Comp. Appl. Math. (2017) 36:79–87
DOI 10.1007/s40314-015-0215-6

On convergence of three iterative methods for solving of the matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = Q$

Vejdi I. Hasanov · Aynur A. Ali

Received: 12 October 2014 / Revised: 12 January 2015 / Accepted: 22 January 2015 /
Published online: 6 February 2015

Abstract In this paper, we give new convergence results for the basic fixed point iteration and its two inversion-free variants for finding the maximal positive definite solution of the matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = Q$, proposed by Long et al. (Bull Braz Math Soc 39:371–386, 2008) and Vaezzadeh et al. (Adv Differ Equ 2013). The new results are illustrated by numerical examples.

Keywords Nonlinear matrix equation · Fixed point iteration · Inversion-free iteration · Convergence rate

Mathematics Subject Classification 65F10 · 65F30 · 65H10 · 15A24

1 Introduction

In this paper, we study the matrix equation

$$X + A^*X^{-1}A + B^*X^{-1}B = Q, \tag{1}$$

where A, B are square matrices and Q is a positive definite matrix. Here, A^* denotes the conjugate transpose of the matrix A. The matrix Eq. (1) can be reduced to

$$Y + C^*Y^{-1}C + D^*Y^{-1}D = I, \tag{2}$$

where I is the identity matrix. Moreover, the Eq. (1) is solvable if and only if the Eq. (2) is solvable. For the first time, the Eqs. (2) and (1) are considered by Long et al. (2008) and Vaezzadeh et al. (2013), respectively. Also, the Eqs. (1) and (2) are appeared as particular cases of the equations in El-Sayed and Ran (2001), Ran and Reurings (2002), He and Long (2010),

Communicated by Jinyun Yuan.

V. I. Hasanov (✉) · A. A. Ali
Faculty of Mathematics and Informatics, Konstantin Preslavsky University of Shumen,
9712 Shumen, Bulgaria
e-mail: v.hasanov@fmi.shu-bg.net

Duan et al. (2011) and Liu and Chen (2011). El-Sayed and Ran (2001) and Ran and Reurings (2002) investigated the equation $X + A^*\mathcal{F}(X)A = Q$. He and Long (2010) and Duan et al. (2011) investigated the equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. Liu and Chen (2011) studied the equation $X^s + A^* X^{-t_1} A + B^* X^{-t_2} B = Q$. Berzig et al. (2012) considered the equation $X = Q - A^* X^{-1} A + B^* X^{-1} B$. Zhou et al. (2013) and Li et al. (2014) investigated the equation $X + A^* \overline{X}^{-1} A = Q$.

Specifically, if $B = 0$, the Eq. (1) reduces to

$$X + A^* X^{-1} A = Q, \tag{3}$$

which has many applications and has been studied recently by several authors (Anderson et al. 1990; Engwerda 1993; Zhan and Xie 1996; Zhan 1996; Guo and Lancaster 1999; Xu 2001; Meini 2002; Sun and Xu 2003; Hasanov and Ivanov 2006; Hasanov 2010).

In this paper, we write $A > 0$ ($A \geq 0$) if A is a Hermitian positive definite (semidefinite) matrix. For Hermitian matrices A and B, we write $A > B$ ($A \geq B$) if $A - B > 0$ ($A - B \geq 0$). A positive definite solutions X_{S} and X_{L} of a matrix equation is called minimal and maximal, respectively, if $X_{\mathrm{S}} \leq X \leq X_{\mathrm{L}}$ for any positive definite solution X of the equation.

Long et al. (2008) presented some conditions for existence of a positive definite solution of (2). They propose two iterative methods: basic fixed point iteration (BFPI) and an inversion-free variant of BFPI for computing the maximal positive definite solution of (2). Vaezzadeh et al. (2013) studied the Eq. (1) and considered inversion-free iterative methods. They give partial generalization of the convergence theorems of Guo and Lancaster (1999). Popchev et al. (2011, 2012) made a perturbation analysis of (1).

