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Accelerated modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems

Baohua Huang[1] · **Changfeng Ma**[1]

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Abstract In this paper, we reformulate a class of nonlinear complementarity problems as the implicit fixed-point equations. We demonstrate accelerated modulus-based matrix splitting iteration method. We show their convergence by assuming that the system matrix is positive definite or the splitting of the system matrix are H_+-compatible splitting and discuss the choice of the optimal parameter. Furthermore, we give two-step accelerated modulus-based matrix splitting iteration method, which may achieve higher computing efficiency. Numerical experiments are presented to show the effectiveness of the method.

Keywords Nonlinear complementarity problems · Modulus-based · Matrix splitting method · Convergence analysis · Numerical experiments

Mathematics Subject Classification 90C33 · 65F10 · 65F50 · 65G40

1 Introduction

In this paper, we consider the following nonlinear complementarity problems:

$$z \geq 0, \quad w = Az + \Psi(z) \geq 0, \quad z^T w = 0, \tag{1.1}$$

where $z = (z_1, z_2, \ldots, z_n)^T \in R^n$ is an unknown vector, $A = (a_{ij}) \in R^{n\times n}$ is a given large and sparse matrix, $\Psi(z)$ is a nonlinear function and the notation "$\geq$" denotes the

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✉ Changfeng Ma
macf@fjnu.edu.cn

1 Fujian Key Laboratory of Mathematical Analysis and Applications, College of Mathematics and Informatics, Fujian Normal University, Fuzhou 350117, China

componentwise defined partial ordering between two vectors and the superscript "T" denotes the transpose of a vector.

When $\Psi(z) = q$ and $q \in R^n$, the nonlinear complementarity problems of the form (1.1) reduce to the linear complementarity problems. Many problems in scientific computing and engineering applications demand to compute solutions of complementarity problem, for example, the free boundary problem of fluid dynamics, contact problem in elasticity, economic transportation; see Ferri and Pang (1997), Cottle et al. (1992). Constructing efficient and feasible iteration methods for solving linear complementarity problem has been received widely attention. For example, the projected iterations (Hadjidimos and Tzoumas 2016; Bai 1996), the matrix muti-splitting iterations (Bai 1999; Bai and Evans 2001, 2002; Bai and Zhang 2013) and the general fixed point iterations (Dong and Jiang 2009; Mangasarian 1997; Noor 1988).

Recently, modulus-based iteration methods are very popular; see, e.g., Bai (2010), Zhang (2011), Zhang and Ren (2013), Xia and Li (2015), Huang and Ma (2016), Xie and Xu (2016), Ma and Huang (2016), Hong and Li (2016), Xu (2015) and Zheng and Yin (2013, 2014). Because these methods avoid the projections of the iterative used in the projected relaxation iterations and the general fixed-point iterations. Frommer and Mayer (1989) researched a modulus-based nonsmooth Newton's method to the equivalent reformulation of the linear complementarity problems and established its locally quadratical convergence theory. Ma and Huang (2016) proposed modified modulus-based matrix splitting iteration method for a class of weakly nondifferentiable nonlinear complementarity problems and studies the convergence property when the system matrices are H_+-matrices. Zheng and Yin (2013, 2014) developed the accelerated modulus-based matrix splitting iteration method for the solution of the large sparse linear complementarity problem and derived the convergence results. Xie and Xu (2016) proved the convergence of two-step modulus-based matrix splitting iteration method for a class of nonlinear complementarity problems.

Inspired by the previous work, in this paper, by reformulating the nonlinear complementarity problem (1.1) as an implicit fixed point equation, we give accelerated modulus-based matrix splitting iteration methods for solving (1.1). The convergence conditions when the system is either a positive definite matrix or an H_+-matrix are presented. Moreover, we discuss the choice of the optimal parameter.

This paper is organized as follows: We first present modulus-based matrix splitting iteration methods for solving a class of nonlinear complementarity problems (1.1) in Sect. 2. The convergence conditions and optimal parameter when the system matrix is a positive definite matrix is presented in Sect. 3. In Sect. 4, we derive the convergence theory when the system is an H_+-matrix. And the optimal parameter of AMAOR method is established in this section. The numerical experiments of the proposed methods are shown and analyzed in Sect. 5 and some conclusions are given in Sect. 6.

2 Accelerated modulus-based matrix splitting iteration method

In this section, by reformulating the problem (1.1) as an implicit fixed point equation based on the splitting of A, we present accelerated modulus-based matrix splitting iteration method.

We first give some definitions, notations and lemmas used in the sequel convergence analysis. For the matrices $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times n}$, we call $A \geq B$ $(A > B)$, if $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$ holds for all $1 \leq i \leq n$, $1 \leq j \leq n$. $|A|$ denotes a nonnegative matrix with entries $|a_{ij}|$. If O is a null matrix and $A \geq O$ $(A > O)$, A is called a nonnegative matrix. We write $\|A\|$ and A^{-1} to denote norm and the inverse of matrix A, respectively. $|x| = (|x_1|, |x_2|, \dots, |x_n|)^T \in R^n$ denotes the absolute value of the vector x . I denotes the identity matrix of the proper size implied by context.

Let $Z^{n\times n}$ denote the set of all real $n \times n$ matrices having all nonpositive off-diagonal entries. A nonsingular matrix $A \in R^{n\times n}$ is called an M-matrix Berman and Plemmons (1979) if $A \in Z^{n\times n}$ and $A^{-1} \geq 0$. Matrix $A \in R^{n\times n}$ is called an H-matrix if its comparison matrix $\langle A\rangle = (\widetilde{a}_{ij}) \in R^{n\times n}$ is an M-matrix, where

$$\widetilde{a}_{ij} = \begin{cases} |a_{ii}|, & \text{for } i = j \\ -|a_{ij}|, & \text{for } i \neq j \end{cases} \quad i, j = 1, 2, \dots, n.$$

In particular, an H-matrix is called an H_+-matrix if the diagonal entries are all positive.

Let $\sigma(A)$ and $\rho(A)$ be the spectrum and the spectral radius of the matrix A, respectively. For a given matrix $A \in R^{n\times n}$, $A = M - N$ is called a splitting of the matrix A if M is nonsingular; a convergence splitting if $\rho(M^{-1}N) < 1$; an M-splitting if M is a nonsingular M-matrix and $N \geq O$; and an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$. Clearly , if $A = M - N$ is an M-splitting and A is a nonsingular M-matrix, then $\rho(M^{-1}N) < 1$, see (Bai 1999).

For the convergence proof, we need the following lemmas:

Lemma 2.1 (Frommer and Mayer 1989) *Let $A, B \in R^{n\times n}$. If A is an M-matrix, $B \in Z^{n\times n}$ and $A \leq B$, then B is an M-matrix.*

Lemma 2.2 (Frommer and Szyld 1992) *Let $A \in R^{n\times n}$ be an H-matrix and $A = D - B$, where D is the diagonal part of the matrix A. Then the following statements hold true:*

1. *A is nonsingular and $|A^{-1}| \leq \langle A\rangle^{-1}$;*
2. *$|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.*

Lemma 2.3 (Huang and Ma 2016) *Let $A = M - N$ be a splitting of the matrix $A \in R^{n\times n}$, and Ω, Γ be $n \times n$ positive diagonal matrices. Then the following statements hold true:*

1. *If (w, z) is a solution of the complementarity problem (1.1), then $x = \frac{1}{2}(\Gamma^{-1}z - \Omega^{-1}w)$ satisfies the implicit fixed point equation*

$$(M\Gamma + \Omega)x = N\Gamma x + (\Omega - A\Gamma)|x| - \Psi(\Gamma(|x| + x)). \tag{2.1}$$

2. *If x satisfies the implicit fixed point Eq. (2.1), then*

$$z = \Gamma(|x| + x), \quad w = \Omega(|x| - x),$$

is a solution of the complementarity problem (1.1).

According to Lemma 2.3, if $A = M_1 - N_1 = M_2 - N_2$ be the splitting of A, we can reformulate the problem (1.1) as the following implicit fixed point equation

$$(M_1 + \Omega)x = N_1x + (\Omega - M_2)|x| + N_2|x| - \gamma\Psi(z), \tag{2.2}$$

where $z = \frac{|x|+x}{\gamma}$, and $\omega = \frac{1}{\gamma}\Omega(|x| - x)$. By using the fixed point equation, we shall establish the following accelerated modulus-based matrix splitting iteration method for solving the problem (1.1).

Algorithm 2.1 Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix $A \in R^{n\times n}$, let Ω be an $n \times n$ positive diagonal matrix and γ be a positive constant. Given an initial vector $x^0 \in R^n$, compute $z^0 = (|x^0| + x^0)/\gamma$. For $k = 0, 1, 2, \ldots$, until the iteration sequence $\{z^k\}_{k=0}^{+\infty}$ is convergence, compute $x^{k+1} \in R^n$ by solving the linear system

$$(M_1 + \Omega)x^{k+1} = N_1x^k + (\Omega - M_2)|x^k| + N_2|x^{k+1}| - \gamma\Psi(z^k) \tag{2.3}$$

and set

$$z^{k+1} = \frac{1}{\gamma}\left(|x^{k+1}| + x^{k+1}\right).$$

Algorithm 2.1 includes the modulus-based matrix splitting iteration method (Xia and Li 2015) with $M_2 = A$ and $N_2 = O$ as its special case. Moreover, with specific choices of the matrix splitting and iteration parameters, Algorithm 2.1 can yield a series of accelerated modulus-based matrix splitting methods. For example, let $A = D - L - U$ with D, $-L$ and $-U$ being the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of A, then

$$M_1 = \frac{1}{\alpha}(D - \beta L), \quad N_1 = \frac{1}{\alpha}[(1-\alpha)D + (\alpha - \beta)L + \alpha U], \quad M_2 = D - U \text{ and } N_2 = L$$

Algorithm 2.1 reduces to the accelerated modulus-based accelerated overrelaxation (AMAOR) iteration method

$$\begin{aligned}(D + \alpha\Omega - \beta L)x^{k+1} = {} & [(1-\alpha)D + (\alpha - \beta)L + \alpha U]x^k + \alpha(\Omega - D + U)|x^k| \\ & + \alpha L|x^{k+1}| - \alpha\gamma\Psi(z^k).\end{aligned}$$

It also gives the accelerated modulus-based successive overrelaxation (AMSOR) iteration method, the accelerated modulus-based Gauss–Seidel (AMGS) iteration method and the accelerated modulus-based Jacobi (AMJ) iteration method when $\alpha = \beta$, $\alpha = \beta = 1$ and $\alpha = 1, \beta = 0$, respectively.

