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Robust Stabilization of Uncertain 2-D Discrete Delayed Systems

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Abstract
This paper is concerned with the stabilization of a class of two-dimensional discrete state-delayed systems described by the Fornasini–Marchesini second local state-space model subject to polytopic uncertainties. A delay-dependent stability criterion is derived for the existence of a memoryless state feedback controller such that the closed-loop system is robustly stable. To match practical issues the influence of finite wordlength nonlinearities in control implementation, delays and parameter uncertainties belonging to a known convex polytope are also discussed. The presented criteria are obtained by employing parameter-dependent Lyapunov–Krasovskii function and slack variables that augments the searching space allowing less conservative solutions. Finally, examples are provided to show the effectiveness of the proposal.

Keywords Two-dimensional system · Robust stabilization · Delay-dependent · Parameter-dependent Lyapunov–Krasovskii functions · Polytopic uncertainty

1 Introduction

A number of relevant processes can be modeled as a two-dimensional system, i.e., a system with two independent variables. Most of the studies found in the literature are based on models such as Roesser (1975), Fornasini and Marchesini (1976, 1978) and Kurek (1985). In many cases the independent variables have a time relation: one related to continuous time variation and the other being the iteration number (Pozdyayev 2015). Some examples concern repetitive processes like pick-and-place tasks performed by robot arms (Cichy et al. 2014), irrigation channels (Knorn and Middleton 2013), traveling waves (Yu et al. 2013) and heating flow processes (Caldeira et al. 2015). Note that, in case of digital controlled systems, both independent variables can be modeled in a discrete-time domain, being one of the focus of this paper.

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The fundamental problems of stability and stabilization of 2-D systems have being investigated during the past two decades. Although these problems are in concept similar to those found in the 1-D framework, the application of respective theory for 2-D systems is much more complicated. A number of key results can be found in the literature including (Chesi and Middleton 2016, 2015; Pozdyayev 2015; Bouagada and Dooren 2013; Gałkowski et al. 2003). Among these works, it is interesting to note that only Chesi and Middleton (2016) handles the robust stabilization. A different approach is proposed in Cichy et al. (2014) where an iterative and robust learning control is employed using non-causal data. Gain-scheduled control of discrete-time 2-D systems has been considered in Osowsky (2013) by means of quadratically parameter-dependent Lyapunov function. In the context of robust filtering design, see (Souza et al. 2010) where $\mathscr{H}_2$ and $\mathscr{H}_\infty$ performance indexes are used to handle Fornasini–Marchesini second model with the aid of a parameter-dependent Lyapunov function.

In practice, models are subject to uncertainties related to parameters errors, drifting, neglected dynamics, etc., which can deteriorate performance or even lead the controlled system to instability. Moreover, the presence of delays adds another source of performance degradation and instability in the control loop (Niculescu 2001). In particular, state delay can arrive in systems from three sources (Gu et al. 2003): sensor delay; it is intentionally introduced in control law (for example in vibration control); or it belongs to the system dynamics and it shows up after the closed loop. Therefore, it is important to incorporate uncertainties and delays in the model of a system. Another practical issue concerns the implementation errors of the controller due to finite word length in digital computers.

Many results have been reported on the stability of uncertain 2-D discrete delayed systems (Chen and Fong 2007; Chen 2010; Dey and Kar 2014; Xu and Yu 2009a; Feng et al. 2010; Xu and Yu 2009b; Paszke et al. 2004). The results can be classified into delay-independent (Paszke et al. 2004) and delay-dependent (Chen and Fong 2007; Chen 2010; Dey and Kar 2014; Xu and Yu 2009a; Feng et al. 2010; Xu and Yu 2009b; Tadepalli et al. 2015a) stability criteria. In general, delay-dependent stability conditions are preferred due to practical reasons: in real systems the delay is usually finite and often bounded in an interval. On the other hand, delay-independent conditions may be of special interest from a mathematical point of view. This kind of condition allows to characterize a stronger stability property, where the finite delay may not affect the stability properties. It is well known that, the delay-dependent stability criteria have the advantage of yielding less conservative results as compared to delay-independent criteria in practical issues. Majority of the results have been obtained for 2-D discrete systems with norm-bounded parametric uncertainties (Dey and Kar 2014; Xu and Yu 2009a; Feng et al. 2010; Paszke et al. 2004; Tadepalli et al. 2015a). When uncertainty is described as norm-bounded, the mathematical tools employed in robust control take it as time-varying. On the other hand, polytopic approach allows to deal with time-invariant uncertainty, leading, in general, to a less conservative stability analysis conditions. Some results for 2-D discrete systems with polytopic uncertainties (Chen and Fong 2007; Hmamed et al. 2008; Xu et al. 2010a, b; Wang and Liu 2013; Xie et al. 2015) may be mentioned.

The problem of robust H_∞ filtering for 2-D discrete state-delayed systems in the presence of polytopic uncertainties has been considered in Chen and Fong (2007); Boukili et al. (2016). A delay-dependent stability criterion was developed for the case where constant delays are present in the state. In Hmamed et al. (2008), a stability criterion employing parameter-dependent Lyapunov function has been obtained for 2-D discrete delay-free systems in the presence of polytopic uncertainties. The problem of non-fragile H_2 and H_∞ filter designs for 2-D discrete delay-free systems has been considered in Xu et al. (2010a) and Xu et al. (2010b). In Benzaouia et al. (2011), the problem of asymptotic stability of discrete 2-D switching systems with state feedback control has been considered. To the best of authors' knowledge, a delay-dependent stability criterion to estimate the controller gain for assuring the stability of the 2-D discrete systems in the presence of time-varying delays and polytopic uncertainties has not been dealt previously.

The other issue discussed in this paper is the stability of discrete systems in the presence of finite wordlength nonlinearities. When a discrete system is implemented using special purpose hardware, errors may arise due to the finite register length. It may happen that while performing arithmetic operations with such hardware the result may exceed the maximum value that can be represented. In other words, overflow may occur. Saturation arithmetic is widely employed as a overflow correction technique, as the nonlinearity arising by employing saturation arithmetic yields less restrictive stability conditions (see, [Chen 2010; Singh 2006; Kar 2007; Singh 2013; Tadepalli and Kandanvli 2016; Tadepalli et al. 2015], and the references cited therein). The stability of discrete systems during the fixed-point implementation of discrete systems is therefore, an important issue. Such a study is motivated by the applications involving finite wordlength implementation of: digital control systems, discrete systems represented by the Darboux equation (Dey and Kar 2014; Tadepalli et al. 2016), networked controlled systems, wireless sensor platforms employing fixed-point digital processors (Sumanasena and Bauer 2011), iterative learning control schemes (Tadepalli et al. 2015a) and so on. The problem of asymptotic stability of discrete 2-D systems described by the Fornasini–Marchesini second local state-space (FMSLSS) model in the presence of saturation finite wordlength nonlinearities and delays was dealt in Chen (2010). To the best of authors' knowledge, the problem of asymptotic stability of discrete 2-D systems described by the FMSLSS model in the presence of saturation finite wordlength nonlinearities, delays and polytopic uncertainties has not been previously studied.

Following are the main contributions of this paper: *(a)* A new delay-dependent stability criterion has been developed for 2-D discrete systems with delay and polytopic uncertainties. *(b)* A new criterion for estimating the controller gain such that the system is robustly stable has been developed. *(c)* A new delay-dependent stability criterion is provided for discrete 2-D systems in the presence of saturation finite wordlength nonlinearities, delays and polytopic uncertainties. *(d)* A parameter-dependent Lyapunov function is employed to achieve less conservative results.

The paper is organized as follows. The system under consideration is described in Sect. 2. Section 3 presents the main results of this paper. Examples illustrating the applicability of the presented results are shown in Sect. 4.

Notations The notations used in this paper are quite standard: $\mathbb{R}^p$ denotes the p-dimensional Euclidean space; $\mathbb{R}^{p\times q}$ is the set of $p \times q$ real matrices; $\mathbf{0}$ represents null matrix or null vector of appropriate dimension; $\boldsymbol{I}$ is the identity matrix of appropriate dimension; $\boldsymbol{B}^T$ stands for the transpose of the matrix (or vector) $\boldsymbol{B}$; $\boldsymbol{B} > \mathbf{0}$ $(\geq \mathbf{0})$ means that $\boldsymbol{B}$ is positive definite (semidefinite) symmetric matrix; $\boldsymbol{B} < \mathbf{0}$ represents that $\boldsymbol{B}$ is negative definite symmetric matrix; $\mathbb{Z}_+$ is the set of nonnegative integers; $\|\cdot\|$ represents any vector or matrix norm; $sup\{\cdot\}$ denotes the supremum or least upper bound of a set; $\boldsymbol{B} = \text{block-diag}\{\boldsymbol{B}_1, \boldsymbol{B}_2\}$ represents a block diagonal matrix $\boldsymbol{B}$ with the elements $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ on the principal diagonal; the symbol '$*$' represents the symmetric terms in a symmetric matrix.

