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# TOPSIS with similarity measure for MADM applied to network selection

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**Abstract** In this article, a new method is introduced to handle fuzzy multi-attribute decision-making problems. The method preserves fuzziness in the preference technique to avoid the drawbacks of defuzzification. The study modifies the technique of order preference by similarity to an ideal solution (TOPSIS) for interval-valued fuzzy numbers. The traditional TOPSIS uses the relative degree of closeness to rank the alternatives. Instead, a similarity measure based on map distance is used for preference. The degree of similarity between each attribute of an alternative and the ideal solution is computed, and a similarity matrix is formed. Then, the total degree of similarity for all the attributes of an alternative is used for ranking. The alternative corresponding to the one norm of the similarity matrix is the best alternative. Thus, the comparison is done on a fuzzy basis to avoid the loss of information due to converting the elements of the weighted normalized decision matrix to crisp values by defuzzification. An illustrative example is given to demonstrate the approach. A practical example in network selection to optimize vertical hand offs is solved where both user preferences and network parameters are treated as interval-valued fuzzy numbers.

**Keywords** Fuzzy multi-criteria decision-making · TOPSIS · Similarity measure · Network selection

**Mathematics Subject Classification** 90B50 · 90C70 · 90C90

## 1 Introduction

The purpose of multi-attribute decision-making (MADM) is to choose the best candidate from a set of alternatives using experts' evaluations of the multiple attributes of the alternatives

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(Chen and Lee 2010). In the process of decision-making, ambiguity and uncertainty are often confronted when evaluating the criteria weights and the alternatives of the problem. Some of the evaluation criteria are subjective and qualitative in nature which hinders expressing the preference using exact numerical values.

Until recently, MADM using type-1 fuzzy sets (T1FSs) attracted many researchers and many studies were introduced. However, T1FSs have a crisp membership function value in the interval [0,1]. Using a crisp membership function can decrease the flexibility and precision of decision-making in an uncertain environment, as it is hard to estimate the exact membership function of fuzzy sets in many situations (Ghorabaee 2016). Currently, MADM methods use more sophisticated fuzzy sets, e.g., interval type-2 fuzzy sets (Chen and Kuo 2017; Qin 2017; Cheng et al. 2016; Mendel 2016; Chen and Hong 2014; Chen and Wang 2013) and intuitionistic fuzzy sets (Das et al. 2016; Garg 2016a, b; Garg and Ansha 2016c; Garg 2017; Jiang et al. 2017; Singh and Garg 2017; Xu and Gou 2017). On the other hand, Shadowed sets as a simplification of fuzzy sets gained a growing interest in recent years. A shadowed set is a fuzzy set with a reduced number of membership degrees. An element with membership degree close to one is approximated to one; an element with membership degree close to zero is approximated to zero. Other elements are placed in a shadowed region (Cai et al. 2017).

Zadeh (1975) introduced type-2 fuzzy sets (T2FSs) as an extension of T1FSs with the capability of representing two types of uncertainties interpersonal and intrapersonal uncertainties (Bakar et al. 2015). They are used in situations where T1FSs fail to express uncertainties (Najariyan et al. 2017). They are characterized by an interval membership function that can provide more degrees of freedom in representing uncertainties in the real world problem (Ghorabaee 2016). When T2FSs are utilized in decision-making, heavy computations are confronted. Consequently, interval type-2 fuzzy sets (IT2FSs) were introduced according to certain simplification assumptions. Interval-valued fuzzy sets (IVFSs) are a special case of IT2FSs.

Extant MADM methods are often applied to data of the same type; they lack the ability to deal adequately with heterogeneous data (Chatterjee and Kar 2017). Recently, granular computing emerged as a promising research area in MADM. It is a structured problem-solving method to deal with information in heterogeneous contexts for decision making. This allows experts having different backgrounds and levels of knowledge (granules) to express their decisions in a more flexible way in accordance with their domain of knowledge (Chatterjee and Kar 2017). For more details on granular computing, the reader is referred to works by Chatterjee and Kar (2017), Meng et al. (2017), Sanchez et al. (2017), Wang et al. (2017) and Xu and Wang (2016). In addition, new techniques are introduced to solve MADM problems in which the decision makers' psychological behaviors are taken into consideration (Liu and You 2017).

The technique of order preference by similarity to an ideal solution (TOPSIS) is a popular approach for multi-attribute/multi-criteria decision-making introduced by Hwang and Yoon (1981) to deal with real-valued data (Rashid et al. 2014). A solution from TOPSIS is defined as the alternative which satisfies being the closest to the positive ideal solution and the farthest from the negative ideal solution (Chu and Lin 2003). TOPSIS also assesses the alternatives and the weighted coefficients which are represented by fuzzy numbers. First, the weighted ratings are defuzzified into crisp values; then a closeness coefficient is defined to determine the ranking order of the alternatives by calculating their distance from both the positive and negative ideal solutions. The conversion of the weighted normalized decision matrix to crisp values by defuzzification was proposed by Chu and Lin (2003) to change a fuzzy

MADM problem into a crisp value (Ilieva 2016). TOPSIS is preferred due to its simplicity and intuitiveness; it doesn't require a lot of computations (Ilieva 2016).

In the previous decade, several modifications have been introduced to fuzzy TOPSIS. Modifications can be either in the defuzzification or in the preference comparison technique. Defuzzification is characterized by being simple and easy, meanwhile, a fuzzy pair-wise comparison is complex and difficult. However, fuzzy pair-wise comparison preserves fuzziness in messages, while defuzzification loses uncertainty of messages (Ilieva 2016). Using interval-valued set concepts, Ashtiani et al. (2009) introduced interval-valued fuzzy TOPSIS to solve MCDM problems in which the weights of the criteria are unequal. Chen and Lee (2010) presented an interval type-2 fuzzy TOPSIS method for fuzzy multiple attributes group-decision making problems based on IT2FS. Rashid et al. (2014) extended TOPSIS using generalized interval-valued trapezoidal fuzzy numbers. Yet, they used an unjustified heuristic expression to calculate the difference between interval-valued trapezoidal fuzzy numbers (Dymova et al. 2015). Dymova et al. (2015) proposed an interval type-2 fuzzy extension of the TOPSIS method using α-cuts representation to avoid the limitations and drawbacks of the extant methods. Ilieva (2016) applied the graded mean integration to defuzzify IT2FS into two crisp values and then compute their average value. Recently, TOPSIS methods using interval-valued fuzzy data are the focus of substantial research (Ilieva 2016).

In this article, a new method to handle fuzzy multi-attribute decision making (MADM) problems is proposed. The method preserves fuzziness in the preference technique to avoid the disadvantages of defuzzification. The method is based on the TOPSIS for interval-valued fuzzy numbers. Although recent research has been devoted to the fuzzy extension of the TOPSIS method, only a few studies handled IT2FSs and the proposed extensions have some limitations and drawbacks (Dymova et al. 2015). In traditional TOPSIS the relative degree of closeness is used to rank the alternatives. Alternatively, a similarity measure based on map distance is used for preference comparison. Keeping the positive and negative ideal solutions fixed values, as the degree of similarity to the positive ideal solution increases, the degree of similarity to the negative ideal solution decreases and vice versa. Therefore, the degree of similarity between each attribute of an alternative and the ideal solution is computed. The total degree of similarity of all the attributes for an alternative is used for preference. Thus, the comparison is done on a fuzzy basis to avoid any loss of information due to the conversion of the elements of the weighted normalized decision matrix to crisp values by defuzzification.

The article is organized as follows. Different types of fuzzy numbers, TOPSIS, and the degree of similarity are presented in Sect. 2. The proposed method is introduced in Sect. 3. A numerical example and a practical example in network selection are solved using the proposed method in Sect. 4. Finally, the conclusion is given in Sect. 5.

