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Distance measures for higher order dual hesitant fuzzy sets

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Abstract In this study, we propose new distance measures for dual hesitant fuzzy sets (DHFSs) in terms of the mean, standard deviation of dual hesitant fuzzy elements (DHFEs), respectively, which overcome some drawbacks of the existing distance measures. Meanwhile, we extend DHFS to its higher order type and refer to it as the higher order dual hesitant fuzzy set (HODHFS). HODHFS is the actual extension of DHFS that enables us to define the membership and non-membership of a given element in terms of several possible generalized type of fuzzy sets (G-Type FSs). The rationale behind HODHFS can be seen in the case that the decision makers are not satisfied by providing exact values for the membership degrees and the non-membership degrees. To indicate HODHFSs have a good performance in decision making, we introduce several distance measures for HODHFSs based on our proposed new distance for dual hesitant fuzzy sets. Finally, we practice our proposed measures for HODHFSs in multi-attribute decision making illustrating their applicability and availability.

Keywords DHFS · Mean · Standard deviation · HODHFS · Distance measure

Mathematics Subject Classification 03E72 · 90B50

1 Introduction

When people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement, that is, there usually exists a hesitation or uncertainty about the degree. Zhu et al. (2012) introduced the definition of DHFS, which is an extension of hesitant fuzzy set (Torra 2010; Torra and Narukawa 2009). DHFSs can better deal

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with the situations that permit both the membership and the non-membership of an element to a given set having a few different values, which can reflect the human's hesitance of not only membership degrees, but also non-membership degrees. Then, a growing number of studies focus on DHFSs. Ye (2014) proposed a correlation coefficient between DHFSs as a new extension of existing correlation coefficients for hesitant fuzzy sets and intuitionistic fuzzy sets and apply it to multiple attribute decision making under dual hesitant fuzzy environments. Wang et al. (2014) first investigated a variety of distance measures and the corresponding similarity measures for dual hesitant fuzzy sets, based on which they presented a TOPSIS approach for the weapon selection problem. After that, Singh (2015) proposed some distance measures based on the geometric distance model, the set-theoretic approach, and the matching functions. However, these existing distance measures for dual hesitant fuzzy sets do not satisfy such fundamental properties as triangle inequality which was stressed by Zhou and Wang et al. (2002) and Singh (2015). Moreover, the existing distance measures only took into account the difference between the membership and non-membership values, but ignored the validity of the values of the hesitant fuzzy elements. Besides, a fatal weakness is that researchers extended the short dual hesitant fuzzy element by adding some values until the membership degrees and non-membership degrees of DHFSs have the same length, which would make the results inaccurate. To overcome such drawbacks, in this paper, we propose some new distance measures by the mean, standard deviation of DHFEs.

In recent years, some extensions of the DHFSs have been developed such as dual hesitant fuzzy rough sets (Zhang et al. 2015) and typical dual hesitant fuzzy sets (Farhadinia 2015). However, DHFSs have their inherent drawbacks, because they express the membership degrees or non-membership degrees of an element to a given set only by crisp numbers. In many practical decision-making problems, the information provided by decision makers might often be described by fuzzy sets instead of crisp numbers or by other fuzzy set extensions instead of intuitionistic fuzzy sets. This makes decision makers uncomfortable to provide exact crisp values or just intuitionistic fuzzy sets for the membership degrees and non-membership degrees. The HODHFS is fit for the situation where the decision makers have a hesitation among several possible membership and non-membership for an element. And it is the actual extension of DHFS encompassing not only fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets and hesitant fuzzy sets, but also the recent extension of DHFSs, called dual interval-valued hesitant fuzzy sets (Farhadinia 2014). Based on the proposed new distance measures for dual hesitant fuzzy sets, we introduce several distance measures for HODHFSs as a way to indicating a good performance in decision making of HODHFSs.

The remainder of the paper is organized as follows: in Sect. 2, we review some basic notions of dual hesitant fuzzy sets, based on which we give some drawbacks about the information measures for them. In Sect. 3, some new distance measures for dual hesitant fuzzy sets are proposed in terms of the mean, standard deviation of dual hesitant fuzzy element. In Sect. 4, we extend DHFS to its higher order type and refer to it as HODHFS, and we also developed a series of distance measures for HODHFSs. In Sect. 5, we apply the proposed distance measures for HODHFSs to multi-attribute decision making. Section 6 is a conclusion of the paper.

2 Preliminaries

In this section, we carry out a brief introduction to some distance measures for DHFSs as a basis of the main body of the paper, and then we point out some drawbacks of the existing distance measures. Let us start by recalling distance measures for DHFS.

2.1 Distance measures for dual hesitant fuzzy sets

Definition 1 (Farhadinia 2014) Let X be a nonempty set. A metric d on X is called distance measure if for any $x, y, z \in X$, the following properties hold:

(i) (Nonnegative) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) (Symmetric) $d(x, y) = d(y, x)$;
(iii) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

Zhu et al. (2012) defined a DHFS, which is an extension of the hesitant fuzzy set, in terms of two functions that return two sets of membership values and non-membership values, respectively, for each element in the domain as follows.

Definition 2 (Zhu et al. 2012) Let X be a fixed set, then a DHFS D on X is defined as

$$D = \{\langle x, h(x), g(x)\rangle | x \in X\}, \tag{1}$$

where $h(x)$ and $g(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set D, respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} \max\{\eta\}$ for $x \in X$. For convenience, the pair $e(x) = \{\langle h(x), g(x)\rangle\}$ is called a DHFE denoted by all $e = \{\langle h, g\rangle\}$.

From the above definition, we can see that DHFS consists of two parts, that is, the membership hesitancy function and the non-membership hesitancy function, supporting a more exemplary and flexible access to assign values for each element in the domain, and can handle two kinds of hesitancy in this situation. The existing sets, including fuzzy sets, intuitionistic fuzzy sets, hesitant fuzzy sets, and fuzzy multisets, can be regarded as special cases of DHFSs.

Distance and similarity measures have attracted plenty of attention in the last few decades due to the fact that they can be applied to many areas such as approximate reasoning (Wang et al. 2002), image processing (Pal and King 1981), medical diagnosis (Szmidt and Kacprzyk 2001) and decision making (Xu 2005; Yager 1988). A distance measure is used for estimating the degree of distance between two sets in the fuzzy set theory, which has received much attention from researchers (Buckley and Hayashi 1993; Candan et al. 2000; Liu and Entropy 1992; Turksen and Zhong 1988). Among them, the most widely used distance measures (Diamond and Kloeden 1994; Kacprzyk 1997; Turksen and Zhong 1988) are the Hamming distance, Euclidean distance, and Hausdorff metric. Later on, the distance and similarity measures about other extensions of fuzzy sets have also been developed (Grzegorzewski 2004; Li and Cheng 2002; Li et al. 2007, 2015a, b; Liang and Shi 2003; Xu 2007, 2010), but there is little research on DHFSs. Consequently, it is very necessary to develop some distance measures under dual hesitant fuzzy environment. Wang et al. (2014) first address this issue by putting forward the axioms for distance measures.

Definition 3 (Wang et al. 2014) Let A and B be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the distance measure between A and B is defined as $d(A, B)$, which satisfies the following properties:

1. $0 \leq d(A, B) \leq 1$;
2. $d(A, B) = 0$ if and only if $A = B$;
3. $d(A, B) = d(B, A)$.

Practically, in most of the cases, the number of values in membership degrees and non-membership degrees may not be equal, i.e., $l(h_A(x_i)) \neq l(h_B(x_i))$ and $m(g_A(x_i)) \neq m(g_B(x_i))$. Let $l_{x_i} = \max\{l(h_A(x_i)), l(h_B(x_i))\}$ and $m_{x_i} = \max\{m(g_A(x_i)), m(g_B(x_i))\}$ for each $x_i \in X$. To find the distance measure between DHFSs, one should extend the shorter one until the membership degrees and non-membership degrees of both DHFSs have the same length. To extend the shorter one, the best way is to add the same value several times in it. In fact, we can extend the shorter one by adding any value in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value of the membership degrees and minimum value of non-membership degrees, while pessimists expect unfavorable outcomes and may add the minimum of value the membership degrees and maximum value of non-membership degrees.

On the basis of Definition 3, Wang et al. (2014) defined the dual hesitant normalized Hamming distance as follows:

Definition 4 Let $A = \{\langle x_i, h_A(x_i), g_A(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x_i, h_B(x_i), g_B(x_i)\rangle | x_i \in X\}$ are two dual hesitant fuzzy sets on $X = \{x_1, x_2, \ldots, x_n\}$. Then, the normalized Hamming distance between A and B is defined as

$$d_{1,\lambda}(A, B) = \sum_{i=1}^{n} \left(\frac{1}{n} \left(\frac{1}{l} \left(\sum_{i=j}^{l_{x_i}} |\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^{\lambda} + \sum_{i=j}^{m_{x_i}} |\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^{\lambda} \right)\right)\right)^{\frac{1}{\lambda}}, \tag{2}$$

where $\lambda > 0$, $l = l_{x_i} + m_{x_i}$, $l_{x_i} = \max\{l_A(x_i), l_B(x_i)\}$ and $m_{x_i} = \max\{m_A(x_i), m_B(x_i)\}$, $l_A(x_i), l_B(x_i)$ and $m_A(x_i), m_B(x_i)$ are the numbers of values of $h_A(x_i), h_B(x_i)$ and $g_A(x_i), g_B(x_i)$, respectively, $\psi_A^{\sigma(j)}(x_i), \psi_B^{\sigma(j)}(x_i)$ and $\phi_A^{\sigma(j)}(x_i), \phi_B^{\sigma(j)}(x_i)$ are the jth largest values of $h_A(x_i), h_B(x_i)$ and $g_A(x_i), g_B(x_i)$, respectively.

