.

For each $i = 1, 2, \ldots, N$, let $A_i : \mathbb{R} \to \mathbb{R}$ be defined by $A_i(x) = \frac{x-(4i+1)}{i}$ and $B : \mathbb{R} \to 2^{\mathbb{R}}$ is defined by $B(x) = \{4x\}$.

It is easy to observe that A_i is i-inverse strongly monotone mapping with $\eta = \min_{i=1,\ldots,N}\{i\} = 1$ and $\bigcap_{i=1}^N (A_i + B)^{-1}(0) = \{1\}$.

Further, let $b_i = \frac{3}{4^i} + \frac{1}{N4^N}$, such that $\sum_{i=1}^N b_i = 1$, for every $i = 1, 2, \ldots, N$. It is easy to check that A_i and B satisfy all the conditions of Theorem 3.1 and $\left(\sum_{i=1}^N b_iA_i + B\right)^{-1}(0) = \bigcap_{i=1}^N (A_i + B)^{-1}(0) = \{1\}$.

Let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ is defined by $T_i(x) = \frac{x+i}{i+1}$, $i = 1, 2, \ldots,$. It is easy to check that $\{T_i\}_{i=1}^\infty$ is infinite family of nonexpansive mapping. For each i, let $\lambda_i = \frac{i}{i+1}$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots,$ with $\sum_{i=1}^\infty \lambda_i < \infty$. Since K_n is

K-mapping generated by $T_1, T_2, \ldots,$ and $\lambda_1, \lambda_2, \ldots$; therefore, we obtain

$$\begin{aligned}
U_0 u_n &= u_n, \\
U_1 u_n &= \frac{1}{2}\left(\frac{U_0 u_n + 1}{2}\right) + \frac{1}{2} U_0 u_n, \\
U_2 u_n &= \frac{2}{3}\left(\frac{U_1 u_n + 2}{3}\right) + \frac{1}{3} U_1 u_n, \\
&\vdots \\
K_n u_n &= U_N u_n = \frac{N}{N+1}\left(\frac{U_{N-1} u_n + N}{N+1}\right) + \frac{1}{N+1} U_{N-1} u_n.
\end{aligned}$$

It is easy to see that $\bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) = \{1\}$. Therefore, it is easy to see that

$$\bigcap_{i=1}^{N} (A_i + B)^{-1}(0) \bigcap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^{N} \mathrm{EP}(F_i) = \{1\}.$$

By Lemma 2.9, we have that $T_{r_n}^{\sum} x$, is a single-valued mapping for each $x \in \mathbb{R}$. Hence, for $r_n > 0$, there exist sequences $\{x_n\}$ and $\{u_n\}$, such that

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in \mathbb{R},$$

which is equivalent to

$$P(y) := \Psi r_n y^2 + (\Psi u_n r_n + u_n - y_n - 3\Psi r_n)y + 3\Psi r_n u_n - u_n^2 - 2\Psi r_n u_n^2 + u_n y_n \geq 0.$$

Since $P(y) = ay^2 + by + c \geq 0$, for all $y \in \mathbb{R}$, then $b^2 - 4ac = (u_n - 3\Psi r_n + 3\Psi r_n u_n - y_n)^2 \leq 0$, which yields $(u_n - 3\Psi r_n + 3\Psi r_n u_n - y_n)^2 = 0$. Therefore, for each $r_n > 0$, it implies that

$$u_n = T_{r_n}^{\sum} y_n = \frac{y_n + 3\Psi r_n}{1 + 3\Psi r_n}. \tag{5.1}$$

By choosing $\alpha_n = r_n = \frac{1}{6n}$, $\beta_n = \frac{18n-3}{30n}$, $\gamma_n = \frac{12n-2}{30n}$, $\theta_n = \frac{1}{12}$ and $s = 0.1$ as $0 < s < 2\eta$, where $\eta = \min_{i=1,\ldots,N}\{\alpha_i\} = 1$. It is clear that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\theta_n\}$ for all $n \geq 1$ satisfy all the conditions of Theorem 3.1. For each $n \in \mathbb{N}$, using (5.1), algorithm (3.1) can be re-written as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ u_n = \frac{y_n + 3\Psi r_n}{1 + 3\Psi r_n} \\ x_{n+1} = \frac{1}{6n} x_n + \frac{18n-3}{30n} K_n u_n + \frac{12n-2}{30n}\left(\frac{u_n - s\sum_{i=1}^{N}\left(\frac{4(u_n - 4i - 1)}{i5^i} + \frac{u_n - 4i - 1}{iN5^N}\right)}{1+4s}\right). \end{cases} \tag{5.2}$$

By taking $x_0 = 2$, $x_1 = 0$ with $N = 2$ and $N = 20$ for $n = 25$ iterations in the algorithm (5.2), we have the numerical results in Table 1 and Fig. 1.

