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Regularization and index reduction for linear differential–algebraic systems

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Abstract In this paper, a necessary and sufficient condition for the existence of a proportional semistate feedback is established such that the closed loop system is regular and of index at most two. The condition is characterized by a rank condition that involves only the original system coefficient matrices. Also, a new rank condition ensuring that a given linear time-invariant descriptor system is regular and of index at most some specific value is also derived. The developed theory is illustrated through physical and numerical examples.

Keywords Descriptor systems · Regularization · Index reduction · Semistate feedback

Mathematics Subject Classification 93B05 · 93B25

1 Introduction

Differential equations are the key to model physical systems. However, very often physical systems of interest are demonstrated by differential equations coupled with some algebraic

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equations. Such examples include the Leontief model of economic systems (Dai 1989b), as well as other applications in economy, biology (Li et al. 2012; Liu et al. 2008, 2009), and engineering (Duan 2010; Moysis et al. 2016; Pantelous et al. 2014; Riaza 2008; Udwadia and Kalaba 1992) to name a few. In this work, we study systems of differential–algebraic equations of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times r}$. Here $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$ represent the semistate vector and the control (input) vector, respectively. Throughout the paper, the set of systems of the form (1) is denoted by $\Sigma_{n,n,r}$ and the set of matrix pairs (E, A), where both the matrices are of the order $m \times n$, is denoted by $\Sigma_{m,n}$. Systems of the form (1) are also popularly known as descriptor or singular systems. We would also prefer to call system (1) as descriptor system. If the matrix E is nonsingular, then the descriptor system transforms to a state space system.

We now recall the Kronecker Canonical Form (KCF) of any matrix pair $(E, A) \in \Sigma_{m,n}$ and related concepts (Gantmacher 1959). These concepts will be incredibly employed in the subsequent development of this work. Corresponding to any matrix pair $(E, A) \in \Sigma_{m,n}$, there exist invertible matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that the pencil $(\lambda E - A)$ can be brought to the KCF

$$U(\lambda E - A)V = \text{block-diag}(0_{\delta\times\delta'}, \lambda E_\eta - A_\eta, \lambda I_f - J_f, \lambda N_\sigma - I_\sigma, \lambda E_\epsilon - A_\epsilon), \tag{2}$$

where δ, δ', f are nonnegative integers; $J_f \in \mathbb{R}^{f\times f}$ is a matrix in Jordan form and η, ϵ, σ are unique multi-indices. Moreover, the block matrices in (2) have the following properties:

1. $\lambda E_\eta - A_\eta$ has a block diagonal structure and each block takes the form
$$\lambda E_{\eta_i} - A_{\eta_i} = \lambda \begin{bmatrix} I \\ 0^T \end{bmatrix} - \begin{bmatrix} 0^T \\ I \end{bmatrix},$$
with order $(\eta_i + 1) \times \eta_i$;
2. N_σ is a nilpotent matrix in Jordan form;
3. $\lambda E_\epsilon^T - A_\epsilon^T$ has the same block structure as $\lambda E_\eta - A_\eta$; naturally, dimensions of their blocks are different.

A system $[E\ A\ B] \in \Sigma_{n,n,r}$ or matrix pair $(E, A) \in \Sigma_{n,n}$ is said to be regular if rank $(\lambda E - A) = n$, where rank represents the maximum rank of the matrix pencil $\lambda E - A$. Notice that if $\delta = \delta' = l(\eta) = l(\epsilon) = 0$ in (2), where $l(\cdot)$ represents the length of multi-indices, then matrix pair (E, A) is regular and the KCF (2) reduces to the Weierstrass Canonical Form (WCF)

$$U(\lambda E - A)V = \text{block-diag}(\lambda I_f - J_f, \lambda N_\sigma - I_\sigma). \tag{3}$$

The regularization and index reduction are two vital concepts in the descriptor systems theory. By regularization of a system $[E\ A\ B] \in \Sigma_{n,n,r}$, we mean that there exists a semistate feedback matrix $K \in \mathbb{R}^{r\times n}$ such that the closed loop system $[E\ A + BK\ B] \in \Sigma_{n,n,r}$ is regular. The regularity is an important property because it ensures the unique solvability of the system (Dai 1989b; Duan 2010). In general, the index of a descriptor system $[E\ A\ B] \in \Sigma_{n,n,r}$ is a nonnegative integer that roughly measures its distance from the set of state space systems. Therefore, analytical and numerical treatment for descriptor systems with higher index (greater than two) is more delicate than that of lower index descriptor systems. There are different notions of index for general time-varying/nonlinear descriptor systems, viz. differentiation index (Campbell and Gear 1995), perturbation index (Hairer and Wanner 1996), tractability index (März 1992), strangeness index (Kunkel and Mehrmann 1994). It is notable that for regular descriptor systems, all these index concepts (except strangeness index) turn out to be the same and is equal to the nilpotency index of N_σ in the WCF (3). Therefore, throughout the paper, index of a regular system (1) means the nilpotency index of the matrix N_σ in the WCF (3).

