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On OM-decomposable sets

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Abstract We introduce and study the family of sets in a finite-dimensional Euclidean space which can be written as the Minkowski sum of an open, bounded, and convex set and a closed convex cone. We establish several properties of the class of such sets, called *OM-decomposable*, some of which extend related properties holding for the class of *Motzkin decomposable* sets (i.e., those for which the convex and bounded set in the decomposition is requested to be closed, hence compact).

Keywords Motzkin decomposable sets · Convex sets · Convex cones

Mathematics Subject Classification 52A07 · 54F65

1 Introduction

It has been known at least since the mid nineteenth century that the set of solutions of a system of linear equations admits an explicit representation, as the sum of a particular solution plus an arbitrary linear combination of the vectors forming a basis of the linear subspace consisting of the solutions of the associated homogeneous system. The corresponding result for systems

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of linear inequalities is much more recent. In his doctoral thesis Motzkin (1936), proved that the set F of solutions of such a system consists of the sums of convex combinations of a finite set of vectors (the *vertices* of F) and nonnegative combinations of another finite set (the *extreme rays* of F). In modern terminology, every (possibly unbounded) convex polyhedron is the Minkowski sum of a convex and compact polytope and a closed and convex cone. This characterization turned out to be quite useful for establishing finite convergence of pivotal algorithms for Linear Programming (e.g., the Simplex Method, see Dantzig 1963), and more specifically, for Quadratic Programming, like Lemke's method, see Cottle et al. (1992).

Motzkin's representation result suggested the consideration of a class of convex sets more general than polyhedra, resulting from removing the "linear" nature of these, while keeping the decomposition aspect. More precisely, those sets in $\mathbb{R}^n$ can be written as the Minkowski sum of a compact and convex set and a closed and convex cone. Such sets were introduced in Goberna et al. (2010a), where they were called *Motzkin decomposable*, or *M-decomposable*, in short, and further studied in Goberna et al. (2010b, 2013), together with the *M-decomposable functions*, namely those whose epigraphs are M-decomposable. The class of M-decomposable sets lies hence in between the classes of closed polyhedra and of closed and convex sets.

M-decomposable sets were considered, without further analysis, in Bair (1976, 1979), where they were called *generalized convex polyhedral*, since it follows from Motzkin's result that polyhedral sets are always M-decomposable. Unfortunately, the same name has been given by other authors to those sets whose nonempty intersection with polytopes is polytopes, which are also called *quasipolyhedral* or *boundedly polyhedral*. Another early reference related to M-decomposability is Klee (1958). In fact, it follows from the representation theorem in Klee (1958) that a set is M-decomposable if and only if its intersection with the orthogonal complement of its affine hull is M-decomposable (see Theorem 6 in Goberna et al. 2010b).

We comment now on some features of M-decomposable sets. It is easy to check that the cone D appearing in the decomposition of an M-decomposable set $F = C + D$ (where C is the compact set), is uniquely determined; it is precisely the *recession cone* $0^+(F)$ of F, namely the set of directions $d \in \mathbb{R}^n$, such that $\{a + td : t \in \mathbb{R}_+\} \subset F$ for any $a \in F$. On the other hand, whenever $D \neq \{0\}$, the compact and convex set C is not uniquely determined; the given set C might be replaced, for instance, by $F \cap B$, where B is any ball in $\mathbb{R}^n$ containing C.

A sizable number of additional properties of such sets and functions were established in the above-mentioned references, and we single out two of them for future reference.

First, M-decomposable sets whose recession cones are pointed admit a *minimal* decomposition, which we define next. We say that a representation $F = C + D$ of an M-decomposable set F is *minimal* when $C \subset C'$ for any convex and compact set $C' \subset \mathbb{R}^n$, such that $F = C' + D$. When F is M-decomposable and D is pointed, there exists such a minimal decomposition, and the corresponding compact and convex set C is itself uniquely determined: it is the closed and convex hull of the extreme points of F (see Theorem 11(ii) in Goberna et al. 2010b).

