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On the periodic solutions of the Michelson continuous and discontinuous piecewise linear differential system

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Abstract Applying new results from the averaging theory for continuous and discontinuous differential systems, we study the periodic solutions of two distinct versions of the Michelson differential system: a Michelson continuous piecewise linear differential system and a Michelson discontinuous piecewise linear differential system. The tools here used can be applied to general nonsmooth differential systems.

Keywords Continuous piecewise linear differential systems · Discontinuous piecewise linear differential systems · Michelson system · Averaging theory · Periodic solutions

Mathematics Subject Classification 34C29 · 34C25 · 37G15

1 Introduction and statement of the main results

Continuous and discontinuous piecewise linear differential systems appear in a natural way in many areas of applied and sciences, for instance, in control theory, in the study of electrical circuits, and in mechanical problems, see the hundreds of references which appear in the book di Bernardo et al. (2008) and in the survey Makarenko and Lamb (2012). Here, we shall study

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the periodic solutions of continuous and discontinuous piecewise linear differential systems using new results on the averaging theory developed for these nonsmooth dynamical systems.

The applications of these new results on averaging theory for studying the periodic solutions of nonsmooth dynamical systems required convenient changes of variable for writing the initial differential system in the normal form of the averaging theory. We shall do these applications to a version of a continuous and a discontinuous piecewise linear differential systems coming from the Michelson differential system. Such piecewise linear differential systems also have been studied by other authors, see Carmona et al. (2012, 2014, 2010, 2015, 2008). We must mention that the ideas and tools used for doing these applications of the averaging theory for studying the periodic solutions of these two Michelson nonsmooth dynamical systems can be applied to general nonsmooth dynamical systems.

The Michelson differential system is given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= c^2 - y - \frac{x^2}{2}, \end{aligned} \tag{1}$$

with $(x, y, z) \in \mathbb{R}^3$ and the parameter $c \geq 0$. The dot denotes derivative with respect to an independent variable t, usually called the time. This system is due to Michelson (1986) for studying the traveling solutions of the Kuramoto–Sivashinsky equation. It also arises in the analysis of the unfolding of the nilpotent singularity of codimension three, see Dumortier et al. (2001) and Freire et al. (2002).

This system has been largely investigated from the dynamical point of view. Michelson (1986) proved that if $c > 0$ is sufficiently large, then system (1) has a unique bounded solution which is a transversal heteroclinic orbit connecting the two finite singularities $(-\sqrt{2}c, 0, 0)$ and $(\sqrt{2}c, 0, 0)$. When c decreases, there will appear a cocoon bifurcation (see Kokubu et al. 2007; -T Lau 1992; Michelson 1986). In Llibre and Zhang (2011), there is an analytical proof of the existence of a zero-Hopf bifurcation for system (1).

In Carmona et al. (2008, 2010), the authors consider a continuous piecewise linear version of Michelson differential system changing the non linear function x^2 in (1) by the piecewise linear function $|x|$. For such system, they proved that some dynamical aspects of the Michelson system remain as the existence of a reversible T-point heteroclinic cycle.

Doing the change of variable $(x, y, z, c) \rightarrow (2\varepsilon X, 2\varepsilon Y, 2\varepsilon Z, 2\varepsilon d)$ with $d \geq 0$ and $\varepsilon > 0$ sufficiently small to the Michelson differential system (1), followed by the change of the function $X^2 \rightarrow |X|$, and denoting again X, Y, and Z by x, y, and z, we obtain the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x|), \end{aligned} \tag{2}$$

that we call the *Michelson continuous piecewise linear differential system*. We note that this system is reversible, because it is invariant under the change of variables $(x, y, z, t) \mapsto (-x, y, -z, -t)$.

Many problems in physics, economics, biology, and applied areas are modeled by discontinuous differential systems, but there exist only few analytical techniques for studying their periodic solutions. In Llibre et al. (2015), the authors extended the averaging theory to the discontinuous differential systems. An improvement of this result for a much bigger class of discontinuous differential systems is given in Llibre and Novaes (2016).

If in the continuous Michelson differential system (1), we change the continuous function $|x|$ by the discontinuous one $|x| + \text{sign}(x)$, where

$$\text{sign}\, x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

we obtain the *Michelson discontinuous piecewise linear differential system* given by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -y + \varepsilon(2d^2 - |x| - \text{sign}\, x). \end{aligned} \tag{3}$$

In this paper, first, we study analytically the periodic solutions of the Michelson continuous and discontinuous piecewise linear differential systems. Thus, our first main result is the following.

