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Stabilization of Asynchronous Switched Systems with Constrained Control

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Abstract This paper investigates the method on stabilizing asynchronous switched linear systems with constrained inputs. Firstly, asynchronous systems mean the asynchronization between the system modes and state feedback controllers. Usually, it takes a period of time to identify which one of the state feedback controllers should be activated in practical application. Next, in consideration of the saturation effect of the controllers, this paper is aimed at stabilizing the systems with constrained inputs by mode-dependent average dwell time method. Besides, unstable subsystems are considered in this paper.

Keywords Stabilization · Switched systems · Mode-dependent average dwell time · Constrained input · Asynchronization

1 Introduction

The past decades have witnessed the fast growing interest in switched systems, which consist of many subsystems and a switching law (Zhao et al. 2015). Switched systems are used in practical applications widely, such as electrical networks (Cong 2016), sampled-data systems (Hetel et al. 2011) and sliding mode control systems (Ullah et al. 2016). Meanwhile, many theories of switched systems have been well established, such as reachability (Chao and Yu 2015), adaptive controller design (Niu et al. 2017) and output tracking control (Niu and Zhao 2013).

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Till now, the stability analysis of switched systems is still a hot field. The mode-dependent average dwell time (MDADT) (Zhao et al. 2012) based on multi-Lyapunov functions has been verified to be a very valid and flexible method. The results get by MDADT are much less conservative than those get by average dwell time. While using MDADT to analyze the stability of switched systems, every subsystem must be Hurwitz stable. Otherwise, a state feedback controller should be designed for the unstable subsystem. Fiacchini and Jungers (2014) and Minh et al. (2011) have studied the method on stabilizing the unstable subsystems. Besides, their researches are based on the assumption that the switching of system modes and state feedback controllers is simultaneous. However, in practical application, it usually takes a period of time to identify which subsystem is activated. And then the certain state feedback controller can be chosen (Liu et al. 2016). This causes the inaccuracy while applying the established theories to practice. Next, when a state feedback controller is designed, the control input is supposed to be set arbitrarily or infinitely. However, the controllers with saturation effect in practice limit this assumption. Over the past two decays, the stabilization study of control system (not switched system) was very abundant. Many excellent theories were established, such as Hu and Lin (2001) and Hu et al. (2002). In the book Hu and Lin (2001), the authors had introduced the solutions to stabilization of control systems with constrained input in detail. After several years, these theories were generalized to switched systems. In Chen et al. (2012), the authors studied stability condition of switched systems based on minimum dwell time method. In Remark 1 of Chen et al. (2012), the authors presented that “we cannot employ average dwell time approach”. In fact, it is incorrect. We can use not only average dwell time approach but also MDADT approach by choosing appropriate initial states. Via this way, we can design a switching signal more flexibly.

Although there are some researches toward stabilization of switched systems with constrained input, to the best of the authors' knowledge, no one considers asynchronous switching and constrained input jointly. For instance, authors in Ding et al. (2015) and Chen et al. (2012) studied the stability of switched systems with constrained input, but they all failed to analyze the asynchronous switching. Next, in Benzaouia et al. (2010), the arbitrary switching condition was studied. However, the results were restrictive to some extent because of the arbitrary switching law. Considering these, we aim at stabilizing the asynchronous switched linear systems with constrained inputs. It should be pointed out that all the subsystems may be unstable. Based on multi-Lyapunov functions, the mode-dependent average dwell time is obtained to guarantee that the system is exponentially stable. Furthermore, a Euclidean ball is found to limit the system states within it. By this method, the system can be stabilized with the constrained input. Moreover, most of the researchers assumed that each subsystem could be stabilized while analyzing the stability of switched systems with constrained input, such as Ding et al. (2015) and Wang and Zhao (2015). In fact, it is not impeccable. Because if certain subsystem state diverges quickly, the subsystem may not be stabilized by controllers with constrained input. In view of this, we present another two theorems (Theorems 2, 3). In the two theorems, the asynchronous switched systems with unstable subsystems can be exponentially stable with constrained control. Finally, while analyzing asynchronous switched systems, most of the researchers suppose that the delay time of state feedback controllers is a constant and equal. In this paper, the delay time of state feedback controllers can be different in different subsystems.