Second, M-decomposable sets whose recession cones are pointed admit an M-decomposition of *truncation type*, meaning a decomposition $F = C + D$ with the following property: there exits a hyperplane H, such that C is the intersection of F with one of the halfspaces determined by H, while the intersection of F with the opposite halfspace consists of a set of halflines emanating from $F \cap H$ (see Lemmas 21 and 29 in Goberna et al. 2013).

M-decomposable sets share some properties of convex sets, but not all. For instance, the Minkowski sum of two convex sets is convex, but the Minkowski sum of M-decomposable sets may fail to be M-decomposable. This is a consequence of the facts that M-decomposable sets are always closed, and the Minkowski sum of closed sets may not be closed. This situation suggests that it might be of interest to relax the closedness assumptions on the sets involved in an M-decomposition, namely the bounded convex set and the cone. The closedness hypothesis on the cone was relaxed in Iusem et al. (2014), where the so-called *M-predecomposable* sets were introduced, meaning those sets of the form $F = C + D$, where C is a convex and compact set and D is a convex cone, not necessarily closed. Several properties of these sets (including some not enjoyed by M-decomposable sets, like the one related to the Minkowski sum) were established in Iusem et al. (2014).

In this paper, we will continue this line of research, keeping the closedness of the cone, but discarding the compactness assumption on the bounded set. More precisely, we will assume that C is an open, bounded, and convex set. Regarding the cone D, for the sake of simplicity, we will fix it as the recession cone of F, i.e., we will consider sets $F \subset \mathbb{R}^n$, such that $F = C + O^+(F)$, where C is open, bounded, and convex, and $O^+(F)$ is the recession cone of F. These sets will be called *OM-decomposable* (OM standing for "Open Motzkin").

We will prove in Sect. 2 that the OM-decomposable sets are precisely those whose closure is M-decomposable. After showing that the Minkowski sum of two OM-decomposable sets is OM-decomposable, we will proceed to establish suitable extensions to OM-decomposable sets of the two above-described results on M-decomposable sets.

2 OM-decomposable sets

We start this section with the formal definition of OM-decomposable sets.

Definition 1 The convex set $\emptyset \neq F \subset \mathbb{R}^n$ is OM-decomposable if there exists a nonempty, open, convex, and bounded set $C \subset \mathbb{R}^n$, such that $F = C + O^+(F)$.

Remark 1 Observe that if $F \subset \mathbb{R}^n$ is OM-decomposable, then F is open and convex.

We start with an elementary property of open and convex sets, needed in for the proof of Theorem 2.

Lemma 1 *Let $\emptyset \neq F \subset \mathbb{R}^n$ be an open and convex set. Then, $O^+(F) = O^+[\mathrm{cl}(F)]$.*

Proof Take $d \in O^+(F)$, $x \in \mathrm{cl}(F)$ and $\lambda \geq 0$. There exists a sequence $\{x^k\} \subset F$ converging to x. Hence, the sequence $\{x^k + \lambda d\}$ is contained in F and it converges to $x + \lambda d \in \mathrm{cl}(F)$. Therefore, $d \in O^+[\mathrm{cl}(F)]$. Now, take $d \in O^+[\mathrm{cl}(F)]$, $x \in F$ and $\lambda \geq 0$. Since F is an open and convex set, we have that $\mathrm{int}[\mathrm{cl}(F)] = F$. Therefore, $x + \lambda d \in F$. The lemma is proved. □

Next, we deal with the relation between OM-decomposable sets and M-decomposable ones. Before establishing the result, we quote a theorem on M-decomposable sets which will be needed in the proof.

Theorem 1 *If F is M-decomposable and $O^+(F)$ is pointed, then there exists an M-decomposition of F, say $F = C + O^+(F)$, of truncation type, meaning that there exists a hyperplane H, such that $C = F \cap H^+$, and $F \cap H^- = (F \cap H) + O^+(F)$, i.e., $F \cap H^-$ is a set of halflines emanating from $F \cap H$.*

Proof See Lemmas 21 and 29 in Goberna et al. (2013). □

Our result on the connection between M- and OM-decomposability is the following.