Bunse-Gerstner et al. (1992) have proved that if the system is I-controllable (impulse controllable), then there exists a semistate feedback such that the closed loop system is regular and of index at most one. For characterization of various types of controllability concepts for descriptor systems, see Dai (1989b), Mishra and Tomar (2016) and Mishra and Tomar (2017) and references therein. A sufficient condition guaranteeing the existence of a feedback such that the closed loop system is regular and of minimal index has been obtained by developing staircase and double staircase condensed forms for system coefficient matrices (Byers et al. 1997). Further, regularization of descriptor systems has also been done via output plus partial semistate derivative feedback (Duan and Zhang 2003). An up-to-date discussion on numerical techniques developed for regularization of descriptor systems can be found in Nichols and Chu (2015). The problem of index reduction for rectangular descriptor systems has also gained the attention of many researchers. See, for example, Berger and Van Dooren (2015), Mishra et al. (2017) and references therein.

In summary, most of the previous works focus on sufficient conditions on the system operators for the existence of a feedback such that the closed loop system is regular with index at most one. This work uncovers some additional features, for instance, it investigates the case when a system does not satisfy such sufficient conditions for regularization. Instead, it satisfies some weaker conditions. Therefore, one natural question arises: Is it possible to design a feedback, under some less restrictive conditions, such that the closed loop system becomes regular and of index at most two? The present article settles this question by establishing a necessary and sufficient condition for the existence of a semistate feedback such that the closed loop system is regular and of index at most two. Also, we have provided a beautiful necessary and sufficient algebraic criterion to check the regularity and the index of a descriptor system simultaneously. The idea of regularizing descriptor systems with index at most two is motivated by the desire to work with systems which have continuous (need not be differentiable) input functions but do not satisfy the conditions required for the existence of a semistate feedback such that the closed loop system is regular and of index at most one. It is notable that the solution of regular descriptor system of index at most two does not contain impulses due to continuous input functions corresponding to any consistent initial condition.

The rest of the paper is organized as follows: results on regularization and index reduction are presented in Sect. 2. Illustrating examples are provided in Sect. 3. Section 4 concludes the paper.

2 Regularization and index reduction

For given matrix pair $(E, A) \in \Sigma_{m,n}$ and $\mu \in \mathbb{N}$, we define the following block matrix

$$E_\mu = \left.\begin{bmatrix} E & & & & \\ A & E & & & \\ & A & E & & \\ & & \ddots & \ddots & \\ & & & A & E \end{bmatrix}\right\} \mu \text{ block rows.} \tag{4}$$

Clearly, $E_\mu \in \mathbb{R}^{\mu m \times \mu n}$ and $E_1 = E$. Writing the matrix E_μ in terms of the KCF, the rank of the matrix E_μ can readily be calculated as

$$\text{rank}\, E_\mu = \mu\, \text{rank}\, E_\eta + \mu\, \text{rank}\, I_f + \mu\, \text{rank}\, E_\epsilon + (\mu - 1)\, \text{rank}\, I_\sigma + \text{rank}\, N_\sigma^\mu \tag{5}$$

The following theorem provides an estimate for the difference of the ranks of two successive $E_{\mu's}$ and also gives a necessary and sufficient condition for the fact that the index of nilpotency of matrix N_σ in (2) is at most μ.

Theorem 1 *For given matrix pair* $(E, A) \in \Sigma_{m,n}$ *and* $\mu \in \mathbb{N}$*, the following inequality holds:*

$$rank\, E_{\mu+1} - rank\, E_\mu \leq rank\, (\lambda E - A). \tag{6}$$

Moreover, in (6), equality holds if and only if the index of nilpotency of the matrix N_σ *in the KCF (2) of pair* (E, A) *is at most* μ.