Notations Throughout this paper, the symbols used are quite standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ represent the n-dimensional Euclidean space and the space of $n \times n$ matrices with real entries, respectively. $\|\cdot\|$ denotes the Euclidean vector norm. For a matrix P_i, $\lambda_i^{\max}(P_i)$ and $\lambda_i^{\min}(P_i)$ are the maximal and minimal eigenvalue of P_i, respectively. $\lambda_{\min}$ is the minimum of $\lambda_i^{\min}$. For two vectors $x, y \in \mathbb{R}^n$, $x \preceq y$ denotes $x_i \le y_i$, $i = 1, 2, \ldots, n$. $R(\cdot)$ denotes the range of a matrix. Δt^s is the total running time of stable subsystems while Δt^u denotes the total running time of unstable subsystems. Next, Δt_i denotes the total running time of the ith subsystem. In this paper, max means the maximum. For example, $T_{\max}$, $\alpha_{\max}$, $\mu_{\max}$ and $\beta_{\max}$ denote the maximum of T_i, α_i, μ_i and β_i, respectively.

2 Preliminary

Consider the switched linear system

$$\dot{x} = A_{\sigma(t)}x + B_{\sigma(t)}u_{\sigma(t)} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\sigma(t) : [0, \infty) \to \mathbb{Z} = \{1, 2, \ldots, N\}$ is the switching law, N is the number of subsystems. For a switching sequence $t_0 < t_1 < \cdots < t_k < \cdots$, $\sigma(t)$ is continuous from right everywhere. Throughout this paper, $\sigma(t_k) = i, \sigma(t_k^-) = \sigma(t_{k-1}) = j, i \neq j$ and $\sigma(t) \in \mathbb{Z}$. So when $t \in [t_k, t_{k+1})$, we say the ith subsystem is activated. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_k}$.

Definition 1 (Liberzon 2003) The equilibrium $x = 0$ of system (1) is globally uniformly exponentially stable (GUES) under certain switching signal if for initial conditions $x(t_0)$, there exist constants $\eta_1 > 0$, $\eta_2 > 0$ such that the solution of the system satisfies

$$\|x(t)\| \le \eta_1 e^{-\eta_2(t-t_0)} \|x(t_0)\|, \quad \forall t \ge t_0.$$

Definition 2 (Zhao et al. 2012) For a switching signal σ and any $t_2 > t_1 > t_0$, let $N_{\sigma i}(t_1, t_2)$ be the switching numbers of the ith subsystem over the interval $[t_1, t_2)$. If $N_{\sigma i}(t_1, t_2) \le N_{0i} + (t_2 - t_1)/\tau_{ai}$ holds, then τ_{ai} is mode-dependent average dwell time and N_{0i} is mode-dependent chatter bound.

Definition 3 (Xie et al. 2013) For a switching signal σ and any $t_2 > t_1 > t_0$, let $N_\sigma^u(t_1, t_2)$ be the switching numbers of unstable subsystems over the interval $[t_1, t_2)$. If $N_\sigma^u(t_1, t_2) \ge N_0^u + \Delta t^u/\tau_a^u$ holds, then τ_a^u is average dwell time of unstable subsystems and N_0^u is called chatter bound.

Remark 1 To introduce the three definitions and the following lemma, the asynchronization between system modes and state feedback controllers is not considered. Besides, Definition 3 is used to analyze the stability of fast switched systems, which means the average dwell time cannot exceed an upper bound value.

Next, a method to stabilize linear systems (not switched linear system) is introduced.

Consider a linear system

$$\dot{x} = Ax + Bu \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $rank(B) = m \le n$, $u \in U \subset \mathbb{R}^m$, U is the admissible control set which is defined by $U = \{u \in \mathbb{R}^m \mid -u_{\min} \preceq u \preceq u_{\max}\}$, besides, $u_{\min}, u_{\max} \in \mathbb{R}^m$ are two vectors only with positive components. Set $K \in \mathbb{R}^{m\times n}$ such that $A - BK$ is Hurwitz matrix. Then the closed-loop system becomes

$$\dot{x} = (A - BK)x \tag{3}$$

Define

$$D = \{x \in \mathbb{R}^n \mid -u_{\min} \preceq Kx \preceq u_{\max}\} \tag{4}$$

Lemma 1 (Ni and Cheng 2012) *For the system matrix A, using Schur unitary triangularization Theorem (Bhatia 1991), there exists an orthogonal matrix* $Q \in \mathbb{R}^{n\times n}$ *such that*

$$Q^T A Q = \begin{pmatrix} A_0 & A_2 \\ 0 & A_1 \end{pmatrix} \tag{5}$$

where A_0 *has all negative real part eigenvalues and* A_1 *has all nonnegative real part eigenvalues.*