Theorem 2 *Let $\emptyset \neq F \subset \mathbb{R}^n$ be an open and convex set. Assume that $O^+(F)$ is pointed. Then, F is OM-decomposable if and only if $\mathrm{cl}(F)$ is M-decomposable.*

Proof $\Longrightarrow$) $F = C + O^+(F)$, implying that $\mathrm{cl}(F) = \mathrm{cl}(C) + \mathrm{cl}[O^+(F)] = \mathrm{cl}(C) + \mathrm{cl}[O^+(\mathrm{cl}(F))] = \mathrm{cl}(C) + O^+[\mathrm{cl}(F)]$. Since C is bounded and convex, we get that $\mathrm{cl}(F)$ is M-decomposable.

$\Longleftarrow$) $\mathrm{cl}(F)$ is M-decomposable, so that there exists a convex, compact set $\overline{C}$, such that $\mathrm{cl}(F) = \overline{C} + O^+[\mathrm{cl}(F)] = \overline{C} + O^+(F)$. Since $O^+(F)$ is a pointed cone, we invoke Theorem 1, concluding that there exists a hyperplane H, such that $\overline{C} \subset H^- \cap \mathrm{cl}(F)$ and $\mathrm{cl}(F) = H^- \cap \mathrm{cl}(F) + O^+(F)$. Since F is an open set, without loss of generality, we can consider that $H^- \cap \mathrm{cl}(F)$ has dimensionality n. By convexity, $\mathrm{int}[H^- \cap \mathrm{cl}(F)] = \mathrm{int}(H^-) \cap F \subset F$. Hence

$$\mathrm{int}\left[H^- \cap \mathrm{cl}(F)\right] + O^+(F) = \mathrm{int}\left(H^- \cap F\right) + O^+(F) \subset F.$$

Now, let us take a point $x \in F$. Suppose that $x \notin \mathrm{int}(H^-) \cap F + O^+(F)$. It follows that:

$$\begin{aligned}\mathrm{cl}(F) &= H^- \cap \mathrm{cl}(F) + O^+[\mathrm{cl}(F)] \\ &= \mathrm{cl}\left[\mathrm{int}\left(H^-\right) \cap F\right] + O^+(F) = \mathrm{cl}\left[\mathrm{int}\left(H^-\right) \cap F + O^+(F)\right],\end{aligned}$$

by convexity and Lemma 1. Hence, $x \in \mathrm{bd}[\mathrm{cl}(F)]$, which is a contradiction. Thus, we obtain that $F = \mathrm{int}(H^-) \cap F + O^+(F)$, which implies that the set F is OM-decomposable. The proof is complete. □

Remark 2 Theorem 2 shows a kind of symmetry between closed M-decomposable and the open OM-decomposable sets.

Remark 3 We remark that a set lying strictly in between an OM-decomposable set and its closure, is neither OM-decomposable nor M-decomposable. Indeed, in view of Remark 1, such set would be in between an open set and its closure, and hence, it would be neither open nor closed, but OM-decomposable sets are open, invoking again Remark 1, and M-decomposable sets are clearly closed.

Next, we show that the Minkowski sum of OM-decomposable sets is OM-decomposable.

