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A family of higher order derivative free methods for nonlinear systems with local convergence analysis

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Abstract A wide general class of Ostrowski's families without memory proposed by Behl et al. (Int J Comput Math 90(2):408–422, 2013) is being extended to solve systems of nonlinear equations. This extension uses multidimensional divided differences of first order. Many more new derivative free iterative families with higher order local convergence are presented. In addition, the proposed iterative family for $\alpha_1 = \mathbb{R} - \{0\}$ and $\alpha_2 = 0$ are special cases of Grau et al. (J Comput Appl Math 237:363–372, 2013) for iterative schemes of fourth and sixth orders. The computational efficiency is compared with some known methods. It is proved that the proposed methods are equally competent with their existing counter parts. Moreover, we present the local convergence analysis of the proposed family of methods based on Lipschitz constants and hypotheses on the divided difference of order one in the more general settings of a Banach space. We expand this way the applicability of these methods, since we used higher derivatives to show convergence of the method in Sect. 3 although such derivatives do not appear in these methods. Numerical experiments are performed which support the theoretical results.

Keywords System of nonlinear equations · Order of convergence · Steffensen's method · Computational efficiency · Derivative free methods

Mathematics Subject Classification MSC 65H10 · 41A58 · 65Y20

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1 Introduction

One of the most basic and earliest problems of numerical analysis concerns with finding efficiently and accurately the approximate solution of a nonlinear system

$$G(X) = 0,$$

where $G(X) = (\widetilde{g}_1(X),\ \widetilde{g}_2(X),\ \ldots,\ \widetilde{g}_t(X))^T$, $X = (x_1,\ x_2,\ \ldots,\ x_t)^T$ and $G : \mathbb{R}^t \to \mathbb{R}^t$ is a sufficiently differentiable vector function. Analytical methods for solving such problems are non-existent, and therefore, it is only possible to obtain approximate solutions, by relying on numerical techniques based on iteration procedures. The most simple and common iterative method for this purpose is the Newton's method (Kelley 2003; Traub 1964), which converges quadratically and is defined by

$$X^{k+1} = X^k - \left\{G'(X^k)\right\}^{-1} G(X^k), \quad k = 0, 1, 2, \ldots,$$

where $\left\{G'(X^k)\right\}^{-1}$ is the inverse of first Fréchet derivative $G'(X^k)$ of the function of $G(X)$. The practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods (Kelley 2003; Traub 1964; Amat et al. 2005, 2008, 2010; Behl et al. 2013; Grau-Sánchez et al. 2014; Ostrowski 1960; Ortega and Rheinboldt 1970; Petković 2011; Sharma and Arora 2014). However, the main practical difficulty associated with this method is to calculate first-order derivative at each step of computation, sometimes which is very difficult and time consuming.

In 2013, Behl et al. (2013) have proposed new optimal families of Ostrowski-like methods for solving scalar nonlinear equations having cubic scaling factor of functions in the correction factor and is given by

$$\begin{aligned} y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\ x_{m+1} &= x_m - \frac{f(x_m)}{f'(x_m)} \left[\frac{(\alpha_1^2 + \alpha_1\alpha_2 - \alpha_2^2) f(x_m) f(y_m) - \alpha_1(\alpha_1 - \alpha_2)\{f(x_m)\}^2}{(\alpha_1 f(x_m) - \alpha_2 f(y_m))((2\alpha_1 - \alpha_2) f(y_m) - (\alpha_1 - \alpha_2) f(x_m))} \right], \end{aligned} \tag{1}$$

where $\alpha_1,\ \alpha_2 \in \mathbb{R}$ but choose α_1 and α_2 such that neither $\alpha_1 = 0$ nor $\alpha_1 = \alpha_2$.

In 1964, Traub (1964) introduced the quadratically convergent scheme defined as

$$X^{k+1} = X^k - [Y^k, X^k; G]^{-1} G(X^k), \tag{2}$$

where $Y^k = X^k + \beta G(X^k)$, $\beta \in \mathbb{R} - \{0\}$. $[Y^k, X^k; G]$ is defined as first-order divided difference of G in t dimensional space as an $t \times t$ matrix with elements

$$\begin{aligned} &[Y^k, X^k; G]_{ij} \\ &\quad = \frac{\widetilde{g}_i(y_1^k, y_2^k, \ldots, y_{j-1}^k, y_j^k, x_{j+1}^k, \ldots, x_t^k) - \widetilde{g}_i(y_1^k, y_2^k, \ldots, y_{j-1}^k, x_j^k, x_{j+1}^k, \ldots, x_t^k)}{y_j^k - x_j^k}, \end{aligned} \tag{3}$$

where $X^k = (x_1^k, \ldots, x_{j-1}^k, x_j^k, x_{j+1}^k, \ldots, x_t^k)$, $Y^k = (y_1^k, \ldots, y_{j-1}^k, y_j^k, y_{j+1}^k, \ldots, y_t^k)$ and $1 \le i, j \le t$ (see Grau-Sánchez et al. 2011; Potrá and Pták 1984). For $\beta = 1$, the Traub's Scheme reduces to Steffensen's method (Steffensen 1933). Inspired from this work, recently many researchers have approximated the derivatives using first-order divided difference operators preserving the local convergence order of iterative methods (Argyros et al. 2015; Ezquerro et al. 2015; Ezquerro and Hernández 2009; Grau-Sánchez and Noguera 2011;

Grau-Sánchez et al. 2013; Sharma and Arora 2013, 2014). In this study, we will construct higher order generalization for several variables of given families of Ostrowski's methods (1) using first-order divided difference operator. For the computational purpose, we used another tool to compute X^{k+1} which is defined as

$$X^{k+1} = X^k - [X^k + G, X^k - G;\ G]^{-1} G(X^k), \quad k = 0, 1, 2, \ldots \tag{4}$$

where $[X^k + G, X^k - G; G] = \big(G(X^k + H^k e^1) - G(X^k - H^k e^1), \ldots, G(X^k + H^k e^t) - G(X^k - H^k e^t)\big)\{H^k\}^{-1}$ with $H^k = \mathrm{diag}(\widetilde{g}_1(X^k), \widetilde{g}_2(X^k), \ldots, \widetilde{g}_t(X^k))$.

The meaning of $X \pm G$ is $X \pm G(X)$. We shall use either notation in this paper.

2 Construction of iterative family

We propose the following modification over iterative scheme (1) as follows:

$$\left.\begin{aligned}
\psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k),\\
\psi_2^k &= \psi_1^k - \eta\, G(\psi_1^k),\\
\psi_3^k &= \psi_2^k - \eta\, G(\psi_2^k),\\
\psi_4^k &= \psi_3^k - \eta\, G(\psi_3^k),\\
&\ \vdots\\
\psi_{i-1}^k &= \psi_{i-2}^k - \eta\, G(\psi_{i-2}^k),\\
\psi_i^k &= \psi_{i-1}^k - \eta\, G(\psi_{i-1}^k).\\
&\text{This relation is true for } i = 2, 3, 4 \ldots n.
\end{aligned}\right\}, \tag{5}$$

Here,

$$\left.\begin{aligned}
\eta &= \tau^{-1}\Big(- (\alpha_2^2 - 2\alpha_1\alpha_2)[\psi_1^k, X^k; G] + (\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2)[X^k + G, X^k - G; G]\Big),\\
\tau &= (2\alpha_1\alpha_2 - \alpha_2^2)[\psi_1^k, X^k; G][\psi_1^k, X^k; G] - \alpha_1\alpha_2[\psi_1^k, X^k; G][X^k + G, X^k - G; G]\\
&\quad + (2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2)[X^k + G, X^k - G; G][\psi_1^k, X^k; G]\\
&\quad - (\alpha_1^2 - \alpha_1\alpha_2)[X^k + G, X^k - G; G][X^k + G, X^k - G; G].
\end{aligned}\right\}. \tag{6}$$

Here α_1 and α_2 are the real parameters. From Eq. (5), the various multi-step methods can be proposed by taking different values of α_1 and α_2 as follows:

(i) For $\alpha_1 = \mathbb{R} - \{0\}$ *and* $\alpha_2 = 0$, first two steps ($i = 2$) of family (5) reduces as follows:

$$\left.\begin{aligned}
\psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k),\\
\psi_2^k &= \psi_1^k - \big\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\big\}^{-1} G(\psi_1^k).
\end{aligned}\right\}.$$

This is a fourth-order iterative scheme derived by Grau-Sánchez et al. (2013).

(ii) For $\alpha_1 = \mathbb{R} - \{0\}$ *and* $\alpha_2 = 0$, first three steps ($i = 3$) of family (5) reads as follows:

$$\left.\begin{aligned}
\psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k),\\
\psi_2^k &= \psi_1^k - \big\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\big\}^{-1} G(\psi_1^k),\\
\psi_3^k &= \psi_2^k - \big\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\big\}^{-1} G(\psi_2^k).
\end{aligned}\right\}.$$

This is a sixth-order iterative scheme derived by Grau-Sánchez et al. (2013).

(iii) For $\alpha_1 = \mathbb{R} - \{0\}$ *and* $\alpha_2 = 0$, first four steps ($i = 4$) of family (5) reduces as follows:

$$\left.\begin{aligned}\psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k),\\ \psi_2^k &= \psi_1^k - \left\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\right\}^{-1} G(\psi_1^k),\\ \psi_3^k &= \psi_2^k - \left\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\right\}^{-1} G(\psi_2^k),\\ \psi_4^k &= \psi_3^k - \left\{2[\psi_1^k, X^k; G] - [X^k + G, X^k - G; G]\right\}^{-1} G(\psi_3^k).\end{aligned}\right\}.$$

This is a new eighth-order iterative scheme.