Consider the linear system (2). Suppose A has r nonnegative real part eigenvalues. Q is an orthogonal matrix satisfying (5), besides,

$$Q^T B = \begin{pmatrix} (B_0)_{(n-r)\times r} & * \\ (B_1)_{r\times r} & * \end{pmatrix}$$

where $*$ *is the element we do not concern. Let* $H = (h_{ij})_{n\times n}$ *be a Hurwitz matrix such that*

$$\begin{pmatrix} H^+ & H^- \\ H^- & H^+ \end{pmatrix} (u_{\max}^T \; u_{\min}^T)^T \preceq 0$$

where

$$H^+(i,j) = \begin{cases} h_{ij} & \text{if } i = j \\ \max(h_{ij}, 0) & \text{if } i \neq j \end{cases}$$
$$H^-(i,j) = \begin{cases} 0 & \text{if } i = j \\ \max(-h_{ij}, 0) & \text{if } i \neq j \end{cases}$$

Then there exists a unique solution to the equation

$$A_1 X - XH = -B_1 \tag{6}$$

Moreover, if $R(B_1) \subset R(X)$*, system (2) with*

$$K = -\begin{pmatrix} 0 & X^{-1} \\ 0 & 0 \end{pmatrix} Q^T$$

is asymptotically stable for all $x_0 \in D$*. The control input* $u = Kx$ *is admissible and domain D in (4) is positively invariant.*

Remark 2 Lemma 1 is used to stabilize linear systems with constrained inputs. We can stabilize all the subsystems according to Lemma 1, firstly, switched system may not be stable with all subsystem stable (the mode-dependent average dwell time must exceed a constant), secondly, how to choose an appropriate initial state? Thirdly, what if the switched system is with asynchronous switching? Finally, what if some of the subsystem cannot be stabilized? So this paper is aimed at solving the four problems.

In what follows, the method of stabilizing switched linear systems with constrained inputs will be presented.

3 Main Results

Consider system (1). $u_i = [u_i^1, u_i^2, \ldots, u_i^{m_i}]^T \in \mathbb{R}^{m_i}$ is the control inputs, besides,

$$-u_i^{\min} \preceq u_i \preceq u_i^{\max}, \quad i \in \mathbb{Z} \tag{7}$$

where $u_i^{\min}$ and $u_i^{\max}$ are only with positive elements. Suppose that system (1) is controllable. According to Lemma 1, feedback matrices K_i and admissible region $D_i = \{x \in \mathbb{R}^n \mid -u_i^{\min} \preceq K_i x \preceq u_i^{\max}\}$ can be obtained. Assume that the switched controllers lag behind system modes for T_i, besides, $T_i < (t_{k+1} - t_k)$. Then system (1) becomes

$$\dot{x} = (A_{\sigma(t)} - B_{\sigma(t)} K_{\sigma(t-T_i)})x \tag{8}$$

Because of asynchronous switching, time interval $[t_k, t_{k+1})$ is divided into matched period (denoted by T_i^+) and mismatched period (denoted by T_i). It should be pointed out that during all the mismatched periods, the system is supposed to be divergent. Let Ω be the biggest Euclidean ball which is centered at the origin and inside the intersection of D_i, $i \in \mathbb{Z}$. For any $z \in \Omega$,

$$d = \max(\|z\|).$$

In what follows, the main results of this paper will be presented.

Theorem 1 *Consider system (8), let* K_i *be obtained in Lemma 1, if there exist constants* $\alpha_i > 0$*,* $\beta_i > 0$*,* $\mu_i \geq 1$ *and two class* κ_∞ *functions* κ_1*,* κ_2 *such that (9), (10), (11), (12) and (13) hold, then system (8) is exponentially stable by admissible control input* $u_i = K_i x$*,* $\forall i \in \mathbb{Z}$*.*

$$\|x(t_0)\|^2 \leq d^2/\psi_1 \tag{9}$$
$$\kappa_1(\|x(t)\|) \leq V_i(x(t)) \leq \kappa_2(\|x(t)\|) \tag{10}$$
$$\dot{V}_i(x(t)) \leq \begin{cases} -\alpha_i V_i(x(t)) & t \in [t_k + T_i, t_{k+1}) \\ \beta_i V_i(x(t)) & t \in [t_k, t_k + T_i) \end{cases} \tag{11}$$
$$V_i(x(t)) \leq \mu_i V_j(x(t)) \tag{12}$$
$$\tau_{ai} > \frac{T_i(\alpha_i + \beta_i) + \ln \mu_i}{\alpha_i} \tag{13}$$

where

$$\psi_1 = \frac{\lambda_{\sigma(t_0)}^{\max}}{\lambda_{\min}} \cdot \exp\{\beta_{\max} T_{\max}\} \cdot \prod_{i=1}^{N} \mu_i^{N_{0i}} \cdot \exp\{T_i(\alpha_i + \beta_i)N_{0i}\}.$$