If some situations cannot allow to extend the shorter one by adding any elements in it to the same length, the following distance measure (Wang et al. 2014) is given:

$$\begin{aligned} d_{2,\lambda}(A, B) = \sum_{i=1}^{n} \Bigg(\frac{1}{n} \Bigg(\Bigg| \frac{1}{l_A(x_i)} \sum_{\psi_A \in h_A(x_i)} \psi_A - \frac{1}{l_B(x_i)} \sum_{\psi_B \in h_B(x_i)} \psi_B \Bigg|^{\lambda} \\ + \Bigg| \frac{1}{m_A(x_i)} \sum_{\phi_A \in g_A(x_i)} \phi_A - \frac{1}{m_B(x_i)} \sum_{\phi_B \in g_B(x_i)} \phi_B \Bigg|^{\lambda} \Bigg)\Bigg)^{\frac{1}{\lambda}}, \end{aligned} \tag{3}$$

where $\lambda > 0$.

In many practical situations, the weight of each element $x_i \in X$ should be taken into account. Thus, the weighted distance measure (Wang et al. 2014) for DHFSs is given as follows:

$$d_{w,\lambda}(A, B) = \sum_{i=1}^{n} \left(w_i \left(\frac{1}{l} \left(\sum_{j=1}^{l_{x_i}} |\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^{\lambda} + \sum_{j=1}^{m_{x_i}} |\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^{\lambda} \right)\right)\right)^{\frac{1}{\lambda}}, \tag{4}$$

where $\lambda > 0$.

To compare the DHFEs, Zhu et al. (2012) give the following comparison law:

Definition 5 (Zhu et al. 2012) Let $e_i = \{\varsigma_{e_i}, \zeta_{e_i}\}(i = 1, 2)$ be any two DHFEs, $s_{e_i} = \frac{1}{l(\varsigma_{e_i})}\Sigma_{\psi\in\varsigma_{e_i}}\psi - \frac{1}{m(\zeta_{e_i})}\Sigma_{\phi\in\zeta_{e_i}}\phi (i = 1, 2)$ the score function of $e_i (i = 1, 2)$ and $p_{e_i} = \frac{1}{l(\varsigma_{e_i})}\Sigma_{\psi\in\varsigma_{e_i}}\psi + \frac{1}{m(\zeta_{e_i})}\Sigma_{\phi\in\zeta_{e_i}}\phi$ $(i = 1, 2)$ the accuracy function of $e_i (i = 1, 2)$, where $l(\varsigma_{e_i})$ and $m(\zeta_{e_i})$ be the number of elements in ς_{e_i} and ζ_{e_i}, respectively, then

(i) if $s_{e_1} > s_{e_2}$, then e_1 is superior to e_2, denoted by $e_1 \succ e_2$;
(ii) if $s_{e_1} = s_{e_2}$, then

1. if $p_{e_1} = p_{e_2}$, then e_1 is equivalent to e_2, denoted by $e_1 \sim e_2$;
2. if $p_{e_1} > p_{e_2}$, then e_1 is superior than to e_2, denoted by $e_1 \succ e_2$.

Based on the comparison law, the following concepts are given by Singh (2015):

Definition 6 (Singh 2015) DHFS $A = \{A(x)|x \in X\}$ is said to be dual hesitant fuzzy subset of DHFS $B = \{B(x)|x \in X\}$, denoted by $A \subseteq B$, if $s_{A(x)} < s_{B(x)}$ for any $x \in X$, and DHFS A is said to be equal to DHFS B, denoted by $A = B$, if $s_{A(x)} = s_{B(x)}$ and $p_{A(x)} = p_{B(x)}$ for any $x \in X$.

Definition 7 (Singh 2015) Let $A = \{h_A, g_A\}$ and $B = \{h_B, g_B\}$ be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the distance measure between A and B is defined as $d(A, B)$, which satisfies the following properties:

(P1) $0 \leq d(A, B) \leq 1$;
(P2) $d(A, B) = 0$ if and only if $A = B$;
(P3) $d(A, B) = d(B, A)$;
(P4) Let C be any DHFS, if $A \subseteq B \subseteq C$, then $d(A, B) \leq d(A, C)$ and $d(B, C) \leq d(A, C)$.

Then the dual hesitant normalized Hamming distance (Singh 2015):

$$d_{dnh} = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)| + \frac{1}{m_{x_i}}\sum_{j=1}^{m_{x_i}}|\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|\right], \quad (5)$$

and a dual hesitant normalized Euclidean distance (Singh 2015):

$$d_{dne} = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^2 + \frac{1}{m_{x_i}}\sum_{j=1}^{m_{x_i}}|\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^2\right)\right]^{\frac{1}{2}}, \quad (6)$$

and a further generalized into a generalized dual hesitant normalized distance (Singh 2015):

$$d_{dgn} = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{1}{l_{x_i}}\sum_{j=1}^{l_{x_i}}|\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^\lambda + \frac{1}{m_{x_i}}\sum_{j=1}^{m_{x_i}}|\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^\lambda\right)\right]^{\frac{1}{\lambda}}, \quad (7)$$

where $\lambda > 0$. $\psi_A^{\sigma(j)}(x_i)$, $\psi_B^{\sigma(j)}(x_i)$ and $\phi_A^{\sigma(j)}(x_i)$, $\phi_B^{\sigma(j)}(x_i)$ are the jth largest values of membership degrees and non-membership degrees of A and B, respectively.

Assume that for $x \in X$, let w_i $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} w_i = 1$ and z_i $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} z_i = 1$ be the weights assigned to membership degrees and non-membership degrees, respectively. Thus,the generalized dual hesitant weighted distance:

$$d_{dgw} = \left[\frac{1}{2} \sum_{i=1}^{n} \left[w_i \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^{\lambda} \right) + z_i \left(\frac{1}{m_{x_i}} \sum_{j=1}^{m_{x_i}} |\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^{\lambda} \right) \right] \right]^{\frac{1}{\lambda}}, \tag{8}$$

precisely, if $\lambda = 1$, then we get a dual hesitant weighted Hamming distance:

$$d_{dwh} = \frac{1}{2} \sum_{i=1}^{n} \left[w_i \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)| \right) + z_i \left(\frac{1}{m_{x_i}} \sum_{j=1}^{m_{x_i}} |\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)| \right) \right], \tag{9}$$

if $\lambda = 2$, then we get a dual hesitant weighted Euclidean distance:

$$d_{dwe} = \left[\frac{1}{2} \sum_{i=1}^{n} \left[w_i \left(\frac{1}{l_{x_i}} \sum_{j=1}^{l_{x_i}} |\psi_A^{\sigma(j)}(x_i) - \psi_B^{\sigma(j)}(x_i)|^{2} \right) + z_i \left(\frac{1}{m_{x_i}} \sum_{j=1}^{m_{x_i}} |\phi_A^{\sigma(j)}(x_i) - \phi_B^{\sigma(j)}(x_i)|^{2} \right) \right] \right]^{\frac{1}{2}}. \tag{10}$$

2.2 Analysis on distance measures for DHFSs

Apparently, (1)–(3) in Definition 3 are the same as (P1)–(P3) in Definition 7 for DHFSs, but (P4) in Definition 7 is different from Definition 3. However, in the metric space theory, a metric space consists of two objects: a nonempty set X and a metric distance d on X. Hence, the metric distance is a fundamental element of metric space. Then, when a new distance measure is proposed, it should satisfy the three properties in Definition 1. Thus, distance measures of hesitant fuzzy sets should also satisfy the property of triangle inequality. It means that (i), (ii), (iii) in Definition 1 can be seen as the basic conditions for distance measures of DHFSs. To analyze the drawbacks of dual hesitant fuzzy sets, we modify this axiomatic definition as follows:

Definition 8 Let M, N and O be three DHFSs on X, then d is called a distance measure for DHFSs if it possesses the following properties:

(d1) $0 \le d(M, N) \le 1$;
(d2) $d(M, N) = 0$ if and only if $M = N$;
(d3) $d(M, N) = d(N, M)$;
(d4) $d(M, N) \le d(M, O) + d(O, N)$;
(d5) if $M \subseteq N \subseteq O$, then $d(M, N) \le d(M, O)$ and $d(N, O) \le d(M, O)$.

Besides, since the existing distance measures only depend on the values of DHFEs, we call them value-based distance measures for DHFSs. The main characteristic of the value-based distance measures is that only the difference between the values of the elements is considered, the distance measures are not precise as they do not consider the volatility of the values of a DHFE. In fact, DHFS focuses on the hesitance of providing the membership and non-membership values, and such hesitance is characterized by both the differences between the values and the volatility of their own values of the elements. So, ignoring the influence of the difference between the volatility of their own values of the elements will lead to unreasonable results.

On the other hand, we find that most of the value-based distance measures for DHFSs (Singh 2015; Wang et al. 2014) (except $d_{2,\lambda}$ in Wang et al. 2014) have been defined depending on the assumptions: where (A1) all the elements in each HFE are rearranged in increasing (or decreasing) order and (A2) the number of values in different HFEs must be indifferent, that is, the different HFEs must have the same length, but $d_{2,\lambda}$ in Wang et al. (2014) are free to the assumptions.

According to the above analysis, we give some concrete drawbacks by some examples in the following. As $d_{1,1}$, $d_{2,1}$, d_{dnh} are special cases of $d_{1,\lambda}$, $d_{2,\lambda}$, d_{dgn}, respectively, and $d_{w,\lambda}$, d_{dgw} are a generalization of $d_{1,\lambda}$, d_{dgn}, respectively, we only focus on analyzing d_{dnh}, d_{dne}, $d_{1,1}$ and $d_{2,1}$.

1. The drawbacks of d_{dnh}, d_{dne} and $d_{1,1}$.

   (i) The first drawback is that these distance measures do not satisfy the property (d4) in Definition 8.