Theorem 1 *For all $d > 0$ and $\varepsilon = \varepsilon(d) > 0$ sufficiently small, the Michelson continuous piecewise linear differential system (2) has a periodic solution of the form:*

$$x(t) = -\pi\, d^2 + \mathcal{O}(\varepsilon), \quad y(t) = \pi\, d^2 \sin t + \mathcal{O}(\varepsilon), \quad z(t) = \pi\, d^2 \cos t + \mathcal{O}(\varepsilon).$$

Moreover, this periodic solution is linearly stable.

We recall that a periodic solution is *asymptotically stable* if all the eigenvalues corresponding to the fixed point of the Poincaré map associated with this solution have negative real part, then this periodic solution is locally asymptotically stable. If one of the eigenvalues has positive real part, the periodic solution is *unstable*. If all the eigenvalues have zero real parts, then we say that the periodic solution is *linearly stable*; in this case, the linear stability does not provide any information on the kind of stability that the periodic solution has when we take into account the nonlinear terms, for more details, see Theorem 11.6 of Verhulst (2000).

Note that the periodic orbit obtained in our Theorem 1 is not reversible, because their dominant terms $(-d^2\pi\,,\ d^2\pi \sin t\,,\ d^2\pi \cos t)$ are not invariant under the change of variables $(x, y, z, t) \mapsto (-x, y, -z, -t)$. Therefore, this periodic orbit has no relation with the periodic orbit of the noose bifurcation studied by the Michelson continuous piecewise linear differential system (1.2) of Carmona et al. (2015) which is reversible. We also note that our Michelson continuous piecewise linear differential system (2) and system (1.2) of Carmona et al. (2015) do not coincide.

Theorem 1 is proved in Sect. 3. Its proof uses an extension of the classical averaging theory for smooth differential systems to continuous differential systems given in Buica et al. (2007) or Llibre et al. (2014).

Theorem 2 *For $\varepsilon > 0$ sufficiently small, the Michelson discontinuous piecewise linear differential system (3) satisfies the following statements.*

(a) *If $(-1 + 2d^2)\pi < 0$, then system (3) has two periodic solutions $(x(t, \varepsilon), r(t, \varepsilon), \theta(t, \varepsilon))$ of the form:*

$$(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon)) = (x_0, r_0 \sin t, r_0 \cos t) + \mathcal{O}(\varepsilon), \tag{4}$$

where

$$r_0 = \frac{2\sqrt{1 - a^2}}{a\sqrt{1 - a^2} + \arcsin\, a}, \qquad x_0 = -r_0(1 + a),$$

and a takes the value of the two unique zeros of the function

$$g(a) = \frac{2a^2 - 2 + \pi a\sqrt{1-a^2}d^2 + \arcsin a\left(\pi d^2 + \arcsin a - a\sqrt{1-a^2}\right)}{a\sqrt{1-a^2} + \arcsin a},$$

in the interval $(-1, 1)$.

(b) *If* $(-1+2d^2)\pi > 0$*, then system* (3) *has a periodic solution of the form* (4) *given by the unique zero of the function* $g(a)$ *in the interval* $(-1, 1)$.

Theorem 2 is proved in Sect. 4.

2 Preliminary

For proving Theorems 1 and 2, we apply two recent results from the averaging theory: one for the continuous piecewise linear differential systems and the other for the discontinuous piecewise linear differential systems. In this section, we present these results and some remarks necessary for their applications.

2.1 Continuous piecewise linear differential systems

From Theorem B of Llibre et al. (2014), we get the following result adapted to the next system (5), see also Theorems 11.5 and 11.6 of Verhulst (2000).

Theorem 3 *Consider the following system:*

$$\dot{x}(t) = F_0(t, x) + \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{5}$$

where $D \subset \mathbb{R}^n$ is an open subset, ε is a small parameter, the functions $F_i : \mathbb{R} \times D \to \mathbb{R}^n$ for $i = 0, 1$ and $R : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are T periodic in the variable t, and for each $t \in \mathbb{R}$, the functions $F_0(t, .) \in C^1$, $F_1(t, .) \in C^0$, $D_x F_0$, and $R \in C^0$ are locally Lipschitz in the second variable. We denote by $x(t, z, \varepsilon)$ the solution of system (5)*, such that $x(0, z, \varepsilon) = z$. Assume that there exists an open and bounded subset V of D with its closure $\overline{V} \subset D$, such that for each $z \in \overline{V}$, the solution $x(t, z, 0)$ is T periodic. We denote by $M_z(t)$ the fundamental matrix solution of the variational equation:*

$$\dot{x}(t) = D_x F_0(t, x(t, z, 0)),$$

associated with the periodic solution $x(t, z, 0)$, such that $M_z(0)$ is the identity.