3 Convergence analysis for the case of positive-definite matrix

In this section, we consider A is positive definite matrix. To this end, we introduce the following functions:

$$\begin{aligned}&\xi_1(\Omega) = \|(\Omega + M_1)^{-1}N_1\|, \quad \xi_2(\Omega) = \|(\Omega + M_1)^{-1}N_2\|, \\ &\xi_3(\Omega) = \|(\Omega + M_1)^{-1}(\Omega - M_1)\|, \quad \xi_4(\Omega) = L\|(\Omega + M_1)^{-1}\|.\end{aligned}$$

Theorem 3.1 *Let $A \in R^{n\times n}$ be a positive definite matrix, and $A = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix A with $M_1 \in R^{n\times n}$ being positive definite matrix. Assume that $\Omega \in R^{n\times n}$ is a positive diagonal matrix, γ is a positive constant and $\Psi(z) : R^n \to R^n$ is a Lipschitz continuous function with the Lipschitz constant L, that is, for any $z_1, z_2 \in R^n$,*

$$\|\Psi(z_1) - \Psi(z_2)\| \le L\|z_1 - z_2\|$$

holds. Let $\varrho(\Omega) = 2[\xi_1(\Omega) + \xi_2(\Omega) + \xi_4(\Omega)] + \xi_3(\Omega)$. If the parameter matrix Ω satisfies $\varrho(\Omega) < 1$, then the iteration sequence $\{z^k\}_{k=0}^{+\infty} \subseteq R^n_+$ generated by Algorithm 2.1 converges to the solution $z^* \in R^n_+$ of the problem (1.1) for any initial vector $x^0 \in R^n$.

Proof Assume that $(z^*, w^*) \in R^n \times R^n$ is a solution of the problem (1.1). By relationship (2.2), we have $z^* = \dfrac{\gamma}{2}(z^* - \Omega^{-1}w^*)$ satisfies the fix point equation

$$(M_1 + \Omega)x^* = N_1x^* + (\Omega - M_2)|x^*| + N_2|x^*| - \gamma\Psi(z^*), \tag{3.1}$$

where $z^* = \frac{|x^*|+x^*}{\gamma}$. Together with Algorithm 2.1, subtracting (3.1) from (2.3), we obtain

$$\begin{aligned}(M_1 + \Omega)(x^{k+1} - x^*) = N_1(x^k - x^*) + (\Omega - M_2)(|x^k| - |x^*|)\\ + N_2(|x^{k+1}| - |x^*|) - \gamma[\Psi(z^k) - \Psi(z^*)].\end{aligned} \tag{3.2}$$

Noticing that M_1 is a positive definite matrix and Ω is a positive diagonal matrix, which follows that $M_1 + \Omega$ is also positive definite matrix; hence $M_1 + \Omega$ is nonsingular. Then, together (3.2) with $A = M_1 - N_1 = M_2 - N_2$ yields

$$\begin{aligned}x^{k+1} - x^* &= (M_1 + \Omega)^{-1}\Big[N_1(x^k - x^*) + (\Omega - M_2)(|x^k| - |x^*|) + N_2(|x^{k+1}| - |x^*|)\Big]\\ &\quad -\gamma(M_1 + \Omega)^{-1}\Big[\Psi(z^k) - \Psi(z^*)\Big]\\ &= (M_1+\Omega)^{-1}\Big[N_1(x^k-x^*) + (\Omega- M_2 + N_2-N_2)(|x^k|-|x^*|)\\ &\quad +N_2(|x^{k+1}| - |x^*|)\Big] - \gamma(M_1 + \Omega)^{-1}[\Psi(z^k) - \Psi(z^*)]\\ &= (M_1+\Omega)^{-1}\Big[N_1(x^k-x^*) + (\Omega- M_1+ N_1-N_2)(|x^k|-|x^*|)\\ &\quad +N_2(|x^{k+1}|-|x^*|)\Big] - \gamma(M_1+\Omega)^{-1}[\Psi(z^k)-\Psi(z^*)]\\ &= (M_1+\Omega)^{-1}\Big[N_1(x^k-x^*)+(\Omega-M_1)(|x^k|-|x^*|)\\ &\quad +N_1(|x^k|-|x^*|)+N_2(|x^{k+1}|-|x^*|)\Big] - \gamma(M_1 + \Omega)^{-1}[\Psi(z^k) - \Psi(z^*)].\end{aligned} \tag{3.3}$$

Taking an arbitrary matrix norm on both sides of (3.3), we have

$$\begin{aligned}\|x^{k+1} - x^*\| &\le 2\|(M_1 + \Omega)^{-1}N_1\|\|x^k - x^*\|\\ &\quad +\|(M_1 + \Omega)^{-1}N_2\|\|x^k - x^*\| + \|(M_1 + \Omega)^{-1}N_2\|\|x^{k+1} - x^*\|\\ &\quad +\|(M_1 + \Omega)^{-1})(\Omega - M_1)\|\|x^k - x^*\|+\gamma\|(M_1+\Omega)^{-1}\|\|\Psi(z^k)-\Psi(z^*)\|\\ &\le 2\|(M_1 + \Omega)^{-1}N_1\|\|x^k - x^*\|+\|(M_1 + \Omega)^{-1}N_2\|\|x^k - x^*\|\\ &\quad +\|(M_1 + \Omega)^{-1}N_2\|\|x^{k+1} - x^*\| + \|(M_1 + \Omega)^{-1})(\Omega - M_1)\|\|x^k - x^*\|\\ &\quad +\gamma L\|(M_1 + \Omega)^{-1}\|\|z^k - z^*\|,\end{aligned} \tag{3.4}$$

where the last inequality uses the fact that $\Psi(z)$ is a Lipschitz continuous function with the Lipschitz constant L. The inequality (3.4) can be rewritten as the following form:

$$\begin{aligned}[1 - \|(M_1 + \Omega)^{-1}N_2\|]\|x^{k+1} - x^*\| \le [\|2(M_1 + \Omega)^{-1}N_1\| + \|(M_1 + \Omega)^{-1}N_2\|\\ +\|(M_1 + \Omega)^{-1})(\Omega - M_1)\|]\|x^k - x^*\|\\ +\gamma L\|(M_1 + \Omega)^{-1}\|\|z^k - z^*\|.\end{aligned}$$

As $\gamma > 0$, we have

$$\begin{aligned}\|z^k - z^*\| &= \left\| \frac{|x^k| + x^k}{\gamma} - \frac{|x^*| + x^*}{\gamma} \right\| \\ &= \frac{1}{\gamma} \| |x^k| + x^k + |x^*| + x^* \| \\ &\le \frac{1}{\gamma} [\| |x^k| - |x^*| \| + \|x^k - x^*\|] \\ &\le \frac{2}{\gamma} \|x^k - x^*\|.\end{aligned}$$

Hence

$$\begin{aligned}[1 - \xi_2(\Omega)]\|x^{k+1} - x^*\| &\le [\|2(M_1 + \Omega)^{-1} N_1\| + \|(M_1 + \Omega)^{-1} N_2\| \\ &\quad + \|(M_1 + \Omega)^{-1})(\Omega - M_1)\|]\|x^k - x^*\| \\ &\quad + 2L\|(M_1 + \Omega)^{-1}\| \|x^k - x^*\| \\ &= [2\xi_1(\Omega) + \xi_2(\Omega) + \xi_3(\Omega) + 2\xi_4(\Omega)]\|x^k - x^*\|.\end{aligned}$$

Thereby, we can obtain

$$\|x^{k+1} - x^*\| \le \frac{2\xi_1(\Omega) + \xi_2(\Omega) + \xi_3(\Omega) + 2\xi_4(\Omega)}{1 - \xi_2(\Omega)} \|x^k - x^*\|,$$

with $\xi_2(\Omega) < 1$. By the fact that

$$\frac{2\xi_1(\Omega) + \xi_2(\Omega) + \xi_3(\Omega) + 2\xi_4(\Omega)}{1 - \xi_2(\Omega)} < 1$$

is equivalent to $\varrho(\Omega) = 2[\xi_1(\Omega) + \xi_2(\Omega) + \xi_4(\Omega)] + \xi_3(\Omega) < 1$, which shows that $\lim_{k \to +\infty} x^k = x^*$. The proof is completed. □

Remark 3.1 If $\Psi(z) = q$, where $q \in R^n$ is a constant vector, then the problem (1.1) reduce to the linear complementarity problem studied in Xu (2015). Because $L = 0$, $\xi_1(\Omega) = \xi(\Omega)$, $\xi_2(\Omega) = \eta(\Omega)$, $\xi_3(\Omega) = \mu(\Omega)$, $\varrho(\Omega) = \delta(\Omega)$, where $\xi(\Omega)$, $\eta(\Omega)$, $\mu(\Omega)$, $\delta(\Omega)$ are defined in Xu (2015). At this case, Theorem 3.1 reduces to Theorem 4.1 in Xu (2015).

Remark 3.2 If we use the norm $\|\cdot\|_{\Omega^{1/2},2}$ is defined by $\|x\|_{\Omega^{1/2},2} = \|\Omega^{1/2}x\|_2$ for a vector $x \in R^n$ and $\|X\|_{\Omega^{1/2},2} = \|\Omega^{1/2}X\Omega^{-1/2}\|_2$ for a matrix $X \in R^{n\times n}$. Let $M_2 = A$, $N_2 = 0$; then $\xi_1(\Omega) = \frac{\sigma_1}{2}$, $\xi_3(\Omega) = \sigma_1$, $\xi_4(\Omega) = \frac{\sigma_3}{2}$, where $\sigma_1, \sigma_2, \sigma_3$ are defined in Xie and Xu (2016). At this case, Theorem 3.1 reduce to Theorem 2.2 in Xie and Xu (2016).