Proof Denote $g_i(t) = e^{\alpha_i t} V_i(t)$. Then

$$\frac{\mathrm{d}g_i(t)}{\mathrm{d}t} = e^{\alpha_i t}\left[\dot{V}_i(t) + \alpha_i V_i(t)\right] \tag{14}$$

From the first inequality in (11), it can be obtained that $g_i(t)$ is decreasing when $t \in [t_k + T_i, t_{k+1})$. Thus,

$$V_{\sigma(t_k)}(x(t_{k+1})) \le e^{-\alpha_i(t_{k+1}-t_k-T_i)} V_{\sigma(t_k)}(x(t_k+T_i)) \quad (15)$$

From the second inequality in (11), when $t \in [t_k, t_k + T_i)$, by using the same method, the following can be got

$$V_{\sigma(t_k)}(x(t_k+T_i)) \le e^{\beta_i T_i} V_{\sigma(t_k)}(x(t_k)) \quad (16)$$

Combine (15) and (16), it follows that

$$V_{\sigma(t_k)}(x(t_{k+1})) \le e^{-\alpha_i T_i^+ + \beta T_i} V_{\sigma(t_k)}(x(t_k)) \quad (17)$$

Multiply both sides of (17) by $\mu_{\sigma(t_{k+1})}$ and then apply (12) to it, we can get

$$V_{\sigma(t_{k+1})}(x(t_{k+1})) \le e^{-\alpha_i T_i^+ + \beta_i T_i} \mu_{\sigma(t_{k+1})} V_{\sigma(t_k)}(x(t_k)) \quad (18)$$

It follows from (18) that,

$$\begin{aligned} V_{\sigma(t_{k+1})}\left(x(t_{k+1})\right) &\le e^{-\alpha_i(T_i^+ + T_i)} \frac{e^{\beta_i T_i^-}}{e^{-\alpha_i T_i^-}} \mu_{\sigma(t_{k+1})} V_{\sigma(t_k)}(x(t_k)) \\ &= e^{-\alpha_i(t_{k+1}-t_k)} e^{T_i(\alpha_i+\beta_i)} \mu_{\sigma(t_{k+1})} V_{\sigma(t_k)}(x(t_k)) \\ &\le e^{-\alpha_i(t_{k+1}-t_k)} e^{T_i(\alpha_i+\beta_i)} \mu_{\sigma(t_{k+1})} \mu_{\sigma(t_k)} V_{\sigma(t_{k-1})}(x(t_k)) \\ &\vdots \\ &\le \prod_{i=1}^{N} \mu_i^{N_{\sigma i}} e^{-\alpha_i \Delta t_i} e^{T_i(\alpha_i+\beta_i) N_{\sigma_i}} V_{\sigma(t_0)}(x(t_0)) \quad (19) \\ &< \exp\{\beta_{\max} T_{\max}\} \\ &\prod_{i=1}^{N} \mu_i^{N_{\sigma i}} e^{-\alpha_i \Delta t_i} e^{T_i(\alpha_i+\beta_i) N_{\sigma i}} V_{\sigma(t_0)}(x(t_0)) \quad (20) \end{aligned}$$

Then Lyapunov functions are set as

$$V_i(x(t)) = x^T P_i x \quad (21)$$

According to the Rayleigh theorem,

$$\lambda_{\min}(P) \le \frac{x^T P x}{x^T x} \le \lambda_{\max}(P) \quad (22)$$

the following can be got from (19) and (22)

$$\begin{aligned} \|x(t_{k+1})\|^2 \le & \frac{\lambda_{\sigma(t_0)}^{\max}}{\lambda_{\sigma(t_{k+1})}^{\min}} \cdot \exp\{\beta_{\max} T_{\max}\} \\ & \cdot \prod_{i=1}^{N} \mu_i^{N_{\sigma i}} e^{-\alpha_i \Delta t_i} e^{T_i(\alpha_i+\beta_i) N_{\sigma_i}} \|x(t_0)\|^2 \quad (23) \end{aligned}$$

Because $N_{\sigma i}(t_1, t_2) \le N_{0i} + \frac{\Delta t_i}{\tau_{ai}}$ in Definitions 2 and (13) hold, (23) can be simplified to

$$\begin{aligned} & \|x(t_{k+1})\|^2 \le \psi_1 \\ & \exp\left\{\sum_{i=1}^{N} \Delta t_i \cdot \left[\frac{\ln \mu_i}{\tau_{ai}} - \alpha_i + \frac{T_i(\alpha_i+\beta_i)}{\tau_{ai}}\right]\right\} \|x(t_0)\|^2 \end{aligned} \quad (24)$$