Example 1 Let $X = \{x\}$, $M = \{\langle x, \{0.1\}, \{0.8, 0.5\}\rangle\}$; $N = \{\langle x, \{0.8, 0.4\}, \{0.1\}\rangle\}$, and $O = \{\langle x, \{0.8, 0.4, 0.3, 0.35\}, \{0.1, 0.2, 0.15, 0.05\}\rangle\}$. Suppose we extend the shorter one by adding the minimum value, then we have

$d_{dnh}(M, N) = 0.525$, $d_{dnh}(M, O) = 0.40625$, $d_{dnh}(O, N) = 0.04375$.
Hence, $d_{dnh}(M, N) > d_{dnh}(M, O) + d_{dnh}(O, N)$.
Similarly, $d_{dne}(M, N) = 0.5545$, $d_{dne}(M, O) = 0.4370$, $d_{dne}(O, N) = 0.0586$.
Hence, $d_{dne}(M, N) > d_{dne}(M, O) + d_{dne}(O, N)$.
$d_{1,1}(M, N) = 0.525$, $d_{1,1}(M, O) = 0.40625$, $d_{1,1}(O, N) = 0.04375$.
Hence, $d_{1,1}(M, N) > d_{1,1}(M, O) + d_{1,1}(O, N)$.

(ii) The second drawback is that we only compare two DHFSs when we extend the shorter DHFEs. Hence, for shorter DHFEs, when we calculate the distance measures between two DHFSs, the numbers of the adding values of DHFEs are different. Then, we actually calculate the distance measures in spaces with different dimensions. It means that we apply different information to obtain distance measures. Apparently, such results are usually incomparable. Moreover, this drawback will lead to the previous two drawbacks.

Example 2 Let $X = \{x\}$, $M(x) = \{\langle\{0.4, 0.3\}, \{0.5, 0.4\}\rangle\}$, $N(x) = \{\langle\{0.6, 0.4, 0.3\}, \{0.2, 0.4\}\rangle\}$ and $O(x) = \{\langle\{0.6, 0.5\}, \{0.2, 0.4, 0.3\}\rangle\}$. When we calculate $d(M, N)$ and $d(M, O)$, $M(x)$ is extended as $\{\langle\{0.4, 0.3, 0.3\}, \{0.5, 0.4\}\rangle\}$ and $\{\langle\{0.4, 0.3\}, \{0.5, 0.4, 0.4\}\rangle\}$, respectively. Thus, we can see that these two extended measures are not equal. So, its meaningless to compare such distance measures.

(iii) The third drawback is that it is not accurate, when we extend the shorter one until the membership degrees and non-membership degrees of both DHFSs have the same length. The number of values in membership degrees and non-membership degrees may not be equal, and we always do not know whether the decision makers are optimistic or not. In fact, the scholars often extend the shorter one by adding any value in it, what has a great effect on the results.

Example 3 Let $X = \{x\}$, $M(x) = \{\langle\{0.4, 0.2\}, \{0.5, 0.4\}\rangle\}$. We can extend $M(x)$ as

1. $\{\langle\{0.4, 0.2, 0.2\}, \{0.5, 0.4, 0.4\}\rangle\}$;
2. $\{\langle\{0.4, 0.2, 0.4\}, \{0.5, 0.4, 0.5\}\rangle\}$.

Obviously, the different extensions lead to different results of the distance measures.

2. The common drawbacks of d_{dnh}, d_{dne}, $d_{1,1}$ and $d_{2,1}$.

(iv) The fourth drawback is that these distance measures do not satisfy the property (d5) in Definition 8.

Example 4 Let $X = \{x\}$, $M = \{\langle x, \{0.6\}, \{0.2\}\rangle\}$; $N = \{\langle x, \{0.4, 0.5\}, \{0.1, 0.2\}\rangle\}$, and $O = \{\langle x, \{0.6\}, \{0.4, 0.3\}\rangle\}$, then $O < N < M$. Suppose we extend the shorter one by adding the minimum value, then we have
$d_{dnh}(M, O) = 0.075$, $d_{dnh}(N, O) = 0.175$.
Hence, $d_{dnh}(M, O) < d_{dnh}(N, O)$.
Similarly, $d_{dne}(M, O) = 0.1118$, $d_{dne}(N, O) = 0.1803$.
Hence, $d_{dne}(M, O) < d_{dne}(N, O)$.
$d_{1,1}(M, O) = 0.1$, $d_{1,1}(N, O) = 0.175$.
Hence, $d_{1,1}(M, O) < d_{1,1}(N, O)$.
$d_{2,1}(M, O) = 0.075$, $d_{2,1}(N, O) = 0.175$.
Hence, $d_{2,1}(M, O) < d_{2,1}(N, O)$.

Remark 1 Although Singh (2015) pointed out that the distance measures should satisfy the property (d5) in Definition 8, Example 4 shows that the dual hesitant normalized Hamming distance and the dual hesitant normalized Euclidean distance presented by him do not satisfy this property.

(v) The fifth drawback is that the formulas of the distance measures are not precise as they do not consider the volatility of the values of a DHFE in DHFSs.

Example 5 Let $X = \{x\}$, $M(x) = \{\langle\{0.5\}, \{0.3\}\rangle\}$, $N(x) = \{\langle\{0.4, 0.6\}, \{0.3\}\rangle\}$, $O(x) = \{\langle\{0.3, 0.7\}, \{0.3\}\rangle\}$ and $P(x) = \{\langle\{0.3\}, \{0.3\}\rangle\}$. Then we have
$d_{dnh}(M, N) = 0.2$, $d_{dnh}(M, P) = 0.2$.
Hence, $d_{dnh}(M, N) = d_{dnh}(M, P)$.
Similarly, $d_{dne}(M, N) = 0.2236$, $d_{dne}(M, P) = 0.2236$.
Hence, $d_{dne}(M, N) = d_{dne}(M, P)$.
$d_{1,1}(M, N) = 0.2$, $d_{1,1}(M, P) = 0.2$.
Hence, $d_{1,1}(M, N) = d_{1,1}(M, P)$.
$d_{2,1}(M, N) = 0$, $d_{2,1}(M, O) = 0$.
Hence, $d_{2,1}(M, N) = d_{2,1}(M, O)$.

As $N(x) \neq P(x) \neq O(x)$, we think $d(M, N) \neq d(M, O) \neq d(M, P)$. The value-based existing distance measures only consider the values of DHFEs but ignore the volatility of DHFEs, which make the measures between DHFFs inaccurate.

3 New distance measures for DHFSs

To overcome some drawbacks analyzed in Sect. 2.2, we will provide a series of new distance measures for DHFSs. Before proposing the new distance, we introduce a new concept as follows:

Definition 9 Let $e = \{\langle \varsigma, \zeta \rangle\}$ be any DHFE. Denote

$$E_m(e) = \frac{\sum_{\psi \in \varsigma} \psi}{l(\psi)}; \tag{11}$$

$$E_n(e) = \frac{\sum_{\phi \in \zeta} \phi}{m(\phi)}; \tag{12}$$

$$S_m(e) = \sqrt{\frac{\sum_{\psi \in \varsigma} (\psi - E_m)^2}{l(\psi)}}; \tag{13}$$

$$S_n(e) = \sqrt{\frac{\sum_{\phi \in \zeta} (\phi - E_n)^2}{m(\phi)}}, \tag{14}$$

where ψ and ϕ are the values of ς and ζ, respectively; $l(\psi)$ and $m(\phi)$ are the number of elements ψ and ϕ. We call $E_m(e)$ and $E_n(e)$ the mean of membership degrees and non-membership degrees of the DHFE e, respectively, $S_m(e)$ and $S_n(e)$ the standard deviation of membership degrees and non-membership degrees of the DHFE e, respectively.

For any DHFE e, $E_m(e)$, $E_n(e)$ reflect the values of the membership degrees and non-membership degrees. When $e = \{\langle \{1\}, \{0\} \rangle\}$, it indicates that the decision maker can determine the precise value of the membership degrees and non-membership degrees without any question, namely there is not any volatility for decision maker to determine the value of the DHFE. When $e = \{\langle \{0\}, \{1\} \rangle\}$, it means that the decision maker gives the opposite evaluation. $S_m(e)$, $S_n(e)$ reflects the degree of volatility for decision maker when they determine the values for the DHFE e. The larger the values of $S_m(e)$ and $S_n(e)$ are, the more volatile the data will be given by the decision maker. When $S_m(e) = S_n(e) = 0$, $l(\psi) = m(\phi) = 1$, namely there is not any volatility for decision maker to determine the value of the DHFE e.

For convenience, we also give the notes as follows:

Let D_1 and D_2 be any DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the mean distance of the membership of a DHFE between D_1 and D_2 is

$$E_m(x_i) = |E_m(D_1(x_i)) - E_m(D_2(x_i))|, \tag{15}$$

the mean distance of the non-membership of a DHFE between D_1 and D_2 is

$$E_n(x_i) = |E_n(D_1(x_i)) - E_n(D_2(x_i))|, \tag{16}$$

the standard deviation distance of the membership of a DHFE between D_1 and D_2 is

$$S_m(x_i) = |S_m(D_1(x_i)) - S_m(D_2(x_i))|, \tag{17}$$

the standard deviation distance of the non-membership of a DHFE between D_1 and D_2 is

$$S_n(x_i) = |S_n(D_1(x_i)) - S_n(D_2(x_i))|. \tag{18}$$

Based on the Definition 9, we present the new distance measures for DHFSs.