In particular, if $M_1 \in R^{n\times n}$ be a symmetric positive definite matrix and $\Omega = \omega I \in R^{n\times n}$ is a scalar matrix, Theorem 3.1 immediately gives another convergence result. For convenience, we introduce some parameters

$$\tau_1 := \|M_1^{-1}N_1\|_2,\ \tau_2 := \|M_1^{-1}N_2\|_2,\ \kappa := \frac{\lambda_{\max}}{\lambda_{\min}},\ L_1 := \frac{L}{\lambda_{\min}},\ \omega_1 = \frac{\omega}{\lambda_{\min}}, \tag{3.5}$$

and

$$\nu(\tau_1, \tau_2) := \frac{[(\tau_1 + \tau_2)\kappa + L_1 - 1] + \sqrt{[(\tau_1 + \tau_2)\kappa + L_1 - 1]^2 + 4\kappa[(\tau_1 + \tau_2) + L_1]}}{2}, \tag{3.6}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of the matrix M_1, respectively. L is the Lipschitz constant of $\Psi(z)$. Combining the parameters in (3.5), (3.6) and Theorem 3.1, the convergence results can be described as follows:

Theorem 3.2 *Let $A \in R^{n\times n}$ be a positive definite matrix and $A = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix A with $M_1 \in R^{n\times n}$ being symmetric positive definite matrix. Assume that $\Omega = \omega I \in R^{n\times n}$ is a positive scalar matrix, γ is a positive constant and $\Psi(z) : R^n \to R^n$ is a Lipschitz continuous function with the Lipschitz constant L. If $L_1 < 1$, then the iteration sequence $\{z^k\}_{k=0}^{+\infty} \subseteq R^n_+$ generated by Algorithm 2.1 converges to the solution $z^* \in R^n_+$ of the problem (1.1) for any initial vector $x^0 \in R^n$, provided that the iterative parameter ω satisfies either of the following conditions:*

1. *when* $0 < \tau_1 + \tau_2 < \frac{1-L_1}{\sqrt{\kappa}}$,
$$\nu(\tau_1, \tau_2) < \omega_1 \leq \sqrt{\kappa};$$
2. *when* $\dfrac{1 - L_1}{\kappa} < \tau_1 + \tau_2 < \frac{1-L_1}{\sqrt{\kappa}}$,
$$\sqrt{\kappa} \leq \omega_1 < \frac{[1 - L_1 - (\tau_1 + \tau_2)]\kappa}{(\tau_1 + \tau_2)\kappa + L_1 - 1};$$
3. *when* $\tau_1 + \tau_2 \leq \frac{1-L_1}{\kappa}$,
$$\omega_1 \geq \sqrt{\kappa}.$$

Proof From Theorem 3.1, we need to derive the condition $\varrho(\Omega) < 1$ with 2-norm. From the properties of spectral norm, the fact that M_1 be a symmetric positive definite matrix and $\tau_1 \geq 0$, $\tau_2 \geq 0$, by directly calculation, we have

$$\begin{aligned}\xi_1(\Omega) &= \|(\omega I + M_1)^{-1} N_1\|_2 = \|(\omega I + M_1)^{-1} M_1 M_1^{-1} N_1\|_2 \\ &\leq \|(\omega I + M_1)^{-1} M_1\|_2 \|M_1^{-1} N_1\|_2 \\ &= \max_{\lambda\in\sigma(M_1)} \frac{\tau_1\lambda}{\omega + \lambda} = \frac{\tau_1\lambda_{\max}}{\omega + \lambda_{\max}}\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\xi_2(\Omega) &= \|(\omega I + M_1)^{-1} N_2\|_2 = \|(\omega I + M_1)^{-1} M_1 M_1^{-1} N_2\|_2 \\ &\leq \|(\omega I + M_1)^{-1} M_1\|_2 \|M_1^{-1} N_2\|_2 \\ &= \max_{\lambda\in\sigma(M_1)} \frac{\tau_2\lambda}{\omega + \lambda} = \frac{\tau_2\lambda_{\max}}{\omega + \lambda_{\max}}\end{aligned} \tag{3.8}$$

and by simple calculations

$$\xi_4(\Omega) = L\|(\omega I + M_1)^{-1}\|_2 = L \max_{\lambda\in\sigma(M_1)} \frac{1}{\omega + \lambda} = \frac{L}{\omega + \lambda_{\min}} \tag{3.9}$$

and

$$\begin{aligned}\xi_3(\Omega) &= \|(\Omega + M_1)^{-1}(\Omega - M_1)\|_2 = \max_{\lambda\in\sigma(M_1)} \frac{|\omega - \lambda|}{\omega + \lambda} \\ &= \max\left\{\frac{|\omega - \lambda_{\min}|}{\omega + \lambda_{\min}}, \frac{|\omega - \lambda_{\max}|}{\omega + \lambda_{\max}}\right\} = \begin{cases}\dfrac{\lambda_{\max} - \omega}{\lambda_{\max} + \omega}, & \text{for } \omega \leq \sqrt{\lambda_{\min}\lambda_{\max}}, \\ \dfrac{\omega - \lambda_{\min}}{\omega + \lambda_{\min}}, & \text{for } \omega \geq \sqrt{\lambda_{\min}\lambda_{\max}}.\end{cases}\end{aligned} \tag{3.10}$$

Together (3.7), (3.8) with (3.9), (3.10) yields

$$
\begin{aligned}
\varrho(\Omega) &= 2[\xi_1(\Omega)+\xi_2(\Omega)+\xi_4(\Omega)]+\xi_3(\Omega)\\
&= 2\left[\frac{\tau_1\lambda_{\max}}{\omega+\lambda_{\max}}+\frac{\tau_2\lambda_{\max}}{\omega+\lambda_{\max}}+\frac{L}{\omega+\lambda_{\min}}\right]+\begin{cases}\dfrac{\lambda_{\max}-\omega}{\lambda_{\max}+\omega}, & \text{for } \omega\le\sqrt{\lambda_{\min}\lambda_{\max}},\\ \dfrac{\omega-\lambda_{\min}}{\omega+\lambda_{\min}}, & \text{for } \omega\ge\sqrt{\lambda_{\min}\lambda_{\max}}.\end{cases}\\
&= \begin{cases}\dfrac{2L}{\omega+\lambda_{\min}}+\dfrac{\lambda_{\max}(2\tau_1+2\tau_2+1)-\omega}{\lambda_{\max}+\omega}, & \text{for } \omega\le\sqrt{\lambda_{\min}\lambda_{\max}},\\ \dfrac{2L+\omega-\lambda_{\min}}{\omega+\lambda_{\min}}+\dfrac{2\lambda_{\max(\tau_1+\tau_2)}}{\omega+\lambda_{\max}}, & \text{for } \omega\ge\sqrt{\lambda_{\min}\lambda_{\max}}.\end{cases}
\end{aligned}
\tag{3.11}
$$

From (3.11), we have

$$
\varrho(\Omega)=\frac{[2L-\lambda_{\min}+\lambda_{\max}+2\tau_1\lambda_{\max}+2\tau_2\lambda_{\max}]\,\omega+\lambda_{\max}\,[2L+\lambda_{\min}+2\tau_1\lambda_{\min}+2\tau_2\lambda_{\min}]-\omega^2}{(\omega+\lambda_{\min})(\omega+\lambda_{\max})}
\tag{3.12}
$$

when $\omega\le\sqrt{\lambda_{\min}\lambda_{\max}}$ and

$$
\varrho(\Omega)=\frac{\omega^2+[2L-\lambda_{\min}+\lambda_{\max}+2\tau_1\lambda_{\max}+2\tau_2\lambda_{\max}]\omega+\lambda_{\max}[2L-\lambda_{\min}+2\tau_1\lambda_{\min}+2\tau_2\lambda_{\min}]}{(\omega+\lambda_{\min})(\omega+\lambda_{\max})}
\tag{3.13}
$$

when $\omega\ge\sqrt{\lambda_{\min}\lambda_{\max}}$.

We first consider the case that $\omega\le\sqrt{\lambda_{\min}\lambda_{\max}}$. Obviously, if $\varrho(\Omega)<1$, relationship (3.12) implies that

$$
\omega^2-[(\tau_1+\tau_2)\lambda_{\max}-\lambda_{\min}+L]w-\lambda_{\max}[(\tau_1+\tau_2)\lambda_{\min}+L]>0.
$$

The above inequality together with (3.5) yields

$$
\omega_1^2-[(\tau_1+\tau_2)\kappa-1+L_1]w_1-\kappa[(\tau_1+\tau_2)+L_1]>0.
\tag{3.14}
$$

This combined with $\omega\le\sqrt{\lambda_{\min}\lambda_{\max}}$ gives that

$$
\nu(\tau_1,\tau_2)<\omega_1\le\sqrt{\kappa}.
$$

With consideration that the obtained upper bound with respect to ω must be not less than the corresponding lower bound, it obtains that $\nu(\tau_1,\tau_2)<\sqrt{\kappa}$ when $0<\tau_1+\tau_2<\frac{1-L_1}{\sqrt{\kappa}}$.

Next, we consider the case that $\omega\ge\sqrt{\lambda_{\min}\lambda_{\max}}$. Clearly, (3.13) implies that

$$
[(\tau_1+\tau_2)\lambda_{\max}-\lambda_{\min}+L]\omega+[(\tau_1+\tau_2)-1]\lambda_{\min}\lambda_{\max}+L\lambda_{\max}<0,
$$

since $\varrho(\Omega)<1$. Using the notation (3.5), the above inequality can be rewritten as

$$
[(\tau_1+\tau_2)\kappa-1+L_1]\omega_1+[(\tau_1+\tau_2)-1]\kappa+L_1\kappa<0.
\tag{3.15}
$$

If $(\tau_1+\tau_2)\kappa-1+L_1>0$, i.e., $\dfrac{1-L_1}{\kappa}<\tau_1+\tau_2$, then we have

$$
\omega_1<\frac{[1-L_1-(\tau_1+\tau_2)]\kappa}{(\tau_1+\tau_2)\kappa+L_1-1}.
$$

This together with $\omega\ge\sqrt{\lambda_{\min}\lambda_{\max}}$ gives that

$$
\sqrt{\kappa}\le\omega_1<\frac{[1-L_1-(\tau_1+\tau_2)]\kappa}{(\tau_1+\tau_2)\kappa+L_1-1}.
$$

Solving the following inequality:

$$\sqrt{\kappa} < \frac{[1 - L_1 - (\tau_1 + \tau_2)]\kappa}{(\tau_1 + \tau_2)\kappa + L_1 - 1},$$

we have $\tau_1 + \tau_2 < \frac{1-L_1}{\sqrt{\kappa}}$. This together with $\frac{1-L_1}{\kappa} < \tau_1 + \tau_2$ yields the second condition.