3 Convergence analysis

We consider the first-order divided difference operator of G on $\mathbb{R}^t$ as a mapping $[\cdot,\ \cdot :\ G] : D \times D \subset \mathbb{R}^t \times \mathbb{R}^t \rightarrow L(\mathbb{R}^t)$, which is defined by Grau-Sánchez and Noguera (2011), Grau-Sánchez et al. (2013)

$$\left[X^k + h, X^k; G\right] = \int_0^1 G'(X^k + uh)\mathrm{d}u, \quad \forall (X^k, h) \in \mathbb{R}^t \times \mathbb{R}^t. \tag{7}$$

Developing $G'(X^k + uh)$ in Taylor's series at X^k and after integrating, one can obtain

$$\int_0^1 G'(X^k + uh)\mathrm{d}u = G'(X^k) + \frac{1}{2}G''(X^k)h + \frac{1}{6}G'''(X^k)h^2 + O(h^3). \tag{8}$$

Taking into account $e^k = X^k - X^*$, we develop $G(X^k)$ and its derivatives in a neighborhood of X^*, where $X^* \in \mathbb{R}^t$ is the solution of system $G(X) = 0$. Assuming that $\Gamma = \left\{G'(X^*)\right\}^{-1}$ exists, one can have

$$G(X^k) = G'(X^*)\left[e^k + A_2\left(e^k\right)^2 + A_3\left(e^k\right)^3 + A_4\left(e^k\right)^4 + A_5\left(e^k\right)^5 + O\left((e^k)^6\right)\right], \tag{9}$$

where $A_i = \frac{1}{i!}\Gamma G^{(i)}(X^*) \in L_i(\mathbb{R}^t, \mathbb{R}^t), \quad i = 2, 3, \ldots$

From Eq. (9), the derivative of $G(X^k)$ can be written as

$$G'(X^k) = G'(X^*)\left[I + 2A_2\left(e^k\right) + 3A_3\left(e^k\right)^2 + 4A_4\left(e^k\right)^3 + 5A_5\left(e^k\right)^4 + O\left((e^k)^5\right)\right], \tag{10}$$

$$G''(X^k) = G'(X^*)\left[2A_2 + 6A_3\left(e^k\right) + 12A_4\left(e^k\right)^2 + 20A_5\left(e^k\right)^3 + O\left((e^k)^4\right)\right], \tag{11}$$

$$\text{and} \quad G'''(X^k) = G'(X^*)\left[6A_3 + 24A_4\left(e^k\right) + O\left((e^k)^2\right)\right], \tag{12}$$

where I is an identity matrix of order t.

Setting $\psi_1^k = X^k + h$ & $\epsilon_1^k = \psi_1^k - X^*$, one can have $h = \psi_1^k - X^k = \epsilon_1^k - e^k$.

By substituting Eqs. (10)–(12) into Eq. (8), one gets

$$[\psi_1^k, X^k; G] = G'(X^*)\left[I + A_2\left(\epsilon_1^k + e^k\right) + A_3\left(\left(\epsilon_1^k\right)^2 + \left(e^k\right)^2 + \epsilon_1^k e^k\right) + O\left((e^k)^3\right)\right]. \tag{13}$$

In our analysis, we have considered the center difference operator

$$[X^k + G, X^k - G; G] = G'(X^*)\left[I + 2A_2\left(e^k\right) + A_3\left(3 + \left\{G'(r)\right\}^2\right)(e^k)^2 + O\left((e^k)^3\right)\right], \tag{14}$$

which we get after replacing ϵ_1^k by $e^k + G(X)$ and e^k by $e^k - G(X)$ in Eq. (13). The convergence of iterative schemes (5) can be proved through the following theorem:

Theorem 1 *Let $X^* \in \mathbb{R}^t$ be solution of the system $G(X) = 0$ and $G : D \subset \mathbb{R}^t \to \mathbb{R}^t$ be sufficiently differentiable in an open neighborhood D of X^* at which $G'(X^*)$ is nonsingular. Then for an initial approximation sufficiently close to X^*, iterative scheme (5) will have $2 \times i$ local order of convergence with error equation*

$$\epsilon_i^k = (-1)^{i+1} \frac{\left(\lambda A_3 - PA_2^2\right)^{i-2}\left(\lambda A_2 A_3 - QA_2^3\right)}{\alpha_1^{i-1}(\alpha_1 - \alpha_2)^{i-1}} (e^k)^{2i} + O\left((e^k)^{2i+1}\right),$$

provided that $\alpha_1 \in \mathbb{R} - \{0\}$, $\alpha_2 \in \mathbb{R}$ *and* $\alpha_1 \neq \alpha_2$, *where* $\lambda = \alpha_1(\alpha_1 - \alpha_2)(1 + \gamma^2)$, $Q = (\alpha_1^2 - 3\alpha_1\alpha_2 + \alpha_2^2)$, $P = (2\alpha_1^2 - 4\alpha_1\alpha_2 + \alpha_2^2)$ & $\gamma = G'(X^*)$.

Proof We shall prove the theorem by induction method.

For $i = 2$, the relation (5) reduces as follows:

$$\left.\begin{aligned} \psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k), \\ \psi_2^k &= \psi_1^k - \eta\, G(\psi_1^k). \end{aligned}\right\}. \tag{15}$$

The inverse operator of Eq. (14) is

$$\begin{aligned} [X^k + G, X^k - G; G]^{-1} = \gamma^{-1}\Big[& I - 2A_2 e^k + \left(4A_2^2 - A_3(3 + \gamma^2)\right)(e^k)^2 \\ & + 2\left((6 + \gamma^2)A_2A_3 - 2(1 + \gamma^2)A_4 - 4A_4^3\right)(e^k)^3 \\ & + O\left((e^k)^4\right)\Big]. \end{aligned} \tag{16}$$

Using (9) and (16) in the first step of Eq. (5), one can get the following error equation:

$$\begin{aligned} \epsilon_1^k = \psi_1^k - X^* &= A_2(e^k)^2 + (-2A_2^2 + (2 + \gamma^2)A_3)(e^k)^3 \\ &+ \left(- (7 + \gamma^2)A_2A_3 + (3 + 4\gamma^2)A_4 + 4A_2^3\right)(e^k)^4 + O\left((e^k)^5\right). \end{aligned} \tag{17}$$

Expanding $G(\psi_1^k)$ by Taylor's series expansion around the solution X^* using (17), one gets

$$G(\psi_1^k) = \gamma\left[\epsilon_1^k + A_2\left(\epsilon_1^k\right)^2 + A_3\left(\epsilon_1^k\right)^3 + O\left((\epsilon_1^k)^4\right)\right]. \tag{18}$$

By substituting Eqs. (13) and (14) in Eq. (6), one can obtain

$$\left.\begin{aligned} \tau &= \gamma^2\alpha_1(\alpha_1 - \alpha_2) + PA_2\gamma^2\left(e^k\right) \\ &\quad + \gamma^2\left[2\alpha_1^2(A_2^2 + A_3) - 2\alpha_1\alpha_2(A_2^2 + (3 + \gamma^2)A_3) + (2 + \gamma^2)\alpha_2^2 A_3\right](e^k)^2 \\ &\quad + O((e^k)^3), \\ \eta &= \frac{1}{\gamma} + \frac{(1 + \gamma^2)A_3\alpha_1(\alpha_1 - \alpha_2) - A_2^2 P}{\gamma\alpha_1(\alpha_1 - \alpha_2)}(e^k)^2 \\ &\quad + \frac{1}{\gamma\alpha_1^2(\alpha_1 - \alpha_2)^2}\left[2(1 + 2\gamma^2)A_4\alpha_1^2(\alpha_1 - \alpha_2)^2 + A_2^3P^2\right. \\ &\quad \left. + 2A_2A_3\alpha_1\left(- 3\alpha_1^3 + 2(\gamma^2 + 5)\alpha_1^2\alpha_2 - 3(\gamma^2 + 3)\alpha_1\alpha_2^2 + (2 + \gamma 2)\alpha_2^3\right)\right](e^k)^3 \\ &\quad + O\left((e^k)^4\right). \end{aligned}\right\}. \tag{19}$$

The second step of Eq. (15) can be rewritten as $\psi_2^k - X^* = \psi_1^k - X^* - \eta\, G(\psi_1^k)$,

$$\Rightarrow \epsilon_2^k = \epsilon_1^k - \eta\, G(\psi_1^k). \tag{20}$$

Putting the values of ϵ_1^k, $G(\psi_1^k)$ *and* η from Eqs. (17)–(19), respectively, Eq. (20) yields

$$\begin{aligned}\epsilon_2^k = \psi_2^k - X^* = &-\frac{\lambda A_2 A_3 - Q A_2^3}{\alpha_1(\alpha_1-\alpha_2)}(e^k)^4\\ &-\frac{1}{(\lambda A_3 - P A_2^2)\alpha_1(\alpha_1-\alpha_2)}\Big[-P(\alpha_2^4 - 10\alpha_2^3\alpha_1 + 26\alpha_1^2\alpha_2^2 - 20\alpha_1^3\alpha_2 + 4\alpha_1^4)A_2^6\\ &+2\alpha_1^3(\alpha_1-\alpha_2)^3(2\gamma^2+1)(\gamma^2+1)A_2A_3A_4 + \alpha_1^3(\alpha_1-\alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3\\ &-2\alpha_1^2(\alpha_1-\alpha_2)^2(2\gamma^2+1)(2\alpha_1^2 - 4\alpha_1\alpha_2 + \alpha_2^2)A_2^3A_4\\ &-\alpha_1^2(\alpha_1-\alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 - 12\alpha_1\alpha_2\gamma^2 + 4\alpha_2^2\gamma^2 + 12\alpha_1^2 - 28\alpha_1\alpha_2 + 8\alpha_2^2)A_2^2A_3^2\\ &+\alpha_1(\alpha_1-\alpha_2)\big(20\alpha_1^4 - 92\alpha_1^3\alpha_2 - 44\alpha_1^3\alpha_2\gamma^2 + 126\alpha_1^2\alpha_2^2 + 74\alpha_1^2\alpha_2^2\gamma^2 - 30\alpha_1\alpha_2^3\gamma^2\\ &-54\alpha_1\alpha_2^3 + 7\alpha_2^4 + 4\alpha_2^4\gamma^2 + 8\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^5 + O\left((e^k)^6\right). \end{aligned} \tag{21}$$

Thus, the iterative family (5) has fourth order of convergence for first two steps.