## 2 Preliminaries

### 2.1 Interval-valued fuzzy numbers

**Definition 2.1.1** (Rashid et al. 2014) A trapezoidal type-1 fuzzy set is denoted by $\tilde{A} = (a_1, a_2, a_3, a_4; w)$, where

$$f_{\tilde{A}}(x) = \begin{cases} w - \frac{a_2 - x}{a_2 - a_1} & \text{when } a_1 < x \leq a_2, \\ w & \text{when } a_2 \leq x \leq a_3, \\ w - \frac{x - a_3}{a_4 - a_3} & \text{when } a_3 \leq x < a_4, \\ 0 & \text{otherwise,} \end{cases}$$

**Definition 2.1.2** (Dymova et al. 2015) A type-2 fuzzy set is defined as follows:

$$\tilde{A} = \int_{\forall x \in \boldsymbol{X}} \int_{\forall u \in J_x \subseteq [0,1]} \mu_{\tilde{A}}(x,u) / (x,u),$$

where $\iint$ denotes the union over all admissible $x$ and $u$ and $\mu_{\tilde{A}}(x,u)$ is a type-2 membership function.

Since T2FSs are three dimensional and require complex and immense computational burdensome operations, IT2FSs were introduced as a special case of generalized type-2 fuzzy sets (Kahraman et al. 2014). IT2FS is defined to be a T2FS with secondary grades equal to 1 (Dymova et al. 2015). It is represented as follows

$$\tilde{A} = \int_{\forall x \in \boldsymbol{X}} \int_{\forall u \in J_x \subseteq [0,1]} 1/(x,u).$$

**Definition 2.1.3** (Kahraman et al. 2014) The upper and lower membership functions of an IT2FS are type-1 membership functions.

**Definition 2.1.4** (Kahraman et al. 2014) Let $\tilde{A}^L$ and $\tilde{A}^U$ be two trapezoidal fuzzy numbers, and let $w_1^L,\ w_2^L,\ w_1^U \,and\, w_2^U$ for $\tilde{A}^L$ and $\tilde{A}^U$, respectively, denote the degrees of confidence of the linguistic opinions. A trapezoidal IT2FS is represented as $\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_1^L, w_2^L), (a_1^U, a_2^U, a_3^U, a_4^U; w_1^U, w_2^U)]$, where

$$a_1^L, a_2^L, a_3^L, a_4^L, a_1^U, a_2^U, a_3^U, \,and\, a_4^U \in R.$$

Trapezoidal IVFS is a special case of trapezoidal IT2FS when $w_1^L = w_2^L$ and $w_1^U = w_2^U$. Thus, a trapezoidal *IVFS* is represented by:

$$\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w^L), (a_1^U, a_2^U, a_3^U, a_4^U; w^U)] \text{ (Rashid et al. 2014).}$$

If $w^L = w^U = 1$, a trapezoidal IVFS is said to be perfectly normal. If $w^U = 1$ and $w^L < 1$ a trapezoidal IVFS is said to be normal (Dymova et al. 2015). A triangular IVFS is a special case of trapezoidal IVFSs when $a_2^L = a_3^L$ and $a_2^U = a_3^U$. If $\tilde{A}^L = \tilde{A}^U$, an IVFS reduces to a T1FS.

**Definition 2.1.5** (Rashid et al. 2014) For two interval-valued fuzzy numbers

$$\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{A}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{A}}^U)] \text{ and}$$
$$\tilde{B} = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{B}}^U), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{B}}^U)]$$
$$\tilde{A} \oplus \tilde{B} = \left[\left(a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; min\left(w_{\tilde{A}}^L, w_{\tilde{B}}^L\right)\right), \left(a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U;\ min\left(w_{\tilde{A}}^U, w_{\tilde{B}}^U\right)\right)\right].$$

**Definition 2.1.6** (Rashid et al. 2014) For two interval-valued fuzzy numbers

$$\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{A}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{A}}^U)] \text{ and}$$
$$\tilde{B} = [(b_1^L, b_2^L, b_3^L, b_4^L; w_{\tilde{B}}^L), (b_1^U, b_2^U, b_3^U, b_4^U; w_{\tilde{B}}^U)]$$
$$\tilde{A} \otimes \tilde{B} = \left[\left(a_1^L b_1^L, a_2^L b_2^L, a_3^L b_3^L, a_4^L b_4^L; min\left(w_{\tilde{A}}^L, w_{\tilde{B}}^L\right)\right), \left(a_1^U b_1^U, a_2^U b_2^U, a_3^U b_3^U, a_4^U b_4^U;\ min\left(w_{\tilde{A}}^U, w_{\tilde{B}}^U\right)\right)\right].$$

**Definition 2.1.7** (Rashid et al. 2014) For an interval-valued fuzzy numbers $\tilde{A} = [(a_1^L, a_2^L, a_3^L, a_4^L; w_{\tilde{A}}^L), (a_1^U, a_2^U, a_3^U, a_4^U; w_{\tilde{A}}^U)]$ and an arbitrary real number $k$

$$
\begin{aligned}
k.\tilde{A} &= \tilde{A}.k \\
&= \begin{cases} \left[\left(k.a_1^L, k.a_2^L, k.a_3^L, k.a_4^L; w_{\tilde{A}}^L\right), \left(k.a_1^U, k.a_2^U, k.a_3^U, k.a_4^U; w_{\tilde{A}}^U\right)\right]; & \textit{if } k \geq 0, \\ \left[\left(k.a_4^L, k.a_3^L, k.a_2^L, k.a_1^L; w_{\tilde{A}}^L\right), \left(k.a_4^U, k.a_3^U, k.a_2^U, k.a_1^U; w_{\tilde{A}}^U\right)\right]; & \textit{if } k \leq 0. \end{cases}
\end{aligned}
$$

## 2.2 TOPSIS method

The classical TOPSIS is based on the idea of selecting the alternative with the shortest distance from the positive ideal solution and the greatest distance from the negative ideal solution. TOPSIS is a useful and practical tool to rank and select among alternatives (Rashid et al. 2014). The method was presented by Hwang and Yoon (1981), later extended to the fuzzy environment by Chen (2000) for T1FSs (Kumar and Garg 2016). Chen and Lee (2010) modified the method for IT2FSs. Although recent research has been devoted to the fuzzy extension of the TOPSIS method, only a few studies handled IT2FSs (Dymova et al. 2015).

Consider a MADM based on $n$ alternatives $X_1, X_2, \ldots, X_n$ and a set of $m$ attributes $f_1, f_2, \ldots, f_m$ in the presence of $k$ decision makers $D_1, D_2, \ldots, D_k$. The set $''F''$ of attributes can be divided into two sets, $F_b$ the set of benefits attribute and $F_c$ the set of cost attributes such that $F_b \cap F_c = \emptyset$. The basics of TOPSIS method can be summarized in the following steps (Chen and Lee 2010):

**Step 1**: (a) Construction of the decision matrix for the $p$th decision maker,

$$
\tilde{\mathbf{D}}_{\mathrm{p}} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & & X_n \end{array} \\ \begin{bmatrix} \tilde{f}_{11}^{\mathrm{p}} & \tilde{f}_{12}^{\mathrm{p}} & \cdots & \tilde{f}_{1\mathrm{n}}^{\mathrm{p}} \\ \tilde{f}_{21}^{\mathrm{p}} & \tilde{f}_{22}^{\mathrm{p}} & & \tilde{f}_{2\mathrm{n}}^{\mathrm{p}} \\ & \vdots & \ddots & \vdots \\ \tilde{f}_{\mathrm{m}1}^{\mathrm{p}} & \tilde{f}_{\mathrm{m}2}^{\mathrm{p}} & \cdots & \tilde{f}_{\mathrm{mn}}^{\mathrm{p}} \end{bmatrix} \end{array}.
$$

(b) Construction of the average decision matrix.