If $(\tau_1 + \tau_2)\kappa - 1 + L_1 \leq 0$, i.e., $\frac{1-L_1}{\kappa} \geq \tau_1 + \tau_2 > 0$. Obviously, (3.15) holds for any $\omega_1 > 0$. This together with the fact that $\omega \geq \sqrt{\lambda_{\min}\lambda_{\max}}$ yields the third condition. □

Remark 3.3 If $\Psi(z) = q$, where $q \in R^n$ is a constant vector, then the problem (1.1) reduces to the linear complementarity problem studied in Xu (2015). Because $L = 0$, then Theorem 3.1 reduces to Theorem 4.2 in Xu (2015).

Let $\mathcal{D}$ be the interval of convergence which has been obtained in Theorem 3.2. Then the optimal ω^* which minimizes the value of $\varrho(\Omega)$ defined as (3.11) in the proof of Theorem 3.2 can be established. Since $\varrho(\Omega)$ is determined by parameter ω, for convenience, denote $f(\omega) = \varrho(\Omega)$. That is

$$f(\omega) = \begin{cases} \dfrac{2L}{\omega + \lambda_{\min}} + \dfrac{\lambda_{\max}(2\tau_1 + 2\tau_2 + 1) - \omega}{\lambda_{\max} + \omega}, & \text{for } \omega \leq \sqrt{\lambda_{\min}\lambda_{\max}}, \\ \dfrac{2L + \omega - \lambda_{\min}}{\omega + \lambda_{\min}} + \dfrac{2\lambda_{\max}(\tau_1 + \tau_2)}{\omega + \lambda_{\max}}, & \text{for } \omega \geq \sqrt{\lambda_{\min}\lambda_{\max}}. \end{cases} \tag{3.16}$$

Based on the relation (3.16) and Theorem 3.2, we have the following theorem:

Theorem 3.3 *With the same notations and interval of convergence for parameter* ω *in Theorem 3.2, we have*

$$\omega^* = \arg\min_{\omega \in \mathcal{D}} f(\omega) = \sqrt{\lambda_{\min}\lambda_{\max}}, \quad f(\omega^*) = \frac{[1 + 2(\tau_1 + \tau_2)]\sqrt{\kappa} + 2L_1 - 1}{\sqrt{\kappa} + 1},$$

where $f(\omega)$ *is defined as (3.16).*

Proof First, we claim that $f(\omega)$ is continuous at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$. From the relation (3.16), it is easy to see that $f(\omega)$ is continuous from the left and the right at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$, and

$$\begin{aligned} f(\omega_+) - f(\omega_-) &= \left[\frac{2L + \sqrt{\lambda_{\min}\lambda_{\max}} - \lambda_{\min}}{\sqrt{\lambda_{\min}\lambda_{\max}} + \lambda_{\min}} + \frac{2\lambda_{\max}(\tau_1 + \tau_2)}{\sqrt{\lambda_{\min}\lambda_{\max}} + \lambda_{\max}}\right] \\ &\quad - \left[\frac{2L}{\sqrt{\lambda_{\min}\lambda_{\max}} + \lambda_{\min}} + \frac{\lambda_{\max}(2\tau_1 + 2\tau_2 + 1) - \sqrt{\lambda_{\min}\lambda_{\max}}}{\lambda_{\max} + \sqrt{\lambda_{\min}\lambda_{\max}}}\right] \\ &= \frac{\sqrt{\lambda_{\min}\lambda_{\max}} - \lambda_{\min}}{\sqrt{\lambda_{\min}\lambda_{\max}} + \lambda_{\min}} - \frac{\lambda_{\max} - \sqrt{\lambda_{\min}\lambda_{\max}}}{\lambda_{\max} + \sqrt{\lambda_{\min}\lambda_{\max}}} \\ &= \frac{(\sqrt{\lambda_{\min}\lambda_{\max}} - \lambda_{\min})^2}{\lambda_{\min}(\lambda_{\max} - \lambda_{\min})} - \frac{(\lambda_{\max} - \sqrt{\lambda_{\min}\lambda_{\max}})^2}{\lambda_{\max}(\lambda_{\max} - \lambda_{\min})} \\ &= 0. \end{aligned}$$

Hence, $f(\omega)$ is continuous at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$.

Next, we will find the minimum value point of $f(\omega)$ in the domain $\mathcal{D}$.

1. If $\omega < \sqrt{\lambda_{\min}\lambda_{\max}}$, that is, $0 < \tau_1 + \tau_2 < \frac{1-L_1}{\sqrt{\kappa}}$, we have

$$\begin{aligned} f'(\omega) &= -\frac{2L}{(\omega+\lambda_{\min})^2} + \frac{-(\omega+\lambda_{\max}) - \lambda_{\max}(2\tau_1+2\tau_2+1)+\omega}{(\omega+\lambda_{\max})^2} \\ &\le -\frac{2\lambda_{\max}(\tau_1+\tau_2+1)+2L}{(\omega+\lambda_{\max})^2} < 0. \end{aligned}$$

With consideration that obtained convergence domain for ω at this case, we know that $f(\omega)$ is a strictly monotonic decreasing function in the interval $(\lambda_{\min}\nu(\tau_1,\tau_2), \sqrt{\lambda_{\min}\lambda_{\max}}]$. Hence $f(\omega)$ attain the minimum value at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$ when $0 < \tau_1 + \tau_2 < \dfrac{1-L_1}{\sqrt{\kappa}}$.

2. If $\omega > \sqrt{\lambda_{\min}\lambda_{\max}}$, it then follows from (3.16) that

$$\begin{aligned} f'(\omega) &= \frac{(\omega+\lambda_{\min}) - (2L+\omega-\lambda_{\min})}{(\omega+\lambda_{\min})^2} - \frac{2\lambda_{\max}(\tau_1+\tau_2)}{(\omega+\lambda_{\max})^2} \\ &= 2\frac{(\lambda_{\min}-L)(\omega+\lambda_{\max})^2 - \lambda_{\max}(\tau_1+\tau_2)(\omega+\lambda_{\min})^2}{(\omega+\lambda_{\max})^2(\omega+\lambda_{\min})^2}. \end{aligned}$$

Let

$$g(\omega) = (\lambda_{\min}-L)(\omega+\lambda_{\max})^2 - \lambda_{\max}(\tau_1+\tau_2)(\omega+\lambda_{\min})^2. \tag{3.17}$$

Then

$$f'(\omega) = \frac{2g(\omega)}{(\omega+\lambda_{\max})^2(\omega+\lambda_{\min})^2}.$$

Now, we simplify (3.17). From (3.5) and (3.17), we have

$$\begin{aligned} g(\omega) &= (\lambda_{\min}-L)(\omega+\lambda_{\max})^2 - \lambda_{\max}(\tau_1+\tau_2)(\omega+\lambda_{\min})^2 \\ &= \omega^2[\lambda_{\min} - L - \lambda_{\max}(\tau_1+\tau_2)] + 2\omega[\lambda_{\max}(\lambda_{\min}-L) - \lambda_{\min}\lambda_{\max}(\tau_1+\tau_2)] \\ &\quad + [\lambda_{\max}^2(\lambda_{\min}-L) - \lambda_{\min}^2\lambda_{\max}(\tau_1+\tau_2)] \\ &= \omega^2\lambda_{\min}[1-L_1-\kappa(\tau_1+\tau_2)] + \omega 2\lambda_{\min}\lambda_{\max}[(1-L_1)-(\tau_1+\tau_2)] \\ &\quad + \lambda_{\min}^2\lambda_{\max}[\kappa(1-L_1)-(\tau_1+\tau_2)] \\ &= a\omega^2 + b\omega + c, \end{aligned}$$

where $a = \lambda_{\min}[1-L_1-\kappa(\tau_1+\tau_2)]$, $b = 2\lambda_{\min}\lambda_{\max}[(1-L_1)-(\tau_1+\tau_2)] > 0$, $c = \lambda_{\min}^2\lambda_{\max}[\kappa(1-L_1)-(\tau_1+\tau_2)] > 0$. If quadratic term coefficient $a \neq 0$, then discriminant of $a\omega^2 + b\omega + c = 0$ is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= 4\lambda_{\min}^2\lambda_{\max}^2[(1-L_1)-(\tau_1+\tau_2)]^2 \\ &\quad -4\lambda_{\min}^3\lambda_{\max}[\kappa(1-L_1)-(\tau_1+\tau_2)][1-L_1-\kappa(\tau_1+\tau_2)] \\ &= 4\lambda_{\min}^3\lambda_{\max}\kappa[(1-L_1)-(\tau_1+\tau_2)]^2 \\ &\quad -4\lambda_{\min}^3\lambda_{\max}[\kappa(1-L_1)-(\tau_1+\tau_2)][1-L_1-\kappa(\tau_1+\tau_2)] \\ &= 4\lambda_{\min}^3\lambda_{\max}(\kappa-1)^2(1-L_1)(\tau_1+\tau_2) \\ &> 0. \end{aligned}$$

Hence, there are two real roots of $a\omega^2 + b\omega + c = 0$, that is,

$$\omega_{(1)} = \frac{-b - \sqrt{\Delta}}{2a}, \quad \omega_{(2)} = \frac{-b + \sqrt{\Delta}}{2a}, \quad \omega_{(1)} + \omega_{(2)} = -\frac{b}{a}.$$

In the sequel, we are going to come up with the results by discussing three cases.

(2i) If $a > 0$, that is,

$$\tau_1 + \tau_2 < \frac{1 - L_1}{\kappa}.$$

From the proof of Theorem 3.2, $f(\omega) < 1$ for $\sqrt{\lambda_{\min}\lambda_{\max}} < \omega$ when $\frac{1-L_1}{\kappa} > \tau_1 + \tau_2$. On the other hand, $\omega_{(1)} < 0$, $\omega_{(1)} < \omega_{(2)}$, $\omega_{(1)} + \omega_{(2)} = -\frac{b}{a} < 0$. As quadratic function $g(\omega)$ opens upward, if $\omega_{(2)} \geq 0$, then certainly $g(0) \leq 0$, which obtains the contradiction with $g(0) = c > 0$. Therefore, $\omega_{(2)} < 0$ and $f(\omega)$ is strictly monotonic increasing in the interval $[\sqrt{\lambda_{\min}\lambda_{\max}}, +\infty)$. Then $f(\omega)$ attains the minimum vale at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$.