We can conclude that the sequence $\{x_n\}$ converges to 1, as shown in Table 1 and Fig. 1. It can also be easily seen that sequence $\{x_n\}$ for $N = 20$ converges more quickly than for $N = 2$.

Figure 2 shows that the sequence generated by our proposed inertial forward–backward method proposed in Theorem 3.1 has a better convergence rate than standard forward–backward method (i.e., at $\theta_n = 0$).

Table 1 Values of $\{x_n\}$ with initial values $x_0 = 2$ and $x_1 = 0$

Iterations	for $N = 2$	for $N = 20$
1	2.000000000000000	2.000000000000000
2	0.000000000000000	0.000000000000000
3	0.396175193050193	0.703954515113034
4	0.565586514836466	0.935116619152469
5	0.738167859284712	0.987425727070499
6	0.817565948385566	0.997738072350744
7	0.882698568776565	0.999614774298055
8	0.918904298444017	0.999937291893665
9	0.946103275118548	0.999990198107920
....	....	....
18	0.998132545998109	1.000000000000013
19	0.998705886266841	1.000000000000011
20	0.999101388201502	1.000000000000003
21	0.999375914282270	1.000000000000001
22	0.999565997388127	1.000000000000000
23	0.999698073723197	1.000000000000000
24	0.999789760416796	1.000000000000000
25	0.999853541198729	1.000000000000000

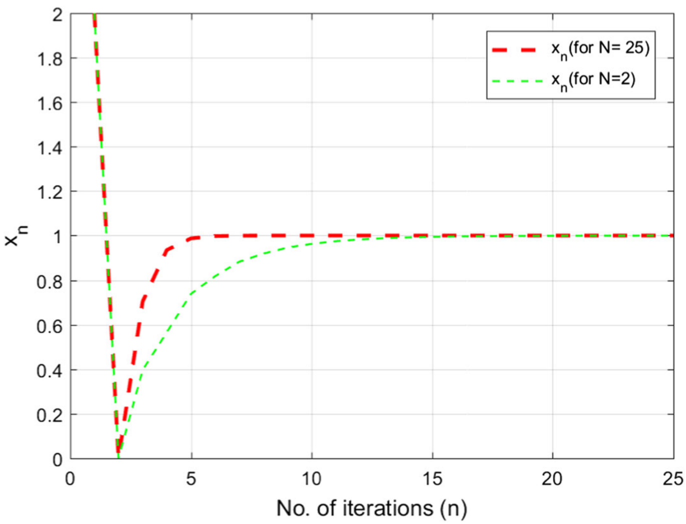

Fig. 1 Convergence of x_n

Example 5.2 Let $\mathbb{R}$ be the set of real numbers. For each $i = 1, 2, \ldots, N$, let $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_i(x, y) = i(y^2 - 3x^2 + 2xy).$$

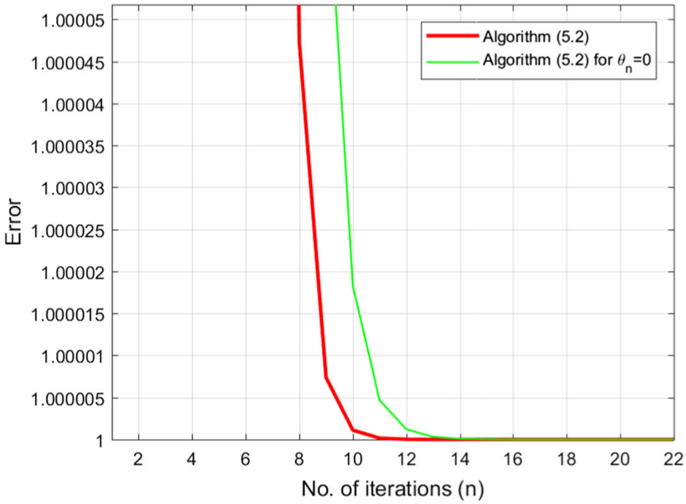

Fig. 2 Error plot for Example 5.1

Furthermore, let $a_i = \frac{4}{5^i} + \frac{1}{N5^N}$, such that $\sum_{i=1}^{N} a_i = 1$, for every $i = 1, 2, \ldots, N$. Then, it is easy to check that $\sum_{i=1}^{N} a_i F_i$ satisfies all the conditions of Theorem 3.1 and $\mathrm{EP}\big(\sum_{i=1}^{N} a_i F_i\big) = \bigcap_{i=1}^{N} \mathrm{EP}(F_i) = \{0\}$.