For $i = n$, the iterative scheme (5) is written as

$$\left.\begin{aligned} \psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1}G(X^k),\\ \psi_2^k &= \psi_1^k - \eta\, G(\psi_1^k),\\ \psi_3^k &= \psi_2^k - \eta\, G(\psi_2^k),\\ \psi_4^k &= \psi_3^k - \eta\, G(\psi_3^k),\\ &\vdots\\ \psi_{n-1}^k &= \psi_{n-2}^k - \eta\, G(\psi_{n-2}^k),\\ \psi_n^k &= \psi_{n-1}^k - \eta\, G(\psi_{n-1}^k). \end{aligned}\right\}. \tag{22}$$

Let us assume the scheme (22) has order of convergence $2 * n$ for first n steps, with error equation

$$\begin{aligned}\epsilon_n^k = \psi_n - X^* = &(-1)^{n+1}\frac{\big(\lambda A_3 - P A_2^2\big)^{n-2}\big(\lambda A_2 A_3 - Q A_2^3\big)}{\alpha_1^{n-1}(\alpha_1-\alpha_2)^{n-1}}(e^k)^{2n}\\ &+(-1)^{n+1}\frac{(\lambda A_3 - P A_2^2)^{n-3}}{\alpha_1^n(\alpha_1-\alpha_2)^n}\Big[-P\big((n-1)\alpha_2^4 + (4-7n)\alpha_2^3\alpha_1 + (15n-4)\alpha_1^2\alpha_2^2\\ &-10n\alpha_1^3\alpha_2 + 2n\alpha_1^4\big)A_2^6 + (2n-2)\alpha_1^3(\alpha_1-\alpha_2)^3(2\gamma^2+1)(\gamma^2+1)A_2A_3A_4\\ &+\alpha_1^3(\alpha_1-\alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3 - 2\alpha_1^2(\alpha_1-\alpha_2)^2(2\gamma^2+1)(n\alpha_1^2 + (2-3n)\alpha_1\alpha_2\\ &+(n-1)\alpha_2^2)A_2^3A_4 - \alpha_1^2(\alpha_1-\alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 - (4n+4)\alpha_1\alpha_2\gamma^2 + 2n\alpha_2^2\gamma^2\\ &+6n\alpha_1^2 - 14n\alpha_1\alpha_2 + 4n\alpha_2^2)A_2^2A_3^2 + \alpha_1(\alpha_1-\alpha_2)\big(10n\alpha_1^4 + (4-48n)\alpha_1^3\alpha_2\\ &-(20n+4)\alpha_1^3\alpha_2\gamma^2 + (72n-18)\alpha_1^2\alpha_2^2 + (34n+2)\alpha_1^2\alpha_2^2\gamma^2 + (6-18n)\alpha_1\alpha_2^3\gamma^2\\ &+(14-34n)\alpha_1\alpha_2^3 + (5n-3)\alpha_2^4 + (3n-2)\alpha_2^4\gamma^2 + 4n\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^{2n+1}\\ &+O\left((e^k)^{2n+2}\right). \end{aligned} \tag{23}$$

Further, for $i = n + 1$, the iterative family (5) is represented as

$$\left.\begin{aligned}
\psi_1^k &= X^{k+1} = X^k - [X^k + G, X^k - G; G]^{-1} G(X^k),\\
\psi_2^k &= \psi_1^k - \eta\, G(\psi_1^k),\\
\psi_3^k &= \psi_2^k - \eta\, G(\psi_2^k),\\
\psi_4^k &= \psi_3^k - \eta\, G(\psi_3^k),\\
&\vdots\\
\psi_{n-1}^k &= \psi_{n-2}^k - \eta\, G(\psi_{n-2}^k),\\
\psi_n^k &= \psi_{n-1}^k - \eta\, G(\psi_{n-1}^k),\\
\psi_{n+1}^k &= \psi_n^k - \eta\, G(\psi_n^k).
\end{aligned}\right\}. \tag{24}$$

Now we shall show that the result is true for $i = n + 1$, i.e., we have to prove that iterative method (24) has $2(n + 1)$ order of convergence for first $n + 1$ steps.

Since we have assumed that the result is true for first n steps, therefore expanding $G(\psi_n^k)$ by Taylor's series around the solution X^*, one gets

$$\begin{aligned}
G(\psi_n^k) \; = \gamma\Bigg[&(-1)^{n+1} \frac{\left(\lambda A_3 - P A_2^2\right)^{n-2}\left(\lambda A_2 A_3 - Q A_2^3\right)}{\alpha_1^{n-1}(\alpha_1 - \alpha_2)^{n-1}} (e^k)^{2n}\\
&+(-1)^{n+1} \frac{(\lambda A_3 - P A_2^2)^{n-3}}{\alpha_1^n(\alpha_1 - \alpha_2)^n}\Big[- P\big((n-1)\alpha_2^4 + (4-7n)\alpha_2^3\alpha_1 + (15n-4)\alpha_1^2\alpha_2^2\\
&-10n\alpha_1^3\alpha_2 + 2n\alpha_1^4\big)A_2^6\\
&+(2n-2)\alpha_1^3(\alpha_1 - \alpha_2)^3(2\gamma^2+1)(\gamma^2+1)A_2A_3A_4\\
&+\alpha_1^3(\alpha_1 - \alpha_2)^3(\gamma^2+2)(\gamma^2+1)^2A_3^3\\
&-2\alpha_1^2(\alpha_1 - \alpha_2)^2(2\gamma^2+1)(n\alpha_1^2 + (2-3n)\alpha_1\alpha_2 + (n-1)\alpha_2^2)A_2^3A_4\\
&-\alpha_1^2(\alpha_1 - \alpha_2)^2(1+\gamma^2)(4\alpha_1^2\gamma^2 - (4n+4)\alpha_1\alpha_2\gamma^2 + 2n\alpha_2^2\gamma^2\\
&+6n\alpha_1^2 - 14n\alpha_1\alpha_2 + 4n\alpha_2^2)A_2^2A_3^2\\
&+\alpha_1(\alpha_1 - \alpha_2)\big(10n\alpha_1^4 + (4-48n)\alpha_1^3\alpha_2 - (20n+4)\alpha_1^3\alpha_2\gamma^2\\
&+(72n-18)\alpha_1^2\alpha_2^2 + (34n+2)\alpha_1^2\alpha_2^2\gamma^2\\
&+(6-18n)\alpha_1\alpha_2^3\gamma^2 + (14-34n)\alpha_1\alpha_2^3 + (5n-3)\alpha_2^4\\
&+(3n-2)\alpha_2^4\gamma^2 + 4n\alpha_1^4\gamma^2\big)A_2^4A_3\Big](e^k)^{2n+1}\\
&+O((e^k)^{2n+2})\Bigg].
\end{aligned} \tag{25}$$

Now last step of Eq. (24) rewritten as

$$\begin{aligned}
\psi_{n+1}^k - X^* &= \psi_n^k - X^* - \eta\, G(\psi_n^k),\\
\Rightarrow \epsilon_{n+1}^k &= \epsilon_n^k - \eta\, G(\psi_n^k).
\end{aligned} \tag{26}$$

Substituting the values of η, ϵ_n^k and $G(\psi_n^k)$ from Eqs. (19), (23) and (25), respectively, in Eq. (26) and after some simplifications, one can get the error equation

$$\epsilon_{n+1}^k = (-1)^{n+2} \frac{\left(\lambda A_3 - P A_2^2\right)^{n-1}\left(\lambda A_2 A_3 - Q A_2^3\right)}{\alpha_1^n(\alpha_1 - \alpha_2)^n} (e^k)^{2n+2} + O\left((e^k)^{2n+3}\right), \tag{27}$$

which shows that the iterative scheme (24) has $2(n+1)$ order of convergence for first $n+1$ steps. That is, the result is true for $i = n+1$. Hence, by induction method one deduces that the result is true $\forall\, i = 2, 3, 4 \ldots n$. □

4 Computational efficiency

For the estimation of efficiency of proposed families, the efficiency index has been used. The efficiency of an iterative method is given by $E = \rho^{1/C}$ Ostrowski (1960) where ρ is the order of convergence and C is the computational cost per iteration. For a system of t nonlinear equations with t variables, the computational cost per iteration is given by

$$C(\nu, t, \ell) = A(t)\nu + P(t, \ell), \tag{28}$$

where A(t) denotes the number of evaluations of scalar functions used in the evaluation of G and $[X, Y; G]$ and $P(t, \ell)$ denotes the number of products needed per iteration. To express the value of $C(\nu, t, \ell)$ in terms of products, a ratio $\nu > 0$ between products and evaluations of scalar functions, and a ratio $\ell \geq 1$ between products and quotients is required.

To compute G in any iterative function, we evaluate t scalar functions $(\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_t)$ and if we compute a divided difference $[X, Y; G]$ then we evaluate $t(t-1)$ scalar functions, where G(X) and G(Y) are computed separately. In addition, for central divided difference operator $[X+G, X-G; G]$, $t(t+1)$ scalar functions are evaluated. We must add t^2 quotient for any divided difference and $5t^2$ products for multiplication of a vector by a scalar. To calculate an inverse linear operator, we solve a linear system where we have $\frac{t(t-1)(2t-1)}{6}$ products and $\frac{t(t-1)}{2}$ quotients in the LU decomposition, $t(t-1)$ products and t quotients in the resolution of two triangular linear system.

For comparison of computational efficiencies of proposed schemes ψ_1^k, ψ_2^k, ψ_3^k and ψ_4^k order of convergence in two, four, six and eight, respectively, the efficiency indices are denoted by CEI_i and computational cost (calculated according to (28)) by C_i. Taking into account the above considerations, one can have

$$C_1 = \frac{t}{6}(2t^2 + 6t\nu + 3t + 9lt + 12\nu + 3\ell - 5) \quad \text{and} \quad CEI_1 = 2^{1/C_1}. \tag{29}$$

$$C_2 = \frac{t}{3}(2t^2 + 6t\nu + 18t + 9lt + 6\nu + 3\ell - 5) \quad \text{and} \quad CEI_2 = 4^{1/C_2}. \tag{30}$$

$$C_3 = \frac{t}{3}(2t^2 + 6t\nu + 21t + 9lt + 9\nu + 6\ell - 8) \quad \text{and} \quad CEI_3 = 6^{1/C_3}. \tag{31}$$

$$C_4 = \frac{t}{3}(2t^2 + 6t\nu + 24t + 9lt + 12\nu + 9\ell - 11) \quad \text{and} \quad CEI_4 = 8^{1/C_4}. \tag{32}$$

4.1 Comparison between efficiencies

To compare the iterative families ψ_i, $1 \leq i \leq 4$, the following ratio can be defined as

$$R_{i,j} = \frac{\log CEI_i}{\log CEI_j} = \frac{\log(\rho_i)C_j}{\log(\rho_j)C_i}.$$

It is clear that if $R_{i,j} > 1$, the iterative method ψ_i is more efficient than ψ_j. Taking into account that the border between two computational efficiencies is given by $R_{i,j} = 1$, this boundary is given by the equation of ν written as a function of ℓ and t, that is $\nu = M_{i,j}(\ell, t)$. Here $\nu > 0$, $\ell \geq 1$ and t is a positive integer $t \geq 2$.