$$
\bar{\tilde{\mathbf{D}}} = \left[\tilde{f}_{\boldsymbol{ij}}\right]_{m \times n},
$$

where $\tilde{f}_{\boldsymbol{ij}} = \left(\frac{\tilde{f}_{ij}^1 \oplus \tilde{f}_{ij}^2 \oplus \cdots \oplus \tilde{f}_{ij}^k}{k}\right)$ is an interval type-2 fuzzy set,

$$
1 \leq i \leq m,\ 1 \leq j \leq n \text{ and } 1 \leq p \leq k.
$$

**Step 2:** (a) Construction of the weighting matrix of the $p^{th}$ decision maker for the attributes,

$$
\tilde{\mathbf{W}}_p = \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & & f_m \end{array} \\ \left[\begin{array}{cccc} \tilde{\mathrm{w}}_1^p & \tilde{\mathrm{w}}_2^p & \ldots & \tilde{\mathrm{w}}_m^p \end{array}\right] \end{array}
$$

(b) Construction of the average weighting matrix

$$
\bar{\tilde{\mathbf{W}}}_{\mathrm{p}} = \left[\tilde{\mathrm{w}}_{\boldsymbol{i}}\right]_{1 \times m},
$$

where $\tilde{w}_{\boldsymbol{i}} = \left(\frac{\tilde{\mathrm{w}}_i^1 \oplus \tilde{\mathrm{w}}_i^2 \oplus \cdots \oplus \tilde{\mathrm{w}}_i^p}{k}\right)$ is an interval type -2 fuzzy set,

$$1 \leq i \leq m \quad \text{and} \quad 1 \leq p \leq k.$$

**Step 3:** Construction of the weighted decision matrix

$$\widetilde{\mathbf{D}}_{\mathrm{w}} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & & X_n \end{array} \\ \begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \cdots & \tilde{v}_{1n} \\ \tilde{v}_{21} & \tilde{v}_{22} & & \tilde{v}_{2n} \\ \vdots & & \ddots & \vdots \\ \tilde{v}_{m1} & \tilde{v}_{m2} & \cdots & \tilde{v}_{mn} \end{bmatrix} \end{array},$$

where $\tilde{v}_{ij} = \tilde{\mathrm{w}}_i \otimes \tilde{f}_{ij}$, $1 \leq i \leq m$ and $1 \leq j \leq n$.
**Step 4**: Calculation of the ranking value of the interval type-2 fuzzy set $\tilde{v}_{ij}$,

$$\bar{\boldsymbol{D}}_{\mathrm{w}}^{\boldsymbol{r}} = \left[\mathrm{rank}\left(\tilde{v}_{ij}\right)\right]_{\boldsymbol{m}\times\boldsymbol{n}}.$$

The ranking value of the interval type-2 fuzzy number $\tilde{v}$ is defined as follows.

$$\begin{aligned} \mathrm{rank}\left(\tilde{v}\right) &= \mathrm{M}_1\left(\tilde{v}^U\right) + \mathrm{M}_1\left(\tilde{v}^L\right) + \mathrm{M}_2\left(\tilde{v}^U\right) + \mathrm{M}_2\left(\tilde{v}^L\right) + \mathrm{M}_3\left(\tilde{v}^U\right) + \mathrm{M}_3\left(\tilde{v}^L\right) \\ &\quad -\frac{1}{4}\left(\mathrm{S}_1\left(\tilde{v}^U\right) + \mathrm{S}_1\left(\tilde{v}^L\right) + \mathrm{S}_2\left(\tilde{v}^U\right) + \mathrm{S}_2\left(\tilde{v}^L\right) + \mathrm{S}_3\left(\tilde{v}^U\right) + \mathrm{S}_3\left(\tilde{v}^L\right)\right. \\ &\quad \left. + \mathrm{S}_4\left(\tilde{v}^U\right) + \mathrm{S}_4\left(\tilde{v}^L\right)\right) + w_1^L + w_2^L + w_1^U + w_2^U, \end{aligned}$$

where $\mathrm{M}_r\left(\tilde{v}^h\right) = \left(a_r^h + a_{(r+1)}^h\right)/2,\ 1 \leq r \leq 3$, is the average value of $a_r^h$ and $a_{(r+1)}^h$,

$\mathrm{S}_r\left(\tilde{v}^h\right) = \sqrt{\frac{1}{2}\sum_{l=r}^{r+1}\left(a_l^h - \frac{1}{2}\sum_{l=r}^{r+1} a_l^h\right)^2},\ 1 \leq r \leq 3,$ is the standard deviation of $a_r^h$ and $a_{r+1}^h$,

$\mathrm{S}_4\left(\tilde{v}^h\right) = \sqrt{\frac{1}{4}\sum_{l=1}^{4}\left(a_l^h - \frac{1}{4}\sum_{l=1}^{4} a_l^h\right)^2},$ is the standard deviation of the elements $a_1^h, a_2^h, a_3^h$ and $a_4^h$, and $h \in \{U,\ L\}$.

**Step 5**: Determination of the positive ideal solution $\tilde{A}^+ = \left\{v_1^+,\ v_2^+,\ \ldots,\ v_m^+\right\}$ and the negative ideal solution $\tilde{A}^+ = \left\{v_1^-,\ v_2^-,\ \ldots,\ v_m^-\right\}$ where

$$v_i^+ = \begin{cases} \max\limits_{1\leq j\leq n}\left\{\mathrm{rank}\left(\tilde{v}_{ij}\right)\right\}, & \textit{if } f_i \in F_b \\ \min\limits_{1\leq j\leq n}\left\{\mathrm{rank}\left(\tilde{v}_{ij}\right)\right\}, & \textit{if } f_i \in F_c \end{cases},$$

and

$$v_i^- = \begin{cases} \min\limits_{1\leq j\leq n}\left\{\mathrm{rank}\left(\tilde{v}_{ij}\right)\right\}, & \textit{if } f_i \in F_b \\ \max\limits_{1\leq j\leq n}\left\{\mathrm{rank}\left(\tilde{v}_{ij}\right)\right\}, & \textit{if } f_i \in F_c. \end{cases}$$

**Step 6**: (a) Calculation of the distance between each alternative and the positive ideal solution, $d^+\left(X_j\right) = \sqrt{\sum_{i=1}^m\left(\mathrm{rank}\left(\tilde{v}_{ij}\right) - v_i^+\right)^2}$.

(b) Calculation of the distance between each alternative and the negative ideal solution $d^{-}\left(X_j\right) = \sqrt{\sum_{i=1}^{m}\left(\text{rank}\left(\tilde{v}_{ij}\right) - v_i^{-}\right)^2}$.

**Step 7**: Calculation of the relative degree of closeness for each alternative to the ideal solution $C\left(X_j\right) = \frac{d^{-}(X_j)}{d^{+}(X_j)+d^{-}(X_j)}$.

**Step 8**: Ranking the alternatives according to the degree of closeness. The greater the value of $C\left(X_j\right)$, the higher the alternative $X_j$ in preference.

The method of Chen and Lee (2010) depends on the defuzzification of the weighted decision matrix to determine the ideal solutions, i.e., the ideal solutions are presented by real values, this leads to the loss of important information and may provide wrong results (Dymova et al. 2015). Therefore, several modifications have been introduced to TOPSIS. Rashid et al. (2014) modified TOPSIS for trapezoidal IVFSs. In their work, they defined a distance measure between two trapezoidal IVFSs and defined the positive and negative ideal solutions as trapezoidal IVFSs, thus preserving fuzziness in information. Dymova et al. (2015) used the $\alpha$-cut representation of IT2FSs to avoid the restrictions on the shapes of IT2FSs which in the real world applications may have more complicated shapes other than triangular or trapezoidal ones.

### 2.3 Degree of similarity

Similarity measures have gained attention due to their wide applications in image processing, pattern recognition and economics (Beg and Rashid 2017). In the past decade, several methods have been proposed to measure the degree of similarity between interval-valued fuzzy numbers. Chen and Chen (2008) proposed a similarity measure based on the center of gravity of the lower and the upper fuzzy numbers. Wei and Chen (2009) combined the concepts of geometric distance, the perimeter, the height and the center of gravity points to measure the degree of similarity. Chen and Chen (2009) introduced similarity measure which considers the similarity of the gravities on the $X$-axis between upper fuzzy numbers, the difference of the spreads between upper fuzzy numbers, the heights of the upper fuzzy numbers, the degree of similarity on the $X$-axis between interval-valued fuzzy numbers, and the gravity on the $Y$-axis between interval-valued fuzzy numbers. In an attempt to overcome similarity measurement problems, Chen and Kao (2010) suggested a new similarity measure based on the standard deviation operator; Chen (2011) also proposed a similarity measure based on the quadratic mean operator.