2 System Description

Consider the following uncertain 2-D discrete system with delays in the state and described by the FMSLSS (Fornasini and Marchesini 1978) model:

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) = &\boldsymbol{A}_1(\lambda)\boldsymbol{x}(i, j+1) + \boldsymbol{A}_2(\lambda)\boldsymbol{x}(i+1, j)\\ &+ \boldsymbol{A}_{d_1}(\lambda)\boldsymbol{x}(i-\alpha(i), j+1)\\ &+ \boldsymbol{A}_{d_2}(\lambda)\boldsymbol{x}(i+1, j-\beta(j))\\ &+ \boldsymbol{B}_1(\lambda)\boldsymbol{u}(i, j+1) + \boldsymbol{B}_2(\lambda)\boldsymbol{u}(i+1, j)\end{aligned} \tag{1a}$$

where $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$ are horizontal coordinate and vertical coordinate, respectively; $\boldsymbol{x}(i, j) \in \mathbb{R}^n$ is the local state vector; $\boldsymbol{u}(i, j) \in \mathbb{R}^m$ is the control input; $\alpha(i)$ and $\beta(j)$ are delays along horizontal and vertical directions, respectively. Assume that $\alpha(i)$ and $\beta(j)$ satisfy

$$1 \le \alpha_l \le \alpha(i) \le \alpha_h, \;\; 1 \le \beta_l \le \beta(j) \le \beta_h, \tag{1b}$$

where α_l and β_l are constant nonnegative integers representing the lower delay bounds along horizontal and vertical directions, respectively; α_h and β_h are constant nonnegative integers representing the upper delay bounds along horizontal and vertical directions, respectively.
The system matrices $\boldsymbol{A}_1(\lambda)$, $\boldsymbol{A}_2(\lambda)$, $\boldsymbol{A}_{d_1}(\lambda)$, $\boldsymbol{A}_{d_2}(\lambda) \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_k(\lambda) \in \mathbb{R}^{n\times m}$ $(k = 1, 2)$ are subject to the uncertainties and belong to the polytope Ω given by

$$\begin{aligned}\Omega = \Bigg\{&\Big[\boldsymbol{A}_1(\lambda), \boldsymbol{A}_2(\lambda), \boldsymbol{A}_{d_1}(\lambda), \boldsymbol{A}_{d_2}(\lambda), \boldsymbol{B}_1(\lambda), \boldsymbol{B}_2(\lambda)\Big]\\ &:= \sum_{s=1}^{p}\lambda_s\Big[\boldsymbol{A}_{1s}, \boldsymbol{A}_{2s}, \boldsymbol{A}_{d_{1s}}, \boldsymbol{A}_{d_{2s}}, \boldsymbol{B}_{1s}, \boldsymbol{B}_{2s}\Big],\\ &\sum_{s=1}^{p}\lambda_s = 1, \lambda_s \ge 0\Bigg\}\end{aligned} \tag{1c}$$

It is assumed (Chen 2010; Dey and Kar 2014; Xu and Yu 2009a; Feng et al. 2010; Xu and Yu 2009b) that system (1) has a finite set of boundary conditions, i.e., there exist two positive integers k_1 and k_2 such that

$$\begin{aligned}&\boldsymbol{x}(i, j) = \boldsymbol{0}, \forall i \ge k_1, j = -\beta_h, -\beta_h+1, \ldots, 0,\\ &\boldsymbol{x}(i, j) = \boldsymbol{u}_{ij}, \forall 0 \le i < k_1, j = -\beta_h, -\beta_h+1, \ldots, 0,\\ &\boldsymbol{x}(i, j) = \boldsymbol{0}, \forall j \ge k_2, i = -\alpha_h, -\alpha_h+1, \ldots, 0,\\ &\boldsymbol{x}(i, j) = \boldsymbol{v}_{ij}, \forall 0 \le j < k_2, i = -\alpha_h, -\alpha_h+1, \ldots, 0\\ &\quad \boldsymbol{u}_{00} = \boldsymbol{v}_{00}\end{aligned} \tag{1d}$$

Under the assumption that the system state is available for feedback, the following memoryless state feedback control law is considered:

$$\boldsymbol{u}(i, j) = \boldsymbol{K}\boldsymbol{x}(i, j) \tag{2}$$

where $\boldsymbol{K} \in \mathbb{R}^{q\times n}$

Applying the state feedback controller (2) to the system (1a) would yield the closed-loop system

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) = &\tilde{\boldsymbol{A}}_1(\lambda)\boldsymbol{x}(i, j+1) + \tilde{\boldsymbol{A}}_2(\lambda)\boldsymbol{x}(i+1, j)\\ &+ \boldsymbol{A}_{d_1}(\lambda)\boldsymbol{x}(i-\alpha(i), j+1)\\ &+ \boldsymbol{A}_{d_2}(\lambda)\boldsymbol{x}(i+1, j-\beta(j))\end{aligned} \tag{3a}$$

where

$$\tilde{\boldsymbol{A}}_1(\lambda) = \big(\boldsymbol{A}_1(\lambda) + \boldsymbol{B}_1(\lambda)\boldsymbol{K}\big),\; \tilde{\boldsymbol{A}}_2(\lambda) = \big(\boldsymbol{A}_2(\lambda) + \boldsymbol{B}_2(\lambda)\boldsymbol{K}\big) \tag{3b}$$

and $\tilde{\boldsymbol{A}}_1(\lambda)$, $\tilde{\boldsymbol{A}}_2(\lambda)$, $\boldsymbol{A}_{d_1}(\lambda)$, $\boldsymbol{A}_{d_2}(\lambda)$ belong to the polytope

$$\begin{aligned}\Theta = \Bigg\{&\Big[\tilde{\boldsymbol{A}}_1(\lambda), \tilde{\boldsymbol{A}}_2(\lambda), \boldsymbol{A}_{d_1}(\lambda), \boldsymbol{A}_{d_2}(\lambda)\Big]\\ &:= \sum_{s=1}^{p}\lambda_s\Big[\tilde{\boldsymbol{A}}_{1s}, \tilde{\boldsymbol{A}}_{2s}, \boldsymbol{A}_{d_{1s}}, \boldsymbol{A}_{d_{2s}}\Big], \sum_{s=1}^{p}\lambda_s = 1, \lambda_s \ge 0\Bigg\}.\end{aligned} \tag{3c}$$

Equation (1a) may be used to represent a broad class of discrete dynamical uncertain systems. For example the control of thermal processes in chemical reactors, heat exchangers and pipe furnaces can be easily formulated in the form of (1a).

Next, a definition that formalizes the asymptotical stability concept for the considered class of 2-D systems is presented.

Definition 1 (Paszke et al. 2004) The system described by (1) is asymptotically stable if $\lim_{l\to\infty} x_l = 0$ for all boundary conditions in (1d), where $x_l = sup\{\|\boldsymbol{x}(i, j)\| : i + j = l, i, j \ge 1\}$.

3 Main Results

Solutions to the fundamental problems of robust stability analysis and robust stabilization of discrete 2-D systems are provided in this section. Moreover, a condition ensuring the robust stability of the closed-loop system under saturation finite wordlength nonlinearities is provided.

3.1 Asymptotic Stability Analysis

The objective of this section is to develop a criterion for the robust stability analysis of the system (3).