Motivated by the work in Long et al. (2008), Vaezzadeh et al. (2013) and Popchev et al. (2011, 2012), we continue to study the fixed point iteration and inversion-free variant of BFPI for solving of (1). In Sect. 2, we give the convergence rate of the BFPI. In Sect. 3, we improve the convergence theorems, proved by Vaezzadeh et al. (2013), of two inversion-free iterative methods. With these methods we obtain the maximal positive definite solution of (1). Some numerical examples are presented to illustrate the convergence behaviour of various algorithms in Sect. 4.

Throughout this paper, we denote by $\|A\|$ and $\rho(A)$ the spectral norm and the spectral radius of a square matrix A, respectively.

2 Basic fixed point iteration

Long et al. (2008) investigated Eq. (2). They propose some iterative algorithms and obtained some conditions for the existence of the positive definite solutions of (2).

We consider the BFPI:

Algorithm 2.1 (Basic fixed point iteration) *Let* $X_0 = Q$. *For* $n = 0, 1, \ldots,$ *compute*

$$X_{n+1} = Q - A^* X_n^{-1} A - B^* X_n^{-1} B.$$

Long et al. (2008) proved that, if Eq. (1) with $Q = I$ has a positive definite solution, then the Algorithm 2.1 defines a monotonically decreasing matrix sequence, which converges to positive definite solution of (1). But the problem of convergence rate in Long et al. (2008) was not considered. It is easy to prove by induction that, if Eq. (1) has a positive definite solution then the Algorithm 2.1 defines a monotonically decreasing matrix sequence, which converges to the maximal positive definite solution X_{L} of (1) for general positive definite matrix Q, i.e.

$$X_0 = Q \geq X_n \geq X_{n+1} \geq X_{\mathrm{L}}, \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} X_n = X_{\mathrm{L}}. \tag{4}$$

We now establish the following result.

Theorem 2.1 *If Eq.* (1) *has a positive definite solution, then for Algorithm* 2.1 *we have*

$$\|X_{n+1} - X_{\mathrm{L}}\| \leq \left(\left\|X_{\mathrm{L}}^{-1}A\right\|^2 + \left\|X_{\mathrm{L}}^{-1}B\right\|^2\right) \|X_n - X_{\mathrm{L}}\|,$$

for all $n \geq 0$.

Proof The proof is similar to Theorem 2.3 in Guo and Lancaster (1999). Since $X_{n+1} = Q - A^*X_n^{-1}A - B^*X_n^{-1}B$ and $X_L = Q - A^*X_{\mathrm{L}}^{-1}A - B^*X_{\mathrm{L}}^{-1}B$, we have

$$\begin{aligned}
X_{n+1} - X_{\mathrm{L}} &= A^*\left(X_{\mathrm{L}}^{-1} - X_n^{-1}\right)A + B^*\left(X_{\mathrm{L}}^{-1} - X_n^{-1}\right)B,\\
&= A^*\left(X_{\mathrm{L}}^{-1} + X_n^{-1} - X_{\mathrm{L}}^{-1}\right)(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}A\\
&\quad + B^*\left(X_{\mathrm{L}}^{-1} + X_n^{-1} - X_{\mathrm{L}}^{-1}\right)(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}B\\
&= A^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}A + B^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}B\\
&\quad - A^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_n^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}A\\
&\quad - B^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_n^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}B.
\end{aligned}$$

Hence,

$$0 \leq X_{n+1} - X_{\mathrm{L}} \leq A^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}A + B^*X_{\mathrm{L}}^{-1}(X_n - X_{\mathrm{L}})X_{\mathrm{L}}^{-1}B$$

and

$$\|X_{n+1} - X_{\mathrm{L}}\| \leq \left(\left\|X_{\mathrm{L}}^{-1}A\right\|^2 + \left\|X_{\mathrm{L}}^{-1}B\right\|^2\right) \|X_n - X_{\mathrm{L}}\|.$$

□

Remark 2.1 If $\|X_{\mathrm{L}}^{-1}A\|^2 + \|X_{\mathrm{L}}^{-1}B\|^2 < 1$ in Theorem 2.1, then the Algorithm 2.1 converges to X_{L} linearly with rate $r \leq \|X_{\mathrm{L}}^{-1}A\|^2 + \|X_{\mathrm{L}}^{-1}B\|^2$. Moreover, if X is a positive definite solution of the Eq. (1) and $\|X^{-1}A\|^2 + \|X^{-1}B\|^2 < 1$, then $X \equiv X_{\mathrm{L}}$.