(2ii) If $a = 0$, that is,

$$\tau_1 + \tau_2 = \frac{1 - L_1}{\kappa}.$$

Then $g(\omega) = b\omega + c > 0$, which is always true for $\omega > \sqrt{\lambda_{\min}\lambda_{\max}}$. On the other hand, from the proof of Theorem 3.2, we know that $f(\omega) < 1$ for all $\omega > \sqrt{\lambda_{\min}\lambda_{\max}}$. Therefore, $f(\omega)$ attains the minimum vale at $\omega = \sqrt{\lambda_{\min}\lambda_{\max}}$ in the interval $[\sqrt{\lambda_{\min}\lambda_{\max}}, +\infty)$.

(2iii) If $a < 0$, that is,

$$\tau_1 + \tau_2 > \frac{1 - L_1}{\kappa}.$$

From the proof of Theorem 3.2, $f(\omega) < 1$ for $\sqrt{\lambda_{\min}\lambda_{\max}} \leq \omega < \frac{[1-L_1-(\tau_1+\tau_2)]\lambda_{\max}}{(\tau_1+\tau_2)\kappa+L_1-1}$ when $\frac{1-L_1}{\kappa} < \tau_1 + \tau_2 < \frac{1-L_1}{\sqrt{\kappa}}$. By using the same analysis as the case (2i), we can conclude that $f(\omega)$ is strictly monotonic increasing in the interval $\left\{\omega | \sqrt{\lambda_{\min}\lambda_{\max}} \leq \omega < \frac{[1-L_1-(\tau_1+\tau_2)]\lambda_{\max}}{(\tau_1+\tau_2)\kappa+L_1-1}\right\}$, which means $\sqrt{\lambda_{\min}\lambda_{\max}}$ is the minimum value point of the function $f(\omega)$.

Sum up the above cases, we know that

$$\omega^* = \arg\min_{\omega \in \mathcal{D}} f(\omega) = \sqrt{\lambda_{\min}\lambda_{\max}}, \quad f(\omega^*) = \frac{[1 + 2(\tau_1 + \tau_2)]\sqrt{\kappa} + 2L_1 - 1}{\sqrt{\kappa} + 1},$$

which completes the proof. □

Notice that Zhang (2011) researched the two-step modulus-based matrix splitting iteration method for linear complementarity problems. Xie and Xu (2016) proposed the two-step modulus-based matrix splitting iteration method for nonlinear complementarity problems. They proved that two-step modulus-based matrix splitting iteration method can achieve higher computing efficiency by utilizing the information contained in the system matrix. Based on Algorithm 2.1, we now present a two-step modulus-based matrix splitting iteration method for (1.1).

Algorithm 3.1 Let $A = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix $A \in R^{n\times n}$, let Ω be an $n \times n$ positive diagonal matrix and γ be a positive constant. Given an initial vector $x^0 \in R^n$, compute $z^0 = (|x^0| + x^0)/\gamma$. For $k = 0, 1, 2, \ldots$, until the iteration sequence $\{z^k\}_{k=0}^{+\infty}$ is convergence, compute $x^{k+1} \in R^n$ by solving the linear system

$$\begin{cases} (M_1+\Omega)x^{k+\frac{1}{2}} = N_1x^k + (\Omega - M_2)|x^k| + N_2|x^{k+\frac{1}{2}}| - \gamma\Psi(z^k) \\ (M_2+\Omega)x^{k+1} = N_2x^{k+\frac{1}{2}} + (\Omega - M_1)|x^{k+\frac{1}{2}}| + N_1|x^{k+1}| - \gamma\Psi(z^{k+\frac{1}{2}}) \end{cases} \tag{3.18}$$

and set

$$z^{k+1} = \frac{1}{\gamma}\left(|x^{k+1}| + x^{k+1}\right).$$

Similar to the proof in Theorem 3.1, we can easily obtain the following convergence theorem:

Theorem 3.4 *Let $A \in R^{n\times n}$ be a positive definite matrix, and $A = M_1 - N_1 = M_2 - N_2$ be two splittings of the matrix A with $M_1, M_2 \in R^{n\times n}$ being positive definite matrix. Assume that $\Omega \in R^{n\times n}$ is a positive diagonal matrix, γ is a positive constant and $\Psi(z): R^n \to R^n$ is a Lipschitz continuous function with the Lipschitz constant L, that is, for any $z_1, z_2 \in R^n$,*

$$\|\Psi(z_1) - \Psi(z_2)\| \le L\|z_1 - z_2\|$$

holds. Let

$$\begin{aligned} &\xi_1(\Omega) = \|(\Omega + M_1)^{-1}N_1\|,\ \xi_2(\Omega) = \|(\Omega + M_1)^{-1}N_2\|, \\ &\xi_3(\Omega) = \|(\Omega + M_1)^{-1}(\Omega - M_1)\|, \\ &\xi_4(\Omega) = L\|(\Omega + M_1)^{-1}\|,\ \xi(\Omega) = \xi_2(\Omega) + \xi_3(\Omega) + 2\xi_1(\Omega) + 2\xi_4(\Omega) \end{aligned}$$

and

$$\begin{aligned} &\eta_1(\Omega) = \|(\Omega + M_2)^{-1}N_2\|,\ \eta_2(\Omega) = \|(\Omega + M_2)^{-1}N_1\|, \\ &\eta_3(\Omega) = \|(\Omega + M_2)^{-1}(\Omega - M_2)\|, \\ &\eta_4(\Omega) = L\|(\Omega + M_2)^{-1}\|,\ \eta(\Omega) = \xi_2(\Omega) + \eta_3(\Omega) + 2\xi_1(\Omega) + 2\xi_4(\Omega). \end{aligned}$$

Let $\varrho(\Omega) = \dfrac{\xi(\Omega) + \eta(\Omega)}{[1 - \xi_2(\Omega)][1 - \eta_2(\Omega)]}$. If the parameter matrix Ω satisfies $\varrho(\Omega) < 1$, then the iteration sequence $\{z^k\}_{k=0}^{+\infty} \subseteq R^n_+$ generated by Algorithm 3.1 converges to the solution $z^* \in R^n_+$ of the problem (1.1) for any initial vector $x^0 \in R^n$.

Proof The proof is the same as that of Theorem 3.1, so we omit here. □

4 Convergence analysis for the case of H_+-matrix

In the following, we consider the convergence analysis of Algorithm 2.1 when the system matrix A is an H_+-matrix. To this end, we suppose that there exists a nonnegative matrix G such that

$$|\Psi(y) - \Psi(z)| \le G|y - z| \tag{4.1}$$

holds for any $y, z \in R^n$.

We shall emphasize that the assumption of Ψ satisfies (4.1) is the same as that in Hong and Li (2016). In Sun and Zeng (2011), the authors assume that Ψ is continuously differentiable monotone. In Ma and Huang (2016), the authors assume that Ψ is Lipschitz continuous diagonal function on R^n, that is, the ith component Ψ_i of Ψ is a function of the ith variable z_i only:

$$\Psi(z) = (\Psi_1(z), \Psi_2(z), \ldots, \Psi_n(z))^T = (\Psi_1(z_1), \Psi_2(z_2), \ldots, \Psi_n(z_n))^T$$

where $z = (z_1, z_2, \ldots, z_n)^T$, and some functions $\Psi_i : R \to R$, for any $y, z \in R^n$ holds that

$$|\Psi_i(y_i) - \Psi_i(z_i)| \le l_i |y_i - z_i|, \ i = 1, 2, \ldots, n,$$

where $l_i \ge 0$ are the Lipschitz constants. By contrast, the assumption here is much weaker.

4.1 Convergence analysis

In this subsection, we will give some convergence theorems.

Theorem 4.1 *Let $A \in R^{n\times n}$ be an H_+-matrix, and $A = M_1 - N_1 = M_2 - N_2$ be two H-compatible splittings of the matrix A, with $M_1 = (m_{ij}^{(1)})$, $M_2 = (m_{ij}^{(2)}) \in R^{n\times n}$. Let $A = D - B$ be a splitting of A with D, $-B$ are the diagonal and the nondiagonal matrices, respectively. Assume that $\Omega \in R^{n\times n}$ is a positive diagonal matrix, and γ is a positive constant. If parameter matrix Ω satisfies $\Omega \ge diag(M_2)$, $\langle A\rangle - G$ and $\Omega + M_1 - |N_2|$ are M-matrix, then the iteration sequence $\{z^k\}_{k=0}^{+\infty} \subseteq R_+^n$ generated by Algorithm 2.1 converges to the solution $z^* \in R_+^n$ of the problem (1.1) for any initial vector $x^0 \in R^n$.*

Proof First, we prove that $M_1 + \Omega$ is a H_+-matrix. Since $A = M_1 - N_1$ is an H-compatible splitting, i.e., $\langle A\rangle = \langle M_1\rangle - |N_1|$, which follows that $|m_{ii}^{(1)}| - |n_{ii}^{(1)}| > 0$. Together with A is an H_+-matrix, we have $a_{ii} = m_{ii}^{(1)} - n_{ii}^{(1)} > 0$; hence $m_{ii}^{(1)} > 0$, $i = 1, 2, \ldots, n$. As $\langle A\rangle = \langle M_1\rangle - |N_1|$ and Ω is a positive diagonal matrix, it holds that

$$\langle A\rangle \le \langle M_1\rangle \le diag(M_1).$$

According to Lemma 2.1, M_1 is an H_+-matrix; hence $M_1 + \Omega$ is an H_+-matrix and it holds from Lemma 2.3 that

$$|(M_1 + \Omega)^{-1}| \le \langle M_1 + \Omega\rangle^{-1} = (\langle M_1\rangle + \Omega)^{-1}.$$

Combining (3.13) and (4.1) yields that

$$\begin{aligned} |x^{k+1} - x^*| &\le |(M_1 + \Omega)^{-1}|\big[|N_1||x^k - x^*| + |\Omega - M_2|||x^k| - |x^*|| + |N_2|||x^{k+1}| - |x^*||\big] \\ &\quad + \gamma|(M_1 + \Omega)^{-1}||\Psi(z^k) - \Psi(z^*)| \\ &\le (\langle M_1\rangle + \Omega)^{-1}\big[|N_1| + |\Omega - M_2| + |N_2|\big]|x^k - x^*| \\ &\quad + (\langle M_1\rangle + \Omega)^{-1}|N_2||x^{k+1} - x^*| + \gamma(\langle M_1\rangle + \Omega)^{-1}G|z^k - z^*|. \end{aligned} \tag{4.2}$$