Denote $\xi_1 = \max_{i \in \mathbb{Z}} \left(\frac{\ln \mu_i}{\tau_{ai}} - \alpha_i + \frac{T_i(\alpha_i+\beta_i)}{\tau_{ai}}\right)$, then (24) becomes

$$\|x(t_{k+1})\|^2 \le \psi_1 \exp\{\xi_1(t-t_0)\} \|x(t_0)\|^2 \quad (25)$$

From Definition 1, it can be obtained that system (8) is exponentially stable under condition (13). Next, since (9) holds and Ω is an Euclidean ball, besides, $D_i, i \in \mathbb{Z}$, are positively invariant, then for any $\sigma(t_k) \in \mathbb{Z}$, state response $x(t) \in \Omega$. Thus, the control input $u_i = K_i x$ is admissible. In conclusion, the system is exponentially stable with constrained control. □

Remark 3 To solve τ_{ai}, the Lyapunov function can be chosen as $V_i(x(t)) = x^T P_i x$, then, (11) and (12) become (26) and (27), respectively.

$$\begin{cases} (A_i - B_i K_i)^T P_i + P_i(A_i - B_i K_i) \le -\alpha_i P_i, & t \in [t_k + T_i, t_{k+1}) \\ (A_i - B_i K_j)^T P_i + P_i(A_i - B_i K_j) \le \beta_i P_i, & t \in [t_k, t_k + T_i) \end{cases} \quad (26)$$

$$P_i \le \mu_i P_j \quad (27)$$

By using LMI toolbox in Matlab, α_i, β_i and μ_i can be got. Then τ_{a_i} can be solved.

Remark 4 As has been supposed that the system is divergent during all the mismatched periods (the divergence speed can be described by β_i) and the delay time (T_i) of state feedback controllers can be different and variable. With β_i and T_i getting larger, ψ_1 is getting larger simultaneously. This means the feasible initial states that can be chosen are shrunken.

As has been assumed that all the subsystems can be stabilized by Lemma 1, but what if some of the subsystems cannot be stabilized? Besides, in practice, it is usually difficult to design some state feedback controllers. Considering this, we suppose that only the subsystems $i \in [1, M]$ can be stabilized, $\Omega \subset \bigcap_{i=1}^{M} D_i$ is the biggest Euclidean ball centered at the origin and the rest of subsystems ($i \in [M+1, N]$) are without state feedback controllers. For any $z \in \Omega$, $d = \max(\|z\|)$. Besides, $V_i^s(x(t))$ and $V_i^u(x(t))$ are Lyapunov functions for stable subsystems and unstable subsystems, respectively. τ_{ai}^s and τ_{ai}^u are MDADT of stable subsystems and unstable subsystems, respectively. Moreover,

$$\Delta_{\max} = \max_{M+1 \le \sigma(t_k) \le N} (t_{k+1} - t_k).$$

Now we will give another two theorems where there are unstable subsystems without state feedback controllers.

Theorem 2 *Consider system* (8)*, let* K_i*,* $i \in [1, M]$ *be obtained in Lemma* 1 *and* $u_i = 0$*,* $\forall i \in [M+1, N]$*. If there exist constants* $\alpha_i > 0$*,* $\alpha^u > 0$*,* $\beta_i > 0$*,* $\beta^u > 0$*,* $\mu_i \geq 1$*,* $0 < \mu^u < 1$ *and two class* κ_∞ *functions* κ_1*,* κ_2 *such that* (28)*,* (29)*,* (30)*,* (31) *and* (32) *hold, then system* (8) *is exponentially stable by admissible control input* $u_i = K_i x$*,* $i \in [1, M]$*.*

$$\|x(t_0)\|^2 \leq d^2/\psi_2 \tag{28}$$

$$\begin{cases} \kappa_1(\|x(t)\|) \leq V_i^s(x(t)) \leq \kappa_2(\|x(t)\|) \\ \kappa_1(\|x(t)\|) \leq V_i^u(x(t)) \leq \kappa_2(\|x(t)\|) \end{cases} \tag{29}$$