Chen et al. (2013) introduced a similarity measure based on the map distance to overcome the limitations of the extant similarity measure methods, e.g., they cannot give the correct degree of similarity between two interval-valued fuzzy numbers in some cases. The results indicated that their method outperforms the existing methods (Chen et al. 2013).

The degree of similarity between two interval-valued trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ based on map distance can be computed as follows (Chen et al. 2013):

**Step 1:** Calculation of the distance values $\Delta a_i$ and $\Delta b_i$.
For the interval-valued trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the distance values between the lower and upper trapezoidal fuzzy numbers are calculated $\Delta a_i = \left|a_i^U - a_i^L\right|$ and $\Delta b_i = \left|b_i^U - b_i^L\right|$, where $i = 1, 2, 3, 4$.

**Step 2:** Calculation of the degree of similarity $S\left(\tilde{A}^{\Delta},\ \tilde{B}^{\Delta}\right)$ between $\Delta a_i$ and $\Delta b_i$.

(a) Calculate the standard deviations $\Delta S_a$ and $\Delta S_b$ between the upper and lower fuzzy numbers.

$$\bar{a}^U = (a_1^U + a_2^U + a_3^U + a_3^U)/4, \quad \bar{a}^L = (a_1^L + a_2^L + a_3^L + a_3^L)/4,$$
$$S_{\tilde{A}^U} = \sqrt{\frac{\sum_{i=1}^4 \left(a_i^U - \bar{a}^U\right)^2}{3}}, \; S_{\tilde{A}^L} = \sqrt{\frac{\sum_{i=1}^4 \left(a_i^L - \bar{a}^L\right)^2}{3}},$$
$$\Delta S_a = \left|S_{\tilde{A}^U} - S_{\tilde{A}^L}\right|.$$

Similarly, calculate $\bar{b}^U,\ \bar{b}^L,\ S_{\tilde{B}^U},\ S_{\tilde{B}^L},\ and\ \Delta S_b.$
(a) Calculate the map distance between the upper and lower trapezoidal fuzzy numbers.

$$T^{\Delta} = \left[\left(2 - \frac{1 + \max\left\{\left|\Delta a_2 - \Delta a_1\right|, \left|\Delta b_2 - \Delta b_1\right|\right\}}{1 + \min\left\{\left|\Delta a_2 - \Delta a_1\right|, \left|\Delta b_2 - \Delta b_1\right|\right\}}\right) + \left(2 - \frac{1 + \max\left\{\left|\Delta a_4 - \Delta a_3\right|, \left|\Delta b_4 - \Delta b_3\right|\right\}}{1 + \min\left\{\left|\Delta a_4 - \Delta a_3\right|, \left|\Delta b_4 - \Delta b_3\right|\right\}}\right)\right]/2.$$

(b) Calculate the degree of similarity $S\left(\tilde{A}^{\Delta}, \tilde{B}^{\Delta}\right) \in [0, 1]$.

$$S\left(\tilde{A}^{\Delta}, \tilde{B}^{\Delta}\right) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 \left(\Delta a_i - \Delta b_i\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|\Delta S_a - \Delta S_b\right|}{2}}\right] \times \left[1 - \frac{\left|w_{\check{A}^L} - w_{\check{B}^L}\right|}{\left|w_{\check{A}^U} + w_{\check{B}^U}\right|}\right] \times T^{\Delta}.$$

**Step 3:** Calculation of the degree of similarity $S\left(\tilde{A}^U,\ \tilde{B}^U\right)$ between $\tilde{A}^U$ and $\tilde{B}^U$.
(a) Calculate the map distance between the upper trapezoidal fuzzy numbers.

$$T^U = \left[\left(2 - \frac{1 + \max\left\{\left|a_2^u - a_1^u\right|, \left|b_2^u - b_1^u\right|\right\}}{1 + \min\left\{\left|a_2^u - a_1^u\right|, \left|b_2^u - b_1^u\right|\right\}}\right) + \left(2 - \frac{1 + \max\left\{\left|a_4^u - a_3^u\right|, \left|b_4^u - b_3^u\right|\right\}}{1 + \min\left\{\left|a_4^u - a_3^u\right|, \left|b_4^u - b_3^u\right|\right\}}\right)\right]/2.$$

(b) Calculate the degree of similarity $S\left(\tilde{A}^U, \tilde{B}^U\right) \in [0, 1]$.

$$S\left(\tilde{A}^U, \tilde{B}^U\right) = \left[1 - \frac{\sqrt{\sum_{i=1}^4 \left(a_i^u - b_i^u\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\left|S_{\tilde{A}^U} - S_{\tilde{B}^U}\right|}{2}}\right] \times \left[\frac{\min\left(w_{\check{A}^U},\ w_{\check{B}^U}\right)}{\max\left(w_{\check{A}^U}, w_{\check{B}^U}\right)}\right] \times T^U.$$

**Step 4**: Calculation of the degree of similarity $S(\tilde{A}, \tilde{B})$ between the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

$$S\left(\tilde{A}, \tilde{B}\right) = \frac{S\left(\tilde{A}^U, \tilde{B}^U\right) \times \left(1 + S\left(\tilde{A}^\Delta, \tilde{B}^\Delta\right)\right)}{2}.$$

The greater the value of $S\left(\tilde{A}, \tilde{B}\right)$ the greater the similarity between $\tilde{A}$ and $\tilde{B}$.

## 3 The proposed TOPSIS

In this section, the modification in TOPSIS using the similarity measure based on map distance is given.

**Step 1:** Construct the fuzzy decision matrix and the average decision matrix,

$$\bar{\tilde{\mathbf{D}}} = \left[\tilde{f}_{\boldsymbol{ij}}\right]_{m\times n}.$$

**Step 2:** Construct the weighting matrix and the average weighting matrix,

$$\bar{\tilde{\mathbf{W}}}_{\mathrm{p}} = \left[\tilde{\mathrm{w}}_{\boldsymbol{i}}\right]_{1\times m}.$$

**Step 3:** Construct the normalized average decision matrix $\bar{\tilde{\mathbf{N}}} = \left[\tilde{n}_{\boldsymbol{ij}}\right]_{m\times n}$.

The normalization process preserves the property that the ranges of the interval fuzzy numbers lie in the interval $[0,\ 1]$. The normalized performance ratings $\tilde{n}_{\boldsymbol{ij}}$ can be calculated as follows:

$$\tilde{n}_{\boldsymbol{ij}} = \left[\left(\frac{f_{1ij}^L}{f_{4j}^+}, \frac{f_{2ij}^L}{f_{4j}^+}, \frac{f_{3ij}^L}{f_{4j}^+}, \frac{f_{4ij}^L}{f_{4j}^+}; w^L\right), \left(\frac{f_{1ij}^U}{f_{4j}^+}, \frac{f_{2ij}^U}{f_{4j}^+}, \frac{f_{3ij}^U}{f_{4j}^+}, \frac{f_{4ij}^U}{f_{4j}^+}; w^U\right)\right],$$

where $i = 1, \ldots, m,\ j \in F_b$ and $f_{4j}^+ = \max_i f_{4ij}^U$.

$$\tilde{n}_{\boldsymbol{ij}} = \left[\left(\frac{f_{1j}^-}{f_{4ij}^L}, \frac{f_{1j}^-}{f_{3ij}^L}, \frac{f_{1j}^-}{f_{2ij}^L}, \frac{f_{1j}^-}{f_{1ij}^L}; w^L\right), \left(\frac{f_{1j}^-}{f_{4j}^U}, \frac{f_{1j}^-}{f_{3ij}^U}, \frac{f_{1j}^-}{f_{2ij}^U}, \frac{f_{1j}^-}{f_{1j}^U}; w^U\right)\right],$$

where $i = 1, \ldots, n,\ j \in F_c$ and $f_{1j}^- = \min_i f_{1ij}^U$

**Step 4:** Construct the weighted normalized decision matrix

$$\tilde{\mathbf{D}}_{\mathrm{w}} = \begin{array}{c} \\ f_1 \\ f_2 \\ \vdots \\ f_m \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & & X_n \end{array} \\ \begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \cdots & \tilde{v}_{1n} \\ \tilde{v}_{21} & \tilde{v}_{22} & & \tilde{v}_{2n} \\ \vdots & & \ddots & \vdots \\ \tilde{v}_{m1} & \tilde{v}_{m2} & \cdots & \tilde{v}_{mn} \end{bmatrix} \end{array},$$

where $\tilde{v}_{ij} = \tilde{\mathrm{w}}_i \otimes \tilde{n}_{ij}$, $1 \le i \le m$ and $1 \le j \le n$.