As $\gamma > 0$, we have

$$\begin{aligned} |z^k - z^*| &= \left|\frac{|x^k| + x^k}{\gamma} - \frac{|x^*| + x^*}{\gamma}\right| \\ &= \frac{1}{\gamma}\big||x^k| + x^k + |x^*| + x^*\big| \\ &\le \frac{1}{\gamma}\big[||x^k| - |x^*|| + |x^k - x^*|\big] \\ &\le \frac{2}{\gamma}|x^k - x^*|. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2), we have

$$\begin{aligned} |x^{k+1} - x^*| &\le (\langle M_1\rangle + \Omega)^{-1}\big[|N_1| + |\Omega - M_2| + 2G\big]|x^k - x^*| \\ &\quad + (\langle M_1\rangle + \Omega)^{-1}|N_2||x^{k+1} - x^*|. \end{aligned} \tag{4.4}$$

By simple calculation, (4.4) can be rewritten as

$$[1-(\langle M_1\rangle+\Omega)^{-1}|N_2|]|x^{k+1}-x^*| \leq (\langle M_1\rangle+\Omega)^{-1}\big[|N_1|+|\Omega-M_2|+2G\big]|x^k-x^*|.$$

Since $\Omega+M_1-|N_2|$ is an M-matrix, $\langle M_1\rangle+\Omega$ is an M-matrix by Lemma 2.1; hence the splitting $\Omega+M_1-|N_2|$ is an M-splitting and $\rho((\langle M_1\rangle+\Omega)^{-1}|N_2|)<1$; thus if $1-(\langle M_1\rangle+\Omega)^{-1}|N_2|$ is an $M-$matrix and its inverse is nonnegative, then

$$\begin{aligned}|x^{k+1}-x^*| &\leq [1-(\langle M_1\rangle+\Omega)^{-1}|N_2|]^{-1}(\langle M_1\rangle+\Omega)^{-1}\big[|N_1|+|\Omega-M_2|+2G\big]|x^k-x^*|\\ &\leq (\Omega+\langle M_1\rangle-|N_2|)^{-1}\big[|N_1|+|\Omega-M_2|+2G\big]|x^k-x^*|.\end{aligned}$$

Let $\widetilde{A}=\widetilde{M}-\widetilde{N}$, $\widetilde{M}=\Omega+\langle M_1\rangle-|N_2|$, $\widetilde{N}=|N_1|+|\Omega-M_2|+2G$, by some calculation, we immediately have

$$\begin{aligned}\widetilde{A}&=\widetilde{M}-\widetilde{N}\\ &=\Omega+\langle M_1\rangle-|N_2|-|N_1|-|\Omega-M_2|-2G\\ &=\Omega-|\Omega-\mathrm{diag}(M_2)|+|M_2|-\mathrm{diag}(M_2)+\langle A\rangle-|N_2|-2G\\ &=\Omega-\mathrm{diag}(M_2)-|\Omega-\mathrm{diag}(M_2)|+2\langle A\rangle-2G\end{aligned}$$

If $\Omega\geq\mathrm{diag}(M_2)$ and $\langle A\rangle-G$ is an M-matrix, the splitting $\widetilde{A}=\widetilde{M}-\widetilde{N}$ is an M-splitting; thus $\rho(\widetilde{M}^{-1}\widetilde{N})<1$ and hence $\lim_{k\to\infty}x^k=x^*$, which completes the proof. □

Analogously, the convergence theorem for AMAOR iteration methods can be established as follows:

Theorem 4.2 *Let $A\in R^{n\times n}$ be an H_+-matrix with $A=D-B$ being a splitting of A, where D, $-B$ are the diagonal and the nondiagonal part of A, respectively. Assume that $\rho:=\rho(\langle A\rangle^{-1}G)<1$, γ is a positive constant, and $\Omega\in R^{n\times n}$ is a positive diagonal matrix satisfying $\Omega\geq D$. Then for any initial vector, the* AMAOR *iteration method is convergent if the parameter α and β satisfying*

$$\alpha<\beta<\frac{1}{\rho},\quad \beta\in[0,\alpha]\cup[\alpha,\alpha\theta_\alpha),$$

where $\theta_\alpha\in[1,+\infty)$ such that

$$\rho(D^{-1}(\theta|L|+|U|+G))=\frac{\alpha+1-|1-\alpha|}{2\alpha} \tag{4.5}$$

Proof From the proof of Theorem 4.1, the AMAOR splitting $A=M_1-N_1=M_2-N_2$ with

$$M_1=\frac{1}{\alpha}(D-\beta L),\quad N_1=\frac{1}{\alpha}[(1-\alpha)D+(\alpha-\beta)L+\alpha U],\quad M_2=D-U \text{ and } N_2=L$$

is convergence when $\widetilde{M}$ is an M-matrix, $\widetilde{N}\geq 0$ and $\widetilde{A}=\widetilde{M}-\widetilde{N}$ is an M-matrix. By some calculation, we have

$$\begin{aligned}\widetilde{M}&=\Omega+\langle M_1\rangle-|N_2|=\Omega+\frac{D}{\alpha}-\frac{\beta}{\alpha}|L|-|L|\\ &=\Omega+\frac{D}{\alpha}-\frac{\alpha+\beta}{\alpha}|L|\end{aligned} \tag{4.6}$$

and

$$\widetilde{N} = |N_1| + |\Omega - M_2| + 2G = \left|\frac{1}{\alpha}[(1-\alpha)D + (\alpha-\beta)L + \alpha U]\right| + |\Omega - D + U| + 2G$$
$$= |\Omega - D| + \frac{|1-\alpha|}{\alpha}D + \frac{|\alpha-\beta|}{\alpha}|L| + 2|U| + 2G. \tag{4.7}$$

Hence

$$\begin{aligned}\widetilde{A} &= \widetilde{M} - \widetilde{N}\\ &= [\Omega + \langle M_1\rangle - |N_2|] - [|N_1| + |\Omega - M_2| + 2G]\\ &= \left[\Omega + \frac{D}{\alpha} - \frac{\alpha+\beta}{\alpha}|L|\right] - \left[|\Omega - D| + \frac{|1-\alpha|}{\alpha}D + \frac{|\alpha-\beta|}{\alpha}|L| + 2|U| + 2G\right]\\ &= \frac{\alpha+1-|1-\alpha|}{\alpha}D - \frac{\alpha+\beta+|\alpha-\beta|}{\alpha}|L| - 2|U| - 2G,\end{aligned}$$

where the last equality uses the condition $\Omega \geq D$. Since $\widetilde{M} \geq \widetilde{A}$ and $\widetilde{M}$ is a Z-matrix, $\widetilde{M}$ is an M-matrix if $\widetilde{A}$ is an M-matrix by Lemma 2.1. And the sufficient conditions for $\widetilde{A}$ to be an M-matrix are $\alpha + 1 - |1-\alpha| > 0$, $\alpha > 0$ and

$$\rho(D^{-1}(\theta|L| + |U| + G)) < \frac{\alpha+1-|1-\alpha|}{2\alpha}, \quad \text{where} \quad \theta = \frac{\alpha+\beta+|\alpha-\beta|}{2\alpha}. \tag{4.8}$$

On the other hand, note that

$$\frac{\alpha+1-|1-\alpha|}{2\alpha} = \begin{cases} 1, & \text{for } \alpha \in (0,1];\\ \frac{1}{\alpha} & \text{for } \alpha \in [1,+\infty),\end{cases}$$

with the maximum value 1 at $\alpha = 1$. Since $\rho := \rho(\langle A\rangle^{-1}G) < 1$, it can be easily verified that $(\alpha+1-|1-\alpha|)/(2\alpha) \in (\rho, 1]$ if $\alpha \in (0, 1/\rho)$. Hence, for any fixed α, there exists $\theta_\alpha \in [1,+\infty)$ such that (4.5) is valid. Therefore, for $\alpha < \beta < \frac{1}{\rho}$, $\beta \in [0,\alpha] \cup [\alpha, \alpha\theta_\alpha)$, the inequality (4.8) is true. □

Similar to the proof of Theorem 4.1, we can easily obtain the following convergence theorem:

Theorem 4.3 *Let $A \in R^{n\times n}$ be an H_+-matrix, and $A = M_1 - N_1 = M_2 - N_2$ be two H-compatible splittings of the matrix A, with $M_1 = (m_{ij}^{(1)})$, $M_2 = (m_{ij}^{(2)}) \in R^{n\times n}$. Let $A = D - B$ be a splitting of A with D, $-B$ are the diagonal and the nondiagonal matrices, respectively. Assume that $\Omega \in R^{n\times n}$ is a positive diagonal matrix, γ is a positive constant. If parameter matrix Ω satisfies $\Omega \geq diag(M_1)$, $\Omega \geq diag(M_2)$, $\langle A\rangle - G$ and $\Omega + M_1 - |N_2|$, $\Omega + M_2 - |N_1|$ are M-matrix, then the iteration sequence $\{z^k\}_{k=0}^{+\infty} \subseteq R_+^n$ generated by Algorithm 3.1 converges to the solution $z^* \in R_+^n$ of the problem (1.1) for any initial vector $x^0 \in R^n$.*

Proof The proof is similar to that of Theorem 4.1, so we omit here. □

4.2 The optimal parameter of AMAOR method

In this subsection, we will discuss the optimal possible AMAOR method by minimizing the associated spectral radius of the iteration matrix $\rho(\widetilde{M}^{-1}\widetilde{N})$. First, we review the following lemma, which can be seen in Marek and Szyld (1990):

Lemma 4.1 (Marek and Szyld 1990) *Let $A_i = M_i - N_i$ be weak nonnegative splittings with $L_i = M_i^{-1}N_i$ and $\rho(L_i) < 1$, $i = 1, 2$. Let $x_i \geq 0$ be such that $L_i x_i = \rho(L_i)x_i$, $i = 1, 2$. Let $A_2^{-1} \geq 0$ and $A_2^{-1} \geq A_1^{-1}$. If either $N_2 x_1 \geq N_1 x_1 \geq 0$ or $N_2 x_2 \geq N_1 x_2 \geq 0$ with $x_2 > 0$, then $\rho(L_1) \leq \rho(L_2)$. Moreover, if $A_2^{-1} > 0$ and if $N_1 \neq N_2$, then $\rho(L_1) < \rho(L_2)$.*