Let the mapping $T_i : \mathbb{R} \to \mathbb{R}$ is defined by $T_i(x) = \frac{ix}{i+1}$, $i = 1, 2, \ldots,$. It is easy to check that $\{T_i\}_{i=1}^{\infty}$ is infinite family of nonexpansive mapping. For each i, let $\lambda_i = \frac{i}{i+1}$ be real numbers, such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots,$ with $\sum_{i=1}^{\infty} \lambda_i < \infty$. Since K_n is K-mapping generated by $T_1, T_2, \ldots,$ and $\lambda_1, \lambda_2, \ldots$; therefore, we obtain

$$
\begin{aligned}
U_0 u_n &= u_n,\\
U_1 u_n &= \left(\frac{1}{2}\right)^2 U_0 u_n + \frac{1}{2} U_0 u_n,\\
U_2 u_n &= \left(\frac{2}{3}\right)^2 U_1 u_n + \frac{1}{3} U_1 u_n,\\
&\vdots\\
K_n u_n &= \left(\frac{N}{N+1}\right)^2 U_{N-1} u_n + \frac{1}{N+1} U_{N-1} u_n.
\end{aligned}
$$

For each $i = 1, 2, \ldots, N$, let a mapping $S_i : \mathbb{R} \to \mathbb{R}$ is defined by

$$
S_i(x) = \begin{cases} -ix, & x \in [0, \infty) \\ x, & x \in (-\infty, 0), \end{cases}
$$

be a finite family of $\frac{i^2-1}{(i+1)^2}$-strictly pseudo-contractive mappings. Furthermore, let $b_i = \frac{7}{8^i} + \frac{1}{N8^N}$, such that $\sum_{i=1}^{N} b_i = 1$, for every $i = 1, 2, \ldots, N$. It is easy to see that $\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) =$

Table 2 Values of $\{x_n\}$ with initial values $x_0 = 4$ and $x_1 = 4.5$

Iterations	for $N = 4$	for $N = 20$
1	4.000000000000000	4.000000000000000
2	4.500000000000000	4.500000000000000
3	1.576972852434431	1.562907563025210
4	1.492661063741572	0.777545434177588
5	0.889217299128382	0.363630189959213
6	0.701950790348523	0.179497233213674
7	0.489846159899863	0.089171093105929
8	0.371993649959412	0.044958545086448
9	0.274087030573037	0.022850080252797
....	....	....
18	0.024288784733220	0.000063921626604
19	0.018776329242411	0.000033703541445
20	0.014537081975031	0.000017799072803
21	0.011269866145140	0.000009413387070
22	0.008747676862031	0.000004984976315
23	0.006797440864107	0.000002643008673
24	0.005287361741074	0.000001402840999
25	0.004116570600244	0.000000745339455

$\{0\}$. Therefore, it is easy to see that

$$\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \bigcap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \bigcap \bigcap_{i=1}^{N} \mathrm{EP}(F_i) = \{0\}.$$

By Lemma 2.9, for each $x \in \mathbb{R}$, a single-valued mapping $T_{r_n}^{\sum} x$ as Example 5.1, can be computed as

$$u_n = T_{r_n}^{\sum} y_n = \frac{y_n}{1 + 4S_1 r_n}, \tag{5.3}$$

where $S_1 = \sum_{i=1}^{N} (\frac{4}{5^i} + \frac{1}{N5^N})i$. By choosing $\alpha_n = r_n = \frac{1}{6n}$, $\beta_n = \frac{18n-3}{30n}$, $\gamma_n = \frac{12n-2}{30n}$, $\theta_n = \frac{1}{20}$, and $s = 0.1$ as $0 < s < 2\eta$, where $\eta = \min_{i=1,\ldots,N}\{\alpha_i\} = 1$. It is clear that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\theta_n\}$ for all $n \geq 1$ satisfy all the conditions of Theorem 4.2 For each $n \in \mathbb{N}$, using (5.3), algorithm (4.2) can be re-written as follows:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ u_n = \frac{y_n}{1+4S_1 r_n} \\ x_{n+1} = \frac{1}{6n} x_n + \frac{18n-3}{30n} K_n u_n + \frac{12n-2}{30n} \Big((1-s)u_n - s \sum_{i=1}^{N} \big(\frac{7}{8^i} + \frac{1}{N8^N}\big) S_i u_n\Big). \end{cases} \tag{5.4}$$