Proof In view of (5), the LHS (left-hand side) of (6) is equal to

$$\text{rank}\, E_\eta + \text{rank}\, I_f + \text{rank}\, I_\sigma + \text{rank}\, E_\epsilon + \text{rank}\, N_\sigma^{\mu+1} - \text{rank}\, N_\sigma^\mu.$$

The RHS (right-hand side) of (6) is equal to

$$\text{rank}\, E_\eta + \text{rank}\, I_f + \text{rank}\, I_\sigma + \text{rank}\, E_\epsilon.$$

Since $\text{rank}\, N_\sigma^{\mu+1} - \text{rank}\, N_\sigma^\mu \leq 0$, the inequality (6) follows. Now, in (6), the equality holds if and only if

$$\text{rank}\, N_\sigma^{\mu+1} = \text{rank}\, N_\sigma^\mu,$$

which is equivalent to the fact that the index of nilpotency of matrix N_σ is at most μ. This completes the proof of the theorem. □

It can be seen that for $\mu = 1$ the above theorem, in equality case, reduces to Proposition 2 of Hou and Muller (1999). The next theorem provides an algebraic criterion to check the regularity and the index of a square descriptor system.

Theorem 2 *The system* $[E\ \ A\ \ B] \in \Sigma_{n,n,r}$ *is regular and of index at most* μ *if and only if*

$$rank\, E_{\mu+1} - rank\, E_\mu = n. \tag{7}$$

Proof Suppose Eq. (7) holds. From Theorem 1, it follows that $\text{rank}(\lambda E - A) = n$. Thus, the system is regular. The converse is also a simple consequence of Theorem 1. □

Remark 1 An alternative proof of Theorem 2 (without use of Theorem 1) is given in Appendix.

Remark 2 It is notable that Eq. (7) for $\mu = 1$ gives that any system is regular and of index at most one if and only if

$$\text{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \text{rank}\, E. \tag{8}$$

The above condition (8) has been derived by Dai (1989a). If we take $E_0 \equiv 0$ (null matrix), then the Eq. (7) also provides a criterion for index zero systems, i.e. state space systems.

It is well known that for any system $[E\ A\ B] \in \Sigma_{n,n,r}$, there exist two invertible matrices $M, N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0 \end{bmatrix},\ MAN = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ \text{and } MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{9}$$

The above decomposition (9) is called dynamic decomposition form for system (1) and matrices M and N may be obtained easily by the singular value decomposition (SVD) of the matrix E. Moreover, in view of (9), one can easily infer that Eq. (8) is equivalent to the invertibility of the matrix A_{22}. Next, we provide an equivalent condition to (7) for $\mu = 2$ in view of dynamic decomposition (9).

Theorem 3 *For $\mu = 2$, the Eq. (7) is equivalent to*

$$rank \begin{bmatrix} A_{22} & 0 \\ A_{21}A_{12} & A_{22} \end{bmatrix} = n - n_0 + rank\, A_{22}, \tag{10}$$

where matrices $A_{ij}(i = 1, 2$ and $j = 1, 2)$ are obtained from (1) using dynamical decomposition (9).

Proof For $\mu = 2$, in view of (9), Eq. (7) is equivalent to

$$\text{rank} \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{11} & A_{12} & I_{n_0} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11} & A_{12} & I_{n_0} & 0 \\ 0 & 0 & A_{21} & A_{22} & 0 & 0 \end{bmatrix} = n + \text{rank} \begin{bmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{11} & A_{12} & I_{n_0} & 0 \\ A_{21} & A_{22} & 0 & 0 \end{bmatrix},$$

which is equivalent to

$$\text{rank} \begin{bmatrix} A_{12} & I_{n_0} & 0 \\ A_{22} & 0 & 0 \\ 0 & A_{21} & A_{22} \end{bmatrix} = n + \text{rank}\, A_{22},$$

which is further equivalent to (10). This completes the proof of the theorem.

□

The above theorem provides an equivalent criterion for any square system to be regular and of index at most two. We now prove the following lemma which is important for the subsequent discussion.

Lemma 1 *For any matrices X, Y, Z of compatible order, the following inequality*

$$rank \left(\begin{bmatrix} X \\ Y \end{bmatrix} Z \right) - rank\, XZ \le rank \begin{bmatrix} X \\ Y \end{bmatrix} - rank\, X, \tag{11}$$

holds.