Theorem 4.4 *Under the assumption of Theorem 4.2 and for any fixed $\Omega = \omega D \geq D$, the spectral radius of the* AMAOR *iteration matrix is a decreasing function for $\beta \in [0, \alpha]$, and an increasing function for $\beta \in [\alpha, \alpha\theta_\alpha)$. Hence, the optimal* AMAOR *method is the* AMSOR *method.*

Proof 1. If $0 \leq \beta_2 \leq \beta_1 \leq \alpha$, from (4.6) and (4.7), we know that the AMAOR methods corresponding to two β's are

$$\widetilde{M}_{\omega,i} = (\omega + \frac{1}{\alpha})D - \frac{\alpha+\beta_i}{\alpha}|L|,\ \widetilde{N}_{\omega,i} = (\omega-1)D + \frac{|1-\alpha|}{\alpha}D + \frac{\alpha-\beta_i}{\alpha}|L| + 2|U| + 2G$$

and hence

$$\widetilde{A}_{\omega,i} = D + \frac{1-|1-\alpha|}{\alpha}D - 2|B| - 2G.$$

Since $\widetilde{A}_{\omega,i}$ is irrelevant with β_i, $\widetilde{N}_{\omega,2} \geq \widetilde{N}_{\omega,1}$, it then follows from Lemma 4.1 that $\rho(\widetilde{M}_{\omega,2}^{-1}\widetilde{N}_{\omega,2}) \geq \rho(\widetilde{M}_{\omega,1}^{-1}\widetilde{N}_{\omega,1})$. Hence, the spectral radius of the AMAOR iteration matrix $\rho(\widetilde{M}_\omega^{-1}\widetilde{N}_\omega)$ is a decreasing function for $\beta \in [0, \alpha]$.

2. If $\alpha \leq \beta_2 \leq \beta_1 < \alpha\theta_\alpha$, then from (4.6) and (4.7), we have

$$\widetilde{M}_{\omega,i} = (\omega + \frac{1}{\alpha})D - \frac{\alpha+\beta_i}{\alpha}|L|,\ \widetilde{N}_{\omega,i} = (\omega-1)D + \frac{|1-\alpha|}{\alpha}D + \frac{\beta_i-\alpha}{\alpha}|L| + 2|U| + 2G$$

and hence

$$\widetilde{A}_{\omega,i} = D + \frac{1-|1-\alpha|}{\alpha}D - \frac{2\beta_i}{\alpha}|L| - 2|U| - 2G.$$

From Theorem 4.2, we know that $\widetilde{M}_{\omega,i} - \widetilde{N}_{\omega,i}$ are M-splittings of nonsingular M-matrices, with $\widetilde{M}_{\omega,i}^{-1}\widetilde{N}_{\omega,i} \geq 0$ and $\rho(\widetilde{M}_{\omega,i}^{-1}\widetilde{N}_{\omega,i}) < 1, i = 1, 2$. Let $\delta = 2\alpha/(\alpha+1-|1-\alpha|)$ and $\vartheta_i = \beta_i/\alpha$. Then

$$\begin{aligned}\widetilde{A}_{\omega,i}^{-1} &= \left(\frac{2}{\delta}D - 2\vartheta_i|L| - 2|U| - 2G\right)^{-1} = \frac{\delta}{2}[I - \delta D^{-1}(\vartheta_i|L| + |U| + G)]^{-1}D^{-1}\\ &= \frac{\delta}{2}\Big[I + \delta D^{-1}(\vartheta_i|L| + |U| + G) + (\delta D^{-1}(\vartheta_i|L| + |U| + G))^2 + \cdots\Big]D^{-1}\\ &\geq 0.\end{aligned}$$

Hence, $\widetilde{A}_{\omega,1}^{-1} \geq \widetilde{A}_{\omega,2}^{-1}$. Since $\widetilde{N}_{\omega,1}^{-1} \geq \widetilde{N}_{\omega,2}^{-1} \geq 0$, then $\widetilde{N}_{\omega,1}^{-1}x_1 \geq \widetilde{N}_{\omega,2}^{-1}x_1 \geq 0$, where x_1 is the eigenvector associated with $\widetilde{M}_{\omega,1}^{-1}\widetilde{N}_{\omega,1}$. It then follows from Lemma 4.1 that $\rho(\widetilde{M}_{\omega,2}^{-1}\widetilde{N}_{\omega,2}) \leq \rho(\widetilde{M}_{\omega,1}^{-1}\widetilde{N}_{\omega,1})$. Hence, the spectral radius of the AMAOR iteration matrix $\rho(\widetilde{M}_\omega^{-1}\widetilde{N}_\omega)$ is a increasing function for $\beta \in [\alpha, \alpha\theta_\alpha)$.

From cases 1 and 2, we can conclude that the optimal AMAOR method is the AMSOR method. The proof is completed.

Theorem 4.5 *Under the assumption of Theorem 4.2 and for any fixed $\Omega = \omega D \geq D$, the spectral radius of the* AMAOR *iteration matrix is an increasing function for $\omega \in [1, +\infty)$. Hence, the optimal parameter of* AMAOR *method is $\omega^* = 1$.*

Proof Since $\Omega = \omega D \geq D$, then $\omega \in [1, \infty)$. If $\omega_1 \geq \omega_2 \geq 1$, then from (4.6) and (4.7), we have

$$\widetilde{M}_{\omega,i} = \left(\omega_i + \frac{1}{\alpha}\right) D - \frac{\alpha+\beta}{\alpha}|L|, \quad \widetilde{N}_{\omega,i} = (\omega_i - 1)D + \frac{|1-\alpha|}{\alpha} D + \frac{\alpha-\beta}{\alpha}|L| + 2|U| + 2, G$$

and hence

$$\widetilde{A}_{\omega,i} = \left(D + \frac{1-|1-\alpha|}{\alpha}\right) D - 2\theta|L| - 2|U| - 2G.$$

Since $\widetilde{A}_{\omega,i}$ is irrelevant with ω_i, $\widetilde{N}_{\omega,1} \geq \widetilde{N}_{\omega,2}$, it then follows from Lemma 4.1 that $\rho(\widetilde{M}_{\omega,1}^{-1}\widetilde{N}_{\omega,1}) \geq \rho(\widetilde{M}_{\omega,2}^{-1}\widetilde{N}_{\omega,2})$. Hence, the spectral radius of the AMAOR iteration matrix $\rho(\widetilde{M}_{\omega}^{-1}\widetilde{N}_{\omega})$ is an increasing function for $\omega \in [1, +\infty]$. Therefore, we can conclude that the optimal parameter is $\omega^* = 1$. □

5 Numerical experiments

In this section, we represent some numerical examples to demonstrate the effectiveness of accelerated modulus-based matrix splitting iteration methods from the aspects of iteration steps (denoted by 'Iter'), elapsed CPU time in seconds (denoted by 'CPU') and the norm of absolute residual vectors (denoted by 'Res'). Here, 'Res' is defined as

$$\text{RES}(z^k) := \|\min(Az^k + \Psi(z^k), z^k)\|_2,$$

where z^k is the kth approximate solution to the problem (1.1), and the minimum is taken componentwise.

All of the tests were run on the Intel (R) Core (TM), where the CPU is 2.40 GHz and the memory is 8.0 GB, the programming language was MATLAB R2015a. The stopping criterion for all methods are $\text{Res}(z^k) \leq 10^{-5}$ or k reaches the maximal number of iteration, e.g., 5000.

5.1 Comparison of Algorithm 2.1 with modulus-based method in Ma and Huang (2016)

In this subsection, we compare our method with modulus-based matrix splitting methods (Ma and Huang 2016). In addition, all initial vectors are chosen to be $x^0 = (1, 1, 1, \ldots, 1)^T \in R^n$, and $\gamma = 1$, $\Omega = \theta D$ are chosen for both accelerated modulus-based matrix splitting methods and modulus-based matrix splitting methods.

For convenience, let $A = D - L - U$ with D, $-L$ and $-U$ being the diagonal, the strictly lower triangular and the strictly upper-triangular matrices of A, then

$$M_1 = \frac{1}{\alpha}(D - \beta L), \quad N_1 = \frac{1}{\alpha}[(1-\alpha)D + (\alpha-\beta)L + \alpha U]$$

Algorithm 2.1 of Ma and Huang (2016) reduces to modulus-based accelerated overrelaxation (MAOR) iteration method

$$(D + \alpha\Omega - \beta L)x^{k+1} = [(1-\alpha)D + (\alpha-\beta)L + \alpha U]x^k + \alpha(\Omega - A)|x^k| - \alpha\gamma\Psi(z^k).$$

It also gives modulus-based successive overrelaxation (MSOR) iteration method, modulus-based Gauss-Seidel (MGS) iteration method and modulus-based Jacobi (MJ) iteration method when $\alpha = \beta$, $\alpha = \beta = 1$ and $\alpha = 1, \beta = 0$, respectively.

In Table 1, the abbreviations of testing methods are listed.

Table 1 Abbreviations of testing methods

Method	Description
MJ	The modulus-based Jacobi method
MGS	The modulus-based Gauss–Seidel method
MSOR	The modulus-based successive overrelaxation method
AMJ	The accelerated modulus-based Jacobi method
AMGS	The accelerated modulus-based Gauss–Seidel method
AMSOR	The accelerated modulus-based overrelaxation method

Example 5.1 We consider the nonlinear complementarity problem (1.1), which is also considered in Xia and Li (2015), for which $A \in R^{n\times n}$ is given as

$$A = \begin{pmatrix} B & -I & O & \cdots & O & O \\ -I & B & -I & \cdots & O & O \\ O & -I & B & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -I \\ O & O & \cdots & \cdots & -I & B \end{pmatrix}, \quad \Psi(z) = \begin{pmatrix} z_1/(1+z_1) \\ z_2/(1+z_2) \\ z_3/(1+z_3) \\ \vdots \\ z_{n-1}/(1+z_{n-1}) \\ z_n/(1+z_n) \end{pmatrix}$$

where $B = \text{tridiag}(-1, 4, -1) \in R^{m\times m}$, $I \in R^{m\times m}$ is a unit matrix and $n = m^2$. It is clear that $A \in R^{n\times n}$ is symmetric positive definite matrix.