3 Inversion-free variants of the basic fixed point iteration

Zhan (1996) proposed an inversion-free variant of the BFPI for the maximal solution of (3) when $Q = I$. Guo and Lancaster (1999) considered this algorithm for general positive definite Q and solved the problem of convergence rate.

Long et al. (2008) investigated Eq. (2). They applied Zhan's idea for (2) and proposed inversion-free variant of the BFPI for the maximal solution of (2). We rewrite their algorithm for general Q, which is applicable directly for (1).

Algorithm 3.1 *Let* $X_0 = Q$, $Y_0 = Q^{-1}$. *For* $n = 0, 1, 2, \ldots$ *compute*

$$\begin{cases} X_{n+1} = Q - A^*Y_nA - B^*Y_nB \\ Y_{n+1} = Y_n(2I - X_nY_n) \end{cases}$$

The convergence of Algorithm 3.1 was established in Long et al. (2008) for $Q = I$. Moreover, Long et al. (2008) derived that, if (1) has a positive definite solution with $Q = I$, for Algorithm 3.1, $X_0 \geq X_1 \geq \cdots, Y_0 \leq Y_1 \leq \cdots$, and $\lim_{n\to\infty} X_n = X_{\mathrm{L}}, \lim_{n\to\infty} Y_n = X_{\mathrm{L}}^{-1}$, where X_{L} is the maximal positive definite solution. For general positive definite matrix Q the convergence properties of Algorithm 3.1 are preserved.

Vaezzadeh et al. (2013) studied the Eq. (1) with $Q = I$ and investigated the problem of convergence rate for Algorithm 3.1. The following result is given in Vaezzadeh et al. (2013).

Theorem 3.1 (Vaezzadeh et al. 2013, Theorem 2) *If matrix Eq. (1) with $Q = I$ has a positive definite solution, for Algorithm 3.1 and any $\epsilon > 0$, we have*

$$\left\| Y_{n+1} - X_{\mathrm{L}}^{-1} \right\| \leq \left(\left\| A X_{\mathrm{L}}^{-1} \right\| + \left\| B X_{\mathrm{L}}^{-1} \right\| + \epsilon \right)^2 \left\| Y_{n-1} - X_{\mathrm{L}}^{-1} \right\| \tag{5}$$

and

$$\| X_{n+1} - X_{\mathrm{L}} \| \leq \left(\|A\|^2 + \|B\|^2 \right) \left\| Y_n - X_{\mathrm{L}}^{-1} \right\| \tag{6}$$

for all n large enough.

We now show that the above result can be improved.

Theorem 3.2 *If matrix Eq.* (1) *has a positive definite solution, then for Algorithm* 3.1 *and any $\epsilon > 0$, we have*

$$\left\| Y_{n+1} - X_{\mathrm{L}}^{-1} \right\| \leq \left(\left\| A X_{\mathrm{L}}^{-1} \right\|^2 + \left\| B X_{\mathrm{L}}^{-1} \right\|^2 + \epsilon \right) \left\| Y_{n-1} - X_{\mathrm{L}}^{-1} \right\| \tag{7}$$

and

$$\| X_{n+1} - X_{\mathrm{L}} \| \leq \left(\|A\|^2 + \|B\|^2 \right) \left\| Y_n - X_{\mathrm{L}}^{-1} \right\| \tag{8}$$

for all n large enough. Moreover, if A and B are nonsingular, then

$$\| X_{n+1} - X_{\mathrm{L}} \| \leq \left(\left\| X_{\mathrm{L}}^{-1} A \right\|^2 + \left\| X_{\mathrm{L}}^{-1} B \right\|^2 + \epsilon \right) \| X_{n-1} - X_{\mathrm{L}} \| \tag{9}$$

for all n large enough.