Proposition 1 *Let the convex set $F \subset \mathbb{R}^n$ be such that $F = C + D$, where $C \subset F$ is a convex, open, and bounded set and D is a convex cone. Then, $D \subseteq O^+(F)$ and $F = C + D = C + O^+(F)$.*

Proof Take any $d \in D$ and $x \in F$. Then, $x = c + h$, with $c \in C$ and $h \in D$. Observe that $x + d = c + (h + d)$ belongs to F, because $h + d \in D$. Therefore, $d \in O^+(F)$ and $D \subseteq O^+(F)$. Hence, $F = C + D \subset F + D \subseteq F + O^+(F) = F$, so that $F = C + D = C + O^+(F)$. □

Corollary 1 *If F, G are OM-decomposable, then $F + G$ is OM-decomposable.*

Proof It follows easily from Proposition 1. □

At this point, a comment is in order on the possibility of considering OM-decomposable sets in an infinite-dimensional space V (e.g., a Hilbert or Banach space), defined as in Definition 1 with V substituting for $\mathbb{R}^n$. It is easy to check that Corollary 1 holds in this context, i.e., the Minkowski sum of infinite-dimensional OM-decomposable sets is indeed OM-decomposable. However, the extension of the reminder of our work to the infinite dimensional setting becomes much more complicated, because a basic component of our analysis will be the relation between an OM-decomposition $C + O^+(F)$ and the decomposition $\mathrm{cl}(C) + O^+(F)$. Now, since $C \subset V$ is assumed to be open, its affine hull is necessarily infinite dimensional when V is infinite dimensional, in which case $\mathrm{cl}(C)$ cannot be compact, i.e., $\mathrm{cl}(C) + O^+(F)$ fails to be an M-decomposition. Of course, one could request the compact set in an M-decomposition the be just closed and bounded, but it turns out to be that in most results on M-decomposable sets proved in Goberna et al. (2010a, b, 2013) (including those which will be invoked in the sequel), the compactness of the bounded component plays an essential role. In other words, to study OM-decomposable sets in infinite-dimensional spaces, one should rework first the theory of M-decomposability in such spaces, relaxing the compactness assumption, e.g., to closedness and boundedness, which is certainly a nontrivial task. For this reason, in this work, we restrict our analysis of OM-decomposability to the finite-dimensional case.

Now, we will extend to OM-decomposable sets some results which hold for M-decomposable sets. We recall that a closed set $F \subset \mathbb{R}^n$ is M-decomposable iff $F = C + O^+(F)$ for some compact and convex set C. We start by introducing some notation and quoting a result on M-decomposable sets.

Given a hyperplane $H = \{x \in \mathbb{R}^n : a^t x = \alpha\}$, with $a \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, we denote as H^+, H^- the two halfspaces determined by H, i.e., $H^+ = \{x \in \mathbb{R}^n : a^t x \geq \alpha\}$, $H^- = \{x \in \mathbb{R}^n : a^t x \leq \alpha\}$.

Theorem 3 *If F is M-decomposable and $O^+(F)$ is pointed, then there exists a minimal M-decomposition of F, meaning an M-decomposition of the form $F = C + O^+(F)$, such that $C \subset C'$ for all M-decomposition $F = C' + O^+(F)$. In fact, C is precisely the closed convex hull of the extreme points of F.*

Proof See Theorem 11 in Goberna et al. (2010b). □

We study next the suitable extensions of Theorems 1 and 3 to the case of OM-decompositions. We start by stating an elementary property.

Proposition 2 *Let $C \subset \mathbb{R}^n$ be an open, bounded, and convex set, $E \subset \mathbb{R}^n$ be a closed and convex set, and $K \subset \mathbb{R}^n$ a closed and convex cone. Then*

(i) $\mathrm{int}(E) + K = \mathrm{int}(E + K)$,
(ii) $\mathrm{cl}(C) + K = \mathrm{cl}(C + K)$.

Proof (i) It is immediate that both sets $\mathrm{int}(E) + K$ and $\mathrm{int}(E + K)$ are open, and that they have the same closure, namely $\mathrm{cl}(E + K)$. Hence, they coincide.
(ii) Observe that $\mathrm{cl}(C + K) = \mathrm{cl}(C) + \mathrm{cl}(K) = \mathrm{cl}(C) + K$, by closedness of K.

□

Now, we state and prove the extension of Theorem 3 to OM-decomposable sets.