Theorem 1 *For $1 \le \alpha_l \le \alpha(i) \le \alpha_h$ and $1 \le \beta_l \le \beta(j) \le \beta_h$, the system (3) with boundary conditions (1d) is asymptotically stable if there exist positive definite symmetric matrices $\boldsymbol{P}_{1s}$, $\boldsymbol{P}_{2s}$, $\boldsymbol{Q}_{1s}$, $\boldsymbol{Q}_{2s}$, $\boldsymbol{Z}_{1s}$, $\boldsymbol{Z}_{2s} \in \mathbb{R}^{n\times n}$ $(s = 1, 2, \ldots, p)$, a matrix $\boldsymbol{X} \in \mathbb{R}^{9n\times n}$, such that*

$$\boldsymbol{\Psi}_s + \boldsymbol{X}\boldsymbol{B}_s + \boldsymbol{B}_s^T\boldsymbol{X}^T < \boldsymbol{0},\ s = 1, 2, \ldots, p \tag{4}$$

where

$$\boldsymbol{B}_s = \left[\boldsymbol{I}\ -\tilde{\boldsymbol{A}}_{1s}\ -\tilde{\boldsymbol{A}}_{2s}\ -\boldsymbol{A}_{d_{1s}}\ -\boldsymbol{A}_{d_{2s}}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\ \boldsymbol{0}\right] \tag{5}$$

$$\boldsymbol{\Psi}_s = \text{block-diag}\Big\{\boldsymbol{M}_s, -\boldsymbol{Q}_{1s}, -\boldsymbol{Q}_{2s}, -\boldsymbol{Z}_{1s}, -\boldsymbol{Z}_{2s}, -\boldsymbol{Z}_{1s}, -\boldsymbol{Z}_{2s}\Big\} \tag{6}$$

and

$$\boldsymbol{M}_s = \begin{bmatrix} \boldsymbol{M}_{11s} & -(\alpha_h+1)\boldsymbol{Z}_{1s} & -(\beta_h+1)\boldsymbol{Z}_{2s} \\ * & \boldsymbol{M}_{22s} & \boldsymbol{0} \\ * & * & \boldsymbol{M}_{33s} \end{bmatrix} \tag{7}$$

$$\boldsymbol{M}_{11s} = \boldsymbol{P}_{1s} + (\alpha_h+1)\boldsymbol{Z}_{1s} + \boldsymbol{P}_{2s} + (\beta_h+1)\boldsymbol{Z}_{2s} \tag{8a}$$
$$\boldsymbol{M}_{22s} = (\alpha_{hl}+1)\boldsymbol{Q}_{1s} - \boldsymbol{P}_{1s} + (\alpha_h+1)\boldsymbol{Z}_{1s} \tag{8b}$$
$$\boldsymbol{M}_{33s} = (\beta_{hl}+1)\boldsymbol{Q}_{2s} - \boldsymbol{P}_{2s} + (\beta_h+1)\boldsymbol{Z}_{2s} \tag{8c}$$

Proof Consider the following parameter-dependent Lyapunov–Krasovskii function:

$$V(\boldsymbol{x}(i,j)) = \bar{V}(\boldsymbol{x}(i,j)) + \hat{V}(\boldsymbol{x}(i,j)) \tag{9}$$

□

where

$$\begin{aligned}\bar{V}(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^T(i,j)\boldsymbol{P}_1(\lambda)\boldsymbol{x}(i,j) \\ &+ \sum_{l=-\alpha(i)}^{-1} \boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i+l,j) \\ &+ \sum_{\theta=2-\alpha_h}^{1-\alpha_l}\sum_{l=-1+\theta}^{-1} \boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i+l,j) \\ &+ \sum_{\theta=-\alpha_h}^{-1}\sum_{l=\theta}^{-1} \boldsymbol{\eta}_1^T(i+l,j)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i+l,j) \\ &+ \sum_{l=-\alpha(i)}^{-1} \boldsymbol{\eta}_1^T(i+l,j)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i+l,j)\end{aligned} \tag{10}$$

$$\begin{aligned}\hat{V}(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^T(i,j)\boldsymbol{P}_2(\lambda)\boldsymbol{x}(i,j) \\ &+ \sum_{r=-\beta(j)}^{-1} \boldsymbol{x}^T(i,j+r)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i,j+r) \\ &+ \sum_{\theta=2-\beta_h}^{1-\beta_l}\sum_{r=-1+\theta}^{-1} \boldsymbol{x}^T(i,j+r)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i,j+r) \\ &+ \sum_{\theta=-\beta_h}^{-1}\sum_{r=\theta}^{-1} \boldsymbol{\eta}_2^T(i,j+r)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i,j+r) \\ &+ \sum_{r=-\beta(j)}^{-1} \boldsymbol{\eta}_2^T(i,j+r)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i,j+r)\end{aligned} \tag{11}$$

and

$$\boldsymbol{\eta}_1(i,j+1) = \boldsymbol{x}(i+1,j+1) - \boldsymbol{x}(i,j+1) \tag{12a}$$
$$\boldsymbol{\eta}_2(i+1,j) = \boldsymbol{x}(i+1,j+1) - \boldsymbol{x}(i+1,j). \tag{12b}$$

Taking the forward difference of the Lyapunov–Krasovskii function along the trajectories of the system (3) yields

$$\begin{aligned}\Delta V(\boldsymbol{x}(i,j)) &= \bar{V}(\boldsymbol{x}(i+1,j+1)) - \bar{V}(\boldsymbol{x}(i,j+1)) \\ &\quad + \hat{V}(\boldsymbol{x}(i+1,j+1)) - \hat{V}(\boldsymbol{x}(i+1,j)) \\ &= \Delta\bar{V}(\boldsymbol{x}(i,j)) + \Delta\hat{V}(\boldsymbol{x}(i,j))\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\Delta\bar{V}(\boldsymbol{x}(i,j)) \\ &\quad = \boldsymbol{x}^T(i+1,j+1)\boldsymbol{P}_1(\lambda)\boldsymbol{x}(i+1,j+1) \\ &\qquad - \boldsymbol{x}^T(i,j+1)\boldsymbol{P}_1(\lambda)\boldsymbol{x}(i,j+1) \\ &\qquad + \boldsymbol{x}^T(i,j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i,j+1) \\ &\qquad - \boldsymbol{x}^T(i-\alpha(i),j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i-\alpha(i),j+1) \\ &\qquad + \alpha_{hl}\boldsymbol{x}^T(i,j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i,j+1) \\ &\qquad - \sum_{l=-\alpha_h+1}^{-\alpha_l} \boldsymbol{x}^T(i+l,j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i+l,j+1) \\ &\qquad + \alpha_h\boldsymbol{\eta}_1^T(i,j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i,j+1) \\ &\qquad - \sum_{l=-\alpha_h}^{-1} \boldsymbol{\eta}_1^T(i+l,j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i+l,j+1) \\ &\qquad + \boldsymbol{\eta}_1^T(i,j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i,j+1) \\ &\qquad - \boldsymbol{\eta}_1^T(i-\alpha(i),j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i-\alpha(i),j+1)\end{aligned} \tag{14}$$

$$\begin{aligned}&\Delta\hat{V}(\boldsymbol{x}(i,j)) \\ &\quad = \boldsymbol{x}^T(i+1,j+1)\boldsymbol{P}_2(\lambda)\boldsymbol{x}(i+1,j+1) \\ &\qquad - \boldsymbol{x}^T(i+1,j)\boldsymbol{P}_2(\lambda)\boldsymbol{x}(i+1,j) \\ &\qquad + \boldsymbol{x}^T(i+1,j)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1,j)\end{aligned}$$

$$
\begin{aligned}
&- \boldsymbol{x}^T(i+1, j-\beta(j))\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1, j-\beta(j)) \\
&+ \beta_{hl}\boldsymbol{x}^T(i+1, j)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1, j) \\
&- \sum_{r=-\beta_h+1}^{-\beta_l} \boldsymbol{x}^T(i+1, j+r)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1, j+r) \\
&+ \beta_h\boldsymbol{\eta}_2^T(i+1, j)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j) \\
&- \sum_{r=-\beta_h}^{-1} \boldsymbol{\eta}_2^T(i+1, j+r)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j+r) \\
&+ \boldsymbol{\eta}_2^T(i+1, j)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j) \\
&- \boldsymbol{\eta}_2^T(i+1, j-\beta(j))\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j-\beta(j)) \quad (15)
\end{aligned}
$$

Now,

$$
\begin{aligned}
&- \sum_{l=-\alpha_h+1}^{-\alpha_l} \boldsymbol{x}^T(i+l, j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i+l, j+1) \\
&\le -\boldsymbol{x}^T(i-\alpha(i), j+1)\boldsymbol{Q}_1(\lambda)\boldsymbol{x}(i-\alpha(i), j+1) \quad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
&- \sum_{l=-\alpha_h}^{-1} \boldsymbol{\eta}_1^T(i+l, j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i+l, j+1) \\
&\le -\boldsymbol{\eta}_1^T(i-\alpha_h, j+1)\boldsymbol{Z}_1(\lambda)\boldsymbol{\eta}_1(i-\alpha_h, j+1) \quad (17)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&- \sum_{r=-\beta_h+1}^{-\beta_l} \boldsymbol{x}^T(i+1, j+r)\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1, j+r) \\
&\le -\boldsymbol{x}^T(i+1, j-\beta(j))\boldsymbol{Q}_2(\lambda)\boldsymbol{x}(i+1, j-\beta(j)) \quad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
&- \sum_{r=-\beta_h}^{-1} \boldsymbol{\eta}_2^T(i+1, j+r)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j+r) \\
&\le -\boldsymbol{\eta}_2^T(i+1, j-\beta_h)\boldsymbol{Z}_2(\lambda)\boldsymbol{\eta}_2(i+1, j-\beta_h) \quad (19)
\end{aligned}
$$