Case 1: **Iterative method ψ_1 verses iterative family ψ_3**

The boundary $R_{3,1} = 1$ expressed by ν written as a function of ℓ and t is

$$M_{3,1} = \frac{(4t^2 + 42t + 18t\ell + 12\ell - 16)\log 2 - (2t^2 + 3t + 9t\ell + 3\ell - 5)\log 6}{\log 6(6t + 12) - \log 2(12t - 48)}. \tag{33}$$

This function has the vertical asymptote for $t = -3.70951$. Note that the numerator of Eq. (33) is negative for $t \geq 25$ and the denominator of Eq. (33) is positive for $t \geq 2$. Consequently, it shows that ν is always positive for $2 \leq t < 25$ and for all $\ell \geq 1$.

So, one can have $CEI_3 > CEI_1, \quad \forall\, \nu > 0,\ \ell \geq 1\ \&\ 2 \leq t < 25.$

Case 2: Iterative method ψ_1 verses iterative family ψ_4

The boundary $R_{4,1} = 1$ expressed by ν written as a function of ℓ and t is

$$M_{4,1} = \frac{-2t^2 + 39t - 9t\ell + 9\ell - 7}{6t + 12}. \tag{34}$$

This function has the vertical asymptote for $t = -2$. Note that the numerator of Eq. (34) is negative for $t > 20$ and the denominator of Eq. (34) is positive for $t \geq 2$. Consequently, it shows that ν is positive for $2 \leq t < 20$ and for all $\ell \geq 1$.

So, one gets $CEI_4 > CEI_1, \quad \forall\, \nu > 0,\ \ell \geq 1\ \&\ 2 \leq t < 20.$

Case 3: Iterative family ψ_2 verses iterative family ψ_3

The boundary $R_{3,2} = 1$ expressed by ν written as a function of ℓ and t is

$$M_{3,2} = \frac{(2t^2 + 21t + 9t\ell + 6\ell - 8)\log 4 - (2t^2 + 18t + 9t\ell + 3\ell - 5)\log 6}{\log 6(6t + 6) - \log 4(6t + 9)}. \tag{35}$$

This function has the vertical asymptote for $t = 0.7095$. Note that the numerator of Eq. (35) is negative for $t \geq 0$ and the denominator of Eq. (35) is positive for $t > 0$. Consequently, it shows that ν is always negative for $t \geq 2$ and for all $\ell \geq 1$.

So, one can get $CEI_3 < CEI_2, \quad \forall\, \nu > 0,\ \ell \geq 1\ \&\ t \geq 2.$

Case 4: Iterative family ψ_2 verses iterative family ψ_4

The boundary $R_{4,2} = 1$ expressed by ν written as a function of ℓ and t is

$$M_{4,2} = \frac{(2t^2 + 24t + 9t\ell + 9\ell - 11)\log 4 - (2t^2 + 18t + 9t\ell + 3\ell - 5)\log 8}{\log 8(6t + 6) - \log 4(6t + 12)}. \tag{36}$$

This function has the vertical asymptote for $t = 1$. Note that the numerator of Eq. (36) is negative for $t \geq 0$ and the denominator of Eq. (36) is positive for $t > 1$. Consequently, it shows that ν is always negative for $t \geq 2$ and for all $\ell \geq 1$.

So, one can obtain $CEI_4 < CEI_2, \quad \forall\, \nu > 0,\ \ell \geq 1\ \&\ t \geq 2.$

Case 5: Iterative family ψ_3 verses iterative family ψ_4

The boundary $R_{4,3} = 1$ expressed by ν written as a function of ℓ and t is

$$M_{4,3} = \frac{(2t^2 + 24t + 9t\ell + 9\ell - 11)\log 6 - (2t^2 + 21t + 9t\ell + 6\ell - 8)\log 8}{\log 8(6t + 9) - \log 6(6t + 12)}. \tag{37}$$

This function has the vertical asymptote for $t = 1.61413$. Note that the numerator of Eq. (37) is positive for $t \geq 1$ and the denominator of Eq. (37) is negative for $t > 1$. Consequently, it shows that ν is always negative for $t \geq 2$ and for all $\ell \geq 1$.

So, one can have $CEI_4 < CEI_3, \quad \forall\, \nu > 0,\ \ell \geq 1\ \&\ t \geq 2.$

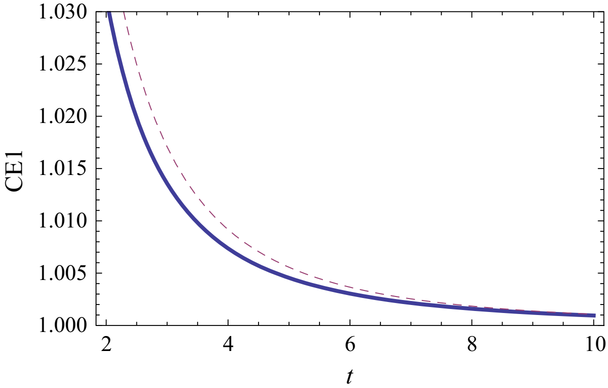

Fig. 1 CEI_1 (dashed line), CEI_3 (thick line) for $t \geq 2$

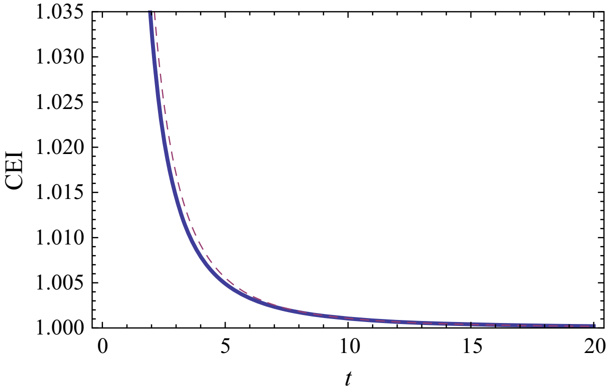

Fig. 2 CEI_1 (dashed line), CEI_4 (thick line) for $t \geq 2$

Theorem 4.1 *For all* $\nu > 0$ *and* $\ell \geq 1$, *we have*

(i) $CEI_3 > CEI_1, \quad for \quad 2 \leq t < 25$ *(see Fig. 1).*
(ii) $CEI_4 > CEI_1, \quad for \quad 2 \leq t < 20$ *(see Fig. 2).*
(iii) $CEI_3 < CEI_2, \quad for \quad t \geq 2$ *(see Fig. 3).*
(iv) $CEI_4 < CEI_2, \quad for \quad t \geq 2$ *(see Fig. 4).*
(v) $CEI_4 < CEI_3, \quad for \quad t \geq 2$ *(see Fig. 5).*

5 Local convergence

In this section, we proposed the local convergence analysis of the proposed family of methods which is based on Lipschitz constants and hypotheses on the divided difference of order one. In this way, we further expand the applicability of the proposed methods, since we used higher derivatives to show convergence of the proposed family in Sect. 3 although such derivatives do not appear in method (5). The local convergence analysis of method (5) is based on some scalar functions and parameters. This analysis is also given for $G : D \subset B \rightarrow B$, in a more general setting than in the previous sections, since B is a Banach space. Let $K_0 > 0, \ K > 0, \ c_0 > 0, \ c > 0, \ c_1 > 0$ and $p = 1, \ 2, \ \ldots,$ be parameters. Define

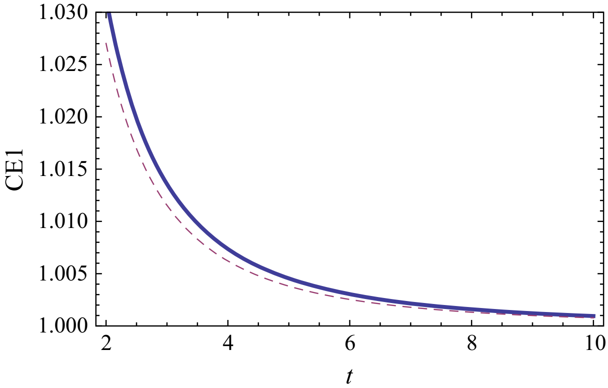

Fig. 3 CEI_2 (dashed line), CEI_3 (thick line) for $t \geq 2$

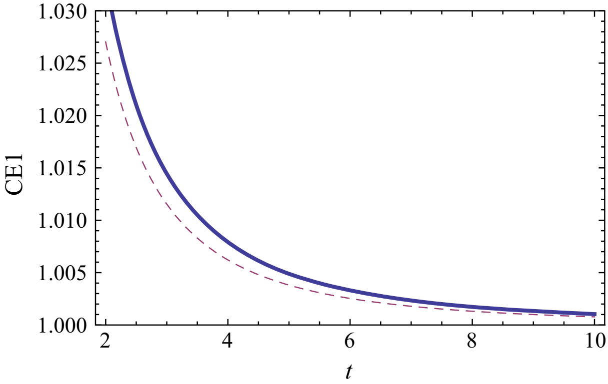

Fig. 4 CEI_2 (dashed line), CEI_4 (thick line) for $t \geq 2$

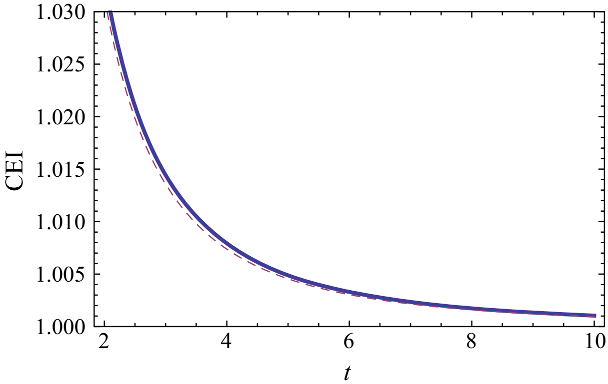

Fig. 5 CEI_3 (dashed line), CEI_4 (thick line) for $t \geq 2$

function g_1 on the interval $[0,\ r_0^-)$ by

$$g_1(s) = \frac{(1+2c_0)Ks}{1-r_0 s},$$

where

$$r_0 = 2(1+c_0)K_0$$

and parameter r_1 by

$$r_1 = \frac{1}{(1+2c_0)K + 2(1+c_0)K_0}.$$

Then, we have that $g_1(r_1) = 1,\ 0 < r_1 < r_0^-$ and for each $s \in [0,\ r_1)$, $0 \le g_1(s) < 1$. Let α_1 and α_2 be real or complex parameters. Define b and $b_i,\ i = 1,\ 2,\ 3,\ 4,\ 5$ by $b_1 = -\alpha_1^2 - \alpha_1\alpha_2,\ b_2 = 2\alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2, b_3 = -\alpha_1\alpha_2,\ b_4 = 2\alpha_1\alpha_2 - \alpha_2^2,\ b_5 = \alpha_1^2 + \alpha_2^2 - 3\alpha_1\alpha_2$ and $b = b_1 + b_2$. Define functions q and h_q in the following way:

$$q(s) = |b|^{-1}\left(|b_1|r_0 s + |b_2|K_0(1+g_1(s))s + \frac{(|b_3|+|b_4|)c_0c_1c}{1-r_0 s}\right),\ b_0 \ne 0$$

and

$$h_q(s) = q(s) - 1.$$

Suppose that

$$(|b_3|+|b_4|)c_0c_1c < |b|, \tag{38}$$

we get by (38) that $h_q(0) = -1 < 0$ and $h_q(s) \to +\infty$ as $s \to \frac{1^-}{r_0}$. It then follows from the intermediate value theorem that function h_q has zeros in the interval $(0,\ r_0^-)$. Let us consider that r_q be the smallest zero among such zero. Moreover, define some functions g_i and h_i on the interval $[0,\ r_q)$ for $i = 2,\ 3,\ \ldots,\ p$ in the following way:

$$\begin{aligned} g_i(s) &= \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|(1-q(s))(1-r_0 s)}\right) g_{i-1}(s) \\ &= \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|(1-q(s))(1-r_0 s)}\right)^{i-1} g_1(s), \\ h_i(s) &= g_i(s) - 1. \end{aligned}$$

Then, we have that $h_i(0) = -1 < 0$ and $h_i(s) \to +\infty$ as $s \to r_q^-$. Denote by $r_i,\ i = 2,\ 3,\ \ldots,\ p$ the smallest zeros of functions g_i on the interval zeros of functions g_i on the interval $(0,\ r_q)$. Notice that $h_i(r_{i-1}) = c\left(|b_2| + \frac{|b_4|c}{|b|(1-r_0 r_{i-1})(1-q(r_{i-1}))}\right) > 0$, which imply that

$$r_n < r_{n-1} < \cdots < r_2. \tag{39}$$

Define

$$r^* = \min\{r_p,\ r_1\}. \tag{40}$$

Then, we have that for each $s \in [0,\ r^*)$

$$0 \le g_i(s) < 1 \text{ and } 0 \le q(t) < 1,\ i = 1,\ 2,\ \ldots,\ p.$$

Let $U(\gamma,\ \rho),\ \bar{U}(\gamma,\ \rho)$ stand, respectively, for the open and closed balls in X with the center $\gamma \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (5) using the preceding notation.

Theorem 2 *Let $G : D \subset B \to B$ be a continuous operator. Suppose that there exists divided difference of order one for operator G, $[\cdot,\ \cdot;\ G] : D^2 \to L(B)$, $X^* \in D$, for which $G'(X^*)^{-1}$ exists, $\alpha_1,\ \alpha_2 \in \mathbb{R}$ (or $\mathbb{C}$), $K_0 > 0,\ K > 0, c_0 > 0,\ c > 0,\ c_1 > 0$ and $p = 1,\ 2,\ 3,\ \ldots$ such that (38) holds and $b \neq 0$ for each $X,\ Y,\ Z \in D$ and $G(X^*) = 0$, $G'(X^*)^{-1} \in L(X)$, $\left\|G'(X^*)^{-1}\right\| \leq c_1,\ b \neq 0$*

$$\left\|G'(X^*)^{-1}\left([X,\ Y;\ G] - G'(X^*)\right)\right\| \leq K_0(\|X - X^*\| + \|Y - X^*\|) \tag{41}$$

$$\left\|G'(X^*)^{-1}\left([X,\ Y;\ G] - [Z,\ X^*;\ G]\right)\right\| \leq K(\|X - Z\| + \|Y - X^*\|) \tag{42}$$

$$\|[X,\ Y;\ G]\| \leq c_0 \tag{43}$$

$$\left\|G'(X^*)^{-1}[X,\ Y;\ G]\right\| \leq c \tag{44}$$

and

$$\bar{U}(X^*,\ (1 + c_0)K_0) \subset D. \tag{45}$$

Then, the sequence generated by method (5) for $X^0 \in U(X^*,\ r^*) - \{X^*\}$ is well defined, remains in $U(X^*,\ r^*)$ and converges to X^*. Moreover, the following estimates hold:

$$\|\psi_i^k - X^*\| \leq g_i(\|X^k - X^*\|)\|X^k - X^*\| < \|X^k - X^*\| < r^*, \tag{46}$$

for each $i = 1,\ 2,\ \ldots,\ p$, where the "g'' functions are defined previously. Furthermore, for $T \in \left[r^*,\ \frac{1}{K_0}\right)$, the limit point X^* is the only solution of Eq. $G(X) = 0$ in $\bar{U}(X^*,\ T) \cap D$.

Proof We shall show estimate (46) holds with the help of mathematical induction. By hypotheses $X^0 \in U(X^*,\ r^*) - \{X^*\}$, (39), (40), and (41), we get that

$$\begin{aligned}
&\left\|G'(X^*)^{-1}\left([X^0 + G,\ X^0 - G; G] - G'(X^*)\right)\right\| \\
&\quad \leq K_0\left(\|X^0 - X^* + G(X^0)\| + \|X^0 - X^* - G(X^0)\|\right) \\
&\quad \leq K_0\left(\|X^0 - X^*\| + \|G(X^0) - G(X^*)\| + \|X^0 - X^*\| + \|G(X^0) - G(X^*)\|\right) \\
&\quad = 2K_0\left(\|X^0 - X^*\| + c_0\|X^0 - X^*\|\right) \\
&\quad = 2K_0(1 + c_0)\|X^0 - X^*\| \\
&\quad = r_0\|X^0 - X^*\| < r_0 r^* < 1.
\end{aligned} \tag{47}$$

Notice that $\|X^0 + G - X^*\| \leq \|X^0 - X^*\| + \|G(X^0) - G(X^*)\| \leq (1 + c_0)\|X^0 - X^*\| < (1 + c_0)r^*$, so $X^0 + G \in \bar{U}(X^*,\ (1 + c_0 r^*)) \subset D$. Similarly, we get that $X^0 - G(X^0) \in D$. Then, it follows from (47) and the Banach Lemma on invertible operators (Argyros 2008; Argyros and Hilout 2013) that ψ_1^0 is well defined by the first sub step of method (5) and

$$\left\|[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}G'(X^*)\right\| \leq \frac{1}{1 - r_0\|X^0 - X^*\|}. \tag{48}$$

We can write by (40) and the first sub step of method (5) that

$$\begin{aligned}
&\psi_1^0 - X^* \\
&\quad = X^0 - X^* - [X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}G(X^0) \\
&\quad = [X^0 + G(X^0),\ X^0 - G(X^0);\ G]^{-1}\left([X^0 + G(X^0),\ X^0 - G(X^0); G](X^0 - X^*) - G(X^0)\right) \\
&\quad = \left([X^0 + G(X^0),\ X^0 - G(X^0);\ G]^{-1}G'(X^*)\right)\left(G'(X^*)^{-1}[X^0 + G(X^0),\right. \\
&\qquad \left. X^0 - G(X^0); G] - [X^0,\ X^0; G]\right)(X^0 - X^*).
\end{aligned} \tag{49}$$

Using (39), (40) (for $i = 1$), (42) and (48), we obtain in turn that

$$\begin{aligned}\|\psi_1^0 - X^*\| &\le \left\|[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1} G(X^*)\right\| \\ &= \left\|G'(X^*)^{-1}\left([X^0 + G(X^0),\ X^0 - G(X^0); G] - [X^0,\ X^*; G]\right)\right\| \|X^0 - X^*\| \\ &\le \frac{(1+2c_0)K}{1-2K_0(1+c_0)}\|X^0 - X^*\| \\ &\le g_1(\|X^0 - X^*\|)\|X^0 - X^*\| < \|X^0 - X^*\| < r^*,\end{aligned} \tag{50}$$

which shows (46) for $k = 0,\ i = 1$ and $\psi_1^0 \in U(X^*,\ r^*)$. Let us define

$$A_0 = b_4[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}[\psi_1^0,\ X^0;\ G] + b_2 I \tag{51}$$

and

$$\begin{aligned}B_0 \ &= b_1[X^0 + G(X^0),\ X^0 - G(X^0);\ G] + b_2[\psi_1^0,\ X^0;\ G] \\ &+ b_3[X^0 + G(X^0),\ X^0 - G(X^0);\ G]^{-1}[\psi_1^0,\ X^0;\ G] \\ &[X^0 + G(X^0),\ X^0 - G(X^0);\ G] \\ &+ b_4[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}[\psi_1^0,\ X^0;\ G]^2,\end{aligned} \tag{52}$$

where the "b" parameters are defined previously. Next, we shall show that $B_0^{-1} \in L(X)$. Using (39), (40), (41), (44), (50) and (52), we get in turn that since $b \ne 0$

$$\begin{aligned}&\left\|\left(bG'(X^*)\right)^{-1}[B_0 - (b_1 + b_2 + b_3)G'(x^*)]\right\| \\ &\le |b|^{-1}\left[|b_1|\|G'(X^*)^{-1}\left([X^0 + G(X^0),\ X^0 - G(X^0);\ G] - G'(X^*)\right)\|\right] \\ &\quad + |b_2|\left\|G'(X^*)^{-1}\left([\psi_1^0,\ X^0;\ G] - G'(X^*)\right)\right\| \\ &\quad + |b_3|\left\|G'(X^*)^{-1}\right\|\left\|[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}G'(X^*)\right\| \\ &\quad \times \left\|G'(X^*)^{-1}[\psi_1^0,\ X^0;\ G]\right\|\left\|[X^0 + G(X^0),\ X^0 - G(X^0);\ G]\right\| \\ &\quad + |b_4|\left\|G'(X^*)^{-1}\right\|\left\|[X^0 + G(X^0),\ X^0 - G(X^0); G]^{-1}G'(X^*)\right\| \\ &\quad \times \left\|G'(X^*)^{-1}[\psi_1^0,\ X^0;\ G]\right\|\left\|[\psi_1^0,\ X^0;\ G]\right\| \\ &\le |b|^{-1}\left[|b_1|r_0\|X^0 - X^*\| + |b_2|K_0\left(\|\psi_1^0 - X^*\| + \|X^0 - X^*\|\right) + \frac{(|b_3| + |b_4|)c_0c_1c}{1 - r_0\|X^0 - X^*\|}\right] \\ &\le |b|^{-1}\left[|b_1|r_0\|X^0 - X^*\| + |b_2|K_0\left(1 + g_1(\|X^0 - X^*\|)\right)\|X^0 - X^*\|\right. \\ &\quad \left. + \frac{(|b_3| + |b_4|)c_0c_1c}{1 - r_0\|X^0 - X^*\|}\right] \\ &\le q(\|X^0 - X^*\|) < q(r^*) < 1.\end{aligned} \tag{53}$$