If $a \in V$ is a zero of the map $\mathcal{F} : \overline{V} \to \mathbb{R}^n$ defined by

$$\mathcal{F}(z) = \int_0^T M_z^{-1}(t) F_1(t, x(t, z))\mathrm{d}t$$

and $\det(D_z\mathcal{F}(a)) \neq 0$, then for $\varepsilon > 0$ sufficiently small, system (5) *has a T-periodic solution $x(t, a_\varepsilon, \varepsilon)$, such that $a_\varepsilon \to a$ as $\varepsilon \to 0$. Moreover, the linear stability type of the periodic solution $x(t, a_\varepsilon, \varepsilon)$ is given by the eigenvalues of the matrix $D_z\mathcal{F}(a)$.*

Note that the stability of the periodic solutions of system (5), when it is applied to the Michelson continuous piecewise linear differential system, can be obtained from the stability of a differential system associated with it. In fact given the continuous system (1), consider a band of amplitude $\varepsilon > 0$ around the plane $x = 0$ and a differentiable extension of the continuous system (1) to this band. Studying the limit of this extended differentiable system when $\varepsilon \to 0$, we conclude that the linear stability of system (1) is given by the eigenvalues of $D_z\mathcal{F}(a)$.

2.2 Discontinuous piecewise linear differential systems

Let $D \subset \mathbb{R}^n$ be an open subset and $h : \mathbb{R} \times D \to \mathbb{R}$ a C^1 function having 0 as regular value. Consider $F^1, F^2 : \mathbb{R} \times D \to \mathbb{R}^n$ continuous functions and $\Sigma = h^{-1}(0)$. We define the Filippov's system as

$$\dot{x}(t) = F(t,x) = \begin{cases} F^1(t,x) & \text{if } (t,x) \in \Sigma^+, \\ F^2(t,x) & \text{if } (t,x) \in \Sigma^-, \end{cases} \tag{6}$$

where $\Sigma^+ = \{(t,x) \in \mathbb{R} \times D : h(t,x) > 0\}$ and $\Sigma^- = \{(t,x) \in \mathbb{R} \times D : h(t,x) < 0\}$.

The manifold Σ is divided in the closure of two disjoint regions, namely, *Crossing region* (Σ^c) and *Sliding region* (Σ^s):

$$\Sigma^c = \{p \in \Sigma : \langle \nabla h(p), (1, F^1(p))\rangle \cdot \langle \nabla h(p), (1, F^2(p))\rangle > 0\},$$
$$\Sigma^s = \{p \in \Sigma : \langle \nabla h(p), (1, F^1(p))\rangle \cdot \langle \nabla h(p), (1, F^2(p))\rangle < 0\}.$$

Consider the differential system associated with system (6):

$$\dot{x}(t) = F(t,x) = \chi_+(t,x)F^1(t,x) + \chi_-(t,x)F^2(t,x), \tag{7}$$

where χ_+, χ_- are the *characteristic functions* defined as

$$\chi_+(t,x) = \begin{cases} 1 & \text{if } h(t,x) > 0, \\ 0 & \text{if } h(t,x) < 0. \end{cases}$$

and

$$\chi_-(t,x) = \begin{cases} 0 & \text{if } h(t,x) > 0, \\ 1 & \text{if } h(t,x) < 0. \end{cases}$$

Systems (6) and (7) do not coincide in Σ, but applying the Filippov's convention for the solutions of systems (6) and (7) (see Filippov 1988) passing through a point $(t,x) \in \Sigma$, we have that these solutions do not depend on the value of $F(t,x)$, so the solutions are the same.

Let $\mathcal{P}$ be the space formed by the periodic solutions of (7). If $\dim \mathcal{P} = \dim D = d$, then the following result follows directly from Theorem B of Llibre and Novaes (2016).

Theorem 4 *Consider the differential system:*

$$\dot{x}(t) = F_0(t,x) + \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon). \tag{8}$$

where

$$F_i(t,x) = \chi_+ F_i^1(t,x) + \chi_- F_i^2(t,x), \text{ for } i = 0, 1, \text{ and}$$
$$R(t,x) = \chi_+ R^1(t,x) + \chi_- R^2(t,x),$$

with $F_i^1 \in C^1$, for $i = 0, 1$, and R^1, R^2 are continuous functions which are Lipschitz in the second variable, and all these functions are T-periodic functions in the variable $t \in \mathbb{R}$.

For $z \in D$ and $\varepsilon > 0$ sufficiently small, denote by $x(t,z,\varepsilon)$ the solution of system (8), such that $x(0,z,\varepsilon) = z$.