Employing (12–19) the following inequality is obtained

$$
\Delta V(\boldsymbol{x}(i, j)) \le \boldsymbol{\xi}^T(i, j)\boldsymbol{\Psi}(\lambda)\boldsymbol{\xi}(i, j) \quad (20)
$$

where

$$
\begin{aligned}
\boldsymbol{\xi}^T(i, j) = \big[&\boldsymbol{x}^T(i+1, j+1)\,\boldsymbol{x}^T(i, j+1)\,\boldsymbol{x}^T(i+1, j) \\
&\boldsymbol{x}^T(i-\alpha(i), j+1)\,\boldsymbol{x}^T(i+1, j-\beta(j)) \\
&\boldsymbol{\eta}_1^T(i-\alpha_h, j+1)\,\boldsymbol{\eta}_2^T(i+1, j-\beta_h) \\
&\boldsymbol{\eta}_1^T(i-\alpha(i), j+1)\,\boldsymbol{\eta}_2^T(i+1, j-\beta(j))\big] \quad (21)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Psi}(\lambda) = \text{block-diag}\Big\{&\boldsymbol{M}(\lambda), -\boldsymbol{Q}_1(\lambda), -\boldsymbol{Q}_2(\lambda), -\boldsymbol{Z}_1(\lambda), \\
&- \boldsymbol{Z}_2(\lambda), -\boldsymbol{Z}_1(\lambda), -\boldsymbol{Z}_2(\lambda)\Big\} \quad (22)
\end{aligned}
$$

and

$$
\boldsymbol{M}(\lambda) = \begin{bmatrix} \boldsymbol{M}_{11}(\lambda) & -(\alpha_h+1)\boldsymbol{Z}_1(\lambda) & -(\beta_h+1)\boldsymbol{Z}_2(\lambda) \\ * & \boldsymbol{M}_{22}(\lambda) & \boldsymbol{0} \\ * & * & \boldsymbol{M}_{33}(\lambda) \end{bmatrix}
$$

$$
\begin{aligned}
\boldsymbol{M}_{11}(\lambda) = \boldsymbol{P}_1(\lambda) + (\alpha_h+1)\boldsymbol{Z}_1(\lambda) + \boldsymbol{P}_2(\lambda)& \\
+ (\beta_h+1)\boldsymbol{Z}_2(\lambda)& \quad (23a)
\end{aligned}
$$

$$
\boldsymbol{M}_{22}(\lambda) = (\alpha_{hl}+1)\boldsymbol{Q}_1(\lambda) - \boldsymbol{P}_1(\lambda) + (\alpha_h+1)\boldsymbol{Z}_1(\lambda) \quad (23b)
$$

$$
\boldsymbol{M}_{33}(\lambda) = (\beta_{hl}+1)\boldsymbol{Q}_2(\lambda) - \boldsymbol{P}_2(\lambda) + (\beta_h+1)\boldsymbol{Z}_2(\lambda) \quad (23c)
$$

From (20), one can observe that $\Delta V(\boldsymbol{x}(i, j)) < 0$ for $\boldsymbol{\xi}(i, j) \neq \boldsymbol{0}$ if $\boldsymbol{\Psi}(\lambda) < \boldsymbol{0}$. Further, $\Delta V(\boldsymbol{x}(i, j)) = 0$ only when $\boldsymbol{\xi}(i, j) = \boldsymbol{0}$. Now, with the help of Definition 1, and following Xu and Yu (2009b), it can be easily shown that $\boldsymbol{x}(i, j) \longrightarrow \boldsymbol{0}$ as $i \longrightarrow \infty$ and/or $j \longrightarrow \infty$ for any boundary conditions satisfying (1d) if $\Delta V(\boldsymbol{x}(i, j)) < 0$. Thus, $\boldsymbol{\Psi}(\lambda) < \boldsymbol{0}$ is a sufficient condition for the asymptotic stability of the system (3). From equation (3a)

$$
\begin{aligned}
\boldsymbol{0} = \boldsymbol{x}(i+1, j+1) - \tilde{\boldsymbol{A}}_1(\lambda)\boldsymbol{x}(i, j+1) - \tilde{\boldsymbol{A}}_2(\lambda)\boldsymbol{x}(i+1, j)& \\
- \boldsymbol{A}_{d_1}(\lambda)\boldsymbol{x}(i-\alpha(i), j+1) - \boldsymbol{A}_{d_2}(\lambda)\boldsymbol{x}(i+1, j-\beta(j))& \quad (24a)
\end{aligned}
$$

or

$$
\boldsymbol{0} = \boldsymbol{B}(\lambda)\boldsymbol{\xi}^T(i, j) \quad (24b)
$$

where

$$
\boldsymbol{B}(\lambda) = [\boldsymbol{I} \; -\tilde{\boldsymbol{A}}_1(\lambda) \; -\tilde{\boldsymbol{A}}_2(\lambda) \; -\boldsymbol{A}_{d_1}(\lambda) \; -\boldsymbol{A}_{d_2}(\lambda)\,\boldsymbol{0}\,\boldsymbol{0}\,\boldsymbol{0}\,\boldsymbol{0}] \quad (24c)
$$

Using (24) and employing the well-known Finsler's Lemma (see Lemma 1 of Miranda and Leite 2011) to (20) the following condition is obtained

$$
\boldsymbol{\Psi}(\lambda) + \boldsymbol{X}(\lambda)\boldsymbol{B}(\lambda) + \boldsymbol{B}^T(\lambda)\boldsymbol{X}^T(\lambda) < \boldsymbol{0} \quad (25)
$$

where $\boldsymbol{X}(\lambda) \in \mathbb{R}^{9n\times n}$.
Assuming an affine structure in λ for matrices $\boldsymbol{P}_1(\lambda)$, $\boldsymbol{P}_2(\lambda)$, $\boldsymbol{Q}_1(\lambda)$, $\boldsymbol{Q}_2(\lambda)$, $\boldsymbol{Z}_1(\lambda)$, $\boldsymbol{Z}_2(\lambda)$, these matrices can be written as $\boldsymbol{P}_1(\lambda) = \sum_{s=1}^{p} \lambda_s P_s$, $\boldsymbol{P}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{P}_{2s}$, $\boldsymbol{Q}_1(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Q}_{1s}$, $\boldsymbol{Q}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Q}_{2s}$, $\boldsymbol{Z}_1(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Z}_{1s}$, $\boldsymbol{Z}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Z}_{2s}$ and $\boldsymbol{X}(\lambda) = \boldsymbol{X}$ and λ belonging to the convex hull. Then, (25) can be recovered as a convex combination of equations in (4). Similarly, M can

be recovered as a convex combination of M_s given in Eq. (7). This completes the proof of Theorem 1.

Remark 1 The use of Finsler's lemma may help in obtaining less conservative results by the introduction of a matrix $\boldsymbol{X}$. The other advantage of using Finsler's lemma is the fact that the synthesis of controller gain $\boldsymbol{K}$ becomes decoupled from the Lyapunov matrices.

Next, a criterion is presented for the existence of a robust feedback gain controller such that the system (1) is asymptotically stable.

3.2 Synthesis of Robust Controller

We define the following theorem for the existence of a robust feedback gain controller.