Proof Let P be an orthogonal matrix such that

$$XP = \begin{bmatrix} X_1 & 0 \end{bmatrix},\ YP = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix},\ \text{and } P^{-1}Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{12}$$

where matrix X_1 is full column rank and matrix partitions are compatible. Further, since X_1 is full column rank and number of columns in X_1 and Y_1 is same, there exists a matrix W such that $Y_1 = WX_1$. Now, we obtain

$$\begin{aligned}
\text{rank}\left(\begin{bmatrix} X \\ Y \end{bmatrix} Z\right) - \text{rank}\, XZ &= \text{rank}\left(\begin{bmatrix} X_1 & 0 \\ Y_1 & Y_2 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\right) - \text{rank}\left(\begin{bmatrix} X_1 & 0 \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\right) \\
&= \text{rank}\begin{bmatrix} X_1 Z_1 \\ Y_1 Z_1 + Y_2 Z_2 \end{bmatrix} - \text{rank}\, X_1 Z_1 \\
&\le n\text{rank}\begin{bmatrix} X_1 Z_1 \\ Y_1 Z_1 \end{bmatrix} + \text{rank}\, Y_2 Z_2 - \text{rank}\, X_1 Z_1 \\
&\qquad (\because \text{rank}\,(R_1 + R_2) \le \text{rank}\, R_1 + \text{rank}\, R_2) \\
&\le \text{rank}\begin{bmatrix} X_1 Z_1 \\ W_1 X_1 Z_1 \end{bmatrix} + \text{rank}\, Y_2 Z_2 - \text{rank}\, X_1 Z_1 \\
&\le \text{rank}\left(\begin{bmatrix} I \\ W \end{bmatrix} X_1 Z_1\right) + \text{rank}\, Y_2 Z_2 - \text{rank}\, X_1 Z_1 \\
&\qquad \left(\because \begin{bmatrix} I \\ W \end{bmatrix} \text{ is full column rank}\right) \\
&\le \text{rank}\, Y_2 Z_2 \\
&\le \text{rank}\, Y_2. \qquad (13)
\end{aligned}$$

Similarly, we obtain

$$\text{rank}\begin{bmatrix} X \\ Y \end{bmatrix} - \text{rank}\, X = \text{rank}\, Y_2. \qquad (14)$$

Thus, the proof is followed by (13) and (14). □

Next, we provide a necessary and sufficient condition on system (1) for the existence of a semistate feedback such that the closed loop system is regular and of index at most two.

Theorem 4 *For given system $[E \;\; A \;\; B] \in \Sigma_{n,n,r}$, there exists a semistate feedback matrix $K \in \mathbb{R}^{r\times n}$ such that $[E \;\; A + BK] \in \Sigma_{n,n}$ is regular and of index at most two if and only if*

$$rank\begin{bmatrix} E & 0 & 0 & 0 & 0 \\ A & B & E & 0 & 0 \\ 0 & 0 & A & B & E \end{bmatrix} = n + rank\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix}. \qquad (15)$$

Proof Let (15) hold. In view of (9), (15) is equivalent to

$$\text{rank}\begin{bmatrix} A_{22} & B_2 & 0 & 0 \\ A_{21}A_{12} & A_{21}B_1 & A_{22} & B_2 \end{bmatrix} = n - n_0 + \text{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix}. \qquad (16)$$

Let P be a matrix such that

$$P\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{A}_{22} & \mathcal{B}_2 \end{bmatrix} \text{ and } P\begin{bmatrix} A_{21}A_{12} & A_{21}B_1 \end{bmatrix} = \begin{bmatrix} \mathfrak{U}_{11} & \mathfrak{U}_{12} \\ \mathfrak{U}_{21} & \mathfrak{U}_{22} \end{bmatrix}, \qquad (17)$$

where matrix partitions are compatible and the matrix $\begin{bmatrix} \mathcal{A}_{22} & \mathcal{B}_2 \end{bmatrix}$ is full row rank. Using Eqs. (16) and (17), we have

$$\text{rank}\begin{bmatrix} A_{22} & B_2 & 0 & 0 \\ \mathfrak{U}_{11} & \mathfrak{U}_{12} & 0 & 0 \\ \mathfrak{U}_{21} & \mathfrak{U}_{22} & \mathcal{A}_{22} & \mathcal{B}_2 \end{bmatrix} = n - n_0 + \text{rank}\begin{bmatrix} \mathcal{A}_{22} & \mathcal{B}_2 \end{bmatrix}, \qquad (18)$$

which is further equivalent to

$$\operatorname{rank}\begin{bmatrix} A_{22} & B_2 \\ \mathfrak{U}_{11} & \mathfrak{U}_{12} \end{bmatrix} = n - n_0 \tag{19}$$