Definition 10 Let D_1 and D_2 be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the new normalized Hamming distance between D_1 and D_2 is defined as

$$d_{dh}(D_1, D_2) = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{E_m(x_i) + E_n(x_i)}{2} + \frac{S_m(x_i) + S_n(x_i)}{2}\right), \tag{19}$$

the new normalized Euclidean distance between D_1 and D_2 is defined as

$$d_{de}(D_1, D_2) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{E_m^2(x_i) + E_n^2(x_i)}{2} + \frac{S_m^2(x_i) + S_n^2(x_i)}{2}\right)\right]^{\frac{1}{2}}, \tag{20}$$

the new normalized generalized distance between D_1 and D_2 is defined as

$$d_{dg}(D_1, D_2) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{E_m^\lambda(x_i) + E_n^\lambda(x_i)}{2} + \frac{S_m^\lambda(x_i) + S_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}}, \tag{21}$$

where $\lambda \geq 1$, $E_m(x_i)$ and $E_n(x_i)$ are the mean distance of the membership degrees and non-membership degrees between D_1 and D_2 satisfying (15) and (16), respectively; $S_m(x_i)$ and $S_n(x_i)$ are the standard deviation distance of the membership degrees and non-membership degrees between D_1 and D_2 satisfying (17) and (18), respectively.

Next, we give an example to show the computational process of the new distance.

Example 6 Let $X = \{x_1, x_2\}$, $M = \{\langle x_1, \{0.8, 0.6\}, \{0.2\}\rangle, \langle x_2, \{0.1\}, \{0.4, 0.8\}\rangle\}$, $N(x) = \{\langle x_2, \{0.4, 0.5\}, \{0.1, 0.2\}\rangle, \langle x_2, \{0.2, 0.4\}, \{0.4, 0.6\}\rangle\}$. Then,

for M, we have

$$E_m(M(x_1)) = \frac{0.6 + 0.8}{2} = 0.7, \quad E_n(M(x_1)) = 0.2,$$

$$S_m(M(x_1)) = \sqrt{\frac{(0.8 - 0.7)^2 + (0.6 - 0.7)^2}{2}} = 0.1, \quad S_n(M(x_1)) = 0;$$

$$E_m(M(x_2)) = 0.1, \quad E_n(M(x_2)) = \frac{0.4 + 0.8}{2} = 0.6,$$

$$S_m(M(x_2)) = 0, \quad S_n(M(x_2)) = \sqrt{\frac{(0.4 - 0.6)^2 + (0.8 - 0.6)^2}{2}} = 0.2.$$

For N, we have

$$E_m(N(x_1)) = \frac{0.4 + 0.5}{2} = 0.45, \quad E_n(N(x_1)) = \frac{0.1 + 0.2}{2} = 0.15,$$

$$S_m(N(x_1)) = \sqrt{\frac{(0.5 - 0.45)^2 + (0.4 - 0.45)^2}{2}} = 0.05,$$

$$S_n(N(x_1)) = \sqrt{\frac{(0.2 - 0.15)^2 + (0.1 - 0.15)^2}{2}} = 0.05;$$

$$E_m(N(x_2)) = \frac{0.2 + 0.4}{2} = 0.3, \quad E_n(N(x_2)) = \frac{0.4 + 0.6}{2} = 0.5,$$

$$S_m(N(x_2)) = \sqrt{\frac{(0.2 - 0.3)^2 + (0.4 - 0.3)^2}{2}} = 0.1,$$

$$S_n(N(x_2)) = \sqrt{\frac{(0.4-0.5)^2+(0.6-0.5)^2}{2}} = 0.1;$$

therefore,

$$\begin{aligned}
d_{dg}(M,N) &= \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{E_m^\lambda(x_i)+E_n^\lambda(x_i)}{2}+\frac{S_m^\lambda(x_i)+S_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}} \\
&= \left[\frac{1}{2\times 2}\left(\frac{|0.7-0.45|^\lambda+|0.2-0.15|^\lambda}{2}+\frac{|0.1-0.05|^\lambda+|0-0.05|^\lambda}{2}\right.\right. \\
&\quad \left.\left.+\frac{|0.1-0.3|^\lambda+|0.6-0.5|^\lambda}{2}+\frac{|0.2-0.1|^\lambda+|0-0.1|^\lambda}{2}\right)\right]^{\frac{1}{\lambda}} \\
&= \left[\frac{1}{4}\left(\frac{0.15^\lambda+0.05^\lambda}{2}+0.05^\lambda+\frac{0.2^\lambda+0.1^\lambda}{2}+0.1^\lambda\right)\right]^{\frac{1}{\lambda}}.
\end{aligned}$$

If we take into account the preferences of the influences of mean and standard deviation of DHFE, then we have the new normalized Hamming distance with preference as follows:

$$d_{dph}(D_1,D_2) = \frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{E_m(x_i)+E_n(x_i)}{2}+\beta\frac{S_m(x_i)+S_n(x_i)}{2}\right), \tag{22}$$

the new normalized Euclidean distance with preference between D_1 and D_2 is defined as

$$d_{dpe}(D_1,D_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{E_m^2(x_i)+E_n^2(x_i)}{2}+\beta\frac{S_m^2(x_i)+S_n^2(x_i)}{2}\right)\right]^{\frac{1}{2}}, \tag{23}$$

the new normalized generalized distance with preference between D_1 and D_2 is defined as

$$d_{dpg}(D_1,D_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{E_m^\lambda(x_i)+E_n^\lambda(x_i)}{2}+\beta\frac{S_m^\lambda(x_i)+S_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}}, \tag{24}$$

where $\alpha+\beta=1$.

If we not only consider the different preferences of the influences of mean and standard deviation of dual hesitant fuzzy element, but also notice the weight of each element $x \in X$, then the weighted distance with preference is shown as follows:

Definition 11 Let D_1 and D_2 be two DHFSs on $X=\{x_1,x_2,\ldots,x_n\}$, then the new weighted normalized Hamming distance with preference between D_1 and D_2 is defined as

$$d_{wdph}(D_1,D_2) = \sum_{i=1}^{n} w_i\left(\alpha\frac{E_m(x_i)+E_n(x_i)}{2}+\beta\frac{S_m(x_i)+S_n(x_i)}{2}\right), \tag{25}$$

the new weighted normalized Euclidean distance with preference between D_1 and D_2 is defined as

$$d_{wdpe}(D_1,D_2) = \left[\sum_{i=1}^{n} w_i\left(\alpha\frac{E_m^2(x_i)+E_n^2(x_i)}{2}+\beta\frac{S_m^2(x_i)+S_n^2(x_i)}{2}\right)\right]^{\frac{1}{2}}, \tag{26}$$

the new weighted normalized generalized distance with preference between D_1 and D_2 is defined as

$$d_{wdpg}(D_1, D_2) = \left[\sum_{i=1}^{n} w_i \left(\alpha \frac{E_m^\lambda(x_i) + E_n^\lambda(x_i)}{2} + \beta \frac{S_m^\lambda(x_i) + S_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}}, \quad (27)$$

where $\alpha + \beta = 1$ and $0 \leq \alpha, \beta \leq 1$, $\lambda \geq 1$, w_i is the weight of $x_i \in X$, $0 \leq w_i \leq 1$, $\sum_{i=1}^{n} w_i = 1$, $E_m(x_i)$ and $E_n(x_i)$ are the mean distance of the membership degrees and non-membership degrees between D_1 and D_2 satisfying (15) and (16), respectively; $S_m(x_i)$ and $S_n(x_i)$ are the standard deviation distance of the membership degrees and non-membership degrees between D_1 and D_2 satisfying (17) and (18), respectively.

Remark 2 To ensure that the distance measures of d_{wdph}, d_{wdpe} and d_{wdpg} satisfy the condition (d5) in Definition 8, the parameter λ should satisfy $\lambda \geq 1$.

Remark 3 If $w_1 = w_2 = \cdots = w_n = \frac{1}{n}$, then d_{wdph}, d_{wdpe} and d_{wdpg} are reduced to d_{dph}, d_{dpe} and d_{dpg}, respectively. If $\alpha = \beta = \frac{1}{2}$, then d_{dph}, d_{dpe} and d_{dpg} are reduced to d_{dh}, d_{de} and d_{dg}, respectively.

To prove the Theorem 1, we give the following lemma:

Lemma 1 (Kuang 2004) *Let* $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in R^n$, *and* $1 \leq \lambda \leq +\infty$. *Then*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^\lambda\right)^{\frac{1}{\lambda}} \leq \left(\sum_{k=1}^{n} |a_k|^\lambda\right)^{\frac{1}{\lambda}} + \left(\sum_{k=1}^{n} |b_k|^\lambda\right)^{\frac{1}{\lambda}}. \quad (28)$$

Theorem 1 *The distance measures* d_{wdph}, d_{wdpe} *and* d_{wdpg} *satisfy the properties* (d1)–(d4) *in Definition* 8.

Proof As d_{wdph} and d_{wdpe} are special cases of d_{wdpg}, we only give the proof of d_{wdpg}.

Assume that D_1, D_2 and D_3 are three DHFSs.