Theorem 2 *The system* (1a) *with delay given by* (1b)*, boundary conditions represented by* (1d)*, is robustly stabilizable with the control law given by* (2) *if there exist positive definite symmetric matrices* $\boldsymbol{P}_{1s}$, $\boldsymbol{P}_{2s}$, $\boldsymbol{Q}_{1s}$, $\boldsymbol{Q}_{2s}$, $\boldsymbol{Z}_{1s}$, $\boldsymbol{Z}_{2s} \in \mathbb{R}^{n\times n}$, *matrices* $\boldsymbol{U} \in \mathbb{R}^{q\times n}$ *and* $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ *such that*

$$\text{block-diag}\left\{\bar{\boldsymbol{M}}_s, -\boldsymbol{Q}_{1s}, -\boldsymbol{Q}_{2s}, -\boldsymbol{Z}_{1s}, -\boldsymbol{Z}_{2s}, -\boldsymbol{Z}_{1s}, -\boldsymbol{Z}_{2s}\right\} < \boldsymbol{0},$$
$$s = 1, 2, \ldots, p \tag{26}$$

where

$$\bar{\boldsymbol{M}}_s = \begin{bmatrix} \bar{\boldsymbol{M}}_{11s} & \bar{\boldsymbol{M}}_{12s} & \bar{\boldsymbol{M}}_{13s} \\ * & \boldsymbol{M}_{22s} & \boldsymbol{0} \\ * & * & \boldsymbol{M}_{33s} \end{bmatrix},$$

$$\bar{\boldsymbol{M}}_{11s} = \boldsymbol{P}_{1s} + (\alpha_h + 1)\boldsymbol{Z}_{1s} + \boldsymbol{P}_{2s} + (\beta_h + 1)\boldsymbol{Z}_{2s} + \boldsymbol{S}^T + \boldsymbol{S}, \tag{27a}$$
$$\bar{\boldsymbol{M}}_{12s} = -(\alpha_h + 1)\boldsymbol{Z}_{1s} - \boldsymbol{S}^T \boldsymbol{A}_{1s}^T - \boldsymbol{U}^T \boldsymbol{B}_{1s}^T, \tag{27b}$$
$$\bar{\boldsymbol{M}}_{13s} = -(\beta_h + 1)\boldsymbol{Z}_{2s} - \boldsymbol{S}^T \boldsymbol{A}_{2s}^T - \boldsymbol{U}^T \boldsymbol{B}_{2s}^T. \tag{27c}$$

Proof Note that

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) = \tilde{\boldsymbol{A}}_1(\lambda)\boldsymbol{x}(i, j+1) + \tilde{\boldsymbol{A}}_2(\lambda)\boldsymbol{x}(i+1, j) \\ + \boldsymbol{A}_{d_1}(\lambda)\boldsymbol{x}(i - \alpha(i), j+1) \\ + \boldsymbol{A}_{d_2}(\lambda)\boldsymbol{x}(i+1, j - \beta(j))\end{aligned} \tag{28}$$

is stable if and only if

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) = \tilde{\boldsymbol{A}}_1^T(\lambda)\boldsymbol{x}(i, j+1) + \tilde{\boldsymbol{A}}_2^T(\lambda)\boldsymbol{x}(i+1, j) \\ + \boldsymbol{A}_{d_1}^T(\lambda)\boldsymbol{x}(i - \alpha(i), j+1) \\ + \boldsymbol{A}_{d_2}^T(\lambda)\boldsymbol{x}(i+1, j - \beta(j))\end{aligned} \tag{29}$$

is stable (Miranda and Leite 2011). In Theorem 1, replacing $\tilde{\boldsymbol{A}}_{1s}$ and $\tilde{\boldsymbol{A}}_{2s}$ by $\tilde{\boldsymbol{A}}_{1s}^T = (\boldsymbol{A}_{1s} + \boldsymbol{B}_{1s}\boldsymbol{K})^T$ and $\tilde{\boldsymbol{A}}_{2s}^T = (\boldsymbol{A}_{2s} + \boldsymbol{B}_{2s}\boldsymbol{K})^T$, respectively and defining $\boldsymbol{X} = [\boldsymbol{S}\, \boldsymbol{0}_{n\times 8n}]^T$ where $\boldsymbol{K}\boldsymbol{S} = \boldsymbol{U}$ one obtains Theorem 2. □

3.3 Stability of Uncertain 2-D Discrete Systems in the Presence of Delay and Saturation Finite Wordlength Nonlinearities

Consider a class of 2-D discrete uncertain systems represented by the FMSLSS model (Fornasini and Marchesini 1978) with delays under the influence of saturation finite wordlength nonlinearities. Specifically, the system under consideration is described by

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) &= \boldsymbol{f}(\boldsymbol{y}(i, j)) \\ &= [f_1(y_1(i, j))\; f_2(y_2(i, j)) \cdots f_n(y_n(i, j))]^T,\end{aligned} \tag{30a}$$

$$\begin{aligned}y(i, j) &= \boldsymbol{A}_1(\lambda)\boldsymbol{x}(i, j+1) + \boldsymbol{A}_2(\lambda)\boldsymbol{x}(i+1, j) \\ &\quad + \boldsymbol{A}_{d_1}(\lambda)\boldsymbol{x}(i - \alpha(i), j+1) \\ &\quad + \boldsymbol{A}_{d_2}(\lambda)\boldsymbol{x}(i+1, j - \beta(j)) \\ &= [y_1(i, j)\; y_2(i, j) \ldots y_n(i, j)]^T,\end{aligned} \tag{30b}$$

where $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$ are horizontal coordinate and vertical coordinate, respectively; $\boldsymbol{x}(i, j) \in \mathbb{R}^n$ is the local state vector; $\alpha(i)$ and $\beta(j)$ are delays along horizontal direction and vertical direction, respectively, satisfying (1b). The system matrices $\boldsymbol{A}_1(\lambda)$, $\boldsymbol{A}_2(\lambda)$, $\boldsymbol{A}_{d_1}(\lambda)$, $\boldsymbol{A}_{d_2}(\lambda) \in \mathbb{R}^{n\times n}$, are subject to the uncertainties and belong to the polytope $\hat{\Omega}$ given by

$$\begin{aligned}\hat{\Omega} = \Bigg\{ &\Big[\boldsymbol{A}_1(\lambda), \boldsymbol{A}_2(\lambda), \boldsymbol{A}_{d_1}(\lambda), \boldsymbol{A}_{d_2}(\lambda)\Big] \\ &:= \sum_{s=1}^{p} \lambda_s \Big[\boldsymbol{A}_{1s}, \boldsymbol{A}_{2s}, \boldsymbol{A}_{d_{1s}}, \boldsymbol{A}_{d_{2s}}\Big], \\ &\sum_{s=1}^{p} \lambda_s = 1, \lambda_s \geq 0 \Bigg\}.\end{aligned} \tag{31}$$

The saturation nonlinearities given by

$$\left.\begin{aligned} f_k(y_k(i, j)) &= y_k(i, j), & |y_k(i, j)| &\leq 1 \\ f_k(y_k(i, j)) &= 1, & y_k(i, j) &> 1 \\ f_k(y_k(i, j)) &= -1, & y_k(i, j) &< -1 \end{aligned}\right\}, \quad k = 1, 2, \ldots, n \tag{32}$$

are under consideration, and the boundary conditions are defined by (1d).

Equations (30)–(32), (1b) and (1d) can be used to represent finite wordlength implementation of discrete systems represented by the Darboux equation (Dey and Kar 2014; Tadepalli et al. 2016), heat exchangers (Tadepalli et al. 2015a), networked controlled systems, wireless sensor platforms employing fixed-point digital processors (Sumanasena and Bauer 2011), iterative learning control schemes (Tadepalli et al. 2015a) and so on.

In what follows, we will establish delay-dependent stability criteria for the asymptotic stability of the system given by (30)–(32), (1b) and (1d).

Theorem 3 *The system represented by (30)–(32), (1b) and (1d) is asymptotically stable if there exist positive definite symmetric matrices* $\boldsymbol{P}_{1s}$, $\boldsymbol{P}_{2s}$, $\boldsymbol{Q}_{1s}$, $\boldsymbol{Q}_{2s}$, $\boldsymbol{Z}_{1s}$, $\boldsymbol{Z}_{2s} \in \mathbb{R}^{n\times n}$ $(s = 1, 2, \ldots, p)$, *such that*

$$\boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{\Gamma}_{11s} & \boldsymbol{\Gamma}_{12s} & \boldsymbol{\Gamma}_{13s} & \boldsymbol{C}^T\boldsymbol{A}_{d1s} & \boldsymbol{C}^T\boldsymbol{A}_{d2s} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \boldsymbol{\Gamma}_{22s} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \boldsymbol{\Gamma}_{33s} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{Q}_{1s} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\boldsymbol{Q}_{2s} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -\boldsymbol{Z}_{1s} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\boldsymbol{Z}_{2s} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & -\boldsymbol{Z}_{1s} & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\boldsymbol{Z}_{2s} \end{bmatrix} < \mathbf{0} \tag{33}$$

where

$$\boldsymbol{\Gamma}_{11s} = \boldsymbol{P}_{1s} + \boldsymbol{P}_{2s} + (\alpha_h + 1)\boldsymbol{Z}_{1s} + (\beta_h + 1)\boldsymbol{Z}_{2s} - (\boldsymbol{C} + \boldsymbol{C}^T) \tag{34a}$$

$$\boldsymbol{\Gamma}_{12s} = \boldsymbol{C}^T\boldsymbol{A}_{1s} - (\alpha_h + 1)\boldsymbol{Z}_{1s} \tag{34b}$$

$$\boldsymbol{\Gamma}_{13s} = \boldsymbol{C}^T\boldsymbol{A}_{2s} - (\beta_h + 1)\boldsymbol{Z}_{2s} \tag{34c}$$