$$\begin{cases} \dot{V}_i^s(x(t)) \leq \begin{cases} -\alpha_i V_i^s(x(t)), & t \in [t_k + T_i, t_{k+1}) \\ \beta_i V_i^s(x(t)), & t \in [t_k, t_k + T_i) \end{cases} \\ \dot{V}_i^u(x(t)) \leq \begin{cases} \alpha^u V_i^u(x(t)), & t \in [t_k + T_i, t_{k+1}) \\ \beta^u V_i^u(x(t)), & t \in [t_k, t_k + T_i) \end{cases} \end{cases} \tag{30}$$

$$\begin{cases} V_i^s(x(t)) \leq \mu_i V_j^s(x(t)) \\ V_i^s(x(t)) \leq \mu_i V_j^u(x(t)) \\ V_i^u(x(t)) \leq \mu^u V_j^s(x(t)) \end{cases} \tag{31}$$

$$\begin{cases} \tau_{ai}^s > \frac{T_i(\alpha_i+\beta_i)+\ln\mu_i}{\alpha_i} \\ \tau_a^u < \frac{-\ln\mu^u}{\gamma} \end{cases} \tag{32}$$

where $\psi_2 = \frac{\lambda_{\sigma(t_0)}^{\max}}{\lambda_{\min}} \cdot \mu_{\max}\exp\{\gamma\Delta_{\max} + \beta_{\max}T_{\max}\}(\mu^u)^{N_0^u} \cdot \prod_{i=1}^{M}\mu_i^{N_{0i}} \cdot \exp\{T_i(\alpha_i+\beta_i)N_{0i}\}$ *is a constant,* τ_a^u *is the average dwell time (except MDADT) of unstable subsystems and* $\gamma = \max(\alpha^u, \beta^u)$*.*

Proof The proof of Theorem 2 is similar to that of Theorem 1. So some part of the proof which has been appeared in Theorem 1 is omitted. It follows from (18) that

$$\begin{aligned} V_{\sigma(t_{k+1})}(x(t_{k+1})) &\leq e^{-\alpha_i T_i^+ + \beta_i T_i}\mu_{\sigma(t_{k+1})}V_{\sigma(t_k)}(x(t_k)) \\ &\vdots \\ &\leq e^{\gamma\Delta t^u}\mu_u^{N_\sigma^u} \cdot \prod_{i=1}^{M} e^{-\alpha_i\Delta t_i}e^{T_i(\alpha_i+\beta_i)N_{\sigma_i}}\mu_i^{N_{\sigma i}}V_{\sigma(t_0)}(x(t_0)) \end{aligned} \tag{33}$$

The Lyapunov functions are set as $V_i(x(t)) = x^T P_i x$ then the following can be got.

$$\|x(t_{k+1})\|^2 \leq \frac{\lambda_{\sigma(t_0)}^{\max}}{\lambda_{\min}} e^{\gamma\Delta t^u}(\mu^u)^{N_\sigma^u} \cdot \prod_{i=1}^{M} e^{-\alpha_i\Delta t_i}e^{T_i(\alpha_i+\beta_i)N_{\sigma_i}}\mu_i^{N_{\sigma_i}}\|x(t_0)\|^2 \tag{34}$$

$$< \frac{\lambda_{\sigma(t_0)}^{\max}}{\lambda_{\min}}\mu_{\max}\exp\{\gamma\Delta_{\max} + \beta_{\max}T_{\max}\}e^{\gamma\Delta t^u}(\mu^u)^{N_{\sigma^u}} \cdot \prod_{i=1}^{M} e^{-\alpha_i\Delta t_i}e^{T_i(\alpha_i+\beta_i)N_{\sigma_i}}\mu_i^{N_{\sigma_i}}\|x(t_0)\|^2 \tag{35}$$

From Definitions 2, 3 and inequality (32), (34) can be simplified to

$$\|x(t_{k+1})\|^2 \leq \psi_2\exp\left\{\sum_{i=1}^{M}\left[\frac{\ln\mu_i + T_i(\alpha_i+\beta_i)}{\tau_{ai}^s} - \alpha_i\right]\cdot\Delta t_i + \left(\frac{\ln\mu^u}{\tau_a^u} + \gamma\right)\cdot\Delta t^u\right\}\|x(t_0)\|^2 \tag{36}$$

Denote $\xi_2 = \max\left(\frac{\ln\mu_i + T_i(\alpha_i+\beta_i)}{\tau_{ai}^s} - \alpha_i, \frac{\ln\mu^u}{\tau_a^u} + \gamma\right)$ Then (36) becomes

$$\|x(t_{k+1})\|^2 \leq \psi_2\exp\{\xi_2(t-t_0)\}\|x(t_0)\|^2 \tag{37}$$

Compared with Definition 1, it can be concluded that system (8) is exponentially stable under condition (32). Next, since (28) holds, and D_i is positively invariant, then from (36), for any $\sigma(t_k) \in \mathbb{Z}$, state response $x(t) \in \Omega$. Thus the control input $u_i = K_i x$, $i \in [1, M]$ is admissible. In conclusion, the system is exponentially stable with constrained control. □

Remark 5 As has been stated, both τ_{ai} and τ_a^u can be calculated by using the LMI toolbox in Matlab. Next, from (31), it should be noticed that once the unstable subsystem is activated, then the following subsystem must be a stable subsystem. In what follows, another theorem is presented where the unstable subsystems can be activated one by one. But its total running time must be limited.