- (d1) Based on Definition 9, we can obtain $0 \leq E_m(d), E_n(d), S_m(d), S_n(d) \leq 1$, then $0 \leq E_m(x_i), E_n(x_i), S_m(x_i), S_n(x_i) \leq 1$. So we have $0 \leq \sum_{i=1}^{n} w_i(\alpha \frac{E_m^\lambda(x_i)+E_n^\lambda(x_i)}{2} + \beta \frac{S_m^\lambda(x_i)+S_n^\lambda(x_i)}{2}) \leq 1$ for $\lambda \geq 1$, $0 \leq w_i \leq 1$, $\sum_{i=1}^{n} w_i = 1$, hence $0 \leq d_{wdpg} \leq 1$.
- (d2) Obviously, if $D_1 = D_2$, then $d_{wdpg}(D_1, D_2) = 0$; on the other hand, if $d_{wdpg}(D_1, D_2) = 0$, for any $x_i \in X$, we have $E_m(x_i) = 0$ and $E_n(x_i) = 0$, then for any $x_i \in X$, $E_m(D_1(x_i)) = E_m(D_2(x_i))$ and $E_n(D_1(x_i)) = E_n(D_2(x_i))$. Thus, $s(D_1(x_i)) = E_m(D_1(x_i)) - E_n(D_1(x_i)) = E_m(D_2(x_i)) - E_n(D_2(x_i)) = s(D_2(x_i))$ and $p(D_1(x_i)) = E_m(D_1(x_i)) + E_n(D_1(x_i)) = E_m(D_2(x_i)) + E_n(D_2(x_i)) = p(D_2(x_i))$ for any $x_i \in X$, from the Definition 2.6, we obtain $D_1 = D_2$.
- (d3) From (15)–(18) and (27), we can easily obtain that $d_{wdpg}(D_1, D_2) = d_{wdpg}(D_1, D_2)$.
- (d4) Based on Lemma 1 and (27), we can obtain $d_{wdpg}(D_1, D_2) \leq d_{wdpg}(D_1, D_3) + d_{wdpg}(D_3, D_2)$.

Remark 4 Since the property (d5) in Definition 8 only depends upon the values of the DHFEs, and has no relation to the volatility of the DHFEs, then the distance measures do not satisfy this property.

Compared to the distance measures given by Singh (2015) and Wang et al. (2014), the new distance has the following advantages:

1. The new distance measures for DHFSs consider the mean, standard deviation of DHFEs, which is another way to calculate distance measures by its own characteristics of elements of a DHFS.
2. As a distance measure, the new distance for DHFSs, which avoids the drawback (v), is more accurate than Farhadinia (2014, 2015)'s as it not only considers the differences between the values of the elements, but also includes the volatility of their own values of the elements.
3. The new distance measures depend on the length of DHFEs. We do not need to extend the shorter one until the membership degrees and non-membership degrees of both DHFSs, which can reduce error of adding different values (see Example 3) and overcome the drawbacks (i), (ii) and (iii).
4. As a basic condition for distance measures in Definition 1, we think the triangle inequality is important to the distance measures of DHFSs. Since we give the analysis on distance measures for DHFSs in Sect. 2.2, the distance measures proposed by Singh (2015) and Wang et al. (2014) only meet the properties d_1, d_2 and d_3. But our new distance can not only meet all basic condition in Definition 1, but also reflect the distance characteristics of DHFS: $0 \leq d \leq 1$.

4 Distance measures for higher order dual hesitant fuzzy sets

This section contains two parts: one is developed to describe the basic definitions of fuzzy set and its new generalization which are referred to as the HODHFS. In fact, the HODHFS is a generalization of DHFS, which is introduced in Sect. 2; the other is to give the distance measures for HODHFSs based on the distance measures for DHFSs proposed in Sect. 3.

4.1 Higher order dual hesitant fuzzy set

An ordinary fuzzy set A in X is defined (Zadeh 1965) as $A = \{\langle x, A(x)\rangle | x \in X\}$, where $A : X \rightarrow [0, 1]$ and the real value $A(x)$ represents the degree of membership of x in A.

Definition 12 (Klir and Yuan 1995) Let X be a fixed set. A generalized type of fuzzy set on X is defined as

$$\tilde{A} = \{\langle x, \tilde{A}(x)\rangle | x \in X\}, \tag{29}$$

where $\tilde{A} : X \rightarrow \psi([0, 1])$. Here, $\psi([0, 1])$ denotes a family of crisp or fuzzy sets that can be defined within the universal set $[0, 1]$.

It is noteworthy that most of the existing extensions of ordinary FS are special cases of G-Type FS, for instance (Klir and Yuan 1995).

- if $\psi([0, 1]) = [0, 1]$, then the $\tilde{A}$ reduces to an ordinary FS;
- if $\psi([0, 1]) = \varepsilon([0, 1])$ denoting the set of all closed intervals, then the G-Type FS $\tilde{A}$ reduces to an IVFS;
- if $\psi([0, 1]) = \mathcal{F}([0, 1])$ denoting the set of all ordinary FSs, then the G-Type FS $\tilde{A}$ reduces to a T2FS;

– if $\psi([0, 1]) = L$ denoting a partially ordered Lattice, then the G-Type FS $\tilde{A}$ reduces to a L-FS.

Based on G-Type FS and DHFS given in Sect. 2, the concept of HODHFS is introduced here to let the membership degrees and non-membership degrees of an element to a given set be expressed by several possible G-Type FSs.

Definition 13 Let X be a fixed set. A HODHFS on $X = \{x_1, x_2, \ldots, x_n\}$ is defined as

$$\tilde{D} = \{\langle x, \tilde{h}(x), \tilde{g}(x)\rangle | x \in X\}, \tag{30}$$

where $\tilde{h}(x)$ and $\tilde{g}(x)$ are two G-Type FSs, denoting the possible membership degrees and non-membership degrees of the element $x \in X$ to the set $\tilde{D}$, respectively, with the conditions: $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$, where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ \in h^+(x) = \cup_{\gamma \in h(x)} \max\{\gamma\}$, and $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} \max\{\eta\}$ for $x \in X$. For convenience, the pair $\tilde{d}(x) = \{\tilde{h}(x), \tilde{g}(x)\}$ is called a higher order dual hesitant fuzzy element (HODHFE) denoted by $\tilde{d} = \{\tilde{h}, \tilde{g}\}$. In this regards, the HODHFS $\tilde{D}$ is also represented as

$$\tilde{D} = \{\langle x, \{\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l(x))}(x)\}, \{\tilde{g}^{(1)}(x), \ldots, \tilde{g}^{(m(x))}(x)\}\rangle | x \in X\},$$

where all $\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l(x))}(x), \tilde{g}^{(1)}(x), \ldots, \tilde{g}^{(m(x))}(x)$ are G-Type FSs on X.

As can be seen from Definition 13, an HODHFS $\tilde{D}$ expresses the membership degrees and non-membership degrees of an element by several possible G-Type FSs instead of several real numbers between 0 and 1 in DHFS. When in many real-world situations assigning exact values to the membership degrees and non-membership degrees do not describe properly the imprecise or uncertain decision information, it seems to be useful for the decision makers to rely on HODHFSs for expressing uncertainty of an element.

Example 7 Let $X = \{x_1, x_2\}$, $\{\tilde{h}(x_1), \tilde{g}(x_1)\} = \{\{[0.3, 0.4], [0.3, 0.35]\}, \{[0.3, 0.4]\}\}$, and $\{\tilde{h}(x_2), \tilde{g}(x_2)\} = \{\{[0.4, 0.6], [0.3, 0.4]\}, \{[0.1, 0.3], [0.2, 0.3]\}\}$ are the HODHFE of $x_i (i = 1, 2, 3)$ to a set $\tilde{D}$, respectively, where G-Type FSs are interval-valued fuzzy set (Turksen and Zhong 1988). Then $\tilde{D}$ can be considered as a HODHFS, i.e., $\tilde{D} = \{\langle x_1, \{[0.3, 0.4], [0.3, 0.35]\}, \{[0.3, 0.4]\}\rangle, \langle x_2, \{[0.4, 0.6], [0.3, 0.4]\}, \{[0.1, 0.3], [0.2, 0.3]\}\rangle\}$.

The example shows that the notion of dual interval-valued hesitant fuzzy set (Farhadinia 2014) are special cases of HOHFSs. A HODHFS $\tilde{D} = \{\langle x, \tilde{h}(x), \tilde{g}(x)\rangle | x \in X\}$ reduces to an dual interval-valued hesitant fuzzy set, when all G-Type FSs $\tilde{h}^{(1)}(x), \ldots, \tilde{h}^{(l(x))}(x), \tilde{g}^{(1)}(x), \ldots, \tilde{g}^{(m(x))}(x)$ for any $x \in X$ are considered as closed intervals of real numbers in [0, 1].

And a simple description of the relationship between HODHFS and the existing sets is given as follows.

$$\text{HODHFS} = \begin{cases} \text{fuzzy set}, & \text{if } \tilde{h} = \{\mu_\alpha \in [0,1]\}, \tilde{g} = \emptyset; \\ \text{intuitionistic fuzzy set}, & \text{if } \tilde{h} = \{\mu_\alpha \in [0,1]\}, \\ & \tilde{g} = \{\nu_\alpha \in [0,1]\}, \mu_\alpha + \nu_\alpha \leq 1; \\ \text{interval-valued fuzzy set}, & \text{if } \tilde{h} = \{[\mu_-, \mu_+] \subset [0,1]\}, \tilde{g} = \emptyset; \\ \text{interval-valued intuitionistic fuzzy set}, & \text{if } \tilde{h} = \{[\mu_-, \mu_+] \subset [0,1]\}, \\ & \tilde{g} = \{[\nu_-, \nu_+] \subset [0,1]\}, \\ & \mu_+ + \nu_+ \leq 1; \\ \text{hesitant fuzzy set}, & \text{if } \tilde{h} = \{\cup_{i=1,2,\ldots,n}\gamma_i | \gamma_i \in [0,1]\}, \\ & \tilde{g} = \emptyset; \\ \text{interval-valued hesitant fuzzy set}, & \text{if } \tilde{h} = \{\{[\gamma_-^i, \gamma_+^i]\} | [\gamma_-^i, \gamma_+^i] \subset [0,1]\}, \\ & \tilde{g} = \emptyset, i = 1, 2, \ldots, n; \\ \text{dual hesitant fuzzy set}, & \text{if } \tilde{h} = \{\cup_{i=1,2,\ldots,n}\gamma_i | \gamma_i \in [0,1]\}, \\ & \tilde{g} = \{\cup_{i=1,2,\ldots,n}\eta_i | \eta_i \in [0,1]\}, \gamma + \eta \leq 1; \\ \text{dual interval-valued hesitant fuzzy set}, & \text{if } \tilde{h} = \{\{[\gamma_-^i, \gamma_+^i]\} | [\gamma_-^i, \gamma_+^i] \subset [0,1]\}, \\ & \tilde{g} = \{\{[\eta_-^i, \eta_+^i]\} | [\gamma_-^i, \gamma_+^i] \subset [0,1]\}, \\ & i = 1, 2, \ldots, n, \gamma_-^i + \eta_+^i \leq 1. \end{cases} \tag{31}$$

Furthermore, among the generalization of ordinary fuzzy set (type-1 fuzzy set), the most widely used extensions are the following: type-2 fuzzy sets whose membership degrees are also fuzzy, that is, instead of being crisp values in [0, 1], the membership degrees are fuzzy sets; intuitionistic fuzzy sets extend fuzzy sets by a hesitancy function, thus the membership takes the form of an interval; the triangular fuzzy numbers extend fuzzy sets to describe the imprecise or uncertain membership degrees of an element to a given set. Thus, we can also get the concepts of type-2 DHFS, intuitionistic DHFS, etc. This implies that HODHFSs are more useful than DHFSs to deal with decision making, clustering, pattern recognition, image processing, etc., when experts have a hesitation among several possible membership degrees and non-membership degrees for an element.

