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Boundary coefficient determination for an eddy current problem based on the potential field method

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Abstract We study a recovery problem for an unknown boundary coefficient relating to one material characteristic in an eddy current field. The field equations are represented in terms of the potential field method ($\boldsymbol{T} - \psi$ method) and can be solved numerically by the nodal finite element method. We introduce a measurement as an additional condition and prove the existence and uniqueness of the weak solution. Further, we present an iteration algorithm for the recovery problem and validate its efficiency by two numerical experiments.

Keywords Eddy current equations · Inverse problem · $\boldsymbol{T} - \psi$ method · Impedance boundary condition

Mathematics Subject Classification 35Q61 · 65M20 · 65M32

1 Introduction

Owing to increasing requirements for high performance devices, it is indispensable to get an accurate evaluation of material characteristics during designing of electromagnetic devices. Usually, these material characteristics include the permeability μ, the conductivity σ and the

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intrinsic impedance η. η plays an important role when the penetration "skin" of the conductor is narrow in comparison with its geometric dimensions. In this case, the electromagnetic fields are closely concentrated near the conductor boundaries and decay very fast in directions normal to these boundaries. To describe these speedy spatial variations accurately, people come up with the idea of impedance boundary condition (cf. Fabrizio and Morro 2000). This boundary condition is based on the local penetration of electromagnetic fields, i.e. at each boundary point, tangential components of electric and magnetic fields (denoted by $\boldsymbol{E}$ and $\boldsymbol{H}$, respectively) are related to each other, which can be expressed mathematically as follows:

$$\boldsymbol{n} \times \boldsymbol{E} = -\eta \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{H}), \tag{1.1}$$

where $\boldsymbol{n}$ is the unit outer normal vector.

Our work in this contribution is to establish a mathematical model to evaluate intrinsic impedance. Let us consider the following time dependent eddy current approximation of Maxwell's equations in a simple-connected bounded convex polyhedron $\Omega \subset \mathbb{R}^3$(cf. Monk 2003):

$$\nabla \times \boldsymbol{H}(\boldsymbol{x}, t) = \sigma \boldsymbol{E}(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times [0, T], \tag{1.2}$$

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) = -\partial_t \mu \boldsymbol{H}(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \Omega \times [0, T]. \tag{1.3}$$

We assume that μ and σ are positive constants. This assumption helps us to build our model as simple as possible with aim to focus our attention on the evaluation.

Maxwell's equations are usually transformed into the $\boldsymbol{E}$ or $\boldsymbol{H}$ equation and solved approximately by edge finite element method. Besides, it can also be changed into potential formulations by means of a decomposition of the field $\boldsymbol{E}$ or $\boldsymbol{H}$ (the so-called $\boldsymbol{A} - \phi$ or $\boldsymbol{T} - \psi$ method). Then, nodal finite elements are used to solve the equation numerically.

There are several advantages for the potential field method. For example, it can deal with the possible discontinuity between different mediums very well and has good numerical accuracy. The method avoids spurious solutions by adding a penalty function term in the dominant equation. Moreover, it also has attractive features including natural coupling to moment and boundary element methods, and global energy conservation. There are many relative works (cf. Chew 2014; Chen et al. 2014; Chovan et al. 2017; Kang et al. 2015 and references therein). Moreover, Mur (1994) gives some results of comparison between the edge element and the nodal element. It claims that the linear nodal element is more accurate than the Nédélec's first-order edge element. With regard to the storage requirements, the Nédélec's first-order edge element is cheaper than the linear nodal element.

Solving electromagnetism problem defined in convex domain or $C^{1,1}$ domain by nodal elements is classical (cf. Kang and Kim 2009; Zeng et al. 2009). Recently, some methods allowing nodal elements to apply in reentrant domain are found, such as weighted method (cf. Costabel and Dauge 2002) and the L^2 projection method (cf. Duan et al. 2009, 2012, 2013a, b, 2016). We only consider the convex domain in this contribution to simply our analysis, and we will handle the reentrant one in the future work.

Now, we transform (1.2), (1.3) into $\boldsymbol{T} - \psi$ formulation. Taking divergence of both sides of (1.2) yields

$$\nabla \cdot (\sigma \boldsymbol{E}) = 0,$$

so there exists a vector potential $\boldsymbol{T}$ such that

$$\boldsymbol{E} = \frac{1}{\sigma} \nabla \times \boldsymbol{T}. \tag{1.4}$$

By substituting (1.4) into (1.2), we have

$$\nabla \times (\boldsymbol{H} - \boldsymbol{T}) = \boldsymbol{0},$$

and then a scalar potential ψ can be introduced satisfying

$$\boldsymbol{H} = \boldsymbol{T} + \nabla \psi. \tag{1.5}$$

Now, taking (1.4), (1.5) into (1.3), we get the following formulation:

$$\partial_t \mu \left(\boldsymbol{T} + \nabla \psi\right) + \nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T}\right) = \boldsymbol{0}. \tag{1.6}$$

The use of nodal elements comes from regularizing (1.6) by adding a penalty term $-\nabla\left(\frac{1}{\sigma}\nabla \cdot \boldsymbol{T}\right)$ (cf. Jin 2014). Take divergence of both sides of (1.6) to get that

$$\nabla \cdot \partial_t \mu \left(\boldsymbol{T} + \nabla \psi\right) = 0,$$

and then add the penalty term in (1.6),

$$\partial_t \mu \left(\boldsymbol{T} + \nabla \psi\right) + \nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T}\right) - \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T}\right) = \boldsymbol{0}.$$

Now, we divide the time interval $[0, T]$ into N sub-intervals $\left[t_{i-1}, t_i\right]$ for $t_i = i\tau$, $i = 1, 2, 3, \ldots, N$, where $\tau = \frac{T}{N}$, $N = 1, 2, 3, \ldots$. Based on backward Euler's method, we get the following time discretization scheme:

$$\begin{aligned}
&\frac{\mu\left(\boldsymbol{T}_i + \nabla \psi_i\right)}{\tau} + \nabla \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T}_i\right) + \nabla \left(\frac{1}{\sigma} \nabla \cdot \boldsymbol{T}_i\right) = \frac{\mu\left(\boldsymbol{T}_{i-1} + \nabla \psi_{i-1}\right)}{\tau},\\
&\nabla \cdot \frac{\mu\left(\boldsymbol{T}_i + \nabla \psi_i\right)}{\tau} = \nabla \cdot \frac{\mu\left(\boldsymbol{T}_{i-1} + \nabla \psi_{i-1}\right)}{\tau},
\end{aligned}$$

We give a pair of nonzero initial value $\boldsymbol{T}_0$ and ψ_0 to begin the computation. For convenience, we eliminate the index i in this paper.