Theorem 3 *Consider system* (8)*, let* K_i*,* $i \in [1, M]$ *be obtained in Lemma* 1*. If there exist constants* $\mu_i \geq 1$*,* $\beta_i > 0$*,* α_i *and two class* κ_∞ *functions* κ_1*,* κ_2 *such that* (38)*,* (39)*,* (40)*,* (41) *and* (42) *hold, then system* (8) *is exponentially stable by admissible control input* $u_i = K_i x$*,* $i \in [1, M]$*.*

$$\|x(t_0)\|^2 \leq d^2/\psi_3 \tag{38}$$

$$\kappa_1(\|x(t)\|) \leq V_i(x(t)) \leq \kappa_2(\|x(t)\|) \tag{39}$$

$$\dot{V}_i(x(t)) \leq \begin{cases} -\alpha_i V_i(x(t)), & t \in [t_k + T_i, t_{k+1}) \\ \beta_i V_i(x(t)), & t \in [t_k, t_k + T_i) \end{cases} \tag{40}$$

$$V_i(x(t)) \leq \mu_i V_j(x(t)) \tag{41}$$

$$\begin{cases} \tau_{ai}^s > \frac{T_i(\alpha_i+\beta_i)+\ln\mu_i}{\alpha_i} \\ \tau_{ai}^u \geq 0 \\ \frac{\Delta t^s}{\Delta t^u} > \frac{\zeta^*-\zeta_2}{\zeta_1-\zeta^*}, \quad (\zeta_1 < \zeta^* < 0) \end{cases} \tag{42}$$

where $\zeta_1 = \max_{1\leq i\leq M}\left(\frac{T_i(\alpha_i+\beta_i)}{\tau_{ai}^s} - \alpha_i + \frac{\ln\mu_i}{\tau_{ai}^s}\right)$ *is a negative constant,* $\zeta_2 = \max_{M+1\leq i\leq N}\left(\frac{T_i(\alpha_i+\beta_i)}{\tau_{ai}^u} - \alpha_i + \frac{\ln\mu_i}{\tau_{ai}^u}\right)$ *is* *a positive constant and* $\psi_3 = \frac{\lambda^{\max}_{\sigma(t_0)}}{\lambda_{\min}}\mu_{\max}\exp\{\gamma\Delta_{\max} + \beta_{\max}T_{\max}\}\prod_{i=1}^{N}\exp\{T_i(\alpha_i+\beta_i)N_{0i}+N_{0i}\ln\mu_i\}$ *is a positive constant.*

Proof The proof of Theorem 3 is similar to that of Theorem 1. So some part of the proof which has been appeared in Theorem 1 is omitted. It follows from (18) that

$$\begin{aligned} V_{\sigma(t_{k+1})}(x(t_{k+1})) &\le e^{-\alpha_i T_i^{+}+\beta_i T_i}\mu_{\sigma(t_{k+1})}V_{\sigma(t_k)}(x(t_k)) \\ &\le e^{-\alpha_i T_i^{+}+\beta_i T_i}\mu_{\sigma(t_{k+1})}\mu_{\sigma(t_k)}V_{\sigma(t_k^{-})}(x(t_k)) \end{aligned} \tag{43}$$

$$\begin{aligned} &\vdots \\ &\le \prod_{i=1}^{M}\mu_i^{N_{\sigma i}}\exp\{-\alpha_i\Delta t_i + T_i(\alpha_i+\beta_i)N_{\sigma i}\} \\ &\cdot \prod_{i=M+1}^{N}\mu_i^{N_{\sigma i}}\exp\{-\alpha_i\Delta t_i + T_i(\alpha_i+\beta_i)N_{\sigma i}\} \end{aligned} \tag{44}$$

$$\begin{aligned} &< \mu_{\max}\exp\{\gamma\Delta_{\max} + \beta_{\max}T_{\max}\} \\ &\prod_{i=1}^{M}\mu_i^{N_{\sigma i}}\exp\{-\alpha_i\Delta t_i + T_i(\alpha_i+\beta_i)N_{\sigma i}\} \\ &\cdot \prod_{i=M+1}^{N}\mu_i^{N_{\sigma i}}\exp\{-\alpha_i\Delta t_i + T_i(\alpha_i+\beta_i)N_{\sigma i}\} \end{aligned} \tag{45}$$