4.2 Distance measures for HODHFSs

In this part, we apply our new information measures for DHFS to introducing distance measures for HODHFSs. In the following, we first give the axiomatic definition of distance measures for HOHFSs based on the analysis on distance measures for DHFS.

Definition 14 Let $\tilde{D}_1 = \{\langle x, \tilde{h}_1(x), \tilde{g}_1(x)\rangle | x \in X\}$, $\tilde{D}_2 = \{\langle x, \tilde{h}_2(x), \tilde{g}_2(x)\rangle | x \in X\}$ and $\tilde{D}_3 = \{\langle x, \tilde{h}_3(x), \tilde{g}_3(x)\rangle | x \in X\}$, be three HODHFSs on X. Then d is called a distance measure for HODHFSs if it possesses the following properties:

(D1) $0 \leq d(\tilde{D}_1, \tilde{D}_2) \leq 1$;
(D2) $d(\tilde{D}_1, \tilde{D}_2) = 0$ if and only if $\tilde{D}_1 = \tilde{D}_2$;
(D3) $d(\tilde{D}_1, \tilde{D}_2) = d(\tilde{D}_2, \tilde{D}_1)$;
(D4) $d(\tilde{D}_1, \tilde{D}_2) \leq d(\tilde{D}_1, \tilde{D}_3) + d(\tilde{D}_3, \tilde{D}_2)$.

As we have introduced HODHFEs of an HODHFS $\tilde{D} = \{\langle x, \tilde{h}(x), \tilde{g}(x)\rangle | x \in X\}$, we pay our attention to the representation of HODHFS $\tilde{D}$ based on its HODHFEs $\langle \tilde{h}(x_1), \tilde{g}(x_1)\rangle, \ldots, \langle \tilde{h}(x_n), \tilde{g}(x_n)\rangle$, i.e.,

$$\tilde{D} = \bigcup_{\langle \tilde{h}, \tilde{g}\rangle \in \tilde{D}} \{\langle \tilde{h}, \tilde{g}\rangle\} = \{\langle \tilde{h}(x_1), \tilde{g}(x_1)\rangle, \ldots, \langle \tilde{h}(x_n), \tilde{g}(x_n)\rangle\},$$

which is of fundamental importance in the study of information measures within the next part of the paper.

Hereafter, the definition of distance measure for HODHFSs is given as follows.

Definition 15 Let $\tilde{D}_1$ and $\tilde{D}_2$ be two HODHFSs on $X = \{x_1, x_2, \ldots, x_n\}$. Then the distance measure for HODHFSs is defined as

$$d(\tilde{D}_1, \tilde{D}_2) = \frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{\tilde{E}_m(x_i) + \tilde{E}_n(x_i)}{2} + \beta\frac{\tilde{S}_m(x_i) + \tilde{S}_n(x_i)}{2}\right), \tag{32}$$

where $\alpha + \beta = 1$, and for each $x_i \in X$,

$$\tilde{E}_m(x_i) = |E_m(\tilde{D}_1(x_i)) - E_m(\tilde{D}_2(x_i))|,\ \tilde{E}_n(x_i) = |E_n(\tilde{D}_1(x_i)) - E_n(\tilde{D}_2(x_i))|, \tag{33}$$

$$\tilde{S}_m(x_i) = |S_m(\tilde{D}_1(x_i)) - S_m(\tilde{D}_2(x_i))|,\ \tilde{S}_n(x_i) = |S_n(\tilde{D}_1(x_i)) - S_n(\tilde{D}_2(x_i))|, \tag{34}$$

$E_m(.,.)$ and $E_n(.,.)$ are the mean of membership degrees and non-membership degrees, respectively; $S_m(.,.)$ and $S_n(.,.)$ are the standard deviation of membership degrees and non-membership degrees, respectively.

As the membership degrees and non-membership degrees are G-Type FSs, $E_m(.,.)$, $E_n(.,.)$ and $S_m(.,.)$, $S_n(.,.)$ are the mean and the standard deviation of G-Type FSs, respectively.

Theorem 2 *Let $E_m(.,.)$, $E_n(.,.)$ and $S_m(.,.)$, $S_n(.,.)$ be the means and standard deviations of G-Type FSs which satisfy the requirements* (D1)–(D4) *listed in Definition* 14. *Then d(.,.) given by* (32) *is a distance measure for HODHFSs.*

Proof Based on Theorem 1 and Definition 15, we can easily get the theorem.

Motivated by the generalized idea provided by Yager (1988), we further extend d(.,.) given by (32) into the generalized HODHFSs distance:

$$d_\lambda(\tilde{D}_1, \tilde{D}_2) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\alpha\frac{\tilde{E}_m^\lambda(x_i) + \tilde{E}_n^\lambda(x_i)}{2} + \beta\frac{\tilde{S}_m^\lambda(x_i) + \tilde{S}_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}}.$$

In most of the real-world applications, the elements in the universe of discourse may have a different importance. This impulses us to consider the weight of each element $x_i \in X$. Assume that the weight of $x_i \in X$ is $w_i (i = 1, \ldots, n)$, and $w_i \in [0, 1]$ with $\sum_{i=1}^{n} w_i = 1$. Then, we get the generalized weighted distance for HODHFSs as follows:

$$d_{w,\lambda}(\tilde{D}_1, \tilde{D}_2) = \left[\sum_{i=1}^{n} w_i\left(\alpha\frac{\tilde{E}_m^\lambda(x_i) + \tilde{E}_n^\lambda(x_i)}{2} + \beta\frac{\tilde{S}_m^\lambda(x_i) + \tilde{S}_n^\lambda(x_i)}{2}\right)\right]^{\frac{1}{\lambda}}.$$

where $\lambda \geq 1$, $\tilde{E}_m(x_i)$, $\tilde{E}_n(x_i)$ and $\tilde{S}_m(x_i)$, $\tilde{S}_n(x_i)$ satisfy (33) and (34), respectively.

5 Applications

In what follows, we demonstrate the practicality and effectiveness of the proposed distance measures applied to higher order dual hesitant fuzzy multi-attribute decision-making problems which can be shown in the following examples.

(i) Notice that an HODHFS reduces to a DHFS, if the G-Type FSs is reduces to the special case of an ordinary FS. Now we consider a weapon selection problem in which alternatives are the weapon packages to be selected and criteria are those attributes under consideration (Wang et al. 2014).

Table 1 Higher order dual hesitant fuzzy decision matrix (Dual hesitant fuzzy decision matrix)

	G_1	G_2	G_3	G_4
A_1	{{0.5, 0.4, 0.3}, {0.4, 0.2}}	{{0.6, 0.5}, {0.3, 0.2, 0.1}}	{{0.8, 0.7, 0.6}, {0.2, 0.1, 0}}	{{0.7}, {0.2}}
A_2	{{0.8, 0.7, 0.6}, {0.2, 0.1}}	{{0.7, 0.6}, {0.3, 0.2, 0.1}}	{{0.7, 0.6, 0.5}, {0.3, 0.2, 0.1}}	{{0.6}, {0.3}}
A_3	{{0.4, 0.3, 0.2}, {0.6, 0.4}}	{{0.6, 0.5}, {0.4, 0.2, 0.1}}	{{0.5, 0.4, 0.3}, {0.4, 0.2, 0.1}}	{{0.9}, {0.1}}
A_4	{{0.4, 0.3, 0.1}, {0.6, 0.5}}	{{0.8, 0.7}, {0.2, 0.1, 0}}	{{0.6, 0.5, 0.4}, {0.3, 0.2, 0.1}}	{{0.7}, {0.3}}

Example 8 (Wang et al. 2014) A computer center desires to select a new information system to improve work productivity. After preliminary screening, four alternatives $A_i(1,2,3,4)$ have remained in the candidate list. There are four attributes $G_j(1,2,3,4)$ that need to be considered and the weight vector of the attributes is $w = \{0.15, 0.25, 0.15, 0.45\}$. One expert evaluates the weapon packages with respect to the attributes, and constructs the following dual fuzzy decision matrix R (Table 1).