Theorem 4 *Assume that F is OM-decomposable and that $O^+(F)$ is pointed. Let C be the closed convex hull of the extreme points of $\mathrm{cl}(F)$. If $\mathrm{int}(C) \neq \emptyset$, then $F = \mathrm{int}(C) + O^+(F)$, and this is a minimal OM-decomposition of F, meaning that $\mathrm{int}(C) \subset C'$ for all OM-decomposition $F = C' + O^+(F)$ of F. If $\mathrm{int}(C) = \emptyset$, then there exists no minimal OM-decomposition of F.*

Proof By Theorem 2, $\mathrm{cl}(F)$ is M-decomposable. By Theorem 3, $\mathrm{cl}(F) = C + O^+(F)$ and this M-decomposition is minimal. We proceed to prove that $F = \mathrm{int}(C) + O^+(F)$. Note that

$$\mathrm{int}(C) + O^+(F) = \mathrm{int}(C + O^+(F)) = \mathrm{int}[\mathrm{cl}(F)] = \mathrm{int}(F) = F,$$

using Proposition 2(i) in the first equality, Theorem 3 in the second one, convexity of F in the third one, and openness of F in the fourth one. The claim is proved. Next, we prove the minimality of the OM-decomposition $F = \mathrm{int}(C) + O^+(F)$. Assume that there exists an open set E, such that $E + O^+(F) = \mathrm{int}(C) + O^+(F)$. Then

$$\mathrm{cl}(E) + O^+(F) = \mathrm{cl}[E + O^+(F)] = \mathrm{cl}[\mathrm{int}(C)] + O^+(F) = \mathrm{cl}(F),$$

using Proposition 2(ii) in the first equality. It follows that $\mathrm{cl}(E) + O^+(F)$ is an M-decomposition of $\mathrm{cl}(F)$. By the minimality property of C (Theorem 3), we get that $C \subset \mathrm{cl}(E)$, so that $\mathrm{int}(C) \subset \mathrm{int}[\mathrm{cl}(E)] = E$, using the fact that E is open and convex in the equality. We have established the minimality of the OM-decomposition $F = \mathrm{int}(C) + O^+(F)$.

Finally, we prove that if $\mathrm{int}(C) = \emptyset$, then there is no minimal OM-decomposition of F. Consider an OM-decomposition of F, i.e., $F = E + O^+(F)$ with E open and bounded. Let A be the affine hull of C (also called the linearity of C, i.e., the intersection of all the affine manifolds containing C). Since $\mathrm{int}(C) = \emptyset$, we get that $\dim(A) < n$. Let $B = A^\perp$ be the orthogonal complement of A, and $\mathrm{relint}(C)$ be the relative interior of C, meaning the interior of C relative to A. For $\varepsilon > 0$, define $\widehat{C}_\varepsilon = \{d + u : d \in \mathrm{relint}(C), u \in B, \|u\| < \varepsilon\}$. Note that $\widehat{C}_\varepsilon$ is homeomorphic to $\mathrm{relint}(C) \times \widehat{B}$, with $\widehat{B} = \{u \in B : \|u\| < \varepsilon\}$. Observe that $\mathrm{relint}(C)$ is open in the relative topology of A, and $\widehat{B}$ is open in the relative topology of B, so that $\mathrm{relint}(C) \times \widehat{B}$ is an open subset of $A \times B$ in the product topology, which is just the standard topology of $\mathbb{R}^n = A \times B$. We conclude that $\widehat{C}_\varepsilon$ is open in $\mathbb{R}^n$ for all $\varepsilon > 0$. Define $\widetilde{C}_\varepsilon = \widehat{C}_\varepsilon \cap E$. Clearly, $\widetilde{C}_\varepsilon$ is open. We will prove that it is nonempty. We claim first that $C \subset \mathrm{cl}(E)$. By Theorem 2, $\mathrm{cl}(F) = \mathrm{cl}(E) + O^+(F)$ is an M-decomposition of $\mathrm{cl}(F)$. Note that the extreme points of $\mathrm{cl}(F)$ belong to $\mathrm{cl}(E)$, because otherwise, they would be of the form $c + y$ with $0 \neq y \in O^+(F)$ and $c \in \mathrm{cl}(E)$, but then $c + y$ belongs to the segment between $c + (1/2)y \in \mathrm{cl}(E) + O^+(F)$ and $c + (3/2)y \in \mathrm{cl}(E) + O^+(F)$, and hence, it cannot be an extreme point. Since $\mathrm{cl}(E)$ is convex, it follows that $C \subset \mathrm{cl}(E)$, establishing the claim. Take a point $z \in \mathrm{relint}(C) \subset \widehat{C}_\varepsilon$. Since $z \in \mathrm{cl}(E)$, there exists a sequence $\{z^k\} \subset E$, such that $\lim_{k\to\infty} z^k = z$. Since $\widehat{C}_\varepsilon$ is open, z^k belongs to $\widehat{C}_\varepsilon$ for large enough k, so that $z^k \in \widehat{C}_\varepsilon \cap E = \widetilde{C}_\varepsilon$, and hence, $\widetilde{C}_\varepsilon$ is nonempty. We claim now that $\widetilde{C}_\varepsilon + O^+(F) = F$ for all $\varepsilon > 0$. Since $\widetilde{C}_\varepsilon \subset E$, we have that