$$\boldsymbol{\Gamma}_{22s} = -\boldsymbol{P}_{1s} + (\alpha_{hl} + 1)\boldsymbol{Q}_{1s} + (\alpha_h + 1)\boldsymbol{Z}_{1s} \tag{34d}$$

$$\boldsymbol{\Gamma}_{33s} = -\boldsymbol{P}_{2s} + (\beta_{hl} + 1)\boldsymbol{Q}_{2s} + (\beta_h + 1)\boldsymbol{Z}_{2s} \tag{34e}$$

and $\boldsymbol{C} = [c_{uv}] \in \mathbb{R}^{n\times n}$ *denotes a matrix (Kar 2007; Tadepalli and Kandanvli 2016) defined by*

$$c_{uu} = \sum_{v=1, v\neq u}^{n} (\alpha_{uv} + \beta_{uv}), \quad u = 1, 2, \ldots, n, \tag{35a}$$

$$c_{uv} = \alpha_{uv} - \beta_{uv}, \quad u, v = 1, 2, \ldots, n \ (u \neq v), \tag{35b}$$

$$\alpha_{uv} > 0, \ \beta_{uv} > 0, \quad u, v = 1, 2, \ldots, n \ (u \neq v), \tag{35c}$$

where for $n = 1$, $\boldsymbol{C}$ *corresponds to a scalar* $\mu > 0$.

Proof Let

$$\boldsymbol{\eta}_1(i, j+1) = \boldsymbol{x}(i+1, j+1) - \boldsymbol{x}(i, j+1) = \boldsymbol{f}(\boldsymbol{y}(i, j)) - \boldsymbol{x}(i, j+1) \tag{36a}$$

$$\boldsymbol{\eta}_2(i+1, j) = \boldsymbol{x}(i+1, j+1) - \boldsymbol{x}(i+1, j) = \boldsymbol{f}(\boldsymbol{y}(i, j)) - \boldsymbol{x}(i+1, j). \tag{36b}$$

By choosing the Lyapunov–Krasovskii function (9)–(11) and employing (13)–(19) and (36) to obtain the forward difference of the Lyapunov–Krasovskii function along the trajectories of the system (30)–(32), the following inequality can be obtained

$$\Delta V(\boldsymbol{x}(i, j)) \leq \hat{\boldsymbol{\xi}}^T(i, j)\boldsymbol{\Gamma}(\lambda)\hat{\boldsymbol{\xi}}(i, j) - \delta \tag{37}$$

where

$$\boldsymbol{\Gamma}(\lambda) = \begin{bmatrix} \boldsymbol{\Gamma}_{11}(\lambda) & \boldsymbol{\Gamma}_{12}(\lambda) & \boldsymbol{\Gamma}_{13}(\lambda) & \boldsymbol{C}^T\boldsymbol{A}_{d1}(\lambda) & \boldsymbol{C}^T\boldsymbol{A}_{d2}(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \boldsymbol{\Gamma}_{22}(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \boldsymbol{\Gamma}_{33}(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{Q}_1(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\boldsymbol{Q}_2(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -\boldsymbol{Z}_1(\lambda) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\boldsymbol{Z}_2(\lambda) & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & -\boldsymbol{Z}_1(\lambda) & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\boldsymbol{Z}_2(\lambda) \end{bmatrix} < \mathbf{0} \tag{38}$$

where

$$\boldsymbol{\Gamma}_{11}(\lambda) = \boldsymbol{P}_1(\lambda) + \boldsymbol{P}_2(\lambda) + (\alpha_h + 1)\boldsymbol{Z}_1(\lambda) + (\beta_h + 1)\boldsymbol{Z}_2(\lambda) - (\boldsymbol{C} + \boldsymbol{C}^T) \quad (39a)$$

$$\boldsymbol{\Gamma}_{12}(\lambda) = \boldsymbol{C}^T \boldsymbol{A}_1(\lambda) - (\alpha_h + 1)\boldsymbol{Z}_1(\lambda) \quad (39b)$$

$$\boldsymbol{\Gamma}_{13}(\lambda) = \boldsymbol{C}^T \boldsymbol{A}_2(\lambda) - (\beta_h + 1)\boldsymbol{Z}_2(\lambda) \quad (39c)$$

$$\boldsymbol{\Gamma}_{22}(\lambda) = -\boldsymbol{P}_1(\lambda) + (\alpha_{hl} + 1)\boldsymbol{Q}_1(\lambda) + (\alpha_h + 1)\boldsymbol{Z}_1(\lambda) \quad (39d)$$

$$\boldsymbol{\Gamma}_{33}(\lambda) = -\boldsymbol{P}_2(\lambda) + (\beta_{hl} + 1)\boldsymbol{Q}_2(\lambda) + (\beta_h + 1)\boldsymbol{Z}_2(\lambda) \quad (39e)$$

$$\begin{aligned}\delta &= \sum_{u=1}^{n} 2[y_u(i,j) - f_u(y_u(i,j))] \times \left[\sum_{v=1, v\neq u}^{n} \{(\alpha_{uv} + \beta_{uv}) f_u(y_u(i,j)) + (\alpha_{uv} - \beta_{uv}) f_v(y_v(i,j))\} \right] \\ &= \boldsymbol{y}^T(i,j)\boldsymbol{C}\boldsymbol{f}(\boldsymbol{y}(i,j)) + \boldsymbol{f}^T(\boldsymbol{y}(i,j))\boldsymbol{C}^T\boldsymbol{y}(i,j) \\ &\quad - \boldsymbol{f}^T(\boldsymbol{y}(i,j))(\boldsymbol{C} + \boldsymbol{C}^T)\boldsymbol{f}(\boldsymbol{y}(i,j)), \end{aligned} \quad (40)$$

$$\begin{aligned}\hat{\boldsymbol{\xi}}^T(i,j) = \big[&\boldsymbol{f}^T(\boldsymbol{y}(i,j))\, \boldsymbol{x}^T(i, j+1)\, \boldsymbol{x}^T(i+1, j) \\ &\boldsymbol{x}^T(i - \alpha(i), j+1)\, \boldsymbol{x}^T(i+1, j - \beta(j)) \\ &\boldsymbol{\eta}_1^T(i - \alpha_h, j+1)\, \boldsymbol{\eta}_2^T(i+1, j - \beta_h) \\ &\boldsymbol{\eta}_1^T(i - \alpha(i), j+1)\, \boldsymbol{\eta}_2^T(i+1, j - \beta(j))\big] \end{aligned} \quad (41)$$

Observe that, for the saturation nonlinearities given by (32) alongwith (35), the quantity δ (see (40)) is nonnegative (Kar 2007; Tadepalli and Kandanvli 2016). From (37), it is clear that $\Delta V(\boldsymbol{x}(i,j)) < 0$ for $\hat{\boldsymbol{\xi}}(i,j) \neq \boldsymbol{0}$ if $\boldsymbol{\Gamma}(\lambda) < \boldsymbol{0}$. Further, $\Delta V(\boldsymbol{x}(i,j)) = 0$ only when $\hat{\boldsymbol{\xi}}(i,j) = \boldsymbol{0}$. With the help of Definition 1, and following Xu and Yu (2009b), it can be easily shown that $\boldsymbol{x}(i,j) \longrightarrow \boldsymbol{0}$ as $i \longrightarrow \infty$ and/or $j \longrightarrow \infty$ for any boundary conditions satisfying (1d) if $\Delta V(\boldsymbol{x}(i,j)) < 0$. Thus, $\boldsymbol{\Gamma}(\lambda) < \boldsymbol{0}$ is a sufficient condition for the asymptotic stability of the system represented by (30)–(32), (1b) and (1d).

It may be observed that the condition (38) is non convex due to the continuous nature of λ. Thus to obtain a convex condition, i.e., Theorem 3, the system matrices and the matrices $\boldsymbol{P}_1(\lambda), \boldsymbol{P}_2(\lambda), \boldsymbol{Q}_1(\lambda), \boldsymbol{Q}_2(\lambda), \boldsymbol{Z}_1(\lambda), \boldsymbol{Z}_2(\lambda)$ are assumed to have a convex formulation, i.e., $\boldsymbol{P}_1(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{P}_{1s}$, $\boldsymbol{P}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{P}_{2s}$, $\boldsymbol{Q}_1(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Q}_{1s}$, $\boldsymbol{Q}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Q}_{2s}$, $\boldsymbol{Z}_1(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Z}_{1s}$, and $\boldsymbol{Z}_2(\lambda) = \sum_{s=1}^{p} \lambda_s \boldsymbol{Z}_{2s}$. This yields equation (33) from (38).This completes the proof of Theorem 3.