The last equality is equivalent to the existence of a matrix $K_2 \in \mathbb{R}^{r\times(n-n_0)}$ such that

$$\operatorname{rank}\begin{bmatrix} A_{22} + B_2K_2 \\ \mathfrak{U}_{11} + \mathfrak{U}_{12}K_2 \end{bmatrix} = n - n_0. \tag{20}$$

That is, the matrix $\begin{bmatrix} A_{22} + B_2K_2 \\ \mathfrak{U}_{11} + \mathfrak{U}_{12}K_2 \end{bmatrix}$ is full column rank. Therefore,

$$\operatorname{rank}\begin{bmatrix} A_{22} + B_2K_2 & 0 \\ \mathfrak{U}_{11} + \mathfrak{U}_{12}K_2 & 0 \\ \mathfrak{U}_{21} + \mathfrak{U}_{22}K_2 & \mathcal{A}_{22} + \mathcal{B}_2K_2 \end{bmatrix} = n - n_0 + \operatorname{rank}(\mathcal{A}_{22} + \mathcal{B}_2K_2) \tag{21}$$

$$\operatorname{rank}\begin{bmatrix} A_{22} + B_2K_2 & 0 \\ P^{-1}\begin{bmatrix}\mathfrak{U}_{11}\\ \mathfrak{U}_{21}\end{bmatrix} + P^{-1}\begin{bmatrix}\mathfrak{U}_{12}\\ \mathfrak{U}_{22}\end{bmatrix}K_2 & P^{-1}\begin{bmatrix}0\\ \mathcal{A}_{22}\end{bmatrix} + P^{-1}\begin{bmatrix}0\\ \mathcal{B}_2\end{bmatrix}K_2 \end{bmatrix} \tag{22}$$

$$= n - n_0 + \operatorname{rank}\left(P^{-1}\begin{bmatrix}0\\ \mathcal{A}_{22}\end{bmatrix} + P^{-1}\begin{bmatrix}0\\ \mathcal{B}_2\end{bmatrix}K_2\right) \tag{23}$$

Respecting (17), the above equality is equivalent to

$$\operatorname{rank}\begin{bmatrix} A_{22} + B_2K_2 & 0 \\ A_{21}(A_{12} + B_1K_2) & A_{22} + B_2K_2 \end{bmatrix} = n - n_0 + \operatorname{rank}(A_{22} + B_2K_2), \tag{24}$$

which is same for index at most two condition for the pair $(E, A + BK)$ provided we choose $K = \begin{bmatrix} 0 & K_2 \end{bmatrix} N^{-1}$.

Conversely, suppose there exists a matrix $K \in \mathbb{R}^{r\times n}$ such that

$$\operatorname{rank}\begin{bmatrix} E & 0 & 0 \\ A + BK & E & 0 \\ 0 & A + BK & E \end{bmatrix} = n + \operatorname{rank}\begin{bmatrix} E & 0 \\ A + BK & E \end{bmatrix}. \tag{25}$$

As per the notations of Lemma 1, let us take

$$X = \begin{bmatrix} E & 0 & 0 & 0 & 0 \\ A & B & E & 0 & 0 \end{bmatrix},\ Y = \begin{bmatrix} 0 & 0 & A & B & E \end{bmatrix},\ \text{and } Z = \begin{bmatrix} I & 0 & 0 \\ K & 0 & 0 \\ 0 & I & 0 \\ 0 & K & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{26}$$

Then, (25) is equivalent to

$$\operatorname{rank}\left(\begin{bmatrix} X \\ Y \end{bmatrix} Z\right) - \operatorname{rank} XZ = n. \tag{27}$$

Respecting the Lemma 1, we obtain

$$\operatorname{rank}\begin{bmatrix} X \\ Y \end{bmatrix} - \operatorname{rank} X \geq n. \tag{28}$$

Now,

$$\operatorname{rank}\begin{bmatrix} X \\ Y \end{bmatrix} \leq \operatorname{rank} X + \operatorname{rank} Y \leq \operatorname{rank} X + n. \tag{29}$$

In view of (28) and (29), the proof follows. This completes the proof of the theorem. □

Remark 3 In Theorem 4.2 of Byers et al. (1997), it has been proved that if the system is regularizable, then there exists a semistate feedback such that the closed loop system is regular with some index that can be determined by calculating the index of a matrix obtained by a decomposition of the system matrices. Nevertheless, calculation of such a decomposition is not a simple task. However, Theorem 4 of the current article requires a simple rank test directly on the system coefficient matrices for the existence of a semistate feedback matrix such that the closed loop system is regular and of index at most two.