**Step 5:** Define the fuzzy positive ideal solution $\tilde{v}^+$ and the fuzzy negative ideal solution $\tilde{v}^-$ where,

$\tilde{v}^+ = [(1, 1, \ldots, 1; 1), (1, 1, \ldots, 1; 1)]$ and $\tilde{v}^- = [(0, 0, \ldots, 0; 1), (0, 0, \ldots, 0; 1)]$

**Step 6:** Find the similarity matrix $\mathbf{S} = \left[S_{ij}\right]$.

Compute the degree of similarity between $\tilde{v}_{ij}$ and the ideal solution using similarity measure based on map distance.

$$S_{ij}^{+} = S\left(\tilde{v}_{ij}, \tilde{v}^{+}\right) = \frac{S\left(\tilde{v}_{ij}^{U}, \tilde{v}^{+^{U}}\right) \times \left(1 + S\left(\tilde{v}_{ij}^{\Delta}, \tilde{v}^{+^{\Delta}}\right)\right)}{2}, \quad \textit{if } f_i \in F_b$$

$$S_{ij}^{-} = S\left(\tilde{v}_{ij}, \tilde{v}^{-}\right) = \frac{S\left(\tilde{v}_{ij}^{U}, \tilde{v}^{-^{U}}\right) \times \left(1 + S\left(\tilde{v}_{ij}^{\Delta}, \tilde{v}^{-^{\Delta}}\right)\right)}{2}, \quad \textit{if } f_i \in F_c.$$

**Step 7:** Calculate the total degree of similarity of each alternative to the ideal solution.

$$S(X_j) = \sum_{i=1}^{m} s_{ij}, \quad \text{for } j = 1, \ldots, n$$

The larger the value of $S\left(X_j\right)$, the higher the preference of the alternative $X_j$. The alternative corresponding to the similarity matrix one norm $\|\mathbf{S}_1\|$ is the best choice, where $\|\mathbf{S}_1\| = \max_{1 \le j \le n} \left( \sum_{i=1}^{m} \left| s_{ij} \right| \right)$.

# 4 Examples

In this section two examples are solved; the first example demonstrates the proposed TOPSIS, and a practical example in network selection to maximize end-users' satisfaction.

## 4.1 Numerical example

This example is due to Chen and Lee (2010). Suppose a company intends to buy cars for high-level managers from three alternatives $X_1$, $X_2$ *and* $X_3$. The decision makers $D_1$, $D_2$, *and* $D_3$ rate cars on four attributes: safety $(f_1)$, price $(f_2)$, appearance $(f_3)$, and performance $(f_4)$. The benefit attributes are the safety, the appearance and the performance, while the cost attribute is the price. Let $X = \{X_1, X_2, X_3\}$ be the set of alternatives, and let $F = \{f_1, f_2, f_3, f_4\}$ be the set of attributes. The decision makers use the linguistic terms: very low [(0,0,0,0.1;1), (0,0,0,0.05;0.9)], low [(0,0.1,0.1,0.3;1), (0.05,0.1,0.1,0.2;0.9)], medium low [(0.1,0.3,0.3,0.5;1), (0.2,0.3,0.3,0.4;0.9)], medium [(0.3,0.5,0.5,0.7;1), (0.4,0.5,0.5,0.6;0.9)], medium high [(0.5,0.7,0.7,0.9;1),(0.6,0.7,0.7,0.8;0.9)], high [(0.7,0.9,0.9,1;1),(0.8,0.9,0.9, 0.95;0.9)] and very high [(0.9,1,1,1;1),(0.95,1,1,1;0.9)]. For more details see Chen and Lee (2010).

**Step 1**: (a) Construction of the decision matrices.

$$\widetilde{\mathbf{D}}_1 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{MH} & \text{H} & \text{VH} \\ \text{H} & \text{MH} & \text{VH} \\ \text{VH} & \text{H} & \text{M} \\ \text{VH} & \text{H} & \text{H} \end{bmatrix} \end{array}, \quad \widetilde{\mathbf{D}}_2 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{H} & \text{MH} & \text{H} \\ \text{VH} & \text{H} & \text{VH} \\ \text{H} & \text{VH} & \text{MH} \\ \text{H} & \text{VH} & \text{VH} \end{bmatrix} \end{array},$$

$$\widetilde{\mathbf{D}}_3 = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{MH} & \text{H} & \text{MH} \\ \text{H} & \text{VH} & \text{H} \\ \text{H} & \text{VH} & \text{MH} \\ \text{H} & \text{H} & \text{VH} \end{bmatrix} \end{array}.$$

(b) Construction of the average decision matrix.

$$\bar{\tilde{\mathbf{D}}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \tilde{f}_{13} \\ \tilde{f}_{21} & \tilde{f}_{22} & \tilde{f}_{23} \\ \tilde{f}_{31} & \tilde{f}_{32} & \tilde{f}_{33} \\ \tilde{f}_{41} & \tilde{f}_{42} & \tilde{f}_{43} \end{bmatrix} \end{array},$$

where $\tilde{f}_{11} = [(0.57, 0.77, 0.77, 0.93; 1), (0.67, 0.77, 0.77, 0.85; 0.9)]$,

$$\begin{aligned}
\tilde{f}_{12} &= [(0.63, 0.83, 0.83, 0.97; 1), (0.73, 0.83, 0.83, 0.9; 0.9)], \\
\tilde{f}_{13} &= [(0.7, 0.87, 0.87, 0.97; 1), (0.78, 0.87, 0.87, 0.92; 0.9)], \\
\tilde{f}_{21} &= [(0.77, 0.93, 0.93, 1; 1), (0.85, 0.93, 0.97, 0.9; 0.9)], \\
\tilde{f}_{22} &= [(0.7, 0.87, 0.87, 0.97; 1), (0.78, 0.87, 0.87, 0.92; 0.9)], \\
\tilde{f}_{23} &= [(0.83, 0.97, 0.97, 1; 1), (0.9, 0.97, 0.97, 0.98; 0.9)], \\
\tilde{f}_{31} &= [(0.77, 0.93, 0.93, 1; 1), (0.85, 0.93, 0.97, 0.9; 0.9)], \\
\tilde{f}_{32} &= [(0.83, 0.97, 0.97, 1; 1), (0.9, 0.97, 0.97, 0.98; 0.9)], \\
\tilde{f}_{33} &= [(0.43, 0.63, 0.63, 0.83; 1), (0.53, 0.63, 0.63, 0.73; 0.9)], \\
\tilde{f}_{41} &= [(0.77, 0.93, 0.93, 1; 1), (0.85, 0.93, 0.97, 0.9; 0.9)], \\
\tilde{f}_{42} &= [(0.83, 0.97, 0.97, 1; 1), (0.9, 0.97, 0.97, 0.98; 0.9)], \\
\tilde{f}_{43} &= [(0.77, 0.93, 0.93, 1; 1), (0.85, 0.93, 0.97, 0.9; 0.9)],
\end{aligned}$$