Then, it follows from (53) that

$$\left\|B_0^{-1}G'(X^*)\right\| \le \frac{1}{|b|\left(1 - q(\|X^0 - X^*\|)\right)}. \tag{54}$$

By (44), (48) and (51), we get that

$$\begin{aligned}\|A_0\| &\le |b_4|\left\|[X^0 + G(X^0),\ X^0 - G(X^0);\ G]^{-1}G'(X^*)\right\|\|G'(X^*)^{-1}[\psi_1^0,\ X^0;\ G]\| + |b_2| \\ &\le \frac{|b_4|c}{1 - r_0\|X^0 - X^*\|} + |b_2|\end{aligned} \tag{55}$$

Then, using (39), (40), (50), (54), (55) and the definition of the "g'' functions, we obtain from the second sub step of method (5) that

$$\begin{aligned}
\|\psi_2^0 - X^*\| &\leq \|\psi_1^0 - X^*\| + \|A_0\|\|B_0^{-1}G'(X^*)\|\|G'(X^*)^{-1}G(\psi_1^0)\| \\
&\leq \|\psi_1^0 - X^*\| + \|A_0\|\|B_0^{-1}G'(X^*)\|\|G'(X^*)^{-1}\left(G(\psi_1^0) - G(X^*)\right)\| \\
&\leq \|\psi_1^0 - X^*\| + \|A_0\|\|B_0^{-1}G'(X^*)\|\|G'(X^*)^{-1}[\psi_1^0,\ X^0;\ G]\|\|\psi_1^0 - X^*\| \\
&\leq \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|\big(1-q(\|X^0-X^*\|)\big)\big(1-r_0(\|X^0-X^*\|)\big)}\right) \\
&\qquad g_1(\|X^0 - X^*\|)\|X^0 - X^*\| \\
&= g_2(\|X^0 - X^*\|)\|X^0 - X^*\| < \|X^0 - X^*\| < r^*,
\end{aligned} \tag{56}$$

which shows (46) for $i = 2,\ k = 0$ and $\psi_2^0 \in U(X^*,\ r^*)$. Similarly, we show

$$\begin{aligned}
\|\psi_3^0 - X^*\| &\leq \left(1 + c|b_2| + \frac{|b_4|c^2}{|b|\big(1-q(\|X^0-X^*\|)\big)\big(1-r_0(\|X^0-X^*\|)\big)}\right) \\
&\qquad g_2(\|X^0 - X^*\|)\|X^0 - X^*\| \\
&= g_3(\|X^0 - X^*\|)\|X^0 - X^*\| < \|X^0 - X^*\| < r^*
\end{aligned}$$

until

$$\begin{aligned}
\|X^1 - X^*\| = \|\psi_n^k - X^*\| &\leq g_n(\|X^0 - X^*\|)\|X^0 - X^*\| \\
&\leq \mu\|X^0 - X^*\| < \|X^0 - X^*\| < r^*,
\end{aligned} \tag{57}$$

where $\mu = g_p(r^*) \in (0,\ 1)$. By simply replacing $\psi_1^0,\ \psi_2^0,\ \ldots, \psi_p^0,\ X^0$ by ψ_1^m, $\psi_2^m,\ \ldots, \psi_p^m,\ X^m$ in the preceding estimates we complete the induction for (46). Then, in view of the estimates $\|X^{m+1} - X^*\| \leq \mu\|X^m - X^*\|$(see (57)), we deduce that $\{X^m\}$ converges to X^* and $X^m \in U(X^*,\ r^*)$ for each $m = 0,\ 1,\ 2,\ \ldots$ Finally, to show the uniqueness part, let $H = [X^*, Y^*; G]$ where $G(Y^*) = 0$ and $Y^* \in \bar{U}(X^*, T)$. Then, using (41), we get that

$$\left|G'(X^*)^{-1}([X^*, Y^*; G] - G'(X^*))\right| \leq K_0\left(\|X^* - Y^*\|\right) < 1. \tag{58}$$

Hence, $H^{-1} \in L(B)$. Then, from the identity $0 = G(X^*) - G(Y^*) = H(X^* - Y^*)$, we conclude that $X^* = Y^*$. □

Remark 5.2 (a) If $X = R^t$ then Theorem 2 specializes in the case studied in the earlier sections.

(b) The convergence of method (5) in the previous sections was shown using hypothesis limit the applicability of method (5). In Argyros et al. (2015), we have presented some examples where the third or higher derivatives do not exist. Therefore, in Example 1, we present another such a case for such equations where method (5) is not applicable. However, in Theorem 2, we only use hypothesis on the divided difference of order one and on $G'(X^*)$, which actually appear in method (5). We expand this way the applicability of method (5). Moreover, we present computable radius of convergence and error radius of convergence and error bounds on the distances involved (see (46)) using only Lipschitz constants.

6 Numerical results

In this section, some numerical problems are considered to illustrate the convergence behavior and computational efficiency of the proposed methods. The computational work and CPU time of all the numerical experiments have been done in the programming package *Mathematica* 7.1 Wolfram (2003) with multiple-precision arithmetic with 2048 digits. The CPU time has been calculated by TimeUsed[] command in *Mathematica* 7.1. For comparison of the computational efficiencies of the proposed schemes (5) $\psi_{2,1}$, $\psi_{3,1}$ which are special cases of Grau-Sánchez et al. (2013) and $\psi_{4,1}$ for $\alpha_1 = \mathbb{R}-\{0\}$ & $\alpha_2 = 0$ are considered. In the same manner, the proposed schemes (5) $\psi_{2,2}$, $\psi_{3,2}$, $\psi_{4,2}$ for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = \pm 10^{-1000}$ and $\psi_{2,3}$, $\psi_{3,3}$, $\psi_{4,3}$ for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$ are denoted and compared with existing schemes of fourth order, namely $M_{4,1}$, $M_{4,2}$ for Sharma and Arora (2013) and seventh order S_7 (Sharma and Arora 2014). To verify the theoretical order of convergence, authors have used the computational order of convergence (COC) (Ezquerro and Hernández 2009).

$$\rho = \frac{ln\frac{\|X^{k+1}-X^*\|}{\|X^k-X^*\|}}{ln\frac{\|X^k-X^*\|}{\|X^{k-1}-X^*\|}}, \quad \text{for each } k = 1, 2, \dots \tag{59}$$

or the approximate computational order of convergence (ACOC) (Ezquerro and Hernández 2009)

$$\rho^* = \frac{ln\frac{\|X^{k+1}-X^k\|}{\|X^k-X^{k-1}\|}}{ln\frac{\|X^k-X^{k-1}\|}{\|X^{k-1}-X^{k-2}\|}}, \quad \text{for each } k = 2,\ 3, \dots \tag{60}$$

Notice that the computational of ρ or ρ^* do not require higher order derivatives to compute the error bounds. According to the definition of the computational cost (28), an estimation of the factors ν is claimed. To do this, one can express the cost of the evaluation of the elementary functions in terms of products which depends on the machine, the software and the arithmetics used (Fousse and Hanrot 2007). In the following table, an estimation of the cost of the elementary functions in number of equivalent products is shown, where running time of one product is measured in milliseconds. For the detail of hardware and software used in the numerical work, the computational cost of quotient with respect to product is $\ell = 3$ is given as follows:

Estimation of computational cost of elementary functions computed with Mathematica 7.1 in a processor Intel(R) Core (TM) i5-2430M CPU @ 2.40 GHz (32-bit machine) Microsoft Windows 7 Ultimate 2009, where $x = \sqrt{3} - 1$ and $y = \sqrt{5}$

Digits	$x * y$	x/y	$\sqrt{x}$	$exn(x)$	$ln(x)$	$sin(x)$	$cos(x)$	$arccos(x)$	$arctan(x)$
2048	0.0301 ms	3	1.5	77	78	78	77	119	118

Example 1 As a motivational example, define function F on $\mathbb{X} = \mathbb{Y} = \mathbb{R}$, $D = [-\frac{1}{\pi},\ \frac{2}{\pi}]$ by

$$F(x) = \begin{cases} x^3 \log(\pi^2 x^2) + x^5 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Table 1 Different radii of convergence

Schemes	Different values of parameters which satisfy Theorem 2					CPU time
	r_1	r_n	r^*	x_0	ρ	
ψ_2 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = ^{-1000}$	0.009951	1.8×10^{-40}	1.8×10^{-40}	0.318309	4.0000	0.108
ψ_2 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009793	0.009793	0.3181	4.0000	0.088
ψ_3 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = ^{-1000}$	0.009951	1.2×10^{-78}	1.2×10^{-78}	0.318309	6.000	1.033
ψ_3 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009633	0.009633	0.31827	6.000	1.088
ψ_4 for $\alpha_1 = \pm 10^{20}$ & $\alpha_2 = ^{-1000}$	0.009951	7.2×10^{-117}	7.2×10^{-117}	0.318309	8.000	1.003
ψ_4 for $\alpha_1 = \pm\sqrt{3}$ & $\alpha_2 = \pm 10^{-2000}$	0.009951	0.009471	0.009471	0.31826	8.000	1.039