The direct problem of this type is usually accompanied by the following standard boundary conditions:

$$\begin{aligned}
&\boldsymbol{n} \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T}\right) = \boldsymbol{0},\\
&\boldsymbol{n} \cdot \boldsymbol{T} = 0,\\
&\psi = 0.
\end{aligned}$$

Similar to Kang and Kim (2009), we can prove that the finite element approximate solution of (1.5) converges on the magnetic field in the case of the potential field formulation with this kind of boundary conditions.

Next, we denote the boundary of Ω as Γ, and split Γ into two complementary, non-empty and non-overlapping parts, $\Gamma = \overline{\Gamma}_{\text{Neu}} + \overline{\Gamma}_{\text{loss}}$. On Γ_{loss}, the impedance boundary condition (1.1) is defined. Considering ψ is homogeneous on the whole boundary, we rewrite (1.1) as:

$$\boldsymbol{n} \times \left(\frac{1}{\sigma} \nabla \times \boldsymbol{T}\right) = \eta \left(\boldsymbol{n} \times \boldsymbol{T}\right) \times \boldsymbol{n}.$$

Above all, considering that μ and σ are positive constants, the $\boldsymbol{T} - \psi$ formulation reads as follows:

- The dominant equations are given by

$$K\left(\boldsymbol{T}+\nabla\psi\right)+\nabla\times\left(\nabla\times\boldsymbol{T}\right)-\nabla\left(\nabla\cdot\boldsymbol{T}\right)=\boldsymbol{f},\quad \text{in}\,\Omega; \tag{1.7}$$

$$K\nabla\cdot\left(\boldsymbol{T}+\nabla\psi\right)=\nabla\cdot\boldsymbol{f},\quad \text{in}\,\Omega, \tag{1.8}$$

  where $K=\frac{\mu\sigma}{\tau}$, $\boldsymbol{f}=\frac{\mu\sigma(\boldsymbol{T}_{i-1}+\nabla\psi_{i-1})}{\tau}$.
- The boundary conditions are

$$\boldsymbol{n}\times\left(\nabla\times\boldsymbol{T}\right)=\boldsymbol{0},\quad \text{on}\,\Gamma_{\text{Neu}}; \tag{1.9}$$

$$\boldsymbol{n}\times\left(\nabla\times\boldsymbol{T}\right)=\lambda\left(\boldsymbol{n}\times\boldsymbol{T}\right)\times\boldsymbol{n},\quad \text{on}\,\Gamma_{\text{loss}}; \tag{1.10}$$

$$\boldsymbol{n}\cdot\boldsymbol{T}=0,\quad \text{on}\,\Gamma; \tag{1.11}$$

$$\psi=0,\quad \text{on}\,\Gamma, \tag{1.12}$$

  where $\lambda=\eta\sigma$.

Our object of interest is to determine the boundary coefficient λ describing the intrinsic impedance. The paper (Slodička and Van Keer 2000) discusses some relevant results on boundary coefficient determination for linear elliptic boundary value problem. Zemanova et al. (2010) provide a measurement function to identify the boundary coefficient in terms of the magnetic field equation; The analysis is made in the space $\boldsymbol{H}(\text{curl},\Omega)$, and Whitney's edge elements are used in the numerical experiment. The aim of this paper is to introduce the mathematical model in terms of the potential field method. We first prove the well-posedness of the problem in a suitable space. Then, we use linear nodal elements to solve the boundary value problem numerically.

The paper is organized as follows. In Sect. 2, we first give some notations and results. Then, we present a recovery problem for an unknown coefficient in the impedance boundary condition based on the $\boldsymbol{T}-\psi$ method. To guarantee the uniqueness of the solution, we introduce a measurement on the boundary as the additional condition. In Sect. 3, we introduce a real function $m(\lambda)$ in terms of the weak solution and study its behavior. Further, we prove the well-posedness of the recovery problem. In Sect. 4, we give two numerical experiments to validate the efficiency of the proposed scheme. Finally, some conclusions are given in the last section.

2 Notation and problem setting

For convenience of presentation, some notations are supplied first, which will be used throughout this paper. Let $L^s(\Omega)(s>1)$ denote the usual s-integrable function space equipped with the norm $\|\cdot\|_{L^s(\Omega)}$. Particularly, $L^2(\Omega)$ is the Hilbert space of square integrable functions equipped with the following inner product and norm:

$$(u,v)_{L^2(\Omega)}:=\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\ \text{ and }\ \|u\|_{L^2(\Omega)}:=\sqrt{(u,u)}.$$

If m is a positive integer, define $H^m(\Omega):=\{v\in L^2(\Omega):D^\xi v\in L^2(\Omega),0<|\xi|\le m\}$ equipped with the norm

$$\|u\|_{H^m(\Omega)}:=\left(\sum_{|\xi|\le m}\left\|D^\xi u\right\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}$$

where ξ represents a non-negative triple index. Moreover, boldface notation expresses vector-valued quantities, such as $\boldsymbol{L}^2(\Omega) := (L^2(\Omega))^3$.

Define the Hilbert space

$$\widetilde{\boldsymbol{H}}_0^1(\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{n} \cdot \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\}.$$

Further, we denote $\boldsymbol{V} := \widetilde{\boldsymbol{H}}_0^1(\Omega) \times H_0^1(\Omega)$ endowed with the sum-norm

$$\|(\boldsymbol{Q}, \phi)\|_{\boldsymbol{V}}^2 := \|\boldsymbol{Q}\|_{\boldsymbol{H}^1(\Omega)}^2 + \|\nabla\phi\|_{\boldsymbol{L}^2(\Omega)}^2.$$

In addition, we show the following lemmas, which will be used in the consequent sections (cf. Kang and Kim 2009). A direct application of these lemmas yields that the $\boldsymbol{V}$-norm is equivalent to the norm $(\|\boldsymbol{Q} + \nabla\phi\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \times \boldsymbol{Q}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \cdot \boldsymbol{Q}\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$.