**Step 2**: (a) Construction of the weighting matrices.

$$\widetilde{\mathbf{W}}_1 = \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} \text{VH} & \text{H} & \text{M} & \text{VH} \end{bmatrix} \end{array}, \widetilde{\mathbf{W}}_2 = \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} \text{H} & \text{VH} & \text{MH} & \text{H} \end{bmatrix} \end{array},$$

$$\widetilde{\mathbf{W}}_3 = \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} \text{VH} & \text{VH} & \text{MH} & \text{H} \end{bmatrix} \end{array}.$$

(b) Construction of the average weighting matrix.

$$\bar{\tilde{\mathbf{W}}} = \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 & \tilde{w}_4 \end{bmatrix} \end{array},$$

where

$$\begin{aligned}
\tilde{w}_1 &= [(0.83, 0.97, 0.97, 1; 1), (0.9, 0.97, 0.97, 0.98; 0.9)], \\
\tilde{w}_2 &= [(0.83, 0.97, 0.97, 1; 1), (0.9, 0.97, 0.97, 0.98; 0.9)], \\
\tilde{w}_3 &= [(0.43, 0.63, 0.63, 0.83; 1), (0.53, 0.63, 0.63, 0.73; 0.9)], \\
\tilde{w}_4 &= [(0.77, 0.93, 0.93, 1; 1), (0.85, 0.93, 0.97, 0.9; 0.9)].
\end{aligned}$$

**Step 3:** No need to normalize since the numbers lie in the interval [0, 1].
**Step 4**: Construction of the weighted decision matrix.

$$\widetilde{\mathbf{D}}_{\mathrm{w}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \tilde{v}_{13} \\ \tilde{v}_{21} & \tilde{v}_{22} & \tilde{v}_{23} \\ \tilde{v}_{31} & \tilde{v}_{32} & \tilde{v}_{33} \\ \tilde{v}_{41} & \tilde{v}_{42} & \tilde{v}_{43} \end{bmatrix} \end{array},$$

where

$$\begin{aligned}
\tilde{v}_{11} &= [(0.47, 0.74, 0.74, 0.93; 1)\ (0.6, 0.74, 0.74, 0.84; 0.9)]\,,\\
\tilde{v}_{12} &= [(0.53, 0.81, 0.81, 0.97; 1)\ (0.66, 0.81, 0.81, 0.89; 0.9)]\,,\\
\tilde{v}_{13} &= [(0.58, 0.84, 0.84, 0.97; 1)\ (0.71, 0.84, 0.84, 0.9; 0.9)]\,,\\
\tilde{v}_{21} &= [(0.64, 0.9, 0.9, 1; 1)\ (0.77, 0.9, 0.9, 0.95; 0.9)]\,,\\
\tilde{v}_{22} &= [(0.58, 0.84, 0.84, 0.97; 1)\ (0.71, 0.84, 0.84, 0.9; 0.9)]\,,\\
\tilde{v}_{23} &= [(0.69, 0.93, 0.93, 1; 1)\ (0.81, 0.93, 0.93, 0.97; 0.9)]\,,\\
\tilde{v}_{31} &= [(0.33, 0.59, 0.59, 0.83; 1)\ (0.45, 0.59, 0.59, 0.71; 0.9)]\,,\\
\tilde{v}_{32} &= [(0.36, 0.61, 0.61, 0.83; 1)\ (0.48, 0.61, 0.61, 0.72; 0.9)]\,,\\
\tilde{v}_{33} &= [(0.19, 0.4, 0.4, 0.69; 1)\ (0.28, 0.4, 0.4, 0.54; 0.9)]\,,\\
\tilde{v}_{41} &= [(0.59, 0.87, 0.87, 0.93; 1)\ (0.72, 0.87, 0.87, 0.93; 0.9)]\,,\\
\tilde{v}_{42} &= [(0.59, 0.87, 0.87, 0.93; 1)\ (0.72, 0.87, 0.87, 0.93; 0.9)]\,,\\
\tilde{v}_{43} &= [(0.64, 0.9, 0.9, 1; 1)\ (0.77, 0.9, 0.9, 0.95; 0.9)]\,,
\end{aligned}$$

**Step 5:** Define the fuzzy positive ideal solution $\tilde{v}^+ = [(1, 1, 1, 1; 1)\ (1, 1, 1, 1; 1)]$ and the fuzzy negative ideal solution $\tilde{v}^- = [(0, 0, 0, 0; 1)\,, (0, 0, 0, 0; 1)]$.
**Step 6:** Construct the similarity matrix $[S_{ij}]$.

$$\begin{array}{l} \\ \text{Safety} \\ \text{Price} \\ \text{Appearance} \\ \text{Performance} \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} S_{11}^+ & S_{12}^+ & S_{13}^+ \\ S_{21}^- & S_{22}^- & S_{23}^- \\ S_{31}^+ & S_{32}^+ & S_{33}^+ \\ S_{41}^+ & S_{42}^+ & S_{43}^+ \end{bmatrix} \end{array} = \begin{array}{l} \\ \text{Safety} \\ \text{Price} \\ \text{Appearence} \\ \text{Performance} \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} 0.2905 & 0.3216 & 0.3565 \\ 0.0633 & 0.0844 & 0.0548 \\ 0.2235 & 0.2411 & 0.1597 \\ 0.3547 & 0.3547 & 0.3934 \end{bmatrix} \end{array}.$$

**Step 7:** Calculate the total degree of similarity of each alternative to the ideal solution.

$$S(X_1) = 0.932,\ S(X_2) = 1.0018 \text{ and } S(X_3) = 0.9644.$$

From the results, $S(X_2) > S(X_3) > S(X_1)$, the ranking is $X_2 > X_3 > X_1$. Then, the best alternative is $X_2$. The ranking of Chen and Lee (2010) is $X_2 > X_1 > X_3$. Despite the best alternative agrees with that of Chen and Lee (2010), the preference differs regarding the first and third alternatives.

The formulas for the degree of similarity given in Sect. 2.3 are remarkably reduced due to setting the positive and negative ideal solution to $\tilde{v}^+ = [(1, 1, 1, 1; 1)\ (1, 1, 1, 1; 1)]$ and $\tilde{v}^- = [(0, 0, 0, 0; 1)\,, (0, 0, 0, 0; 1)]$. For example, when calculating the degree of similarity between the positive ideal solution and $\tilde{v}_{11} = [(0.47, 0.74, 0.74, 0.93; 1)\ (0.6,\ 0.74,\ 0.74, 0.84; 0.9)]$, the following results are obtained.