Then, we have that

$$F'(x) = 2x^2 - x^3 \cos\left(\frac{1}{x}\right) + 3x^2 \log(\pi^2 x^2) + 5x^4 \sin\left(\frac{1}{x}\right),$$
$$F''(x) = -8x^2 \cos\left(\frac{1}{x}\right) + 2x(5 + 3\log(\pi^2 x^2)) + x(20x^2 - 1)\sin\left(\frac{1}{x}\right)$$

and

$$F'''(x) = \frac{1}{x}\left[(1 - 36x^2)\cos\left(\frac{1}{x}\right) + x\left(22 + 6\log(\pi^2 x^2) + (60x^2 - 9)\sin\left(\frac{1}{x}\right)\right)\right].$$

One can easily find that the function $F'''(x)$ is unbounded on $\mathbb{D}$ at the point $x = 0$. Therefore, the results before Sect. 5 cannot apply to show the convergence of method (5). In particular, hypotheses on the third derivative of function F or even higher are assumed to prove convergence of method (5) in Sect. 3. However, according to this section, we just need the hypotheses on first order. Moreover, we have

$$K = K_0 = \frac{80 + 16\pi + (\pi + 12\log 2)\pi^2}{2\pi + 1}, \quad c_1 = \frac{\pi^3}{2\pi + 1},$$
$$c = \frac{8}{\pi(2\pi + 1)(10 + \pi + (1 + 3\log 2)\pi^2)}, \quad c_0 = \frac{8}{[10 + \pi + (1 + 3\log 2)\pi^2]\pi^4}$$

and our required zero is $X^* = \frac{1}{\pi}$. We obtain different radii of convergence, COC (ρ) and n in Table 1.

Other such examples can be found in Argyros et al. (2015).

Example 2 Considering mixed Hammerstein integral equation (see [Ortega and Rheinboldt (1970), pp. 19–20]).

$x(s) = 1 + \frac{1}{5}\int_0^1 G(s,t)x(t)^3 \mathrm{d}t$ where $x \in C[0,1]$; $s, t \in [0,1]$ and the kernel G is

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

To transform the above equation into a finite-dimensional problem using Gauss Legendre quadrature formula given as $\int_0^1 f(t)\mathrm{d}t \simeq \sum_{j=1}^{8} w_j f(t_j)$, where the abscissas t_j and the weights w_j are determined for $t = 8$ by Gauss–Legendre quadrature formula. Denoting the approximations of $x(t_i)$ by x_i $(i = 1, 2, \ldots, 8)$, one gets the system of nonlinear equations $5x_i - 5 - \sum_{j=1}^{8} a_{ij} x_j^3 = 0$, where $i = 1, 2, \ldots, 8$

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i), \ j \leq i, \\ w_j t_i (1 - t_j), i < j, \end{cases}$$

where the abscissas t_j and the weights w_j are known and given in the following table for $t = 8$.

Abscissas and weights of Gauss–Legendre quadrature formula for $t = 8$

j	t_j	w_j
1	0.01985507175123188415821957 . . .	0.05061426814518812957626567 . . .
2	0.10166676129318663020422303 . . .	0.11119051722668723527217800 . . .
3	0.23723379504183550709113047 . . .	0.15685332293894364366898110 . . .
4	0.40828267875217509753026193 . . .	0.18134189168918099148257522 . . .
5	0.59171732124782490246973807 . . .	0.18134189168918099148257522 . . .
6	0.76276620495816449290886952 . . .	0.15685332293894364366898110 . . .
7	0.89833323870681336979577696 . . .	0.11119051722668723527217800 . . .
8	0.98014492824876811584178043 . . .	0.05061426814518812957626567 . . .

In addition, $(t, \nu) = (8, 11)$ are the values used in Eqs. (29)–(32). The convergence of the methods towards the root
$X^* = (1.00209624503115679 \ldots, 1.00990031618748877 \ldots, 1.01972696099317687 \ldots, 1.02643574303062052 \ldots, 1.02643574303062052 \ldots, 1.01972696099317687 \ldots, 1.00990031618748877 \ldots, 1.00209624503115679 \ldots)^T$ is tested in Table 2.

Example 3 (see Sharma and Arora 2013)

$$G(x_1, x_2) = \begin{cases} (x_1 - 1)^4 + e^{-x_2} - x_2^2 + 3x_2 + 1, \\ 4sin(x_1 - 1) - log(x_1^2 - x_1 + 1) - x_2^2. \end{cases}$$

Table 2 Performance of various iterative schemes at initial value $(0.85, 0.85, \ldots, 0.85)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ	C	CEI	CPU time
$M_{4,1}$	6.15 (− 5)	1.07 (− 19)	1.03 (− 78)	4.000	3296	1.000420	0.333
$M_{4,2}$	6.23 (− 5)	1.15 (− 19)	1.44 (− 78)	4.000	3376	1.000410	0.236
$\psi_{2,1}$	2.46 (− 5)	1.62 (− 21)	3.28 (− 86)	4.000	2896	1.000478	0.344
$\psi_{2,2}$	2.46 (− 5)	1.62 (− 21)	3.28 (− 86)	4.000	2896	1.000478	0.326
$\psi_{2,3}$	2.46 (− 5)	1.62 (− 21)	3.28 (− 86)	4.000	2896	1.000478	0.293
$\psi_{3,1}$	3.74 (− 7)	1.64 (− 42)	1.28 (− 254)	6.000	3064	1.000584	0.601
$\psi_{3,2}$	3.74 (− 7)	1.64 (− 42)	1.28 (− 254)	6.000	3064	1.000584	0.636
$\psi_{3,3}$	3.74 (− 7)	1.64 (− 42)	1.28 (− 254)	6.000	3064	1.000584	0.615
S_7	5.27(− 11)	1.48(− 79)	2.14(− 559)	7.000	8328	1.000233	2.042
$\psi_{4,1}$	5.69 (− 9)	8.85 (− 71)	3.50 (− 565)	8.000	3232	1.000643	0.592
$\psi_{4,2}$	5.69 (− 9)	8.85 (− 71)	3.50 (− 565)	8.000	3232	1.000643	0.628
$\psi_{4,3}$	5.69 (− 9)	8.85 (− 71)	3.50 (− 565)	8.000	3232	1.000643	0.617

Table 3 Performance of various iterative schemes at initial guess $(1.2,\ -1.2)^T$

Scheme	$\|X^{(1)}-X^*\|$	$\|X^{(2)}-X^*\|$	$\|X^{(3)}-X^*\|$	ρ	C	CEI	CPU time
$M_{4,1}$	1.09 (− 18)	1.71 (− 73)	1.04 (− 292)	4.000	1500	1.00092	0.282
$M_{4,2}$	1.34 (− 17)	1.19 (− 69)	4.49 (− 278)	4.001	1508	1.00091	0.283
$\psi_{2,1}$	4.18 (− 21)	2.81 (− 82)	5.79 (− 327)	4.000	1508	1.00091	0.319
$\psi_{2,2}$	4.18 (− 21)	2.81 (− 82)	5.79 (− 327)	4.000	1508	1.00091	0.296
$\psi_{2,3}$	4.18 (− 21)	2.81 (− 82)	5.79 (− 327)	4.000	1508	1.00091	0.298
$\psi_{3,1}$	5.04 (− 11)	2.56 (− 62)	4.33 (− 370)	6.000	1756	1.00102	1.398
$\psi_{3,2}$	5.04 (− 11)	2.56 (− 62)	4.33 (− 370)	6.000	1756	1.00102	1.434
$\psi_{3,3}$	5.04 (− 11)	2.56 (− 62)	4.33 (− 370)	6.000	1756	1.00102	1.442
S_7	2.27 (− 6)	3.23 (− 41)	5.41 (− 285)	6.999	3470	1.00056	1.486
$\psi_{4,1}$	3.70 (− 18)	9.94 (− 140)	2.69 (− 1112)	8.000	2004	1.00103	1.388
$\psi_{4,2}$	3.70 (− 18)	9.94 (− 140)	2.69 (− 1112)	8.000	2004	1.00103	1.445
$\psi_{4,3}$	3.70 (− 18)	9.94 (− 140)	2.69 (− 1112)	8.000	2004	1.00103	1.428

To calculate computational cost and efficiency indices the values $(t, \nu) = (2, 120)$ are used in Eqs. (29)–(32). The convergence of the methods towards the root $X^* = (1.271384307950131633\ldots, 0.88081907310266102\ldots)^T$ is tested in Table 3.

Example 4 (see Grau et al. 2007) Consider the following boundary value problem:

$$y'' + y^3 = 0,\ \ y(0) = 0, y(1) = 1.$$

Further, assume the partition of the interval [0, 1], which is defined as follows:

$$x_0 = 0 < x_1 < x_2 < x_3 < \cdots < x_n, \quad \text{where } x_i = x_0 + ih,\ \ h = \frac{1}{n}.$$

Let us define $y_0 = y(x_0) = 0,\ \ y_1 = y(x_1),\ \ \ldots,\ \ y_{n-1} = y(x_{n-1}),\ \ y_n = y(x_n) = 1.$

The following discretization for the second derivative is used:

$$y_k'' = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2}, \quad k = 1,\ 2,\ \ldots,\ n-1,$$

which reduces to a system of nonlinear equations of order $n - 1$

$$y_{k-1} - 2y_k + y_{k+1} + h^2 y_k^3 = 0, \quad k = 1,\ 2,\ \ldots,\ n-1.$$

The solution of this system $X^* = (0.0207113\ldots, 0.0414227\ldots,0.0621341\ldots, 0.0828453\ldots, 0.1035564\ldots, 0.1242670\ldots,0.1449769\ldots, 0.1656856\ldots,0.1863926\ldots,0.2070970\ldots, 0.2277981\ldots 0.2484946\ldots, 0.2691852\ldots,0.2898683\ldots, 0.3105421\ldots,0.3312043\ldots, 0.3518526\ldots, 0.3724841\ldots,0.3930958\ldots,0.4136841\ldots,0.4342452\ldots,0.4547747\ldots, 0.4752682\ldots, 0.4957203\ldots,0.5161257\ldots,0.5364781\ldots,0.5567712\ldots,0.5769980\ldots, 0.5971509\ldots, 0.6172219\ldots,0.6372025\ldots, 0.6570837\ldots,0.6768557\ldots,0.6965086\ldots,0.7160316\ldots, 0.7354134\ldots, 0.7546422\ldots,0.7737059\ldots,0.7925915\ldots,0.8112857\ldots, 0.8297745\ldots, 0.8480437\ldots,0.8660785\ldots,0.8838634\ldots,0.9013829\ldots,0.9186208\ldots, 0.9355607\ldots, 0.9521858\ldots, 0.9684789\ldots, 0.9844228\ldots)^T$ by taking $n = 51$ and the values $(t, \nu) = (50, 4)$ used in Eqs. (29)–(32) are tested in Table 4.