Then, TOPSIS steps can be outlined as follows:

Step 1. Construct the decision matrix R: (see Wang et al. 2014).

$$R = \begin{bmatrix} \{\{0.5,0.4,0.3\},\{0.4,0.2\}\} & \{\{0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.8,0.7,0.6\},\{0.2,0.1,0\}\} & \{\{0.7\},\{0.2\}\} \\ \{\{0.8,0.7,0.6\},\{0.2,0.1\}\} & \{\{0.7,0.6\},\{0.3,0.2,0.1\}\} & \{\{0.7,0.6,0.5\},\{0.3,0.2,0.1\}\} & \{\{0.6\},\{0.3\}\} \\ \{\{0.4,0.3,0.2\},\{0.6,0.4\}\} & \{\{0.6,0.5\},\{0.4,0.2,0.1\}\} & \{\{0.5,0.4,0.3\},\{0.4,0.2,0.1\}\} & \{\{0.9\},\{0.1\}\} \\ \{\{0.4,0.3,0.1\},\{0.6,0.5\}\} & \{\{0.8,0.7\},\{0.2,0.1,0\}\} & \{\{0.6,0.5,0.4\},\{0.3,0.2,0.1\}\} & \{\{0.7\},\{0.3\}\} \end{bmatrix} \tag{35}$$

Step 2. Construct the weighted decision matrix: (see Wang et al. 2014).

$$V = \begin{bmatrix} \left\{\begin{matrix}\{0.0987,0.0738,0.0521\},\\ \{0.8716,0.0.7855\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.2047,0.1591\},\\ \{0.7401,0.6687,0.5623\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.2145,0.1652,0.1284\},\\ \{0.7855,0.7080,0\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.4183\},\\ \{0.4847\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{0.2145,0.1652,0.1284\},\\ \{0.7855,0.7080\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.2599,0.2047\},\\ \{0.7401,0.6687,0.5623\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.1652,0.1284,0.0988\},\\ \{0.8348,0.7855,0.7080\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.3379\},\\ \{0.5817\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{0.0738,0.0521,0.0329\},\\ \{0.9262,0.8716\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.2047,0.1591\},\\ \{0.7953,0.6687,0.5623\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.1284,0.0738,0.0521\},\\ \{0.8716,0.7855,0.7080\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.6452\},\\ \{0.3548\}\end{matrix}\right\} \\ \left\{\begin{matrix}\{0.0738,0.0521,0.0157\},\\ \{0.9262,0.9013\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.3313,0.2599\},\\ \{0.6687,0.5623,0\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.1284,0.0988,0.0738\},\\ \{0.8348,0.7855,0.7080\}\end{matrix}\right\} & \left\{\begin{matrix}\{0.4183\},\\ \{0.5817\}\end{matrix}\right\} \end{bmatrix} \tag{36}$$

Step 3. Determine the positive ideal and the negative ideal solutions: (see Wang et al. 2014).

$$\begin{aligned} A^+ &= \left[\left\{\begin{matrix}\max_i\{\gamma_{i1}\},\\ \min_i\{\eta_{i1}\}\end{matrix}\right\} \left\{\begin{matrix}\max_i\{\gamma_{i2}\},\\ \min_i\{\eta_{i2}\}\end{matrix}\right\} \left\{\begin{matrix}\max_i\{\gamma_{i3}\},\\ \min_i\{\eta_{i3}\}\end{matrix}\right\} \left\{\begin{matrix}\max_i\{\gamma_{i4}\},\\ \min_i\{\eta_{i4}\}\end{matrix}\right\}\right] \\ &= \left[\left\{\begin{matrix}0.2145,\\ 0.7080\end{matrix}\right\} \left\{\begin{matrix}3313,\\ 0\end{matrix}\right\} \left\{\begin{matrix}0.2145,\\ 0\end{matrix}\right\} \left\{\begin{matrix}0.6452,\\ 0.3548\end{matrix}\right\}\right] \\ A^- &= \left[\left\{\begin{matrix}\min_i\{\gamma_{i1}\},\\ \max_i\{\eta_{i1}\}\end{matrix}\right\} \left\{\begin{matrix}\min_i\{\gamma_{i2}\},\\ \max_i\{\eta_{i2}\}\end{matrix}\right\} \left\{\begin{matrix}\{\min_i\{\gamma_{i3}\},\\ \max_i\{\eta_{i3}\}\end{matrix}\right\} \left\{\begin{matrix}\min_i\{\gamma_{i4}\},\\ \max_i\{\eta_{i4}\}\end{matrix}\right\}\right] \\ &= \left[\left\{\begin{matrix}0.0157,\\ 0.9262\end{matrix}\right\} \left\{\begin{matrix}0.1591,\\ 0.7953\end{matrix}\right\} \left\{\begin{matrix}0.0521,\\ 0.8716\end{matrix}\right\} \left\{\begin{matrix}0.3379,\\ 0.5817\end{matrix}\right\}\right] \end{aligned}$$

Step 4. Measure the distance of alternatives to the positive ideal and the negative ideal points:

$$S_i^+ = \sum_{j=1}^{4} d(v_{ij}, A_j^+); \quad S_i^- = \sum_{j=1}^{4} d(v_{ij}, A_j^-);$$

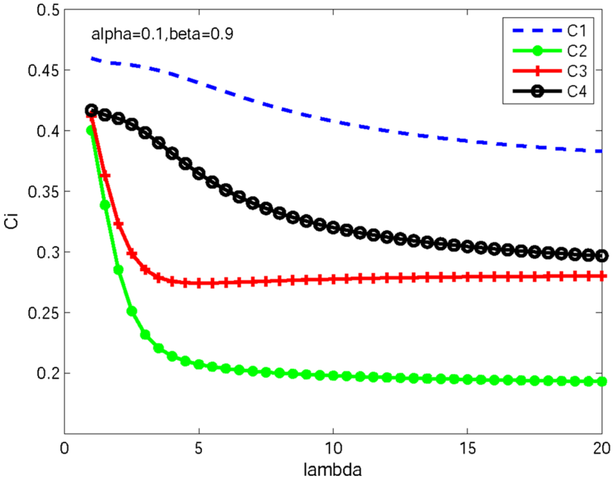

Fig. 1 The results of Ci

where $i = 1, \ldots, 4$, $A_j^+ \in A^+$, $A_j^- \in A^-$, $v_{ij} = \{\gamma_{ij}, \eta_{ij}\}$ and $v_{ij} \in V$.
Here we can use (24) and v_{ij}, A_j^+ and A_j^- are related in step 2 and step 3 in the study by Wang et al. (2014).

Step 5. Calculate the relative closeness to the ideal solution.

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}.$$

Step 6. Rating of each alternative.

As a comparison to the example in Wang et al. (2014), we consider different value of λ. We choose $\lambda = [1, 20]$, as its result is typical. Furthermore, it is possible to analyze how different values of the attitudinal characters α and β change the results as α and β reflect the different preferences of the influences of mean and standard deviation of dual hesitant fuzzy element. Then we give three figures as follows:

From Fig. 1, if $\alpha = 0.1$, $\beta = 0.9$, we can find that the ranking of the four alternatives is $A_1 \succ A_4 \succ A_3 \succ A_2$ and the best choice is A_1.

From Fig. 2, if $\alpha = 0.5$, $\beta = 0.5$, we can find that,

1. when $\lambda \in [1, 1.6088]$, the ranking of the four alternatives is $A_1 \succ A_3 \succ A_2 \succ A_4$ and the best choice is A_1;
2. when $\lambda \in (1.6088, 20]$, the ranking of the four alternatives is $A_1 \succ A_3 \succ A_4 \succ A_2$ and the best choice is A_1.

From Fig. 3, if $\alpha = 0.9$, $\beta = 0.1$, we can find that,

1. when $\lambda \in [1, 5.4231]$, the ranking of the four alternatives is $A_1 \succ A_3 \succ A_2 \succ A_4$ and the best choice is A_1;
2. when $\lambda \in (5.4231, 20]$, the ranking of the four alternatives is $A_1 \succ A_3 \succ A_4 \succ A_2$ and the best choice is A_1.

As can be seen from the results, the proposed distance measures can provide the decision makers with more choices as the different values of the parameter α (or β) are given according

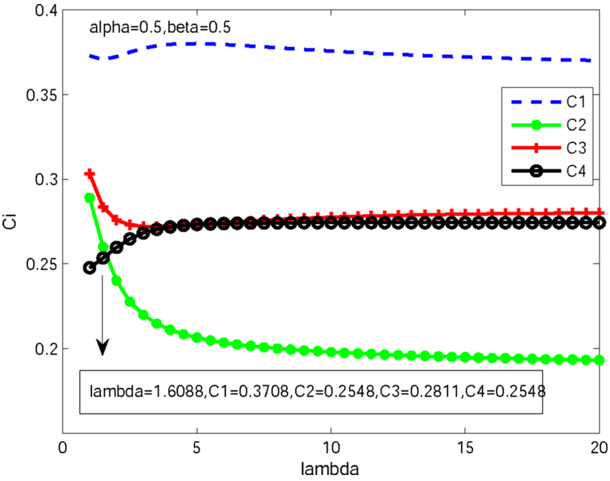

Fig. 2 The results of Ci

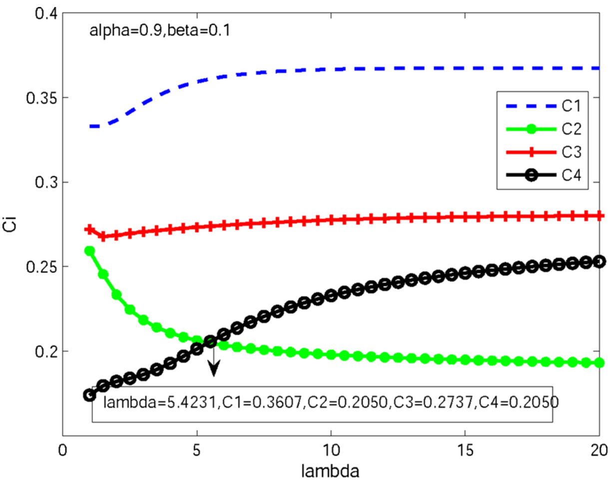

Fig. 3 The results of Ci

to the decision makers' attitudes. Compared to the results in Wang et al. (2014), the ranking of the four alternatives is different although the best choice is always A_1. It implies that our proposed distance measures are applicable and available.