By taking $x_0 = 4$, $x_1 = 4.5$ with $N = 4$ and $N = 20$ for $n = 25$ iterations in the algorithm (5.4), we have the numerical results in Table 2 and Fig. 3.

We can conclude that the sequence $\{x_n\}$ converges to 0, as shown in Table 2 and Fig. 3. It can also be easily seen that sequence $\{x_n\}$ for $N = 20$ converges more quickly than for $N = 4$.

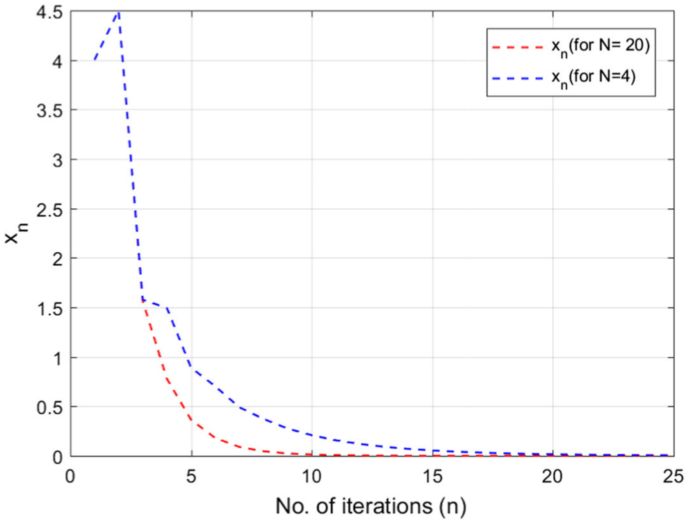

Fig. 3 Convergence of x_n

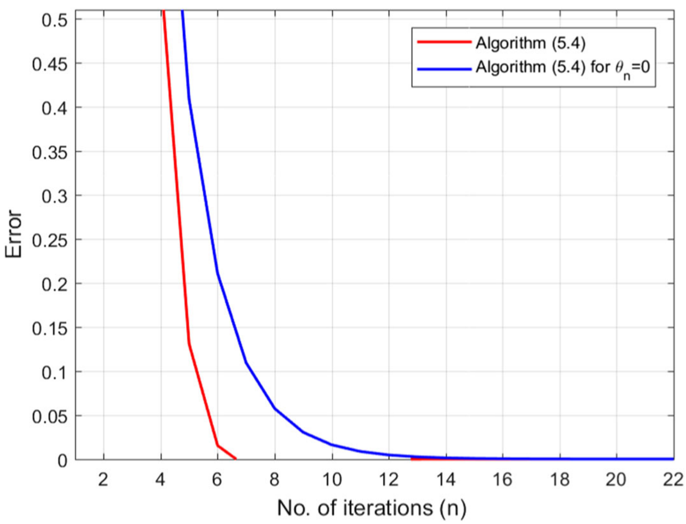

Fig. 4 Error plot for Example 5.2

Figure 4, shows that the sequence generated by our proposed inertial forward–backward method proposed in Theorem 4.2 has a better convergence rate than forward–backward method (i.e., at $\theta_n = 0$).

6 Conclusion

In this work, we established weak convergence result for finding a common element of the fixed point sets of a infinite family of nonexpansive mappings and the solution sets of a combination of equilibrium problems and combination of inclusion problems. It has been illustrated by an example with different choices that our proposed method involving the inertial term converges faster than usual projection method. Finally, we discussed some applications of modified inclusion problems in finding a common element of the set of fixed points of a infinite family of strictly pseudo-contractive mappings and the set of solution of equilibrium problem supported by numerical result. The method and results presented in this paper generalize and unify the corresponding known results in this area (see Cholamjiak 1994; Dong et al. 2017; Khan et al. 2018; Khuangsatung and Kangtunyakarn 2014).

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