Proof In the proof of Theorem 3.1 (Vaezzadeh et al. 2013) obtained the expression

$$\begin{aligned} X_{\mathrm{L}}^{-1} - Y_{n+1} = &\left(X_{\mathrm{L}}^{-1} - Y_n \right) X_{\mathrm{L}} \left(X_{\mathrm{L}}^{-1} - Y_n \right) + Y_n A^* \left(X_{\mathrm{L}}^{-1} - Y_{n-1} \right) A Y_n \\ &+ Y_n B^* \left(X_{\mathrm{L}}^{-1} - Y_{n-1} \right) B Y_n. \end{aligned} \tag{10}$$

The inequality (7) follows from (10) since $\|Y_n - X_{\mathrm{L}}^{-1}\| \leq \|Y_{n-1} - X_{\mathrm{L}}^{-1}\|$ and $\lim Y_n = X_{\mathrm{L}}^{-1}$. The inequality (8) follows from

$$X_{n+1} - X_{\mathrm{L}} = A^* \left(X_{\mathrm{L}}^{-1} - Y_n \right) A + B^* \left(X_{\mathrm{L}}^{-1} - Y_n \right) B \tag{11}$$

So, from (10) and (11) follows:

$$X_{\mathrm{L}}^{-1} - Y_n = \left(X_{\mathrm{L}}^{-1} - Y_{n-1} \right) X_{\mathrm{L}} \left(X_{\mathrm{L}}^{-1} - Y_{n-1} \right) + Y_{n-1}(X_{n-1} - X_{\mathrm{L}}) Y_{n-1} \tag{12}$$

If A and B are nonsingular, from (11) and (12) we have

$$\begin{aligned} X_{n+1} - X_{\rm L} = & A^*Y_{n-1}(X_{n-1} - X_{\rm L})Y_{n-1}A + B^*Y_{n-1}(X_{n-1} - X_{\rm L})Y_{n-1}B \\ & + A^*\left(X_{\rm L}^{-1} - Y_{n-1}\right)AA^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)A \\ & + B^*\left(X_{\rm L}^{-1} - Y_{n-1}\right)BB^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)B. \end{aligned}$$

Hence,

$$\begin{aligned} \|X_{n+1} - X_{\rm L}\| \le & \left(\|Y_{n-1}A\|^2 + \|Y_{n-1}B\|^2\right)\|X_{n-1} - X_{\rm L}\| \\ & + \left\|A^*\left(X_{\rm L}^{-1} - Y_{n-1}\right)A + B^*\left(X_{\rm L}^{-1} - Y_{n-1}\right)B\right\| \\ & \times\left(\left\|A^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)A\right\| + \left\|B^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)B\right\|\right) \\ = & \left(\|Y_{n-1}A\|^2 + \|Y_{n-1}B\|^2\right)\|X_{n-1} - X_{\rm L}\| \\ & + \left(\left\|A^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)A\right\| + \left\|B^{-1}X_{\rm L}\left(X_{\rm L}^{-1} - Y_{n-1}\right)B\right\|\right) \\ & \times\|X_n - X_{\rm L}\|. \end{aligned}$$

Therefore, since $\|X_n - X\| \le \|X_{n-1} - X\|$ and $\lim Y_n = X_{\rm L}^{-1}$, (9) is satisfied for all n large enough. □

Remark 3.1 According the Theorem 3.1 for the linear convergence of the Algorithm 3.1 is guaranteed if $(\|AX_{\rm L}^{-1}\| + \|BX_{\rm L}^{-1}\|)^2 < 1$. But, according our result (Theorem 3.2) is necessarily $\|AX_{\rm L}^{-1}\|^2 + \|BX_{\rm L}^{-1}\|^2 < 1$. It is obvious that

$$\left\|AX_{\rm L}^{-1}\right\|^2 + \left\|BX_{\rm L}^{-1}\right\|^2 < \left(\left\|AX_{\rm L}^{-1}\right\| + \left\|BX_{\rm L}^{-1}\right\|\right)^2.$$

Hence, there are matrices A, B and maximal solution $X_{\rm L}$ of the Eq. (1), for which $\|AX_{\rm L}^{-1}\|^2 + \|BX_{\rm L}^{-1}\|^2 < 1$ and $(\|AX_{\rm L}^{-1}\| + \|BX_{\rm L}^{-1}\|)^2 > 1$, see Examples 4.1 and 4.2.