Define the averaged function

$$\mathcal{F}(z) = \int_0^T M(s,z)^{-1} F_1(s, x(s,z,0))\mathrm{d}s$$

where $x(s,z,0)$ *is a periodic solution of* (8) *with* $\varepsilon = 0$*, such that* $x(0,z,0) = z$*, and* $M(s,z)$ *is the fundamental matrix of the variational system* $\dot{y} = D_x F_0(t, x(t,z,0))y$ *associated with the unperturbed system evaluated on the periodic solution* $x(s,z,0)$*, such that* $M(0,z) = Id$*. Moreover, we assume the following hypotheses.*

(H_-) There exists an open bounded subset $C \subset D$*, such that, for* ε *sufficiently small, every orbit starting in* C *reaches the set of discontinuity only at its crossing region.*
(H_+) For $a \in C$ *with* $\mathcal{F}(a) = 0$*, there exists a neighborhood* $U \subset C$ *of* a*, such that* $\mathcal{F}(z) \neq 0$*, for all* $z \in \bar{U}\backslash\{a\}$ *and* $\det(D_z\mathcal{F}(a)) \neq 0$.

Then, for $\varepsilon > 0$ *sufficiently small, there exists a* T*-periodic solution* $x(t, a_\varepsilon, \varepsilon)$ *of* (8)*, such that* $a_\varepsilon \to a$ *as* $\varepsilon \to 0$*. Moreover, the linear stability of the periodic solution* $x(t,\varepsilon)$ *is given by the eigenvalues of the matrix* $D_z\mathcal{F}(a)$.

The same arguments for computing the kind of stability of the obtained periodic solution from the eigenvalues of the Jacobian matrix for the continuous piecewise differential systems also work for the discontinuous piecewise differential systems.

3 Proof of Theorem 1

Doing to the Michelson continuous piecewise linear differential system (2) the change to cylindrical coordinates $x = x$, $y = r\sin\theta$ and $z = r\cos\theta$, the system becomes

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon(2d^2 - |x|)\cos\theta, \\ \dot{\theta} &= 1 - \frac{\varepsilon}{r}(2d^2 - |x|)\sin\theta. \end{aligned} \tag{9}$$

Taking θ as the new independent variable, we can write the previous differential system as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\theta} &= x' = r\sin\theta + \varepsilon(2d^2 - |x|)\sin^2\theta + \mathcal{O}(\varepsilon^2), \\ \frac{\mathrm{d}r}{\mathrm{d}\theta} &= r' = \varepsilon(2d^2 - |x|)\cos\theta + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{10}$$

The unperturbed system is

$$\begin{aligned} x' &= r\sin\theta, \\ r' &= 0. \end{aligned} \tag{11}$$

For each (x_0, r_0), the solution $(x(\theta, x_0, r_0), r(\theta, x_0, r_0))$, such that $x(0, x_0, r_0), r(0, x_0, r_0))$ $= (x_0, r_0)$ is

$$(x(\theta, x_0, r_0), r(\theta, x_0, r_0)) = (x_0 + r_0(1 - \cos\theta), r_0),$$

which is 2π periodic for all $(x_0, r_0) \neq (0, 0)$. When $r_0 = 0$, we have a straight line of equilibrium points.

Now, note that the function $F_0(\theta, (x, r)) = (r \sin\theta, 0)$ is C^∞ and in particular C^1 and that the function

$$F_1(\theta, (x, r)) = (2d^2 - |x|)(\sin^2\theta, \cos\theta)$$

is C^0, and both are Lipschitz. Therefore, the differential system (9) satisfies the assumptions of Theorem 3. Then, by Theorem 3, we need to calculate the averaged function:

$$\mathcal{F}(x_0, r_0) = \int_0^{2\pi} M(\theta)^{-1} F_1(\theta, x(\theta, (x_0, r_0)))\mathrm{d}\theta,$$

where

$$M(\theta) = \begin{pmatrix} 1 & 1 - \cos\theta \\ 0 & 1 \end{pmatrix}$$

is the fundamental matrix of the variational differential system associated with system (11) evaluated on the periodic solution $(x_0 + r_0(1 - \cos\theta), r_0)$, such that $M(0)$ is the identity matrix. Therefore, we have

$$\begin{aligned} \mathcal{F}(x_0, r_0) &= \int_0^{2\pi} (2d^2 - |x_0 + r_0(1 - \cos\theta)|) \begin{pmatrix} 1 & \cos\theta - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta \\ &= \int_0^{2\pi} (2d^2 - |x_0 + r_0(1 - \cos\theta)|) \begin{pmatrix} 1 - \cos\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta \\ &= \int_0^{2\pi} g(\theta)\mathrm{d}\theta. \end{aligned}$$