Remark 4 It is now a well-known fact that if any system $[E\ A\ B] \in \Sigma_{n,n,r}$ is I-controllable, then there exists a semistate feedback such that $[EA + BK]$ is regular and of index at most one. It is easy to see that condition (15) is milder than I-controllability. Thus, if any system is I-controllable, then automatically condition (15) is satisfied. It is easy to see that in such cases if the algorithm given in proof of Theorem 4 is performed, then closed loop system will automatically be of index at most one. The remark is illustrated in Examples 1 and 3 in the next section.

3 Illustrating examples

Example 1 A general constrained mechanical system can be modelled as

$$\dot{x}_1(t) = x_2(t) \tag{30a}$$

$$\dot{x}_2(t) = Cx_1(t) + Dx_2(t) + H^T x_3(t) + Gu_1(t) \tag{30b}$$

$$0 = Hx_1(t) + u_2(t) \tag{30c}$$

Here, $x_1(t)$ is the position, $x_2(t)$ the velocity, (30c) is a physical constraint, and $H^T x_3(t)$ is the force caused by the constraint. $Gu_1(t)$ is the applied force and $u_2(t)$ allows the adjustment of the constraint. The system (30) can be written in the abstract form (1), if we take

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0 & I & 0 \\ C & D & H^T \\ H & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 0 \\ G & 0 \\ 0 & I \end{bmatrix},$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},\ \text{and } u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

For numerical purpose, we take the matrices C, D, H, and G as follows:

$$C = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},\ D = \begin{bmatrix} 0.25 & 1 \\ 1 & 0.25 \end{bmatrix},\ H = \begin{bmatrix} 1 & -1 \end{bmatrix},\ \text{and } G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Notice that $x_1(t)$, $x_2(t)$ are 2×1 and $x_3(t)$ is 1×1. For the sake of simplicity, let us denote

$$x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix} \quad \text{and} \quad x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}.$$

Then the matrices E, A and B can be rewritten as

$$E=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix},\ A=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\-2&1&0.25&1&1\\1&-2&1&0.25&-1\\1&-1&0&0&0\end{bmatrix},\ \text{and } B=\begin{bmatrix}0&0\\0&0\\1&0\\1&0\\0&1\end{bmatrix}. \tag{31}$$

This Example has been taken from Mishra et al. (2016). It can be checked that the system (31) is regular and of index three as (7) is satisfied for the least value of $\mu = 3$. Moreover, the system (31) is I-controllable, *i.e.*

$$\text{rank}\begin{bmatrix}E&0&0\\A&B&E\end{bmatrix} = n + \text{rank}\, E. \tag{32}$$

So, system (31) automatically satisfies (15). Now, applying Theorem 4, we obtain

$$K = \begin{bmatrix}0&0&0&0&0\\0&0&0&0&1\end{bmatrix}. \tag{33}$$

Now, it can be seen that the closed loop system $[E\ \ A+BK] \in \Sigma_{n,n}$ is regular and of index one.

Example 2 Let the system (1) be represented by the following matrices E, A, and B as

$$E=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix},\ A=\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\-2&1&0.25&1&1\\1&-2&1&0.25&-1\\1&-1&0&0&0\end{bmatrix},\ \text{and } B=\begin{bmatrix}1&1\\0&0\\0&0\\0&0\\0&0\end{bmatrix}. \tag{34}$$

Note that the system (34) is same as the system (31) except that we have changed the matrix B so that it satisfies (15) but does not satisfy (32). The index of the system (34) is three. Now, applying Theorem 4, we obtain

$$\begin{bmatrix}A_{22}\\ \mathfrak{U}_{11}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \text{ and } \begin{bmatrix}B_2\\ \mathfrak{U}_{12}\end{bmatrix} = \begin{bmatrix}0&0\\0&1\end{bmatrix}. \tag{35}$$

So, matrix $K_2 = \begin{bmatrix}0\\1\end{bmatrix}$ fulfils the purpose. Hence, a desired feedback matrix K can be given the same as (33). Now, it can be seen that the closed loop system $[E\ \ A+BK] \in \Sigma_{n,n}$ is regular and of index two.