Vaezzadeh et al. (2013) proposed modification of Algorithm 3.1 with $Q = I$ and investigated the problem of convergence rate. For general positive definite matrix Q this algorithm takes the following form:

Algorithm 3.2 *Let* $X_0 = Q$, $Y_0 = Q^{-1}$. *For* $n = 0, 1, \ldots$, *compute*

$$\begin{cases} Y_{n+1} = Y_n(2I - X_nY_n) \\ X_{n+1} = Q - A^*Y_{n+1}A - B^*Y_{n+1}B \end{cases}$$

We denote that Algorithm 3.2 is generalization of Guo and Lancaster algorithm for (3) proposed in Guo and Lancaster (1999).

Vaezzadeh et al. (2013), Theorem 3 derived that, if (1) has a positive definite solution with $Q = I$, for Algorithm 3.2, $X_0 \ge X_1 \ge \cdots$, $Y_0 \le Y_1 \le \cdots$, and $\lim_{n\to\infty} X_n = X_{\rm L}$, $\lim_{n\to\infty} Y_n = X_{\rm L}^{-1}$. Vaezzadeh et al. (2013) for convergence rate the following result is given.

Theorem 3.3 (Vaezzadeh et al. 2013, Theorem 4) *If matrix Eq.* (1) *with* $Q = I$ *has a positive definite solution, for Algorithm* 3.2 *and any* $\epsilon > 0$, *then we have*

$$\left\|Y_{n+1} - X_{\rm L}^{-1}\right\| \le \left(\left\|AX_{\rm L}^{-1}\right\| + \left\|BX_{\rm L}^{-1}\right\| + \epsilon\right)^2 \left\|Y_n - X_{\rm L}^{-1}\right\| \qquad (13)$$

and

$$\|X_n - X_{\mathrm{L}}\| \leq (\|A\|^2 + \|B\|^2) \left\| Y_n - X_{\mathrm{L}}^{-1} \right\| \tag{14}$$

for all n large enough.

For general positive definite matrix Q the convergence properties of Algorithm 3.2 are preserved. We now show that the above result can be improved.

Theorem 3.4 *If matrix Eq.* (1) *has a positive definite solution, then for Algorithm* 3.2 *and any* $\epsilon > 0$, *we have*

$$\left\| Y_{n+1} - X_{\mathrm{L}}^{-1} \right\| \leq \left(\left\| A X_{\mathrm{L}}^{-1} \right\|^2 + \left\| B X_{\mathrm{L}}^{-1} \right\|^2 + \epsilon \right) \left\| Y_n - X_{\mathrm{L}}^{-1} \right\| \tag{15}$$

and

$$\|X_n - X_{\mathrm{L}}\| \leq (\|A\|^2 + \|B\|^2) \left\| Y_n - X_{\mathrm{L}}^{-1} \right\| \tag{16}$$

for all n large enough. Moreover, if A and B are nonsingular, then

$$\|X_{n+1} - X_{\mathrm{L}}\| \leq \left(\left\| X_{\mathrm{L}}^{-1} A \right\|^2 + \left\| X_{\mathrm{L}}^{-1} B \right\|^2 + \epsilon \right) \|X_n - X_{\mathrm{L}}\| \tag{17}$$

for all n large enough.

Proof The proof is similar to that of Theorem 3.2. □

4 Numerical experiments

In this section, we present some numerical examples to show the effectiveness of the new result for convergence rate of the considered inversion-free methods. We consider examples, which are modification of the examples in Long et al. (2008) and Vaezzadeh et al. (2013) and compare the Algorithm 2.1 (BFPI), Algorithm 3.1 (FIFV-BFPI) and Algorithm 3.2 (SIFV-BFPI). For the stopping criterion we take

$$\|X_n - X_{n-1}\|_\infty \leq 10^{-10},$$

where $\|A\|_\infty = \max_i \sum_{j=1}^m |a_{ij}|$ for a complex $m \times m$ matrix A.