1. $\Delta a_1 = \left|a_1^U - a_1^L\right| = 0.13$, $\Delta a_2 = \left|a_2^U - a_2^L\right| = 0$, $\Delta a_3 = \Delta a_2 = 0$ *and* $\Delta a_4 = \left|a_4^U - a_4^L\right| = 0.09$. $\Delta b_i$ is always equal to zero.
2. The standard deviations $\Delta S_a$ between the upper and lower fuzzy numbers, $\bar{a}^U = 0.72$, $\bar{a}^L = 0.73$, $S_{\tilde{A}^U} = \sqrt{\frac{\sum_{i=1}^{4}\left(a_i^U - \bar{a}^U\right)^2}{3}} = 0.1892$, $S_{\tilde{A}^L} = \sqrt{\frac{\sum_{i=1}^{4}\left(a_i^L - \bar{a}^L\right)^2}{3}} = 0.0987$ and $\Delta S_a = \left|S_{\tilde{A}^U} - S_{\tilde{A}^L}\right| = 0.0905$. $\Delta S_b$ is always equal to zero.
   Since $|\Delta b_2 - \Delta b_1| = |\Delta b_4 - \Delta b_3| = 0$, $T^\Delta$ reduces to
   $$T^\Delta = \left[(1 - |\Delta a_2 - \Delta a_1|) + (1 - |\Delta a_4 - \Delta a_3|)\right]/2 = 0.89.$$
   Also, $\Delta b_i = 0$, $\Delta S_b = 0$, $w_{\breve{B}^L} = 1$ and $w_{\breve{B}^U} = 1$. Then, $S\left(\tilde{A}^\Delta, \tilde{B}^\Delta\right)$ reduces to
   $$S\left(\tilde{A}^\Delta, \tilde{B}^\Delta\right) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}(\Delta a_i)^2}}{2}\right] \times \left[1 - \sqrt{\frac{\Delta S_a}{2}}\right] \times \left[1 - \frac{\left|1 - w_{\breve{A}^L}\right|}{\left|1 + w_{\breve{A}^U}\right|}\right] \times T^\Delta = 0.6130.$$
3. Since $\left|b_2^u - b_1^u\right| = \left|b_4^u - b_3^u\right| = 0$, $T^U$ reduces to
   $$T^U = \left[\left(1 - \left|a_2^u - a_1^u\right|\right) + \left(1 - \left|a_4^u - a_3^u\right|\right)\right]/2 = 0.77.$$
   We also have $S_{\tilde{B}^U} = 0$ *and* $w_{\breve{B}^U} = 1$. Then $S\left(\tilde{A}^U, \tilde{B}^U\right)$ reduces to
   $$S\left(\tilde{A}^U, \tilde{B}^U\right) = \left[1 - \frac{\sqrt{\sum_{i=1}^{4}\left(a_i^u - b_i^u\right)^2}}{2}\right] \times \left[1 - \sqrt{\frac{S_{\tilde{A}^U}}{2}}\right] \times w_{\breve{A}^U} \times T^U = 0.3602.$$
4. Finally, we get $S\left(\tilde{A}, \tilde{B}\right) = \frac{S\left(\tilde{A}^U, \tilde{B}^U\right) \times \left(1 + S\left(\tilde{A}^\Delta, \tilde{B}^\Delta\right)\right)}{2} = 0.2905.$

### 4.2 Practical example

The 4G mobile terminals roam freely across several wireless systems; as a result, they continuously undergo vertical handoffs (VHOs) (Chamodrakas and Martakos 2011). In a VHO process, a mobile station diverts its current point of attachment to a different network due to degradation or complete loss of signal and/or deterioration of the provided quality of service (QoS) (Mehbodniya et al. 2013). Network selection is the backbone of the VHO process. The selection of the network must consider both user preferences and various attributes including QoS and monetary cost (Chamodrakas and Martakos 2011). QoS parameters include but not restricted to: Received Signal Strength (RSS), bandwidth (BW), throughput, delay, jitter, and security. The selected network must fulfill end-users service requests while keeping the overall satisfaction at a high level. The wrong selection may lead to undesirable conditions, e.g., weak QoS, network congestions, blocked calls, resulting in unsatisfied users (Mehbodniya et al. 2013). Therefore, the decision to select the best network from various networks to optimize VHOs is crucial.

A wireless environment is dynamic in nature; it is characterized by its inherent uncertainty and imprecise parameters and constraints. Due to this vagueness, a fuzzy approach for system design seems to yield better results when used in such environments (Mehbodniya et al. 2013). The simultaneous optimization of all network criteria may be impossible due to the conflict

**Table 1** Linguistic terms of the weights and their corresponding IVFS

| Linguistic terms | Interval-valued fuzzy sets |
|---|---|
| Very low (VL) | ((0, 0.1, 0.1, 0.3; 1), (0.05, 0.1, 0.1, 0.25; 0.9)) |
| Low (L) | ((0.1, 0.3, 0.3, 0.5; 1), (0.15, 0.3, 0.3, 0.45; 0.9)) |
| Fair (F) | ((0.3, 0.5, 0.5, 0.7; 1), (0.35, 0.5, 0.5, 0.65; 0.9)) |
| High (H) | ((0.5, 0.7, 0.7, 0.9; 1), (0.55, 0.7, 0.7, 0.85; 0.9)) |
| Very high (VH) | ((0.7, 0.9, 0.9, 1; 1), (0.75, 0.9, 0.9, 0.95; 0.9)) |

**Table 2** Linguistic terms of the ratings and their corresponding IVFS

| Linguistic terms | Interval-valued fuzzy sets |
|---|---|
| Absolutely low (AL) | ((0, 0.2, 0.2, 0.4; 1) , (0.05, 0.2, 0.2, 0.35; 0.9)) |
| Medium low (ML) | ((0.2, 0.4, 0.4, 0.6; 1) , (0.25, 0.4, 0.4, 0.55; 0.9)) |
| Medium (M) | ((0.4, 0.6, 0.6, 0.8; 1) , (0.45, 0.6, 0.6, 0.75; 0.9)) |
| Medium high (MH) | ((0.6, 0.8, 0.8, 1; 1) , (0.65, 0.8, 0.8, 0.95; 0.9)) |
| Absolutely high (AH) | ((0.8, 1, 1, 1; 1) , (0.85, 0.95, 0.95, 0.95; 0.9)) |

among them, e.g. BW and energy consumption (EC). Thus, fuzzy TOPSIS is used to aggregate them in a consistent and theoretically robust technique (Chamodrakas and Martakos 2011).

The following example is adopted from Chamodrakas and Martakos (2011) and Mehbodniya et al. (2013) with some modifications. Both the user provided preference and the network parameters are treated as interval-valued fuzzy numbers which are more suitable for representing uncertainties.

A mobile terminal integrates three access network interfaces: Wireless Local Area Network (WLAN) ($X_1$), Worldwide Interoperability for Microwave Access (WiMAX) ($X_2$) and Universal Mobile Telecommunication System (UMTS) ($X_3$). The best network selection is based on three QoS attributes BW ($f_1$), delay ($f_2$), and energy consumption ($f_3$) for four different QoS applications in the mobile context: voice (V), video conferencing (VC), video streaming (VS) and web browsing (WB). The linguistic assessments of the user preferences for the attributes per application and the linguistic ratings of the attributes are given directly in the solution steps. The interval-valued fuzzy numbers corresponding to the linguistic assessments are given in Table 1. The interval-valued fuzzy numbers corresponding to the linguistic ratings are given in Table 2.

**Step 1:** Construction of the decision matrix $\left[f_{ij}\right]$ for each QoS application.

$$\widetilde{\mathbf{V}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{AH} & \text{AH} & \text{AH} \\ \text{AH} & \text{MH} & \text{AL} \\ \text{AL} & \text{ML} & \text{MH} \end{bmatrix} \end{array}, \quad \widetilde{\mathbf{VC}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{AH} & \text{M} & \text{AL} \\ \text{AH} & \text{MH} & \text{AL} \\ \text{AL} & \text{ML} & \text{MH} \end{bmatrix} \end{array},$$

$$\widetilde{\mathbf{VS}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{AH} & \text{AH} & \text{AH} \\ \text{AH} & \text{AH} & \text{AH} \\ \text{AH} & \text{MH} & \text{AH} \end{bmatrix} \end{array}, \quad \widetilde{\mathbf{WB}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \text{AH} & \text{AH} & \text{AH} \\ \text{AH} & \text{AH} & \text{AH} \\ \text{AH} & \text{MH} & \text{AH} \end{bmatrix} \end{array}.$$

(b) Construction of the average decision matrix.