Example 3 Let the system (1) be represented by the following matrices E, A, and B as

$$E = \begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix},\ A = \begin{bmatrix}0&0&0\\1&0&0\\0&0&1\end{bmatrix},\ \text{and } B = \begin{bmatrix}0\\1\\1\end{bmatrix}. \tag{36}$$

In contrast to previous examples, the system (36) is irregular but satisfies (15). Hence, applying Theorem 4, we get

$$\begin{bmatrix}A_{22}\\ \mathfrak{U}_{11}\end{bmatrix} = 0 \text{ and } \begin{bmatrix}B_2\\ \mathfrak{U}_{12}\end{bmatrix} = -1. \tag{37}$$

Hence, a desired feedback matrix K can be given as

$$K = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}. \tag{38}$$

Thus, the closed loop system $[E\ A + BK] \in \Sigma_{n,n}$ is regular and of index one because (7) is satisfied for the least value of $\mu = 1$. This is because the system (36) satisfies (32) also.

Example 4 Let the system (1) be represented by the following matrices E, A, and B as

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \text{and } B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{39}$$

The only difference in this Example in context of previous Example 3 is that here the matrix B has been changed a bit. Since the system (39) satisfies (15), applying Theorem 4, we obtain a feedback matrix K as

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{40}$$

Now, it can be seen that the closed loop system $[E\ A + BK] \in \Sigma_{n,n}$ is regular and of index two.

4 Concluding remarks

A simple unified algebraic criterion involving the system coefficient matrices has been presented to check the regularity and the index of a square descriptor system. It is always desirable to reduce the index of descriptor systems to avoid the complexity in the numerical solution. In this connection, we have presented a necessary and sufficient condition for the existence of a semistate feedback such that the closed loop system is regular and of index at most two. The proposed condition is milder than the existing conditions in the literature for the existence of a semistate feedback such that the closed loop system is regular and of index at most one. The developed theory has been illustrated by several examples.

Appendix

An alternative proof of Theorem 2

Let the system be square and satisfy (7). Then, we have

$$\text{rank}\, E_\mu + \text{rank} \begin{bmatrix} E & A \end{bmatrix} - \text{rank}\, E_\mu \geq n, \tag{41}$$

which is equivalent to rank $\begin{bmatrix} E & A \end{bmatrix} = n$. That is, the matrix $\begin{bmatrix} E & A \end{bmatrix}$ is full row rank which ensures the non-existence of zero blocks in the KCF. Hence, $\delta = \delta' = 0$ in Kronecker canonical form.

Now, it can be checked that (7) is equivalent to

$$|\eta| + |\epsilon| + \text{rank}\, N_\sigma^{\mu+1} = n - |\sigma| - f + \text{rank}\, N_\sigma^{\mu}. \tag{42}$$

Before proceeding further, let us mark the following observations:

$$\begin{aligned} |\eta| + |\epsilon| + l(\epsilon) + f + |\sigma| &= n, \\ \text{and, } |\eta| + l(\eta) + |\epsilon| + f + |\sigma| &= n, \\ \Rightarrow |\eta| + |\epsilon| + l(\epsilon) = |\eta| + l(\eta) + |\epsilon| &= n - |\sigma| - f, \\ \text{So, } |\eta| + |\epsilon| < n - |\sigma| - f &\text{ if } |\eta| + |\epsilon| \neq 0. \end{aligned}$$

Here, it is notable that all the parameters $|\eta|,\ l(\eta),\ |\epsilon|,\ l(\epsilon),\ f,\ |\sigma|,\ n$ are nonnegative. Now, suppose that the system is not regular and since the system is square, the matrices E_ϵ and E_η are not void in the Kronecker canonical form. This implies that $|\eta|,\ |\epsilon| \neq 0$ and hence, $|\eta| + |\epsilon| < n - |\sigma| - f$. Also, we know that rank $N_\sigma^{\mu+1} \leq$ rank N_σ^{μ}. Therefore, (42) does not hold. In case the system is regular, the matrices E_ϵ and E_η are void and hence (42) is equivalent to

$$\text{rank } N_\sigma^{\mu+1} = \text{rank } N_\sigma^{\mu}, \tag{43}$$

which is equivalent to the fact that the system has index at most μ. This completes the proof of the theorem.

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