Lemma 2.1 *Let Ω be a bounded convex polyhedron. Then, there exists a constant $C > 0$ such that*

$$\|\boldsymbol{Q} + \nabla\phi\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \times \boldsymbol{Q}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \cdot \boldsymbol{Q}\|_{L^2(\Omega)}^2 \geq C\,\|(\boldsymbol{Q}, \phi)\|_{\boldsymbol{V}}^2, \quad \forall\,(\boldsymbol{Q}, \phi) \in \boldsymbol{V}.$$

Lemma 2.2 *There exists a constant $C > 0$ such that*

$$\begin{aligned}&(\boldsymbol{P} + \nabla\varphi, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega)} + (\nabla \times \boldsymbol{P}, \nabla \times \boldsymbol{Q})_{\boldsymbol{L}^2(\Omega)} + (\nabla \cdot \boldsymbol{P}, \nabla \cdot \boldsymbol{Q})_{L^2(\Omega)} \\ &\quad \leq C\,\|(\boldsymbol{P}, \varphi)\|_{\boldsymbol{V}}\,\|(\boldsymbol{Q}, \phi)\|_{\boldsymbol{V}}, \ \forall\,(\boldsymbol{P}, \varphi), (\boldsymbol{Q}, \phi) \in \boldsymbol{V}.\end{aligned}$$

Now, we can define the weak solution to (1.7)–(1.12) in suitable spaces. First, we assume that

$$\begin{aligned}&0 < K_{\min} \leq K \leq K_{\max} \quad \text{a.e. in } \Omega, \\ &\boldsymbol{f} \in \boldsymbol{L}^2(\Omega).\end{aligned} \tag{2.1}$$

We take the scalar product of (1.7) with $\boldsymbol{Q} \in \widetilde{\boldsymbol{H}}_0^1(\Omega)$ and the scalar product of (1.8) with $\psi \in H_0^1(\Omega)$. Afterwards, we integrate the results over Ω and apply Green's theorems. Then, the resulting formulations are added together. The following variational formulation can be stated.

Problem 1 Find $(\boldsymbol{T}, \psi) \in \boldsymbol{V}$, such that

$$\begin{aligned}&K\,(\boldsymbol{T} + \nabla\psi, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega)} + (\nabla \times \boldsymbol{T}, \nabla \times \boldsymbol{Q})_{\boldsymbol{L}^2(\Omega)} + (\nabla \cdot \boldsymbol{T}, \nabla \cdot \boldsymbol{Q})_{L^2(\Omega)} \\ &\quad + \lambda\,(\boldsymbol{n} \times \boldsymbol{T}, \boldsymbol{n} \times \boldsymbol{Q})_{L^2(\Gamma_{loss})} = (\boldsymbol{f}, \boldsymbol{Q} + \nabla\phi)_{\boldsymbol{L}^2(\Omega)}, \ \forall\,(\boldsymbol{Q}, \phi) \in \boldsymbol{V}.\end{aligned} \tag{2.2}$$

According to the Lax–Milgram theorem, Problem 1 has an infinite number of solutions depending on the free positive parameter λ at Γ_{loss}. From now on, we denote $(\boldsymbol{T}_\lambda, \psi_\lambda)$ to show the relationship between λ and the solution to Problem 1. To guarantee the uniqueness of $(\lambda, (\boldsymbol{T}_\lambda, \psi_\lambda))$, we need an additional condition independent to (2.2).

In this contribution, the uniqueness is guaranteed by means of the following tangential component measurement along Γ_{loss}:

$$\int_{\Gamma_{\text{loss}}} |\boldsymbol{n} \times \boldsymbol{H}_\lambda|^2 = \int_{\Gamma_{\text{loss}}} |\boldsymbol{n} \times \boldsymbol{T}_\lambda|^2 = M > 0. \tag{2.3}$$

Remark 2.1 Note that $\nabla\psi$ should be included as the integrand in the measurement. We omit this term since ψ is homogeneous on the whole boundary. In Sect. 4, we give the complete $\boldsymbol{T} - \psi$ formulation of the measurement and verify its effect on the uniqueness of the solution by two experiments.

3 Well-posedness

In this section, we will prove some properties for the solution $(\lambda, (\boldsymbol{T}_\lambda, \psi_\lambda))$ first.

The following lemma gives the uniform estimate of $(\boldsymbol{T}_\lambda, \psi_\lambda)$ in the $\boldsymbol{V}$-norm and its trace with respect to $\lambda > 0$.

Lemma 3.1 *Let the assumption* (2.1) *be satisfied. Then, there exists a positive constant* C *such that*

$$\|(\boldsymbol{T}_\lambda, \psi_\lambda)\|_{\boldsymbol{V}}^2 + \lambda \|\boldsymbol{n} \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Gamma_{loss})}^2 \le C$$

Proof Taking $\boldsymbol{Q} = \boldsymbol{T}_\lambda$, $\phi = \psi_\lambda$ in (2.2), using Cauchy inequality, Young inequality and Lemma 2.2, we obtain

$$\begin{aligned} &K \|\boldsymbol{T}_\lambda + \nabla\psi_\lambda\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \cdot \boldsymbol{T}_\lambda\|_{L^2(\Omega)}^2 + \lambda \|\boldsymbol{n} \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Gamma_{\text{loss}})}^2 \\ &\quad \le \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{T}_\lambda + \nabla\psi_\lambda\|_{\boldsymbol{L}^2(\Omega)} \le C \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)} \|(\boldsymbol{T}_\lambda, \psi)\|_{\boldsymbol{V}} \\ &\quad \le C_\varepsilon \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}^2 + \varepsilon \|(\boldsymbol{T}_\lambda, \psi)\|_{\boldsymbol{V}}^2 . \end{aligned} \tag{3.1}$$

On the other hand, considering Lemma 2.1, we get

$$K \|\boldsymbol{T}_\lambda + \nabla\psi_\lambda\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla \cdot \boldsymbol{T}_\lambda\|_{L^2(\Omega)}^2 \ge C \|(\boldsymbol{T}_\lambda, \psi_\lambda)\|_{\boldsymbol{V}}^2 .$$

From the above two inequalities, we have

$$C \|(\boldsymbol{T}_\lambda, \psi_\lambda)\|_{\boldsymbol{V}}^2 + \lambda \|\boldsymbol{n} \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Gamma_{\text{loss}})}^2 \le C_\varepsilon \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}^2 + \varepsilon \|(\boldsymbol{T}_\lambda, \psi)\|_{\boldsymbol{V}}^2$$

Considering a suitable ε concludes the proof.