We use the following notations:

- k is the smallest number of iteration, such that the stopping criterion is satisfied;
- $\mathrm{res}(X_k) = \|X_k + A^* X_k^{-1} A + B^* X_k^{-1} B - Q\|_\infty$;
- $r_1 = \|X_{\mathrm{L}}^{-1} A\|^2 + \|X_{\mathrm{L}}^{-1} B\|^2$—convergence rate of Algorithm 2.1 (BFPI);
- $r_{2y} = (\|A X_{\mathrm{L}}^{-1}\| + \|B X_{\mathrm{L}}^{-1}\|)^2$—convergence "semi"-rate of Y_n in Algorithm 3.1 (FIFV-BFPI) and convergence rate of Y_n in Algorithm 3.2 (SIFV-BFPI) given by Veazzadeh et al. [see (5) and (13)];
- $r_{3y} = \|A X_{\mathrm{L}}^{-1}\|^2 + \|B X_{\mathrm{L}}^{-1}\|^2$ and $r_{3x} = r_1$ are convergence "semi"-rate of Y_n and X_n in Algorithm 3.1, respectively. Moreover, r_{3y} and r_{3x} are convergence rate of Y_n and X_n in Algorithm 3.2, respectively [see (17)];
- $\varepsilon_x(r) = r - \frac{\|X_n - X_{\mathrm{L}}\|}{\|X_{n-1} - X_{\mathrm{L}}\|}; \varepsilon'_x(r) = r - \frac{\|X_n - X_{\mathrm{L}}\|}{\|X_{n-2} - X_{\mathrm{L}}\|}$;
- $\varepsilon_y(r) = r - \frac{\|Y_n - X_{\mathrm{L}}^{-1}\|}{\|Y_{n-1} - X_{\mathrm{L}}^{-1}\|}; \varepsilon'_y(r) = r - \frac{\|Y_n - X_{\mathrm{L}}^{-1}\|}{\|Y_{n-2} - X_{\mathrm{L}}^{-1}\|}$.

In our case for Algorithm 3.1 convergence "semi"-rate means that:

$\|Y_{n+1} - X_L^{-1}\| \leq (r_{2y} + \epsilon)\|Y_{n-1} - X_L^{-1}\|$ is satisfied [see (5)];
$\|Y_{n+1} - X_L^{-1}\| \leq (r_{3y} + \epsilon)\|Y_{n-1} - X_L^{-1}\|$ is satisfied [see (7)];
$\|X_{n+1} - X_L\| \leq (r_{3x} + \epsilon)\|X_{n-1} - X_L\|$ is satisfied [see (9)].

Convergence rate of Algorithm 3.1 is approximately square root of the "semi"-rate.

Example 4.1 Consider the Eq. (1) with

$$A = \frac{1}{10}\begin{pmatrix} 0.10 & -1.50 & -2.59 \\ 0.15 & 2.12 & -0.64 \\ 0.25 & -0.69 & 1.39 \end{pmatrix}, \quad B = \frac{1}{10}\begin{pmatrix} 1.60 & -0.25 & 0.20 \\ -0.25 & -2.88 & -0.60 \\ 0.04 & -0.16 & -1.20 \end{pmatrix},$$
$$Q = \frac{1}{2}I + 2A^*A + 2B^*B.$$

Now, for Example 4.1 the maximal solution is $X_L = \frac{1}{2}I$, and $r_1 = r_{3y} = r_{3x} = 0.7537$ and $r_{2y} = 1.5063$. In Table 1 are given the numbers of iteration k, for which the stopping criterion is satisfied, the norm $\|X_k - X_{k-1}\|_\infty$ and $\text{res}(X_k)$ for the three algorithms.

The rest of our numerical results are reported in Table 2.