Note that $g(\theta) = g(-\theta)$ and $g(\theta)$ are 2π periodic. Therefore

$$\int_0^{2\pi} g(\theta)\mathrm{d}\theta = \int_{-\pi}^{\pi} g(\theta)\mathrm{d}\theta = 2\int_0^{\pi} g(\theta)\mathrm{d}\theta.$$

To calculate this integral, we need to study the zeros of the function $G(\theta) = x_0 + r_0(1 - \cos\theta)$.

As $G(\theta) = 0$ if and only if $\theta = \pm \arccos(\frac{x_0 + r_0}{r_0})$ and the function $\arccos(x)$ takes real values when $x \in [-1, 1]$, we have to consider the following three cases.

Case 1 $x_0 \le -2r_0$ or equivalently $\frac{x_0+r_0}{r_0} \le -1$. Then, $r_0 + x_0 - r_0\cos\theta \le 0$ in $[0, \pi]$.
Case 2 $x_0 \in (-2r_0, 0)$ or equivalently $|\frac{x_0+r_0}{r_0}| < 1$. Then

(i) $r_0 + x_0 - r_0\cos\theta < 0$ if $\theta \in (0, \arccos(\frac{r_0+x_0}{r_0}))$;
(ii) $r_0 + x_0 - r_0\cos\theta > 0$ if $\theta \in (\arccos(\frac{r_0+x_0}{r_0}), \pi)$.

Case 3 $x_0 \ge 0$ or equivalently $\frac{x_0+r_0}{r_0} \ge 1$. Then, $r_0 + x_0 - r_0\cos\theta \ge 0$ in $[0, \pi]$.

In the computation of the integral $\int_0^\pi g(\theta)\mathrm{d}\theta$, we distinguish the previous three cases.
Case 1 In this case, the averaged function is

$$\begin{aligned} \mathcal{F}(x_0, r_0) &= 2\int_0^{2\pi} (2d^2 - |x_0 + r_0(1 - \cos\theta)|) \begin{pmatrix} 1 - \cos\theta \\ \cos\theta \end{pmatrix} \mathrm{d}\theta \\ &= 2\pi(4d^2 + 3r_0 + 2x_0, -r_0), \end{aligned}$$

whose unique zero is $(x_0, r_0) = (-2d^2, 0)$. Since this initial condition corresponds to an equilibrium point of the unperturbed system (11), the averaging theory in this case does not provide periodic solutions.

Case 3 Analogously to Case 1, we have

$$\begin{aligned}\mathcal{F}(x_0, r_0) &= 2\int_0^{2\pi} (2d^2 - |x_0 + r_0(1-\cos\theta)|)\begin{pmatrix}1-\cos\theta\\ \cos\theta\end{pmatrix}\mathrm{d}\theta\\ &= 2\pi(4d^2 - 3r_0 - 2x_0, r_0),\end{aligned}$$

whose unique zero is $(x_0, r_0) = (2d^2, 0)$. The conclusion follows as in Case 1.

Case 2 Here

$$\begin{aligned}\mathcal{F}(x_0, r_0) &= 2\int_0^{\arccos\left(\frac{r_0+x_0}{r_0}\right)} (2d^2 + (x_0 + r_0(1-\cos\theta)))\begin{pmatrix}1-\cos\theta\\ \cos\theta\end{pmatrix}\mathrm{d}\theta\\ &\quad +2\int_{\arccos\left(\frac{r_0+x_0}{r_0}\right)}^{\pi} (2d^2 - (x_0 + r_0(1-\cos\theta)))\begin{pmatrix}1-\cos\theta\\ \cos\theta\end{pmatrix}\mathrm{d}\theta\\ &= \begin{pmatrix}f_1(x_0, r_0)\\ f_2(x_0, r_0)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}f_1(x_0, r_0) &= 4\pi d^2 - \frac{\sqrt{-x_0(2r_0+x_0)}}{r_0}(6r_0 + 2x_0)\\ &\quad -2(3r_0 + 2x_0)\arcsin\left(\frac{r_0+x_0}{r_0}\right),\\ f_2(x_0, r_0) &= 2(r_0 + x_0)\frac{\sqrt{-x_0(2r_0+x_0)}}{r_0}\\ &\quad +2r_0\arcsin\left(\frac{r_0+x_0}{r_0}\right).\end{aligned}$$