To make use of the measurement (2.3), we define a real value function $m : [0, \infty) \to [0, \infty)$ given by

$$m(\lambda) = \|\boldsymbol{n} \times \boldsymbol{T}_\lambda\|_{\boldsymbol{L}^2(\Gamma_{\text{loss}})}^2 .$$

This function $m(\lambda)$ is defined in terms of the weak solution to (2.2) depending on the free positive parameter λ.

First, let us study the continuous behavior of the introduced function $m(\lambda)$.

Lemma 3.2 (Continuity) *Let* (2.1) *be satisfied. Then, the function* $m(\lambda)$ *is continuous on* $(0, \infty)$.

Proof $|m(\lambda) - m(\lambda + \varepsilon)| \to 0\,(\varepsilon \to 0)$ needs to be shown according to the definition of continuity. For any $\lambda > 0$, we choose a small parameter ε satisfying $|\varepsilon| < \lambda$. Subtracting (2.2) from (2.2) for $\lambda = \lambda + \varepsilon$, we obtain

$$\begin{aligned} &K\left((\boldsymbol{T}_{\lambda+\varepsilon} + \nabla\psi_{\lambda+\varepsilon}) - (\boldsymbol{T}_\lambda + \nabla\psi_\lambda), \boldsymbol{Q} + \nabla\phi\right)_{\boldsymbol{L}^2(\Omega)} + \left(\nabla \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \nabla \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \left(\nabla \cdot (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \nabla \cdot \boldsymbol{Q}\right)_{L^2(\Omega)} + \lambda \left(\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \boldsymbol{n} \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} \\ &\quad + \varepsilon \left(\boldsymbol{n} \times \boldsymbol{T}_{\lambda+\varepsilon}, \boldsymbol{n} \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0. \end{aligned} \tag{3.2}$$

This can be written equivalently as:

$$\begin{aligned} &K\left((\boldsymbol{T}_{\lambda+\varepsilon} + \nabla\psi_{\lambda+\varepsilon}) - (\boldsymbol{T}_\lambda + \nabla\psi_\lambda), \boldsymbol{Q} + \nabla\phi\right)_{\boldsymbol{L}^2(\Omega)} + \left(\nabla \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \nabla \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Omega)} \\ &\quad + \left(\nabla \cdot (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \nabla \cdot \boldsymbol{Q}\right)_{L^2(\Omega)} + (\lambda + \varepsilon) \left(\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda), \boldsymbol{n} \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} \\ &\quad + \varepsilon \left(\boldsymbol{n} \times \boldsymbol{T}_\lambda, \boldsymbol{n} \times \boldsymbol{Q}\right)_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0. \end{aligned} \tag{3.3}$$

Summing up (3.2) and (3.3) and choosing $\boldsymbol{Q} = \boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda$, $\phi = \psi_{\lambda+\varepsilon} - \psi_\lambda$, we obtain

$$\begin{aligned} &2K \left\| (\boldsymbol{T}_{\lambda+\varepsilon} + \nabla\psi_{\lambda+\varepsilon}) - (\boldsymbol{T}_\lambda + \nabla\psi_\lambda) \right\|^2_{\boldsymbol{L}^2(\Omega)} + 2 \left\| \nabla \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \right\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\quad + 2 \left\| \nabla \cdot (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \right\|^2_{L^2(\Omega)} + (2\lambda + \varepsilon) \left\| \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \right\|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \\ &\quad + \varepsilon \left(\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} + \boldsymbol{T}_\lambda), \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \right)_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} = 0 \end{aligned} \tag{3.4}$$

Using Lemma 3.1 for the last term on the left, we deduce

$$\begin{aligned} &\left| \varepsilon \left(\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} + \boldsymbol{T}_\lambda), \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \right)_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \right| \\ &\quad = |\varepsilon| \left| \| \boldsymbol{n} \times \boldsymbol{T}_{\lambda+\varepsilon} \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} - \| \boldsymbol{n} \times \boldsymbol{T}_\lambda \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \right| \le C\, |\varepsilon| \left| \frac{1}{\lambda+\varepsilon} + \frac{1}{\lambda} \right| \\ &\quad \le C \frac{|\varepsilon|}{\lambda} \to 0 (\varepsilon \to 0). \end{aligned} \tag{3.5}$$

Thus, the absolute value of the sum of the first four terms in (3.4) tends to 0 for $\varepsilon \to 0$. From the non-negativity of each of these terms, Lemmas 2.1 and 2.2, we have

$$\| (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda, \psi_{\lambda+\varepsilon} - \psi_\lambda) \|_{\boldsymbol{V}} \to 0 \text{ and } \| \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \|_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \to 0\, (\varepsilon \to 0)$$

Recalling Cauchy inequality and Lemma 3.1, the following yields

$$\begin{aligned} |m(\lambda+\varepsilon) - m(\lambda)| &= \left| \| \boldsymbol{n} \times \boldsymbol{T}_{\lambda+\varepsilon} \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} - \| \boldsymbol{n} \times \boldsymbol{T}_\lambda \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \right| \\ &= \left| (\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} + \boldsymbol{T}_\lambda), \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda))_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \right| \\ &\le \| \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} + \boldsymbol{T}_\lambda) \|_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \| \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \|_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \\ &\le \frac{C}{\lambda} \| \boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda) \|_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \\ &\to 0 (\varepsilon \to 0), \end{aligned}$$

which concludes the proof. □

Next, the decreasing behavior of the function $m(\lambda)$ is shown.

Lemma 3.3 (Decreasing nature). *Let* (2.1) *be satisfied. Moreover suppose* $\varepsilon > 0$ *and* $\lambda > 0$. *Then* $m(\lambda + \varepsilon) \le m(\lambda)$.