$$\tilde{\bar{\mathbf{D}}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \tilde{f}_{13} \\ \tilde{f}_{21} & \tilde{f}_{22} & \tilde{f}_{23} \\ \tilde{f}_{31} & \tilde{f}_{32} & \tilde{f}_{33} \end{bmatrix} \end{array},$$

where

$$\begin{aligned}
\tilde{f}_{11} &= [(0.8, 1, 1, 1; 1), (0.85, 0.95, 0.95, 0.95; 0.9)], \\
\tilde{f}_{12} &= [(0.75, 0.95, 0.95, 1; 1), (0.8, 0.9125, 0.9125, 0.95; 0.9)], \\
\tilde{f}_{13} &= [(0.6, 0.8, 0.8, 0.85; 1), (0.65, 0.7625, 0.7625, 0.8; 0.9)], \\
\tilde{f}_{21} &= [(0.8, 1, 1, 1; 1), (0.85, 0.95, 0.95, 0.95; 0.9)], \\
\tilde{f}_{22} &= [(0.7, 0.9, 0.9, 1; 1), (0.75, 0.875, 0.875, 0.95; 0.9)], \\
\tilde{f}_{23} &= [(0.4, 0.6, 0.6, 0.7; 1), (0.45, 0.575, 0.575, 0.65; 0.9)], \\
\tilde{f}_{31} &= [(0.4, 0.6, 0.6, 0.7; 1), (0.45, 0.6, 0.6, 0.75; 0.9)], \\
\tilde{f}_{32} &= [(0.4, 0.6, 0.6, 0.8; 1), (0.45, 0.6, 0.6, 0.75; 0.9)], \\
\tilde{f}_{33} &= [(0.7, 0.9, 0.9, 1; 1), (0.75, 0.875, 0.875; 0.9)].
\end{aligned}$$

**Step 2**: (a) Construction of the weighting matrices.

$$\widetilde{\mathbf{W}}_{\mathrm{v}} = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} \mathrm{L} & \mathrm{VH} & \mathrm{VH} \end{bmatrix} \end{array}, \widetilde{\mathbf{W}}_{\mathrm{vc}} = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} \mathrm{H} & \mathrm{VH} & \mathrm{VH} \end{bmatrix} \end{array}, \widetilde{\mathbf{W}}_{\mathrm{vs}} = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} \mathrm{H} & \mathrm{L} & \mathrm{VH} \end{bmatrix} \end{array},$$

$$\widetilde{\mathbf{W}}_{\mathrm{wb}} = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} \mathrm{L} & \mathrm{L} & \mathrm{VH} \end{bmatrix} \end{array}.$$

(b) Construction of the average weighting matrix.

$$\tilde{\bar{\mathbf{W}}} = \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} \tilde{w}_1 & \tilde{w}_2 & \tilde{w}_3 \end{bmatrix} \end{array},$$

where

$$\begin{aligned}
\tilde{w}_1 &= [(0.3, 0.5, 0.5, 0.7; 1), (0.35, 0.5, 0.5, 0.65; 0.9)], \\
\tilde{w}_2 &= [(0.4, 0.6, 0.6, 0.75; 1), (0.45, 0.6, 0.6, 0.7; 0.9)], \\
\tilde{w}_3 &= [(0.7, 0.9, 0.9, 1; 1), (0.75, 0.9, 0.9, 0.95; 0.9)],
\end{aligned}$$

**Step 3:** No need for normalization since the numbers lie in the interval [0, 1].
**Step 4:** Construction of the average-weighted decision matrix.

$$\widetilde{\mathbf{D}}_{\mathrm{w}} = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \end{array} \\ \begin{bmatrix} \tilde{v}_{11} & \tilde{v}_{12} & \tilde{v}_{13} \\ \tilde{v}_{21} & \tilde{v}_{22} & \tilde{v}_{23} \\ \tilde{v}_{31} & \tilde{v}_{32} & \tilde{v}_{33} \end{bmatrix} \end{array}$$

$$\begin{aligned}
\tilde{v}_{11} &= [(0.24, 0.5, 0.5, 0.7; 1)\ (0.2975, 0.4750, 0.4750, 0.6175; 0.9)], \\
\tilde{v}_{12} &= [(0.2250, 0.4750, 0.4750, 0.7; 1)\ (0.28, 0.4562, 0.4562, 0.6175; 0.9)],
\end{aligned}$$

$$\tilde{v}_{13} = [(0.18, 0.4, 0.4, 0.5950; 1)\ (0.2275, 0.3812, 0.3812, 0.52; 0.9)]\,,$$
$$\tilde{v}_{21} = [(0.32, 0.6, 0.6, 0.75; 1)\ (0.3825, 0.57, 0.57, 0.665; 0.9)]\,,$$
$$\tilde{v}_{22} = [(0.28, 0.54, 0.54, 0.75; 1)\ (0.3375, 0.5250, 0.5250, 0.6650; 0.9)]\,,$$
$$\tilde{v}_{23} = [(0.16, 0.36, 0.36, 0.525; 1)\ (0.2025, 0.345, 0.345, 0.455; 0.9)]\,,$$
$$\tilde{v}_{31} = [(0.28, 0.54, 0.54, 0.7; 1)\ (0.3375, 0.5175, 0.5175, 0.6175; 0.9)]\,,$$
$$\tilde{v}_{32} = [(0.28, 0.54, 0.54, 0.8; 1)\ (0.3375, 0.54, 0.54, 0.7125; 0.9)]\,,$$
$$\tilde{v}_{33} = [(0.49, 0.81, 0.81, 1; 1)\ (0.5625, 0.7875, 0.7875, 0.9025; 0.9)]\,.$$

**Step 4:** Define the fuzzy positive ideal solution $\tilde{v}^{+} = [(1, 1, 1, 1; 1)\,, (1, 1, 1, 1; 1)]$ and the fuzzy negative ideal solution $\tilde{v}^{-} = [(0, 0, 0, 0; 1)\,, (0, 0, 0, 0; 1)]$.

The QoS attributes is divided into two main categories: downward attributes and upward attributes (Chamodrakas and Martakos 2011). The utility of upward attributes rises as their value gets higher, while the utility of downward attributes rises as their value gets lower. Therefore, for upward attributes, the degree of similarity to the positive ideal solution is measured, while the degree of similarity to the negative ideal solution is measured for the downward attributes.

**Step 6:** Calculate the similarity matrix $\left[S_{ij}\right]$.

$$\begin{array}{c} \\ BW \\ D \\ EC \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathit{WLAN} & \mathit{WiMAX} & \mathit{UMTS} \end{array} \\ \begin{bmatrix} S^{+}_{11} & S^{+}_{12} & S^{+}_{13} \\ S^{-}_{21} & S^{-}_{22} & S^{-}_{23} \\ S^{-}_{31} & S^{-}_{32} & S^{-}_{33} \end{bmatrix} \end{array} = \begin{array}{c} \\ BW \\ D \\ EC \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \mathit{WLAN} & \mathit{WiMAX} & \mathit{UMTS} \end{array} \\ \begin{bmatrix} 0.2101 & 0.1930 & 0.1627 \\ 0.1962 & 0.2017 & 0.3517 \\ 0.2210 & 0.1813 & 0.0849 \end{bmatrix} \end{array}$$

**Step 7:** Calculate the total degree of similarity of each alternative to the ideal solution.

$$S(X_1) = 0.6242,\ S(X_2) = 0.5814 \quad \text{and} \quad S(X_3) = 0.5911.$$

The results reveal that WLAN is the best alternative followed by UMTS then WiMAX. WLAN outperforms both WiMAX and UMTS in the bandwidth and in the energy consumption, though being the least in the packet delay. Despite its worst performance in the bandwidth and the energy consumption, the high compensation of the packet delay characteristics of UMTS gives it a higher rating than WiMAX.

## 5 Conclusion

In this article, a new TOPSIS was introduced to handle fuzzy multi-attribute decision-making (MADM)problems. The proposed TOPSIS uses similarity measure based on map distance to preserve fuzziness in the preference technique to avoid the drawbacks of defuzzification. The conventional TOPSIS uses the relative degree of closeness to rank the alternatives. Alternatively, the degree of similarity between each attribute of an alternative and the ideal solution is computed and the similarity matrix is formed. The alternative corresponding to the similarity matrix one-norm is the best alternative. Thus, the comparison is done on a fuzzy basis to avoid the loss of information due to the conversion of the elements of the weighted normalized decision matrix to crisp values by defuzzification. A numerical example was given to clarify the method, and a practical example in network selection to optimize VHOs was solved taking both user preferences and network parameters as interval-valued fuzzy numbers.

The future research will study the possibility of modifying TOPSIS for IT2FSs in a similar manner by an appropriate degree of similarity.

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