To solve the system $f_1(x_0, r_0) = f_2(x_0, r_0) = 0$, we do the change of variables $x_0 \to X$, where $x_0 = -r_0 - Xr_0$ with $-1 < X < 1$, recall that $|\frac{x_0+r_0}{r_0}| < 1$. Then, the system becomes

$$\begin{aligned}2\pi d^2 + \left((X-2)\sqrt{1-X^2} + (1-2X)\arcsin X\right) r_0 = 0,\\ r_0\left(X\sqrt{1-X^2} + \arcsin X\right) = 0.\end{aligned}$$

Since r_0 must be positive, from the second equation, it follows that $X = 0$, and from the first that $r_0 = \pi d^2$. Therefore, we have the solution $(x_0, r_0) = (-\pi d^2, \pi d^2)$.

The Jacobian of the map (f_1, f_2) evaluated at $(x_0, r_0) = (-\pi d^2, \pi d^2)$ is 4. It follows from Theorem 3 and for any $d > 0$ and $\varepsilon = \varepsilon(d) > 0$ sufficiently small that system (10) has a periodic solution $\varphi(\theta, \varepsilon) = (x(\theta, \varepsilon), r(\theta, \varepsilon)) = (-d^2\pi + \mathcal{O}(\varepsilon), d^2\pi + \mathcal{O}(\varepsilon))$. Moreover, the eigenvalues of the Jacobian matrix of the map (f_1, f_2) at the solution $(-d^2\pi, d^2\pi)$ are $\pm 2i$, so the periodic solution is linearly stable.

Now, we must identify the periodic solution of system (2) which corresponds to the periodic solution found. Going back to system (9) with the independent variable t, we obtain the periodic solution:

$$(x(t, \varepsilon), r(t, \varepsilon), \theta(t, \varepsilon)) = (-d^2\pi, d^2\pi, t(\mathrm{mod}\, 2\pi)) + \mathcal{O}(\varepsilon).$$

Finally, coming back to system (2), we find the periodic solution:

$$(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon)) = (-d^2\pi\ ,\ d^2\pi \sin t\ ,\ d^2\pi \cos t) + \mathcal{O}(\varepsilon).$$

This concludes the proof of Theorem 1.

4 Proof of Theorem 2

Doing the change to cylindrical coordinates $x = x$, $y = r\sin\theta$, and $z = r\cos\theta$, the Michelson discontinuous piecewise linear differential system becomes

$$\begin{aligned} \dot{x} &= r\sin\theta, \\ \dot{r} &= \varepsilon(2d^2 - |x| - \operatorname{sign} x)\cos\theta, \\ \dot{\theta} &= 1 - \frac{\varepsilon}{r}(2d^2 - |x| - \operatorname{sign} x)\sin\theta. \end{aligned} \tag{12}$$

Now, taking as new independent variable, the angle θ we get the system:

$$\begin{aligned} x' &= r\sin\theta + \varepsilon(2d^2 - |x| - \operatorname{sign} x)\sin^2\theta + \mathcal{O}(\varepsilon^2), \\ r' &= \varepsilon(2d^2 - |x| - \operatorname{sign} x)\cos\theta + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{13}$$

where the prime denotes the derivative with respect to θ. This differential system satisfies the assumptions of Theorem 4, so we shall apply it for finding some of its periodic solutions. Using the notation of Theorem 4, we have that

$$F_1(\theta, x, r) = (2d^2 - |x| - \operatorname{sign} x)\begin{pmatrix} \sin^2\theta \\ \cos\theta \end{pmatrix}.$$

As in Theorem 1, the unperturbed system is given by (11) and the fundamental matrix is $M(\theta) = \left(\begin{smallmatrix} 1 & 1-\cos\theta \\ 0 & 1 \end{smallmatrix}\right)$. Then, by Theorem 4, we need to calculate

$$\mathcal{F}(x_0, r_0) = \int_0^{2\pi} M(\theta)^{-1} F_1(\theta, x(\theta, (x_0, r_0)), r(\theta, (x_0, r_0)))\mathrm{d}\theta = \int_0^{2\pi} g(\theta)\mathrm{d}\theta.$$

where

$$g(\theta) = \left(2d^2 - |x_0 + r_0(1-\cos\theta)| - \operatorname{sign}(x_0 + r_0(1-\cos\theta))\right)\begin{pmatrix} 1-\cos\theta \\ \cos\theta \end{pmatrix}.$$

Since $g(\theta)$ is 2π-periodic and $g(\theta) = g(-\theta)$, then

$$\int_0^{2\pi} g(\theta)\mathrm{d}\theta = 2\int_0^{\pi} g(\theta)\mathrm{d}\theta.$$

As in the study of the continuous differential system, we separate the calculation of the averaged function corresponding to the discontinuous system (13) in the same three cases that appear in the proof of Theorem 1.