Example 4.2 Consider the Eq. (1) with

$$A = \frac{1}{70}\begin{pmatrix} 40 & 25 & 23 & 35 & 66 \\ 25 & 32 & 27 & 45 & 21 \\ 23 & 27 & 28 & 16 & 24 \\ 35 & 45 & 16 & 52 & 65 \\ 66 & 21 & 24 & 65 & 69 \end{pmatrix}, \quad B = \frac{1}{70}\begin{pmatrix} 11 & 21 & 23 & 25 & 32 \\ 21 & 31 & 60 & 42 & 33 \\ 23 & 60 & 34 & 18 & 26 \\ 25 & 42 & 18 & 44 & 30 \\ 32 & 33 & 26 & 30 & 50 \end{pmatrix},$$

Table 1 Numerical results of Example 4.1

Algorithm	k	$\|X_k - X_{k-1}\|_\infty$	$\text{res}(X_k)$
BFPI	31	8.0713e−11	4.0537e−11
FIFV-BFPI	60	6.6362e−11	1.1674e−10
SIFV-BFPI	32	7.2509e−11	7.2509e−11

Table 2 Numerical results of Example 4.1

	BFPI	FIFV-BFPI			SIFV-BFPI		
n	$\varepsilon_x(r_1)$	$\varepsilon'_y(r_{2y})$	$\varepsilon'_y(r_{3y})$	$\varepsilon'_x(r_{3x})$	$\varepsilon_y(r_{2y})$	$\varepsilon_y(r_{3y})$	$\varepsilon_x(r_{3x})$
1	0.4526	*	*	*	0.5624	−0.1901	0.4136
2	0.3414	0.9317	0.1791	0.3332	0.8964	0.1438	0.2072
5	0.2622	1.0426	0.2900	0.3011	1.0111	0.2585	0.2544
10	0.2518	1.0136	0.2610	0.2628	1.0046	0.2521	0.2518
20	0.2515	1.0045	0.2519	0.2519	1.0041	0.2515	0.2515
30	0.2515	1.0041	0.2515	0.2515	1.0041	0.2515	0.2515
31	0.2515	1.0041	0.2515	0.2515	1.0041	0.2515	0.2515
32	*	1.0041	0.2515	0.2515	1.0041	0.2515	0.2515
$k > 32$	*	1.0041	0.2515	0.2515	*	*	*

Table 3 Numerical results of Example 4.2

Algorithm	k	$\|X_k - X_{k-1}\|_\infty$	$\mathrm{res}(X_k)$
BFPI	56	$8.2800e{-}11$	$5.5176e{-}11$
FIFV-BFPI	108	$8.2826e{-}11$	$1.5294e{-}10$
SIFV-BFPI	57	$8.1894e{-}11$	$8.1894e{-}11$

Table 4 Numerical results of Example 4.2

	BFPI	FIFV-BFPI			SIFV-BFPI		
n	$\varepsilon_x(r_1)$	$\varepsilon'_y(r_{2y})$	$\varepsilon'_y(r_{3y})$	$\varepsilon'_x(r_{3x})$	$\varepsilon_y(r_{2y})$	$\varepsilon_y(r_{3y})$	$\varepsilon_x(r_{3x})$
1	0.3495	*	*	*	0.6124	−0.1028	0.3259
2	0.2213	0.8458	0.1306	0.2173	0.7368	0.0216	0.0918
5	0.1111	1.3130	0.5978	0.1650	1.1938	0.4786	0.1051
10	0.0826	0.8208	0.1056	0.1085	0.7983	0.0831	0.0829
20	0.0787	0.7969	0.0817	0.0824	0.7939	0.0787	0.0787
30	0.0786	0.7942	0.0790	0.0791	0.7938	0.0786	0.0786
40	0.0786	0.7939	0.0787	0.0787	0.7938	0.0786	0.0786
56	0.0786	0.7938	0.0786	0.0786	0.7938	0.0786	0.0786
57	*	0.7938	0.0786	0.0786	0.7938	0.0786	0.0786
$k > 57$	*	0.7938	0.0786	0.0786	*	*	*

$$Q = X + A^*X^{-1}A + B^*X^{-1}B, \text{ where } X = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

We have for Example 4.2 the maximal solution $X_{\mathrm{L}} = X$, $r_1 = r_{3x} = r_{3y} = 0.7450$ and $r_{2y} = 1.4602$. Our numerical results are reported in Tables 3 and 4.

Acknowledgements The authors are grateful to editor and the reviewers for their helpful comments and suggestions. This paper is supported by the Project BG051PO 00l-3.3.06-0003 "Building and steady development of PhD students, post-PhD and young scientists in the areas of the natural, technical and mathematical sciences". The Project is realized by the financial support of the Operative Program "Development of the human resources" of the European social fund of the European Union.