Table 4 Results of various iterative schemes at initial value $(0.6, 0.6, \ldots, 0.6)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ	C	CEI	CPU time
$M_{4,1}$	4.67 (− 4)	4.68 (− 18)	6.16 (− 74)	4.000	944	1.00146	5.217
$M_{4,2}$	4.99 (− 4)	7.35 (− 18)	4.59 (− 73)	4.000	992	1.00139	9.672
$\psi_{2,1}$	4.89 (− 4)	5.33 (− 18)	1.04 (− 73)	4.000	980	1.00141	9.421
$\psi_{2,2}$	4.89 (− 4)	5.33 (− 18)	1.04 (− 73)	4.000	980	1.00141	9.544
$\psi_{2,3}$	4.89 (− 4)	5.33 (− 18)	1.04 (− 73)	4.000	980	1.00141	7.255
$\psi_{3,1}$	4.27 (− 6)	3.03 (− 39)	5.31 (− 238)	5.975	1064	1.00168	11.486
$\psi_{3,2}$	4.27 (− 6)	3.03 (− 39)	5.31 (− 238)	5.975	1064	1.00168	13.934
$\psi_{3,3}$	4.27 (− 6)	3.03 (− 39)	5.31 (− 238)	5.975	1064	1.00168	14.113
S_7	1.09 (− 7)	3.22 (− 57)	1.94 (− 297)	7.000	2375	1.00000	39.167
ψ_4	3.73 (− 8)	9.06 (− 69)	1.68 (− 553)	7.983	1058	1.00196	11.836
ψ_4	3.73 (− 8)	9.06 (− 69)	1.68 (− 553)	7.983	1058	1.00196	15.429
ψ_4	3.73 (− 8)	9.06 (− 69)	1.68 (− 553)	7.983	1058	1.00196	13.394

Table 5 Performance of various iterative schemes at initial guess $(0.083, 0.083, \ldots, 0.083)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ	C	CEI	CPU time
$M_{4,1}$	3.39 (− 16)	9.53 (− 67)	5.95 (− 269)	4.000	15336	1.00000	2.589
$M_{4,2}$	3.39 (− 16)	9.53 (− 67)	5.95 (− 269)	4.000	15380	1.00000	2.852
$\psi_{2,1}$	5.81 (− 16)	9.19 (− 67)	5.74 (− 265)	3.999	11132	1.00001	2.053
$\psi_{2,2}$	5.81 (− 16)	9.19 (− 67)	5.74 (− 265)	3.999	11132	1.00001	2.464
$\psi_{2,3}$	5.81 (− 16)	9.19 (− 67)	5.74 (− 265)	3.999	11132	1.00001	2.217
$\psi_{3,1}$	2.09 (− 22)	1.11 (− 135)	2.42 (− 815)	6.000	11414	1.00001	7.354
$\psi_{3,2}$	2.09 (− 22)	1.11 (− 135)	2.42 (− 815)	6.000	11414	1.00001	7.476
$\psi_{3,3}$	2.09 (− 22)	1.11 (− 135)	2.42 (− 815)	6.000	11414	1.00001	7.191
S_7	4.13 (− 34)	5.39 (− 248)	3.46 (− 1745)	7.000	47602	1.00000	37.715
$\psi_{4,1}$	7.53 (− 29)	2.34 (− 231)	1.35 (− 1851)	8.000	11696	1.00001	13.906
$\psi_{4,2}$	7.53 (− 29)	2.34 (− 231)	1.35 (− 1851)	8.000	11696	1.00001	16.121
$\psi_{4,3}$	7.53 (− 29)	2.34 (− 231)	1.35 (− 1851)	8.000	11696	1.00001	15.264

Example 5 (see Grau-Sánchez et al. 2013)

$$cos^{-1}(x_i) - \sum_{j=1, i \neq j}^{20} (x_j - x_i) = 0, \quad i = 1, 2, 3 \ldots 20,$$

where $(t, \nu) = (20, 119)$ are the values used in Eqs. (29)–(32). Solution of this problem is $X^* = (0.08266851975958913\ldots, 0.08266851975958913\ldots, \ldots, 0.08266851975958913\ldots)^T$ and comparisons of the method are displayed in Table 5.

Example 6 Considering the gravity flow discharge chute problem (see [Burden and Faires (2014), pp. 646]).

$$G_i = \begin{cases} \dfrac{\sin x_{i+1}}{v_{i+1}}(1 - \mu w_{i+1}) - \dfrac{\sin x_i}{v_i}(1 - \mu w_i) = 0, & 1 \leq i \leq 19, \\ \Delta y \Sigma_{i=1}^{20} tan\ x_i - X = 0, & i = 20, \end{cases}$$

Table 6 Results of various iterative schemes at initial value $(0.75, 0.75, \ldots, 0.75)^T$

Scheme	$\|X^{(1)} - X^*\|$	$\|X^{(2)} - X^*\|$	$\|X^{(3)} - X^*\|$	ρ	C	CEI	CPU time
$M_{4,1}$	2.36 (– 4)	2.95 (– 17)	6.13 (– 69)	4.001	11232	1.00001	7.864
$M_{4,2}$	2.32 (– 4)	2.70 (– 16)	1.24 (– 63)	4.000	11276	1.00001	7.924
$\psi_{2,1}$	1.33 (– 5)	8.85 (– 22)	1.69 (– 86)	3.999	82592	1.00001	7.875
$\psi_{2,2}$	1.33 (– 5)	8.85 (– 22)	1.69 (– 86)	3.999	82592	1.00001	7.878
$\psi_{2,3}$	1.33 (– 5)	8.85 (– 22)	1.69 (– 86)	3.999	82592	1.00001	7.823
$\psi_{3,1}$	9.14 (– 12)	2.76 (– 70)	1.39 (– 421)	5.997	84728	1.00002	25.334
$\psi_{3,2}$	9.14 (– 12)	2.76 (– 70)	1.39 (– 421)	5.997	84728	1.00002	24.632
$\psi_{3,3}$	9.14 (– 12)	2.76 (– 70)	1.39 (– 421)	5.997	84728	1.00002	19.553
S_7	4.42 (– 28)	5.85 (– 197)	3.43 (– 804)	6.989	34332	1.00000	47.809
$\psi_{4,1}$	1.77 (– 20)	6.52 (– 163)	2.71 (– 1303)	8.041	86804	1.00002	24.259
$\psi_{4,2}$	1.77 (– 20)	6.52 (– 163)	2.71 (– 1303)	8.041	86804	1.00002	30.806
$\psi_{4,3}$	1.77 (– 20)	6.52 (– 163)	2.71 (– 1303)	8.041	86804	1.00002	28.747

where $v_i^2 = v_0^2 + 2gi\Delta y - 2\mu\Delta y\Sigma_{j=1}^{20}\frac{1}{cos\ x_j},\ 1 \leq i \leq 20\ and\ w_i = -\Delta y v_i \Sigma_{j=1}^{20}\frac{1}{v_j^3 cos\ x_j}$, $1 \leq i \leq 20$.

Here, $v_0 = 0$ initial velocity of the granular material, $X = 2$ the x-coordinate the end of the chute, $\mu = 0$ the friction force, $g = 32.17 ft/s^2$ gravitational force and $\Delta y = 0.2$ has been considered. The solution of this system $X^* = (0.14062\ldots, 0.19954\ldots, 0.24522\ldots, 0.28413\ldots, 0.31878\ldots, 0.35045\ldots, 0.37990\ldots, 0.40763\ldots, 0.43398\ldots, 0.45920\ldots, 0.48348\ldots, 0.50697\ldots, 0.52980\ldots, 0.55205\ldots, 0.57382\ldots, 0.59516\ldots, 0.61615\ldots, 0.63683\ldots, 0.65726\ldots, 0.67746\ldots)^T$ and the values $(t, \nu) = (20, 84.8)$ used in Eqs. (29)–(32) are tested in Table 6.

In Tables 2, 3, 4, 5, and 6, $\|X^{(k)} - X^*\|$ shows the errors of approximations to the corresponding solutions of Examples 3–6, (ρ) the computational order of convergence and C_i the computational costs given by Eqs. (29)–(32) in terms of products and the computational efficiencies CEI , where $\widetilde{b}(-a)$ denoted by $\widetilde{b} \times 10^{-a}$. The numerical results in Tables 2, 3, 4, 5, and 6 demonstrate that proposed methods work more efficiently with less error as compared to existing methods, namely $M_{4,1},\ M_{4,2}$ and S_7. In addition, the higher order methods not only works on simple experiment, it also works on application-oriented problems as shown in Examples 4 and 6.

7 Concluding remarks

In this work, we have proposed several families of Ostrowski's method for solving nonlinear systems. The new families are completely derivative free, and therefore, suited to those problems in which derivatives require lengthy computations. A development of an inverse first-order divided difference operator for multivariable function is applied to prove the convergence order of proposed methods. Moreover, the fourth- and sixth-order methods proposed by Grau-Sánchez et al. (2013) have been recovered as the special cases of the presented families. Further, the computational efficiency index is used to compare the efficiency of these different proposed families. Computational results have conformed robust and efficient character of the proposed families. We have also presented local convergence analysis based on divided differences of order one and Lipschitz constants. This way we expand the applicability of method (5), since in Sect. 3, we have to use hypotheses on high order derivatives to obtain convergence which may not exist (Argyros et al. 2015). Some numerical experimentations have also being carried out for a number of problems and results are found to be at a par with those presented here. Thus, the new methods are very suitable and applicable to solve nonlinear systems.

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