Case 1 In this subcase, $r_0 + x_0 - r_0\cos\theta < 0$ in $[0, \pi]$. Then, the averaged function is

$$\mathcal{F}(x_0, r_0) = \pi\left(2 + 4d^2 + 3r_0 + 2x_0, r_0\right).$$

This function has no zeros with $r_0 > 0$, so the averaging theory in this case does not detect any periodic solution.

Case 2 Now, we have

$$\mathcal{F}(x_0, r_0) = \pi(-2 + 4d^2 - 3r_0 - 2x_0, r_0).$$

The conclusion follows as in Case 1.

Case 3 Here, $r_0 + x_0 - r_0 \cos\theta < 0$ when $\theta \in (0, \arccos(\frac{r_0+x_0}{r_0}))$, and $r_0 + x_0 - r_0 \cos\theta > 0$, when $\theta \in (\arccos(\frac{r_0+x_0}{r_0}), \pi)$, so

$$\mathcal{F}(x_0, r_0) = \begin{pmatrix} f_1(x_0, r_0) \\ f_2(x_0, r_0) \end{pmatrix},$$

with

$$\begin{aligned} f_1(x_0, r_0) &= \sqrt{-\frac{x_0(2r_0 + x_0)}{r_0^2}}(3r_0 + x_0 + 2) + \frac{\pi}{2}\left(4d^2 - 3r_0 - 2x_0 + 2\right) \\ &\quad +(3r_0 + 2x_0 + 2)\arccos\left(\frac{r_0 + x_0}{r_0}\right), \\ f_2(x_0, r_0) &= \frac{\pi r_0}{2} - \arccos\left(\frac{r_0 + x_0}{r_0}\right) r_0 + (r_0 + x_0 + 2)\sqrt{-\frac{x_0(2r_0 + x_0)}{r_0^2}}. \end{aligned}$$

Again, for solving the system $f_1(x_0, r_0) = f_2(x_0, r_0) = 0$, we do the change of variables $x_0 \to X$, where

$$x_0 = -r_0 - Xr_0, \tag{14}$$

with $-1 < X < 1$. Then, the system becomes

$$\begin{aligned} 2\pi d^2 + \sqrt{1 - X^2}((X - 2)r_0 - 2) + (2 - 2Xr_0 + r_0)\arcsin X &= 0, \\ \sqrt{1 - X^2}(2 - Xr_0) - r_0 \arcsin X &= 0. \end{aligned}$$

From the second equation, we get

$$r_0 = \frac{2\sqrt{1 - X^2}}{X\sqrt{1 - X^2} + \arcsin X}. \tag{15}$$

Substituting r_0 in the first equation we obtain $g(X) = 0$, where $g(X)$ is

$$\frac{2X^2 - 2 + \pi X\sqrt{1 - X^2}d^2 + \arcsin X\left(\pi d^2 + \arcsin X - X\sqrt{1 - X^2}\right)}{X\sqrt{1 - X^2} + \arcsin X}.$$

It is easy to compute that

$$\begin{aligned} \lim_{X \searrow -1} g(X) &= (-1 + 2d^2)\pi, \\ \lim_{X \nearrow 0} g(X) &= +\infty, \\ \lim_{X \searrow 0} g(X) &= -\infty, \\ \lim_{X \nearrow 1} g(X) &= (1 + 2d^2)\pi. \end{aligned}$$

Therefore, by continuity of the function $g(X)$ in the intervals $(-\infty, 0)$ and $(0, \infty)$, it follows that $g(X)$ has one zero in the interval $(-\infty, 0)$ if $(-1 + 2d^2)\pi < 0$ and that $g(X)$ always has one zero in the interval $(0, \infty)$. Moreover, since the derivative $g'(X) > 0$ in those two intervals, such zeros are the unique zeros of the function $g(X)$.