On the other hand, optimists often focus on the size of the values but ignore the volatility of the values, while pessimists expect no volatility and, therefore, they focus on the volatility of the values. So, the values of the parameter α (or β) can be treated as the optimistic or pessimistic levels. According to Figs. 1, 2 and 3, we conclude that the decision makers who have a positive perception of the prospects could choose a larger value for the parameter α, whereas the decision makers who are pessimistic could choose a smaller value for the parameter α.

Now, we analyze how different values of the parameter λ change the results of $C_i (i = 1, 2, 3, 4)$ from Figs. 1, 2 and 3. When λ is assigned different values between 0 and 20,

Table 2 Higher order dual hesitant fuzzy decision matrix (dual interval-valued hesitant fuzzy decision matrix)

	G_1	G_2
A_1	{{[0.5, 0.4], [0.3, 0.1]}, {[0.3, 0.2]}}	{{[0.6, 0.5]}, {[0.1, 0]}}
A_2	{{[0.8, 0.7], [0.6, 0.4]}, {{[0.1, 0]}}	{{[0.7, 0.6]}, {[0.3, 0.2]}}
A_3	{{[0.4, 0.3]}, {{[0.6, 0.4]}}	{{[0.6, 0.5]}, {[0.4, 0.1]}}
A_4	{{[0.4, 0.3], [0.1, 0]}, {{[0.6, 0.5]}}	{{[0.8, 0.7]}, {[0.2, 0.1]}}
	G_3	G_4
A_1	{{[0.7, 0.5]}, {[0.2, 0.1], [0.3, 0]}}	{{[0.4, 0.4]}, {[0.2, 0.1]}}
A_2	{{[0.8, 0.7], [0.6, 0.4}, {[0.2, 0.1]}}	{{[0.6, 0.6]}, {[0.3, 0.2]}}
A_3	{{[0.5, 0.4], [0.3, 0.3]}, {[0.4, 0.1]}}	{{[0.9, 0.8]}, {[0.1, 0]}}
A_4	{{[0.6, 0.2], [0.5, 0.4]}, {[0.3, 0.2]}}	{{[0.7, 0.4]}, {[0.3, 0]}}

Fig. 1 demonstrates that all of the values obtained with $C_i(i = 1, 2, 3, 4)$ decrease, but the values obtained with $C_i(i = 1, 2, 3, 4)$ in Figs. 2 and 3 do not always decrease, which come to a conclusion that the monotonicity of $C_i(i = 1, 2, 3, 4)$ is related to α (or β). For example, with the increase of λ, when $\alpha = 0.1$, C_4 is decreasing but when $\alpha = 0.9$, C_4 is increasing. The reason for this result is that the different preferences of the mean and standard deviation have a great influence on the value of d_{wdpg}, which is used to calculate the values of $C_i(i = 1, 2, 3, 4)$. But when λ ($\lambda \geq 1$) continues to increase, we find that the values of $C_i(i = 1, 2, 3, 4)$ tend to be stable regardless of the size of α (or β). In fact, it is easy to prove that the limits of $C_i(i = 1, 2, 3, 4)$ exist and have nothing to do with the parameter α (or β).

(ii) As discussed previously in Sect. 4, it is difficult for the decision makers to provide exact values for the membership degrees and of non-membership degrees an element to a given set like those values considered in the form of DHFEs in Zhu et al. (2012). One way to overcome this difficulty is to describe the membership degrees and non-membership degrees by a HODHFE in which the membership degrees and non-membership degrees are considered as fuzzy sets. Next, we examine again the problem discussed in the example, but with a higher order hesitant fuzzy decision matrix in which G-Type FSs are in the form of closed intervals.

Example 9 Consider the multi-attribute decision-making problem in Example 8. Suppose that all possible evaluations for an alternative under the attributes are contained in a HOHFS. The results evaluated by the decision makers are the elements of a higher order dual hesitant fuzzy decision matrix, shown as follows (Table 2).

We let the full HODHFS $\tilde{D}^* = \{\langle x, \{[1, 1]\}, \{[0, 0]\}\rangle | x \in X\}$ be the ideal alternative. Using the generalized weighted distance for HODHFSs to calculate the deviations between each alternative and the ideal alternative $\tilde{D}^*$, the ranking of all alternatives can be obtained. For example, the deviation between the alternative $\tilde{D}_i$ and the ideal alternative $\tilde{D}^*$ is calculated as follows:

$$d_{w,\lambda}(\tilde{D}_i, \tilde{D}^*) = \left[\sum_{j=1}^{n} w_j \left(\alpha \frac{\tilde{E}_m^{\lambda}(x_{ij}) + \tilde{E}_n^{\lambda}(x_{ij})}{2} + \beta \frac{\tilde{S}_m^{\lambda}(x_{ij}) + \tilde{S}_n^{\lambda}(x_{ij})}{2}\right)\right]^{\frac{1}{\lambda}}.$$

Table 3 Results of $d_{\lambda,w}(\tilde{D}_i, \tilde{D}^*)$

	A_1	A_2	A_3	A_4	Rankings
$\lambda = 4$	0.4015	0.2786	0.3651	0.4091	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\lambda = 6$	0.4600	0.3064	0.4285	0.4860	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\lambda = 8$	0.4955	0.3235	0.4685	0.5401	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\lambda = 10$	0.5199	0.3350	0.4961	0.5800	$A_2 \succ A_3 \succ A_1 \succ A_4$
$\lambda = 20$	0.5800	0.3622	0.5622	0.6789	$A_2 \succ A_3 \succ A_1 \succ A_4$

where

$$\tilde{E}_m(x_{ij}) = |E_m(\tilde{D}_i(x_{ij})) - E_m(\tilde{D}^*(x_{ij}))|, \tilde{E}_n(x_{ij}) = |E_n(\tilde{D}_i(x_{ij})) - E_n(\tilde{D}^*(x_{ij}))|,$$
$$\tilde{S}_m(x_{ij}) = |S_m(\tilde{D}_i(x_{ij})) - S_m(\tilde{D}^*(x_{ij}))|, \tilde{S}_n(x_{ij}) = |S_n(\tilde{D}_i(x_{ij})) - S_n(\tilde{D}^*(x_{ij}))|.$$

As a special case of G-Type FSs, the mean of a set of closed interval $\{[a_k, b_k]\}(k = 1, 2, \ldots, n)$ is defined as

$$E(\cup[a_k, b_k]) = \frac{1}{n}\sum_{k=1}^{n}\frac{a_k + b_k}{2};$$

and the variance of the set of closed interval is defined as

$$S(\cup[a_k, b_k]) = \sqrt{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{a_k + b_k - 2E(\cup[a_k, b_k])}{2}\right)^2}.$$

For convenience, we choose $\alpha = \beta = 0.5$. Then, the deviation between the other alternatives $\tilde{D}_i (i = 1, 2, 3, 4)$ and the ideal alternative $\tilde{D}^*$ are obtained as (Table 3)

As the parameter λ changes, we obtain different results. From the results, we can see that the rankings obtained by the distance measures $d_{\lambda,w}(\tilde{D}_i, \tilde{D}^*)$ and the arguments are kept fixed. The decision makers can choose the value of λ according to their preferences.

Furthermore, it is possible to analyze how different values of the attitudinal character λ change the results. To obtain the more specific results, we give an image which can reflect the changes of results with λ shown in Fig. 4.

From Fig. 4, we can find that,

1. when $\lambda \in [1, 1.3060]$, the ranking of the four alternatives is $A_3 \succ A_2 \succ A_1 \succ A_4$;
2. when $\lambda \in (1.3060, 20]$, the ranking of the four alternatives is $A_2 \succ A_3 \succ A_1 \succ A_4$.

6 Conclusion

In this study, we first review some distance measures of DHFSs and give some drawbacks about the information measures, based on which we propose some new distance measures for DHFSs in terms of the mean, standard deviation of dual hesitant fuzzy element, respectively. Meanwhile, we extend the DHFS to its higher order type and refer to it as the HODHFS. HODHFS is the actual extension of DHFS encompassing not only fuzzy sets, intuitionistic fuzzy sets, Type-2 fuzzy sets, hesitant fuzzy sets, but also the recent extension of DHFS, called interval-valued hesitant fuzzy sets. The rationale behind HODHFS can be seen in the case that the decision makers are not satisfied by providing exact values for the membership degrees

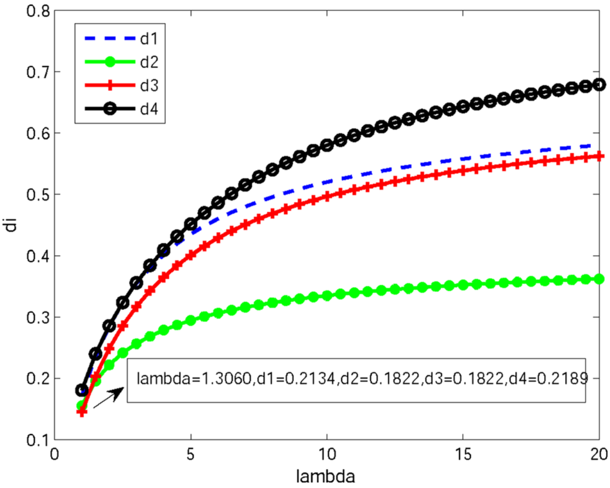

Fig. 4 The results of *di*

and the non-membership degrees. And we also developed a series of distance measures for HODHFSs and employed them to solve the higher order hesitant fuzzy multi-attribute decision-making problems. In the future, we may consider the study of aggregation operators in the higher order dual hesitant fuzzy set for handling multiple attribute decision making with higher order dual hesitant fuzzy information.

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