Proof The first four terms in (3.4) are non-negative and $\varepsilon > 0$, thus, from the last term we obtain

$$m(\lambda+\varepsilon) = \| \boldsymbol{n} \times \boldsymbol{T}_{\lambda+\varepsilon} \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \le \| \boldsymbol{n} \times \boldsymbol{T}_\lambda \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} = m(\lambda).$$

□

At last, the asymptotic character of the function $m(\lambda)$ is given.

Lemma 3.4 (Asymptotic character) *Let* (2.1) *be satisfied. Then it holds that* $m(\lambda) \to 0\, (\lambda \to \infty)$.

Proof From Lemma 3.1, we have

$$\lambda \| \boldsymbol{n} \times \boldsymbol{T}_\lambda \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})} \le C, \quad \forall \lambda > 0.$$

Thus, the proof is directly concluded by

$$m(\lambda) = \frac{\lambda \| \boldsymbol{n} \times \boldsymbol{T}_\lambda \|^2_{\boldsymbol{L}^2(\Gamma_{\mathrm{loss}})}}{\lambda} \to 0\, (\lambda \to \infty).$$

□

Now, we give the main theorem in this contribution, which shows the well-posedness of (2.2) and (2.3).

Theorem 3.5 *Let* (2.1) *be satisfied and assume* $\lambda > 0$. *Then for any* $0 < M < \lim_{\lambda \to 0^+} m(\lambda)$, *there exists a unique weak solution to the inverse problem* (2.2) *and* (2.3).

Proof The existence of a weak solution is directly guaranteed by Lemmas 3.1–3.4, so we only need to show its uniqueness.

Suppose there exist two solutions. Then, one of the three following cases can occur:

1. Let $(\lambda, (\boldsymbol{T}, \psi))$ and $\left(\widetilde{\lambda}, (\boldsymbol{T}, \psi)\right)$ be two different solutions to (2.2) and (2.3). Subtracting (2.2) for both solutions from each other and setting $\boldsymbol{Q} = \boldsymbol{T}$, $\phi = \psi$, we get

$$(\lambda - \widetilde{\lambda}) \, \|\boldsymbol{n} \times \boldsymbol{T}\|^2_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0.$$

Hence, $\|\boldsymbol{n} \times \boldsymbol{T}\|^2_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0$. It is a contradiction between this result and $M > 0$.

2. Now let $(\lambda, (\boldsymbol{T}, \psi))$ and $\left(\lambda, \left(\widetilde{\boldsymbol{T}}, \widetilde{\psi}\right)\right)$ be two solutions to (2.2) and (2.3). Using the same steps as in previous case, but setting $\boldsymbol{Q} = \boldsymbol{T} - \widetilde{\boldsymbol{T}}$, $\phi = \psi - \widetilde{\psi}$, we obtain

$$\begin{aligned} K \left\| (\boldsymbol{T} + \nabla \psi) - \left(\widetilde{\boldsymbol{T}} + \nabla \widetilde{\psi}\right) \right\|^2_{\boldsymbol{L}^2(\Omega)} + \left\| \nabla \times \left(\boldsymbol{T} - \widetilde{\boldsymbol{T}}\right) \right\|^2_{\boldsymbol{L}^2(\Omega)} \\ + \left\| \nabla \cdot \left(\boldsymbol{T} - \widetilde{\boldsymbol{T}}\right) \right\|^2_{L^2(\Omega)} + \lambda \left\| \boldsymbol{n} \times \left(\boldsymbol{T} - \widetilde{\boldsymbol{T}}\right) \right\|^2_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0. \end{aligned}$$

Recalling (2.1), Lemma 2.1 and $\lambda > 0$, the last relations imply $(\boldsymbol{T}, \psi) = \left(\widetilde{\boldsymbol{T}}, \widetilde{\psi}\right)$.

3. Finally let $(\lambda, (\boldsymbol{T}_\lambda, \psi_\lambda))$ and $(\lambda + \varepsilon, (\boldsymbol{T}_{\lambda+\varepsilon}, \psi_{\lambda+\varepsilon}))$ with $\varepsilon > 0$ be different solutions to (2.2) and (2.3). Resulting from (3.4) and considering that each solution satisfies (2.3) yields

$$\begin{aligned} 2K \, \|(\boldsymbol{T}_{\lambda+\varepsilon} + \nabla \psi_{\lambda+\varepsilon}) - (\boldsymbol{T}_\lambda + \nabla \psi_\lambda)\|^2_{\boldsymbol{L}^2(\Omega)} + 2 \, \|\nabla \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda)\|^2_{\boldsymbol{L}^2(\Omega)} \\ +2 \, \|\nabla \cdot (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda)\|^2_{L^2(\Omega)} + (2\lambda + \varepsilon) \, \|\boldsymbol{n} \times (\boldsymbol{T}_{\lambda+\varepsilon} - \boldsymbol{T}_\lambda)\|^2_{\boldsymbol{L}^2(\Gamma_{\text{loss}})} = 0 \end{aligned} \tag{3.6}$$

This equation contradicts with $(\boldsymbol{T}_\lambda, \psi_\lambda) \neq (\boldsymbol{T}_{\lambda+\varepsilon}, \psi_{\lambda+\varepsilon})$.

The proof is done on the basis of these three cases. □

4 Numerical experiments

This section will provide a recovery algorithm, which is tested by two numerical experiments.

Let Ω be a unit cube in $\mathbb{R}^3$. The boundary Γ is split into two pieces as follows: the loss boundary condition is prescribed on the face $z = 0$ and $z = 1$, and other boundary conditions are considered on the other faces, as shown in Fig. 1.

We design an algorithm based on our analysis to the following test problem. Note that we change the condition on Γ_{loss} because ψ_λ is inhomogeneous.