Since for each zero of $g(X)$, we have a unique zero (x_0, r_0) of system $f_1(x_0, r_0) = f_2(x_0, r_0) = 0$ (see (15) and (14)), and we obtain two solutions of the system $f_1(x_0, r_0) = f_2(x_0, r_0) = 0$ if $(-1 + 2d^2)\pi \leq 0$, and only one if $(-1 + 2d^2)\pi > 0$. Computing the Jacobian of the map $(f_1(x_0, r_0), f_2(x_0, r_0))$ at (15) and (14), we get

$$\frac{X^4 - 5X^2 + 6X\sqrt{1 - X^2}\arcsin X + \left(4X^2 - 1\right)(\arcsin X)^2 + 4}{1 - X^2} \geq 4,$$

if $X \in (-1, 1)$. Therefore, by Theorem 4, we obtain two periodic solutions of the differential system (13) if $(-1 + 2d^2)\pi < 0$, and one periodic solution if $(-1 + 2d^2)\pi > 0$. Such periodic solutions are of the form:

$$(x(\theta, \varepsilon), r(\theta, \varepsilon)) = (x_0 + \mathcal{O}(\varepsilon), r_0 + \mathcal{O}(\varepsilon)),$$

where x_0 and r_0 are given by (15) and (14) when X is a zero of $g(X)$.

Going back to system (12) with the independent variable t, we obtain the periodic solution:

$$(x(t, \varepsilon), r(t, \varepsilon), \theta(t, \varepsilon)) = (x_0, r_0, t(\text{mod } 2\pi)) + \mathcal{O}(\varepsilon).$$

Finally, coming back to system (2), we find the periodic solution:

$$(x(t, \varepsilon), y(t, \varepsilon), z(t, \varepsilon)) = (x_0, r_0 \sin t, r_0 \cos t) + \mathcal{O}(\varepsilon).$$

This concludes the proof of Theorem 2.

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References

Buica A, Françoise JP, Llibre J (2007) Periodic solutions of nonlinear periodic differential systems with a small parameter. Commun Pure Appl Anal 6:103–111

Carmona V, Fernandez-García S, Fernandez-Sánchez F, García-Medina E, Teruel AE (2012) Reversible periodic orbits in a class of 3D continuous piecewise linear systems of differential equations, Nonlinear Analysis. Theory Methods Appl 75:5866–5883

Carmona V, Fernandez-García S, Fernandez-Sánchez F, García-Medina E, Teruel AE (2014) Noose bifurcation and crossing tangency in reversible piecewise linear systems. Nonlinearity 27:585–606

Carmona V, Fernandez-Sánchez F, García-Medina E, Teruel A (2010) Existence of homoclinic connections in continuous piecewise linear systems. Chaos 20:013124

Carmona V, Fernandez-Sánchez F, García-Medina E, Teruel AE (2015) Noose structure and bifurcations of periodic orbits in reversible three-dimensional piecewise linear differential systems. J Nonlinear Sci 25:1209–1224

Carmona V, Fernandez-Sánchez F, Teruel A (2008) Existence of a reversible T-point heteroclinic cycle in a piecewise linear version of the Michelson system. SIAM J Appl Dyn Syst 7:1032–1048

di Bernardo M, Budd CJ, Champneys AR, Kowalczyk P (2008) Piecewise-smooth dynamical systems. Theory and applications. Applied Mathematical Sciences, vol 163. Springer, London

Dumortier F, Ibañez S, Kokubu H (2001) New aspects in the unfolding of the nilpotent singularity of codimension three. Dyn Syst 16:63–95

Filippov AF (1988) Differential equations with discontinuous righthand side. Mathematics and Its Applications. Kluwer, Dordrecht

Freire E, Gamero E, Rodriguez-Luis AJ, Algaba A (2002) A note on the triple-zero linear degeneracy: normal forms, dynamical and bifurcation behaviors of an unfolding. Int J Bifur Chaos 12:2799–2820

Kokubu H, Wilczak D, Zgliczyński P (2007) Rigorous verification of cocoon bifurcations in the Michelson system. Nonlinearity 20:2147–2174

Lau Y-T (1992) The 'cocoon' bifurcation in three-dimensional systems with two fixed points. Int J Bifur Chaos 2:543–558

Llibre J, Novaes D (2016) On the periodic solutions of discontinuous piecewise differential systems **(preprint)**

Llibre J, Novaes D, Teixeira MA (2014) Higher order averaging theory for finding periodic solutions via Brouwer degree. Nonlinearity 27:563–583

Llibre J, Novaes D, Teixeira MA (2015) On the birth of limit cycles for non-smooth dynamical systems. Bull Sci Math 139:229–244

Llibre J, Zhang X (2011) On the Hopf-zero bifurcation of the Michelson system. Nonlinear Anal Real World Appl 12:1650–1653

Makarenko O, Lamb JSW (2012) Dynamics and bifurcations of nonsmooth systems: a survey. Phys D 241:1826–1844

Michelson D (1986) Steady solutions for the Kuramoto–Sivashinsky equation. Phys D 19:89–111

Verhulst F (2000) Nonlinear differential equations and dynamical systems, 2nd edn. Springer, Berlin