$$\widetilde{C}_\varepsilon + O^+(F) \subset E + O^+(F) = F. \tag{1}$$

On the other hand, note that $C \subset \mathrm{cl}(\widehat{C}_\varepsilon)$. Since $C \subset \mathrm{cl}(E)$, we have that $C \subset \mathrm{cl}(\widehat{C}_\varepsilon \cap E) = \mathrm{cl}(\widetilde{C}_\varepsilon)$, so that

$$\mathrm{cl}(F) = C + O^+(F) \subset \mathrm{cl}\left(\widetilde{C}_\varepsilon\right) + O^+(F) = \mathrm{cl}\left(\widetilde{C}_\varepsilon + O^+(F)\right), \tag{2}$$

using Proposition 2(ii) in the last equality. Since F and $\widetilde{C}_\varepsilon + O^+(F)$ are both convex, taking interior in both sides of (2), we obtain that $F \subset \widetilde{C}_\varepsilon + O^+(F)$, which, in view of (1), gives $F = \widehat{C}_\varepsilon + O^+(F)$, establishing the claim. We have shown that $\widetilde{C}_\varepsilon + O^+(F)$ is an OM-decomposition of F for all $\varepsilon > 0$. If there existed a minimal OM-decomposition of F, say of the form $F = C^* + O^+(F)$ with C^* open and bounded, we would have that $C^* \subset \cap_{\varepsilon>0}\widetilde{C}_\varepsilon \subset \cap_{\varepsilon>0}\widehat{C}_\varepsilon$, but it follows easily from the definition of $\widehat{C}_\varepsilon$ that $\cap_{\varepsilon>0}\widehat{C}_\varepsilon = \emptyset$, completing the proof. □

We give next an example of an OM-decomposable set for which there exists no minimal OM-decomposition.

Example 1 Take $F = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, y > 0\}$.

It is clear that the set of extreme points of $\mathrm{cl}(F)$ is $\{(0, 0), (2, 0)\}$, so that the closed and convex hull of the extreme points of F is the set $C = \{(x, 0) : 0 \le x \le 2\}$, which has empty interior in $\mathbb{R}^2$. Consider the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(1, t)$, with $t \in (0, 1)$, and let U_t be the interior of such triangle. It is easy to check that $O^+(F) = \{(0, s) : s \ge 0\}$, and that F admits the OM-decomposition $F = U_t + O^+(F)$ for all $t > 0$. Since $\cap_{t>0} U_t = \emptyset$, there exists no minimal OM-decomposition of F.