In Table 2, the iteration steps, the CPU time and the residual norms for the modulus-based matrix splitting iterative methods and the accelerated modulus-based matrix splitting iterative methods for Example 5.1 are listed. Here, $\Omega = 3D$.

Example 5.2 We consider the nonlinear complementarity problem (1.1), which is also considered in Xia and Li (2015), for which $A \in R^{n\times n}$ is given as

$$A = \begin{pmatrix} B & -0.5I & O & \cdots & O & O \\ -1.5I & B & -0.5I & \cdots & O & O \\ O & -1.5I & B & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -0.5I \\ O & O & \cdots & \cdots & -1.5I & B \end{pmatrix}, \quad \Psi(z) = \begin{pmatrix} \arctan(z_1) \\ \arctan(z_2) \\ \arctan(z_3) \\ \vdots \\ \arctan(z_{n-1}) \\ \arctan(z_n) \end{pmatrix}$$

where $B = \text{tridiag}(-1.5, 4, -0.5) \in R^{m\times m}$, $I \in R^{m\times m}$ is a unit matrix and $n = m^2$.

In Table 3, the iteration steps, the CPU time and the residual norms for the modulus-based matrix splitting iterative methods and the accelerated modulus-based matrix splitting iterative methods for Example 5.2 are listed. Here, $\Omega = 5D$.

From Tables 2 and 3, we can easily see that the iteration steps and CPU time of six methods increase with the increasing of the problem size $n = m^2$. Moreover, it is observed that the accelerated modulus-based Jacobi, Gauss–Seidel and SOR methods require less iteration steps and CPU time than modulus-based Jacobi, Gauss–Seidel and SOR methods, respectively. Among all these methods, accelerated modulus-based SOR use the least iteration steps and CPU time.

Table 2 Numerical comparison of the testing methods for Example 5.1

Dim	AMJ	AMGS	AMSOR ($\alpha = 1.1$)	MJ (Ma and Huang 2016)	MGS (Ma and Huang 2016)	MSOR (Ma and Huang 2016) ($\alpha = 1.1$)
10						
Iter	90	77	67	105	91	81
CPU	0.0635	0.0421	0.0496	0.0758	0.0671	0.0464
Res	9.6702e−06	8.3346e−06	7.8586e−06	9.0118e−06	9.9860e−06	9.9085e−06
20						
Iter	107	91	79	124	108	96
CPU	0.2790	0.3040	0.2400	0.4966	0.4388	0.2609
Res	9.0981e−06	8.6464e−06	8.7699e−06	9.6261e−06	9.4407e−06	9.7077e−06
30						
Iter	112	95	83	131	114	101
CPU	1.1024	1.1832	0.9564	1.8709	1.8275	1.0165
Res	9.9360e−06	9.8676e−06	8.9840e−06	8.9458e−06	8.8153e−06	9.6358e−06
40						
Iter	115	98	85	134	117	104
CPU	3.5515	3.7975	3.1215	6.1575	6.3057	3.2774
Res	9.9818e−06	9.0927e−06	9.4378e−06	9.6041e−06	8.8703e−06	9.1423e−06
50						
Iter	118	100	87	137	119	106
CPU	10.4885	10.9576	9.5996	18.2196	18.2629	9.6400
Res	8.6334e−06	8.6815e−06	8.5221e−06	8.8551e−06	8.9713e−06	8.8933e−06

Table 3 Numerical comparison of the testing methods for Example 5.2

Dim	AMJ	AMGS	AMSOR ($\alpha = 1.1$)	MJ (Ma and Huang 2016)	MGS (Ma and Huang 2016)	MSOR (Ma and Huang 2016) ($\alpha = 1.1$)
10						
Iter	108	96	84	121	109	97
CPU	0.0516	0.0638	0.0580	0.1031	0.1059	0.0698
Res	9.0708e−06	8.5169e−06	8.8913e−06	9.2302e−06	9.3026e−06	9.8687e−06
20						
Iter	138	120	105	158	140	125
CPU	0.4374	0.3793	0.3458	0.5915	0.6075	0.3920
Res	9.9031e−06	8.8532e−06	8.6854e−06	9.6449e−06	8.9895e−06	8.8985e−06
30						
Iter	151	130	113	174	153	136
CPU	1.4505	1.5736	1.4345	2.4640	2.3986	1.3866
Res	9.9388e−06	8.9735e−06	9.5939e−06	9.9180e−06	9.1593e−06	9.7347e−06
40						
Iter	158	135	118	182	159	142
CPU	4.8699	5.3597	4.6832	8.3842	8.4592	4.5782
Res	9.2650e−06	9.0515e−06	8.7616e−06	9.6424e−06	9.5366e−06	9.3497e−06
50						
Iter	162	138	120	187	163	145
CPU	14.6275	15.4142	13.2463	25.4466	24.8870	13.2054
Res	9.2285e−06	9.3080e−06	9.7948e−06	9.3905e−06	9.5002e−06	9.9474e−06

Table 4 Comparison of parameter α for Example 5.3

m	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1.0$	$\alpha = 1.1$	$\alpha = 1.2$	$\alpha = 1.3$
10						
MSOR						
Iter	20	18	17	18	20	22
CPU	0.0243	0.0103	0.0109	0.0126	0.0109	0.0153
AMSOR						
Iter	12	10	8	9	10	11
CPU	0.0231	0.0081	0.0076	0.0064	0.0074	0.0090
20						
MSOR						
Iter	21	19	17	19	21	23
CPU	0.0689	0.0609	0.0487	0.0687	0.0732	0.0755
AMSOR						
Iter	13	10	8	9	10	11
CPU	0.0571	0.0437	0.0301	0.0368	0.0385	0.0429
30						
MSOR						
Iter	22	19	18	19	21	23
CPU	0.2566	0.2209	0.2255	0.2308	0.2613	0.2716
AMSOR						
Iter	13	11	9	9	10	11
CPU	0.2423	0.1569	0.1321	0.1326	0.1537	0.1566
40						
MSOR						
Iter	22	20	18	19	21	24
CPU	0.7978	0.7210	0.6448	0.6929	0.7608	0.8560
AMSOR						
Iter	13	11	9	9	10	11
CPU	0.6321	0.5341	0.4614	0.4503	0.4899	0.5347

5.2 The optimal parameter of AMSOR method

In this subsection, we consider the optimal AMAOR method, that is, AMSOR method. First, we determine the optimal iteration parameter α, which is obtained experimentally by minimizing the corresponding iteration steps. Moreover, we determine the optimal iteration parameter ω in the AMGS method to illustrate the conclusion of Sect. 4. Finally, we choose the initial vector as $x^0 = (0, 0, \ldots, 0)^T \in R^n$, and $\gamma = 1$ in this subsection.

Example 5.3 We consider the nonlinear complementarity problem (1.1), for which $A \in R^{n\times n}$ is given as

Table 5 Results of AMGS with different ω for Example 5.4

m	$\omega=1$	$\omega=2$	$\omega=3$	$\omega=4$	$\omega=5$
10					
Iter	18	31	46	62	77
CPU	0.0294	0.0173	0.0327	0.0441	0.0515
20					
Iter	20	34	50	66	82
CPU	0.0864	0.1132	0.1681	0.2332	0.2690
30					
Iter	22	36	52	68	84
CPU	0.3174	0.4667	0.6436	0.8725	1.0838
40					
Iter	22	36	52	69	85
CPU	0.9983	1.6766	2.2798	5.3235	3.7466
50					
Iter	22	36	53	69	85
CPU	2.8024	6.6500	6.8752	11.9720	14.7316

$$A = \begin{pmatrix} B & -0.5I & O & \cdots & O & O \\ -1.5I & B & -0.5I & \cdots & O & O \\ O & -1.5I & B & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -0.5I \\ O & O & \cdots & \cdots & -1.5I & B \end{pmatrix},$$

where $B = \text{tridiag}(-1.5, 4, -0.5) \in R^{m\times m}$, $I \in R^{m\times m}$ is a unit matrix and $n = m^2$, $\Psi(z) = h(z) - Az^*$, here $h(z) = (\sqrt{z_1^2 + 0.01}, \ldots, \sqrt{z_n^2 + 0.01})^T$, $z^* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T$. In this example, we choose $\Omega = D + I$.

In Table 4, the number of iteration steps and the elapsed CPU time in seconds are listed for two methods when the parameter α varies from 0.8 to 1.3 for Example 5.3. From Table 4, it is observed that for Example 5.3 the optimal parameter $\alpha^* = 1.0$ for MSOR method and AMSOR method.

Example 5.4 We consider the nonlinear complementarity problem (1.1), for which $A \in R^{n\times n}$ is given as

$$A = \begin{pmatrix} B & -I & -I & & \\ & B & -I & \ddots & \\ & & B & \ddots & -I \\ & & & \ddots & -I \\ & & & & B \end{pmatrix},$$

where $B = \text{tridiag}(-1, 4, -1) \in R^{m\times m}$, $I \in R^{m\times m}$ is a unit matrix and $n = m^2$, $\Psi(z) = h(z) - Az^*$, here $h(z) = (\arctan(z_1), \ldots, \arctan(z_n))^T$, $z^* = (1, 2, 1, 2, \ldots, 1, 2, \ldots)^T$. In this example, we fix $\alpha = 1$. That is, we consider the AMGS method.

In Table 5, we list the iteration steps and CPU time of AMGS method with iteration parameter $\omega = 1, 2, 3, 4, 5$. From Table 5, it is further confirmed that the iteration steps and CPU time increase as problem size increases. At the same time, we find that the AMGS method with $\omega = 1$ requires the least iteration steps and CPU time compared to other iteration parameters. This result verifies the analysis in Theorem 4.5.

6 Conclusions

The accelerated modulus-based matrix splitting iteration methods for the solution of a class of nonlinear complementarity problem is presented. The proposed method not only computationally more convenient to use because of storage requirement, but it is also faster than the modulus-based matrix splitting methods. We show their convergence by assuming that the system matrix is positive definite or the splitting of the system matrix are H_+-compatible splitting. Also, we discuss the optimal parameter. Furthermore, we give two-step accelerated modulus-based matrix splitting iteration method, which may achieve higher computing efficiency. In addition, we present numerical examples, which demonstrate that accelerated modulus-based matrix splitting iteration method is efficient.

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