The criteria presented are in the form of linear matrix inequality (LMI) which can be solved easily in the MATLAB environment employing YALMIP 3.0 (Löfberg 2004) parser alongwith SeDuMi 1.21 (Sturm 1999) solver.

Remark 2 It is well known that delay-dependent criteria generally have a higher computational burden as compared to delay-independent criteria. So it is important to estimate the computational burden of the presented criteria.
Computational burden depends on the solver used and can be calculated by knowing the number of rows (L) and the number of decision variables (M) of the LMI. For Theorem 1, the number of rows $L_1 = 15np$ and the number of decision variables $M_1 = 3n(n+1)p + 9n^2$. For Theorem 2, the number of rows $L_2 = 15np$ and the number of decision variables $M_2 = 3n(n+1)p + pn + n^2$, while for Theorem 3, number of rows $L_3 = 2n^2 + (15p - 2)n$ and number of decision variables $M_3 = (3p+2)n^2 + (3p-2)n$. Using SeDuMi solver, for a given number of rows L and a given number of decision variables M the numerical complexity is proportional to $M^2 L^{\frac{5}{2}} + L^{\frac{7}{2}}$ (Sturm 1999).

Remark 3 The criterion presented in Theorem 3 is able to determine the asymptotic stability of 2-D discrete systems in the presence of saturation finite wordlength nonlinearities, delays in the state and uncertainties modeled as belonging to a polytope. To the best of authors' knowledge the study of discrete-time 1-D systems in the presence of finite wordlength nonlinearities and polytopic uncertainties has not been explicitly done, although numerous results exist for such 1-D systems employing norm-bounded modeling of uncertainties. In this light, the methodology used for the proof of Theorem 3 may be employed for obtaining results for finite wordlength implementation of discrete-time 1-D systems.

Next, we present a criterion for the stability of a class of systems represented by (30) in the absence of uncertainties.

Corollary 1 *Consider the system represented by (30)–(32), (1b) and (1d) in the absence of uncertainties, that is the system becomes*

$$\begin{aligned}\boldsymbol{x}(i+1, j+1) &= \boldsymbol{f}(\boldsymbol{y}(i,j)) \\ &= [f_1(y_1(i,j))\, f_2(y_2(i,j)) \cdots f_n(y_n(i,j))]^T, \end{aligned} \quad (42a)$$

$$\begin{aligned} y(i,j) &= \boldsymbol{A}_1\boldsymbol{x}(i, j+1) + \boldsymbol{A}_2\boldsymbol{x}(i+1, j) \\ &\quad + \boldsymbol{A}_{d_1}\boldsymbol{x}(i - \alpha(i), j+1) \\ &\quad + \boldsymbol{A}_{d_2}\boldsymbol{x}(i+1, j - \beta(j)) \\ &= [y_1(i,j)\, y_2(i,j) \ldots y_n(i,j)]^T. \end{aligned} \quad (42b)$$

Then, the system represented by (42), (1b) and (1d) is asymptotically stable if there exists positive definite symmetric matrices $\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Z}_1, \boldsymbol{Z}_2 \in \mathbb{R}^{n\times n}$*, such that*

$$\hat{\boldsymbol{\Gamma}} = \begin{bmatrix} \hat{\boldsymbol{\Gamma}}_{11} & \hat{\boldsymbol{\Gamma}}_{12} & \hat{\boldsymbol{\Gamma}}_{13} & \boldsymbol{C}^T\boldsymbol{A}_{d1} & \boldsymbol{C}^T\boldsymbol{A}_{d2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & \hat{\boldsymbol{\Gamma}}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \hat{\boldsymbol{\Gamma}}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{Q}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\boldsymbol{Q}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -\boldsymbol{Z}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\boldsymbol{Z}_2 & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & -\boldsymbol{Z}_1 & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\boldsymbol{Z}_2 \end{bmatrix} < \mathbf{0} \quad (43)$$

where

$$\hat{\boldsymbol{\Gamma}}_{11} = \boldsymbol{P}_1 + \boldsymbol{P}_2 + (\alpha_h + 1)\boldsymbol{Z}_1 + (\beta_h + 1)\boldsymbol{Z}_2 - (\boldsymbol{C} + \boldsymbol{C}^T) \quad (44a)$$

$$\hat{\boldsymbol{\Gamma}}_{12} = \boldsymbol{C}^T\boldsymbol{A}_1 - (\alpha_h + 1)\boldsymbol{Z}_1 \quad (44b)$$

$$\hat{\boldsymbol{\Gamma}}_{13} = \boldsymbol{C}^T\boldsymbol{A}_2 - (\beta_h + 1)\boldsymbol{Z}_2 \quad (44c)$$

$$\hat{\boldsymbol{\Gamma}}_{22} = -\boldsymbol{P}_1 + (\alpha_{hl} + 1)\boldsymbol{Q}_1 + (\alpha_h + 1)\boldsymbol{Z}_1 \quad (44d)$$

$$\hat{\boldsymbol{\Gamma}}_{33} = -\boldsymbol{P}_2 + (\beta_{hl} + 1)\boldsymbol{Q}_2 + (\beta_h + 1)\boldsymbol{Z}_2 \quad (44e)$$

Proof Consider the Lyapunov function

$$V(\boldsymbol{x}(i,j)) = V_1(\boldsymbol{x}(i,j)) + V_2(\boldsymbol{x}(i,j)) \quad (45)$$

where

$$\begin{aligned} V_1(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^T(i,j)\boldsymbol{P}_1\boldsymbol{x}(i,j) \\ &+ \sum_{l=-\alpha(i)}^{-1} \boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_1\boldsymbol{x}(i+l,j) \\ &+ \sum_{\theta=2-\alpha_h}^{1-\alpha_l} \sum_{l=-1+\theta}^{-1} \boldsymbol{x}^T(i+l,j)\boldsymbol{Q}_1\boldsymbol{x}(i+l,j) \\ &+ \sum_{\theta=-\alpha_h}^{-1} \sum_{l=\theta}^{-1} \boldsymbol{\eta}_1^T(i+l,j)\boldsymbol{Z}_1\boldsymbol{\eta}_1(i+l,j) \\ &+ \sum_{l=-\alpha(i)}^{-1} \boldsymbol{\eta}_1^T(i+l,j)\boldsymbol{Z}_1\boldsymbol{\eta}_1(i+l,j) \quad (46) \\ V_2(\boldsymbol{x}(i,j)) &= \boldsymbol{x}^T(i,j)\boldsymbol{P}_2\boldsymbol{x}(i,j) \\ &+ \sum_{r=-\beta(j)}^{-1} \boldsymbol{x}^T(i,j+r)\boldsymbol{Q}_2\boldsymbol{x}(i,j+r) \\ &+ \sum_{\theta=2-\beta_h}^{1-\beta_l} \sum_{r=-1+\theta}^{-1} \boldsymbol{x}^T(i,j+r)\boldsymbol{Q}_2\boldsymbol{x}(i,j+r) \\ &+ \sum_{\theta=-\beta_h}^{-1} \sum_{r=\theta}^{-1} \boldsymbol{\eta}_2^T(i,j+r)\boldsymbol{Z}_2\boldsymbol{\eta}_2(i,j+r) \\ &+ \sum_{r=-\beta(j)}^{-1} \boldsymbol{\eta}_2^T(i,j+r)\boldsymbol{Z}_2\boldsymbol{\eta}_2(i,j+r). \quad (47) \end{aligned}$$

Now, following the steps similar to the proof of Theorem 3 one can easily obtain the conditions presented in Corollary 1. □

Remark 4 A criterion for the system represented by (42) is also presented in Chen (2010). It may be observed that for the criterion presented in Theorem 1 of Chen (2010), the number of rows $L_4 = 24n$ and the number of decision variables is $M_4 = 9n^2 + 6n$, whereas for Corollary 1 the number of rows $L_5 = 13n + 2n^2$ and the number of decision variables is $M_5 = 5n^2 + n$, where n is the system order. On comparison one can easily arrive at the conclusion that Corollary 1 has a significantly smaller computational burden as compared to Theorem 1 of Chen (2010).

4 Examples

To demonstrate the effectiveness of the proposed criteria the following examples are presented.