References

Anderson WN, Morley TD, Trapp GE (1990) Positive solutions to $X = A - BX^{-1}B^*$. Linear Algebra Appl 134:53–62

Berzig M, Duan X, Samet B (2012) Positive definite solutions of the matrix equation $X = Q - A^*X^{-1}A + B^*X^{-1}B$ via Bhaskar–Lakshmikantham fixed point theorem. Math Sci. doi:10.1186/2251-7456-6-27

Duan X, Li C, Liao A (2011) Solutions and perturbation analysis for the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$. Appl Math Comput 218:4458–4466

El-Sayed SM, Ran ACM (2001) On an iteration method for solving a class of nonlinear matrix equations. SIAM J Matrix Anal Appl 23:632–645

Engwerda JC (1993) On the existence of a positive definite solution of the matrix equation $X + A^TX^{-1}A = I$. Linear Algebra Appl 194:91–108

Guo CH, Lancaster P (1999) Iterative solution of two matrix equations. Math Comput 68:1589–1603

Hasanov VI, Ivanov IG (2006) On two perturbation estimates of the extreme solutions to the matrix equations $X \pm A^*X^{-1}A = Q$. Linear Algebra Appl 413:81–92

Hasanov VI (2010) Notes on two perturbation estimates of the extreme solutions to the equations $X \pm A^*X^{-1}A = Q$. Appl Math Comput 216:1355–1362

He Y, Long J (2010) On the Hermitian positive definite solution of the nonlinear matrix equation $X + \sum_{i=1}^{m} A_i^*X^{-1}A_i = I$. Appl Math Comput 216:3480–3485

Li Z, Zhou B, Lam J (2014) Towards positive definite solutions of a class of nonlinear matrix equations. Appl Math Comput 237:546–559

Liu A, Chen G (2011) On the Hermitian positive definite solutions of nonlinear matrix equation $X^s + A^*X^{-t_1}A + B^*X^{-t_2}B = Q$. Math Probl Eng. Article ID 163585

Long J, Hu X, Zhang L (2008) On the Hermitian positive definite solution of the matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = I$. Bull Braz Math Soc 39:371–386

Meini B (2002) Efficient computation of the extreme solutions of $X + A^*X^{-1}A = Q$ and $X - A^*X^{-1}A = Q$. Math Comput 71:1189–1204

Popchev I, Petkov P, Konstantinov M, Angelova V (2011) Condition numbers for the matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = I$. Comptes Rendus de L'Academie Bulgare des Sciences 64:1679–1688

Popchev I, Petkov P, Konstantinov M, Angelova V (2012) Perturbation bounds for the nonlinear matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = I$. In: Lirkov I, Margenov S, Waśniewski J (eds) Large-scale scientific computing 2011. Lecture notes in computer science, vol 7116. Springer Berlin Heidelberg, pp 155–162

Ran ACM, Reurings MCB (2002) On the nonlinear matrix equation $X + A^*\mathcal{F}(X)A = Q$: solution and perturbation theory. Linear Algebra Appl 346:15–26

Sun JG, Xu SF (2003) Perturbation analysis of the maximal solution of the matrix equation $X + A^*X^{-1}A = P$ II. Linear Algebra Appl 362:211–228

Vaezzadeh S, Vaezpour S, Saadati R, Park C (2013) The iterative methods for solving nonlinear matrix equation $X + A^*X^{-1}A + B^*X^{-1}B = Q$. Adv Differ Equ. doi:10.1186/1687-1847-2013-229

Xu SF (2001) Perturbation analysis of the maximal solution of the matrix equation $X + A^*X^{-1}A = P$. Linear Algebra Appl 336:61–70

Zhan X, Xie J (1996) On ehe matrix equation $X + A^TX^{-1}A = I$. Linear Algebra Appl 147:337–342

Zhan X (1996) Computing the extremal positive definite solution of a matrix equation. SIAM J Sci Comput 247:337–345

Zhou B, Cai G, Lam J (2013) Positive definite solutions of the nonlinear matrix equation $X + A^*\overline{X}^{-1}A = I$. Appl Math Comput 219:7377–7391