Problem 2 Find $(\lambda, (\boldsymbol{T}_\lambda, \psi_\lambda))$ satisfying

$$\begin{aligned} &(\boldsymbol{T}_\lambda + \nabla \psi_\lambda) + \nabla \times (\nabla \times \boldsymbol{T}_\lambda) - \nabla (\nabla \cdot \boldsymbol{T}_\lambda) = \boldsymbol{f}, \quad \text{in } \Omega, \\ &\nabla \cdot ((\boldsymbol{T}_\lambda + \nabla \psi_\lambda)) = \nabla \cdot \boldsymbol{f}, \quad \text{in } \Omega, \\ &\boldsymbol{n} \times \nabla \times \boldsymbol{T}_\lambda = \boldsymbol{g}_1, \quad \text{on } \Gamma_{\text{Neu}}, \\ &\boldsymbol{n} \times (\nabla \times \boldsymbol{T}_\lambda) - \lambda \, (\boldsymbol{n} \times (\boldsymbol{T}_\lambda + \nabla \psi_\lambda)) \times \boldsymbol{n} = \boldsymbol{g}_2, \quad \text{on } \Gamma_{\text{loss}}, \\ &\boldsymbol{n} \cdot \boldsymbol{T}_\lambda = h_1, \quad \text{on } \Gamma, \end{aligned}$$

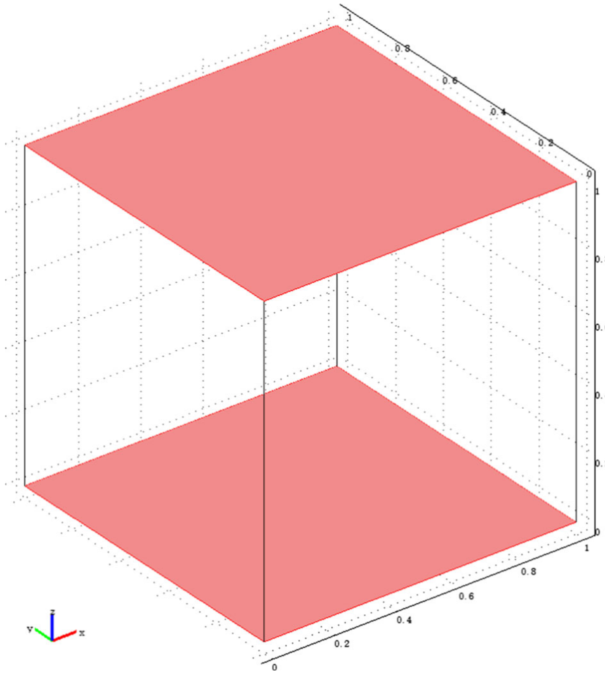


Fig. 1 Sketch of the domain and boundary conditions of the model. The red surfaces are the loss boundary

$$\psi_\lambda = h_2, \quad \text{on } \Gamma,$$
$$\int_{\Gamma_{\text{loss}}} |\boldsymbol{n} \times (\boldsymbol{T}_\lambda + \nabla\psi_\lambda)|^2 = M.$$

Correspondingly, we define the function m as:

$$m\,(\lambda) = \|\boldsymbol{n} \times (\boldsymbol{T}_\lambda + \nabla\psi_\lambda)\|^2_{\boldsymbol{L}^2(\Gamma_{\text{loss}})}.$$

The nodal finite element method is used to solve the boundary value problem for each given λ, and the following iteration scheme is used to determine the coefficient λ,

$$\lambda_k^{(n)} = \lambda_{k-1} - \left(\frac{1}{2}\right)^{n-1} \frac{m\,(\lambda_{k-1})}{m'\,(\lambda_{k-1})}, \quad n = 1, 2, 3, \ldots$$

where

$$m'\,(\lambda) = \frac{m\,(\lambda + h) - m\,(\lambda - h)}{2h}$$

with $h = 0.005$. We start with $\lambda_0 = 0.01$, and $\lambda_k = \lambda_k^{(n)}$ when $m(\lambda_k^{(n)}) > M$. The iteration ends if $|m(\lambda_k) - M| < 10^{-6}$.

Experiment 4.1 The data functions $\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2, h_1, h_2$, and $M = 5.333$ are defined such that

$$\boldsymbol{T}_\lambda = \begin{pmatrix} z - y \\ x - z \\ y - x \end{pmatrix}, \ \psi_\lambda = x + y + z, \ \lambda = 1.24$$

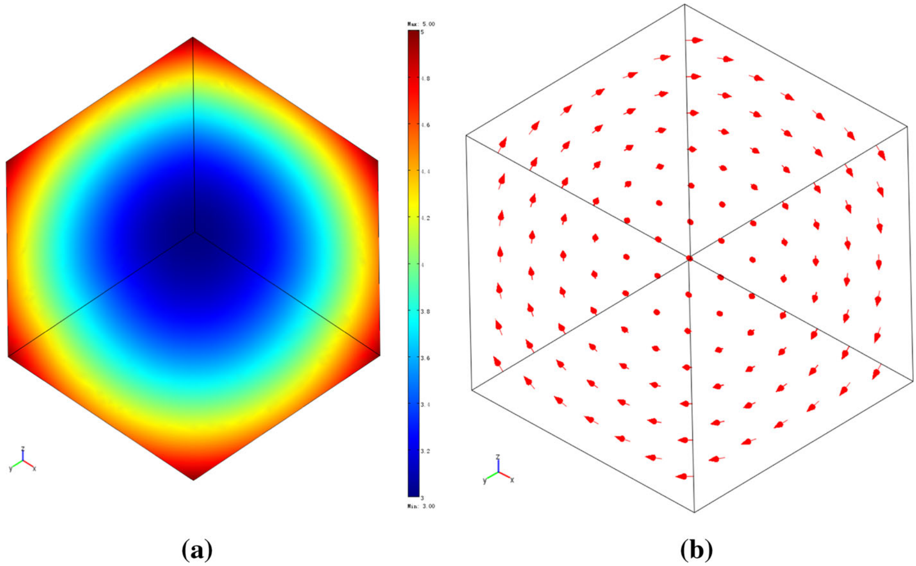


Fig. 2 Exact solution in Experiment 4.1. **a** $\|\boldsymbol{H}_\lambda\|$. **b** $\boldsymbol{H}_\lambda$

are the exact solutions (see Fig. 2), namely

$$\boldsymbol{H}_\lambda = \boldsymbol{T}_\lambda + \nabla\psi_\lambda = \begin{pmatrix} z-y+1 \\ x-z+1 \\ y-x+1 \end{pmatrix}.$$

Figure 3 shows the behavior of the function, which is consistent with the theory. Its three properties (continuity, monotonicity and asymptotic character) ensure the existence of a unique solution for any amount of M on the boundary.

Figure 4 and Table 1 show the convergence of the iteration, which has stopped after five iterations.