Example 1 Consider the system (3) with system matrices (Tadepalli et al. 2016)

$$\tilde{\boldsymbol{A}}_1 = \begin{bmatrix} 0.02 & 1 \\ -0.09 & 0.15 \end{bmatrix}, \tilde{\boldsymbol{A}}_2 = \begin{bmatrix} 0.2 & 0 \\ 0.26 & 0.12 \end{bmatrix},$$
$$\boldsymbol{A}_{d_1} = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A}_{d_2} = \begin{bmatrix} 0.02 & 0 \\ 0 & -0.12 \end{bmatrix} \quad (48)$$

The system matrices $\tilde{\boldsymbol{A}}_1$, $\tilde{\boldsymbol{A}}_2$, $\boldsymbol{A}_{d_1}$, $\boldsymbol{A}_{d_2}$ are assumed to be under the influence of the multiplicative uncertainty $(1+\rho)$ where $\rho \geq 0$. This results in a polytopic uncertainty representation with the vertices

$$\tilde{\boldsymbol{A}}_{1s} = \tilde{\boldsymbol{A}}_1(1 + (-1)^s\rho) \quad (49a)$$

$$\tilde{\boldsymbol{A}}_{2s} = \tilde{\boldsymbol{A}}_2(1 + (-1)^s\rho) \quad (49b)$$

$$A_{d_{1s}} = A_{d_1}(1 + (-1)^s \rho) \quad (49c)$$
$$A_{d_{2s}} = A_{d_2}(1 + (-1)^s \rho), \ s = 1, 2. \quad (49d)$$

Now, by using $\rho = 0.01$, $\alpha_l = 3$, $\alpha_h = 7$, $\beta_l = 2$ and employing Theorem 1, the upper delay bound $\beta_h = 11$ is obtained for the asymptotic stability of the system (3).

Example 2 Consider the system (1) with the parameters (Peng and Hua 2015)

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0.1 \\ 0.2 & 0.65 \end{bmatrix}, A_{d_1} = \begin{bmatrix} 0 & 0.25 \\ -0.2 & 0 \end{bmatrix},$$
$$A_{d_2} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0.4 & 0.1 \\ 0.8 & 0 \end{bmatrix}. \quad (50)$$

The time-varying state delays are expected to satisfy $1 \le \alpha(i) \le 20, 3 \le \beta(j) \le 11$. It is easy to show that the system described by (50) can be represented by (49). Further, we consider $\rho = 0$ for representing a system in the absence of uncertainties. Using Theorem 2 we find that the system under consideration can be stabilized by a state feedback controller with $K = \begin{bmatrix} -0.1102 & -0.5480 \\ -0.6198 & 1.4798 \end{bmatrix}$.

By employing Theorem 2 in Peng and Hua (2015), the above system was found to be stabilizable using a suitable state feedback gain. But the advantage of the proposed Theorem 2 over the Theorem 2 in Peng and Hua (2015) is a significant reduction in the computational burden. It may also be noted that, by taking a nonzero value of ρ the effect of uncertainties may also be studied employing Theorem 2 which is an added advantage as the criterion presented in Peng and Hua (2015) does not take into account the effect of uncertainties.

Example 3 Consider a system represented by the following Darboux's equation (Xu and Yu 2009a)

$$\frac{\partial\theta(x,t)}{\partial x} = -\frac{\partial\theta(x,t)}{\partial t} - a_0\theta(x,t) - a_1\theta(x,t-\tau) + bu(x,t) \quad (51)$$

where $\theta(x,t)$ is the temperature at space $x \in [0, x_f]$ and time $t \in [0, \infty)$, τ is the time delay, a_0 and a_1 are real coefficients. Equation (51) may be used to describe thermal processes in chemical reactors, heat exchangers and pipe furnaces (Xu and Yu 2009a, b), etc. Equation (51) can be transformed into the 2-D FMSLSS model (Xu and Yu 2009b) with

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & (1 - \frac{\Delta t}{\Delta x} - a_0\Delta t) \end{bmatrix},$$
$$A_{d_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_{d_2} = \begin{bmatrix} 0 & 0 \\ 0 & -a_1\Delta t \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ b\Delta t \end{bmatrix}. \quad (52)$$

Let the system be under the influence of multiplicative uncertainty $(1+\rho)$, where $\rho = 0.01$. The various parameters of the system are assumed to be $\Delta t = 0.1$, $\Delta x = 0.4$, $a_0 = 5$, $a_1 = 1.2$ and $b = 1$. Employing YALMIP 3.0 Löfberg (2004) parser and SeDuMi 1.21 Sturm (1999) solver it is observed with the help of Theorem 1 that the above system under open loop (with u(i,j) = 0) is infeasible for $\beta_h = 18$, $\alpha_l = 3$, $\alpha_h = 9$ and $\beta_l = 2$.

Now, it remains to determine the closed-loop stability of the above system by employing a suitable controller. Using Theorem 2, it is observed that the closed-loop system is stable for the delay bounds $\alpha_l = 3$, $\alpha_h = 9$, $\beta_l = 2$ and $\beta_h = 18$ with a control law $u(i,j) = Kx(i,j)$ where $K = [-3.0332 - 1.0164]$. Table 1 shows how a delay-dependent robust control performance can be achieved by employing a suitable control law. It may be noted that it was possible to obtain a suitable controller for $\beta_h \to \infty$.

Table 1 Delay-dependent robust control performance

β_h	Controller gain K
10	[− 3.0171 − 1.0811]
18	[− 3.0332 − 1.0164]
30	[− 2.9608 − 1.2079]
50	[− 2.9245 − 1.3113]

Example 4 In this example we present the effect of finite wordlength implementation of the system (52).

Consider the system represented by (52) with $\Delta t = 0.1$, $\Delta x = 0.4$, $a_0 = 5$, $a_1 = 1.2$ and $b = 1$ implemented with a suitable control law $u(i,j) = Kx(i,j)$ where for example K = [-3.0332 -1.0164]. Now, under finite wordlength implementation, i.e., saturation nonlinearities (finite wordlength implementation of such system has been widely considered in the literature, see for example Dey and Kar (2014) and Tadepalli et al. (2016) and under the influence of uncertainties it remains to be found that for what values of delay ranges the above system is stable. Assuming $\alpha_l = 3, \alpha_h = 9, \beta_l = 2$ and multiplicative uncertainty $(1 + \rho)$, where $\rho = 0.01$ our objective is to determine the upper bound of $\beta(j)$, i.e., β_h. By iteratively solving it was observed that Theorem 3 was able to determine the asymptotic stability of the system under consideration for $\beta_h = 44$.

Example 5 In this example we compare the criterion presented in Corollary 1 with an existing criterion (Theorem 1) in Chen (2010). Consider a system represented by (42) with the parameters Chen (2010)

$$A_1 = \begin{bmatrix} -0.24 & -0.01 \\ 0.2 & 0.13 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.22 & 0.2 \\ 0.29 & -0.14 \end{bmatrix},$$

$$A_{d_1} = \begin{bmatrix} 0.1 & 0.03 \\ -0.05 & 0.12 \end{bmatrix}, \quad A_{d_2} = \begin{bmatrix} 0.05 & 0.08 \\ 0.12 & 0.1 \end{bmatrix} \tag{53}$$

and under the influence of saturation nonlinearities. It is observed that Corollary 1 is able to test the stability of the above system for $2 \le \alpha(i) \le 11$ and $3 \le \beta(j) \le 5$. Theorem 1 in Chen (2010) was also able to determine the asymptotic stability of the above system for the same delay range but with a heavier computational burden (see, Remark 4). By assuming $\alpha_l = \alpha_h$ and $\beta_l = \beta_h$ Corollary 1 can also be employed for the cases of systems with constant delays.

Recently, a criterion was proposed in Kokil (2017) for the study of 2-D discrete systems described by FMSLSS second model with constant delays and saturation nonlinearities. In Kokil (2017) additional restrictions are imposed on the system matrices A_1, A_2, A_{d_1} and A_{d_2}, i.e., the sum of modulus of all the elements in the first row of A_1, A_2, A_{d_1} and A_{d_2} must be greater than 1 and the sum of modulus of all the elements in the second row of A_1, A_2, A_{d_1} and A_{d_2} must be less than 1 in order to test the stability. Due to the above restrictions the system considered in this example falls outside the application scope of the criterion presented in Kokil (2017). Thus, Corollary 1 is advantageous as it does not employ any restrictions on the system matrices.

5 Conclusion

This paper presented a criterion (Theorem 1) for the robust stability analysis of uncertain 2-D discrete systems described by the FMSLSS model with delays. A criterion (Theorem 2) was proposed for the existence of a robust feedback gain controller such that the 2-D discrete systems under consideration are asymptotically stable. Finally, a new delay-dependent stability criterion (Theorem 3) is proposed for the asymptotic stability of a class of uncertain 2-D discrete systems in the presence of saturation finite wordlength nonlinearities and delays. Using examples it is shown that the criteria presented are advantageous in terms of computational burden and do not present any restrictions on the system matrices for their study. The presented results can be extended for the study of guaranteed cost control problem for multidimensional systems.

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