Next, we extend Theorem 1 to OM-decompositions. We start by introducing a suitable definition of an OM-decomposition of truncation type.

Definition 2 An OM-decomposition of truncation type of an OM-decomposable set F is an OM-decomposition $F = C + O^+(F)$, such that there exists a hyperplane H satisfying:

(i) $C = F \cap \mathrm{int}(H^+)$,
(ii) $F \cap H^- = (F \cap H) + O^+(F)$.

We will need an elementary result.

Proposition 3 *Assume that the hyperplane $H = \{x \in \mathbb{R}^n : a^t x = \alpha\}$ induces an M-decomposition of truncation type of an M-decomposable set E. If $\beta < \alpha$, then the hyperplane $\widehat{H} = \{x \in \mathbb{R}^n : a^t x = \beta\}$ also induces an M-decomposition of truncation type of E.*

Proof First, we must check that $E \cap \widehat{H}^+$ is compact and convex. It is certainly closed and convex, since both E and $\widehat{H}^+$ are closed and convex. We prove next that it is bounded. Suppose, for the sake of contradiction, that it is unbounded. Since $E \cap H^+$ is bounded, because H induces an M-decomposition of truncation type, we obtain that $E \cap \{x \in \mathbb{R}^n : \beta \le a^t x \le \alpha\}$ is unbounded, and hence, being convex, it contains a halfline, say with direction $u \in O^+(E)$. It follows that u is a direction in H, but then $E \cap H^+$ is unbounded, a contradiction. Hence, $E \cap \widehat{H}^+$ is bounded. Now, $E \cap \widehat{H}^- \subset E \cap H^-$. Therefore, $E \cap \widehat{H}^-$ also consists of a set of halflines emanating from $E \cap \widehat{H}$ (each one of which is contained in the corresponding halfline of $E \cap H^-$ of the same direction, emanating from $E \cap H$). □

The extension of Theorem 1 is the following.

Theorem 5 *If F is OM-predecomposable and $O^+(F)$ is pointed, then there exists an OM-decomposition of F, say $F = C + O^+(F)$, of truncation type.*

Proof By Theorem 2, $\mathrm{cl}(F)$ is M-decomposable. By Theorem 1, $\mathrm{cl}(F)$ admits an M-decomposition of truncation type, say the one given by the hyperplane $H = \{x \in \mathbb{R}^n : a^t x = \alpha\}$ ($a \in \mathbb{R}^n, \alpha \in \mathbb{R}$). Take any $\beta < \alpha$, and consider the hyperplane $\widehat{H} := \{x \in \mathbb{R}^n : a^t x = \beta\}$. By Proposition 3, $\widehat{H}$ induces an M-decomposition of $\mathrm{cl}(F)$ of truncation type. We must check that $\widehat{H}$ also induces an OM-decomposition of F of truncation type. It is clear that $F \cap \widehat{H}^-$ is a set of halflines emanating from $F \cap \widehat{H}$, and that $F \cap \mathrm{int}(\widehat{H}^+)$ is open. Since $F \cap \mathrm{int}(\widehat{H}^+) \subset \mathrm{cl}(F) \cap \widehat{H}^+$, which is a compact set, we obtain that $F \cap \mathrm{int}(\widehat{H}^+)$ is bounded. It only remains to see that $F \cap \mathrm{int}(\widehat{H}^+)$ is nonempty (indeed, this is the only reason for switching from H to $\widehat{H}$; the set $F \cap \mathrm{int}(H^+)$ might be empty). Note that $\mathrm{cl}(F) \cap H \neq \emptyset$, because H induces an M-decomposition of truncation type of $\mathrm{cl}(F)$. Take $x \in \mathrm{relint}[\mathrm{cl}(F) \cap H] = F \cap H$. Then, $a^t x = \alpha > \beta$, meaning that $x \in \mathrm{int}(\widehat{H}^+)$. Thus, $x \in F \cap \mathrm{int}(\widehat{H}^+)$, completing the proof. □

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