For the last approximations, the errors are

$$\left|\lambda - \lambda_{\text{app}}\right| = 1.9663 \times 10^{-7}, \quad \left\|\boldsymbol{H}_\lambda - \boldsymbol{H}_{\lambda\text{app}}\right\|_{\boldsymbol{L}^2(\Omega)} = 1.7012 \times 10^{-10}.$$

Experiment 4.2 The data functions $\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2, h_1, h_2$, and $M = 3.345$ are defined such that

$$\boldsymbol{T}_\lambda = \begin{pmatrix} \sin x - x\cos y \\ \sin y - y\cos z \\ \sin z - z\cos x \end{pmatrix}, \ \psi_\lambda = \sin x + \sin y + \sin z, \ \lambda = 12.4$$

are the exact solutions (see Fig. 5), namely

$$\boldsymbol{H}_\lambda = \boldsymbol{T}_\lambda + \nabla\psi_\lambda = \begin{pmatrix} \sin x - x\cos y + \cos x \\ \sin y - y\cos z + \cos y \\ \sin z - z\cos x + \cos z \end{pmatrix}$$

Figure 6 also shows the behavior of the function, and Fig. 7 and Table 2 show the convergence of the iteration, which has stopped after seven iterations. The errors obtained for the last approximations are

$$\left|\lambda - \lambda_{\text{app}}\right| = 3.773 \times 10^{-4}, \ \left\|\boldsymbol{H}_\lambda - \boldsymbol{H}_{\lambda\text{app}}\right\|_{\boldsymbol{L}^2(\Omega)} = 3.133 \times 10^{-7}.$$

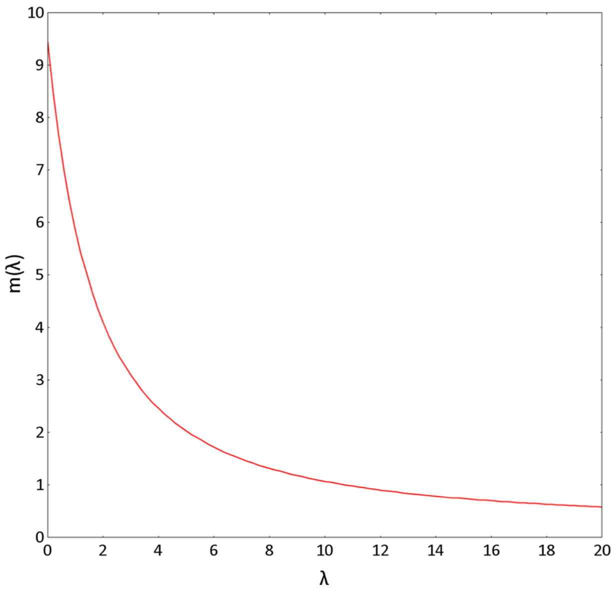


Fig. 3 Numerically obtained graph of the function $m(\lambda)$ in Experiment 4.1

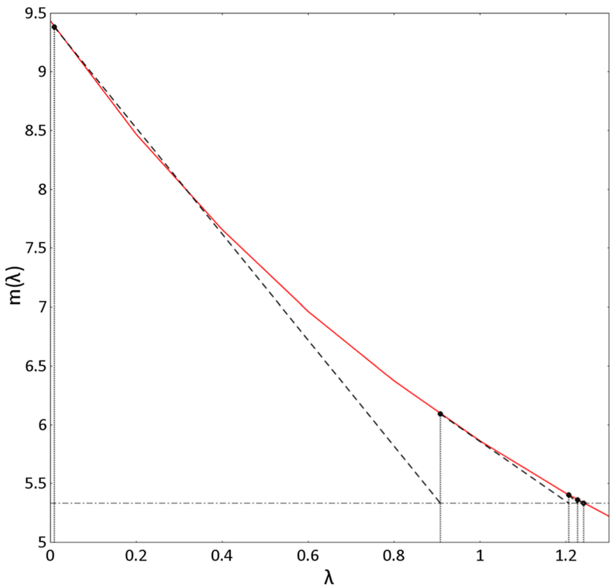


Fig. 4 The convergence of the iteration in Experiment 4.1

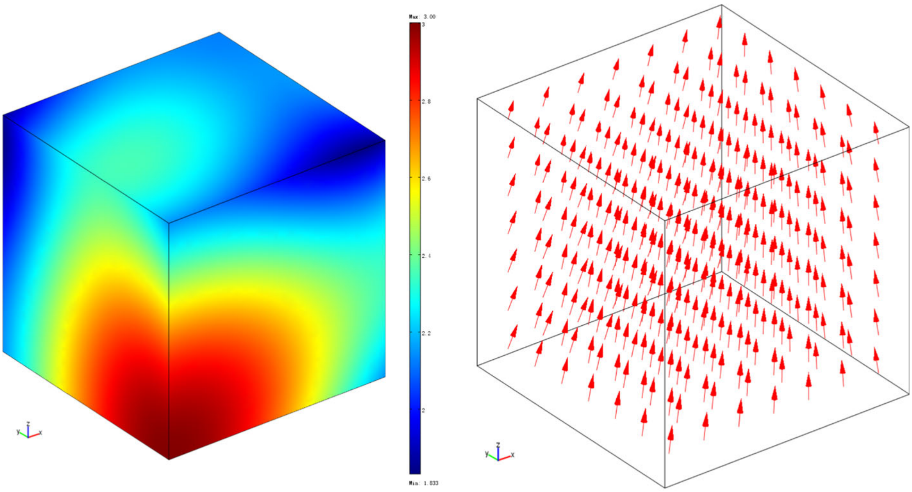
5 Conclusions

We study a recovery problem for an unknown coefficient relating to one material characteristic for an eddy current problem based on the $\boldsymbol{T} - \psi$ method. The coefficient is derived from the impedance boundary condition. Moreover, a loss boundary condition is introduced as an

Table 1 Precision of the iteration in Experiment 4.1

The situation from Fig. 4

Iter.	λ	$m(\lambda)$	Error in %
1	0.010	9.381	99.2
2	0.907	6.090	26.8
3	1.206	5.402	2.71
4	1.227	5.360	1.07
5	1.240	5.333	0.00



(a) (b)

Fig. 5 Exact solution in Experiment 4.2. **a** $\|\boldsymbol{H}_\lambda\|$. **b** $\boldsymbol{H}_\lambda$

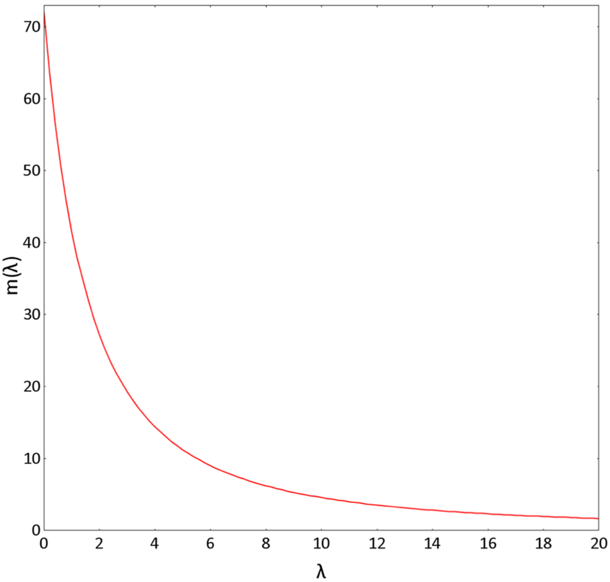


Fig. 6 Numerically obtained graph of the iron loss function $m(\lambda)$ in Experiment 4.2

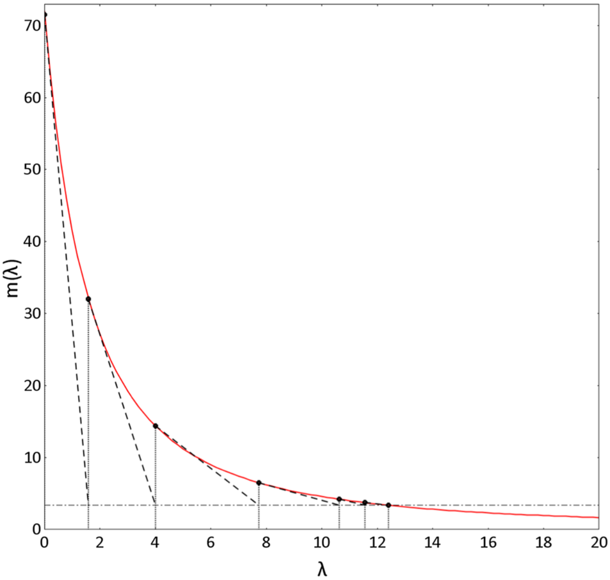


Fig. 7 The convergence of the iteration in Experiment 4.2

Table 2 Precision of the iteration in Experiment 4.2

Iter.	λ	m (λ)	Error in %
1	0.010	71.53	99.9
2	1.590	32.03	87.2
3	4.005	14.39	67.7
4	7.724	6.492	37.7
5	10.62	4.183	14.3
6	11.55	3.707	6.82
7	12.40	3.345	0.00

The situation from Fig. 7

additional condition. Later, a real function m (λ) is defined in terms of the weak solution. Its continuity, monotonicity and asymptotic characters ensure the existence of a unique solution for any amount on the loss boundary condition. According to these characters, we prove that the recovery problem is well posed. Finally, we design a numerical iteration algorithm and validate its effectiveness for inhomogeneous problem by two numerical experiments.

References

Chen T, Kang T, Lu G, Wu L (2014) A $(\boldsymbol{T}, \psi)$-ψ_e decoupled scheme for a time-dependent multiply-connected eddy current problem. Math Methods Appl Sci 37(3):343–359

Chew WC (2014) Vector potential electromagnetic theory with generalized gauge for inhomogeneous anisotropic media. arXiv preprint arXiv:1406.4780

Chovan J, Geuzaine C, Slodička M (2017) $\boldsymbol{A} - \phi$ formulation of a mathematical model for the induction hardening process with a nonlinear law for the magnetic field. Comput Methods Appl Mech Eng 321:294–315

Costabel M, Dauge M (2002) Weighted regularization of maxwell equations in polyhedral domains. Numer Math 93(2):239–277
Duan H-Y, Jia F, Lin P, Tan RCE (2009) The local L^2 projected C^0 finite element method for maxwell problem. SIAM J Numer Anal 47(2):1274–1303
Duan H-Y, Lin P, Tan RCE (2012) C^0 elements for generalized indefinite maxwell equations. Numer Math 122(1):61–99
Duan H-Y, Lin P, Tan RCE (2013) Analysis of a continuous finite element method for $\boldsymbol{H}$ (curl, div)-elliptic interface problem. Numer Math 123(4):671–707
Duan H-Y, Lin P, Tan RCE (2013) Error estimates for a vectorial second-order elliptic eigenproblem by the local L^2 projected C^0 finite element method. SIAM J Numer Anal 51(3):1678–1714
Duan H-Y, Lin P, Tan RCE (2016) A finite element method for a curlcurl-graddiv eigenvalue interface problem. SIAM J Numer Anal 54(2):1193–1228
Fabrizio M, Morro A (2003) Electromagnetism of continuous media. Oxford University Press, Oxford
Jin J (2014) The finite element method in electromagnetics, 3rd edn. Wiley-IEEE Press, New York
Kang T, Chen T, Wang Y, Kim KI (2015) A $\boldsymbol{T}$-ψ formulation with the penalty function term for the 3d eddy current problem in laminated structures. Appl Math Comput 271:618–641
Kang T, Kim KI (2009) Fully discrete potential-based finite element methods for a transient eddy current problem. Computing 85(4):339–362
Monk P (2003) Finite element methods for Maxwell's equations. Clarendon Press, Oxford
Mur G (1994) Edge elements, their advantages and their disadvantages. IEEE Trans Magn 30(5):3552–3557
Slodička M, Van Keer R (2000) Determination of the convective transfer coefficient in elliptic problems from a non-standard boundary condition. In: Marylka J, Tuma M, Sembere J (eds) Simulation, modelling, and numerical analysis : SIMONA 2000. Technical University Liberec, Liberec, pp 13–20
Zemanova V, Slodička M, Dupré L (2010) Determination of a material constant in the impedance boundary condition for electromagnetic fields. J Comput Appl Math 234(7):2062–2068
Zeng Z, Udpa L, Udpa SS, Chan MSC (2009) Reduced magnetic vector potential formulation in the finite element analysis of eddy current nondestructive testing. IEEE Trans Magn 45(3):964–967