The Lyapunov functions are set as $V_i(x(t)) = x^T P_i x$ then the following can be got,

$$\|x(t_{k+1})\|^2 \le \psi_3\exp\{\Delta t^s\zeta_1 + \Delta t^u\zeta_2\}\|x(t_0)\|^2 \tag{46}$$

Because $\zeta_1 < \zeta^* < 0$ holds, (46) becomes,

$$\|x(t_{k+1})\|^2 \le \psi_3\exp\{\zeta^*(t-t_0)\}\|x(t_0)\|^2 \tag{47}$$

So (42) can verify exponential stability of the system. Besides, (38) makes the control input admissible. This completes the proof of the theorem. □

Remark 6 In Theorem 1, ψ_1 will get larger with T_i and β_i getting larger. In Theorems 2 and 3, ψ_2 and ψ_3 not only have a relationship with T_i and β_i, but also have a relationship with $\Delta_{\max}$ and γ. In this way, the trajectory of state response with unstable subsystems can remain in the feasible region all the time.

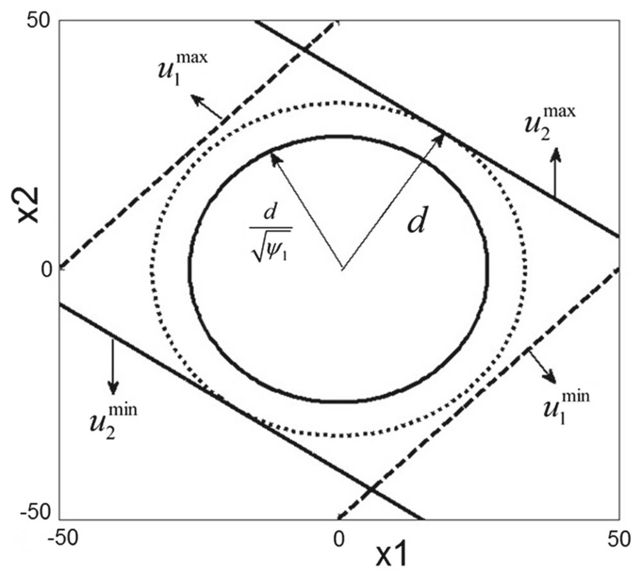

Fig. 1 The feasible region of initial states

4 Numerical Examples

Example 1 Consider system (8) with

$$A_1 = \begin{bmatrix} -1.4402 & -3.36 \\ -3.52 & -1.28 \end{bmatrix},\ B_1 = \begin{bmatrix} -3.36 \\ 3.52 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -2.928 & 2.88 \\ 2.24 & -1.488 \end{bmatrix},\ B_2 = \begin{bmatrix} 1.92 \\ 2.24 \end{bmatrix}$$

$T_1 = T_2 = 0.1$ s. $N_{01} = N_{02} = 0$. According to Lemma 1, feedback matrices can be chosen as: $K_1 = [-1\ 1]$, $K_2 = [1\ 1.5]$.

By solving the LMIs in (26) and (27), we can get $\alpha_1 = 9.5999$, $\beta_1 = 0.224$, $\mu_1 = 1.4455$, $\alpha_2 = 9.6938$, $\beta_2 = 0.3198$, $\mu_2 = 1$ (μ_2 is smaller than one so it is enlarged to one.), $\psi_1 = 1.7211$. According to (13), $\tau_{a1} > 0.1023$, $\tau_{a2} > 0.1033$. The control input is constrained with $-50 \le u_1 \le 50$ and $-40 \le u_2 \le 40$. Because $u_i = K_i x$, we can get the biggest Euclidean ball, which is plotted in a dotted line in Fig. 1. The circle plotted in a solid line illustrates the feasible region of initial states. We choose $x(t_0) = [-20\ 17.5601]$, $\tau_{a1} = 0.12$ s, $\tau_{a2} = 0.13$ s. Figure 2 illustrates the state responses, while Fig. 3 is the switching signal. Next, we choose $\tau_{a1} = 0.19$ s, $\tau_{a2} = 0.2$ s, Fig. 4 illustrates the state responses, while Fig. 5 is the switching signal.

Example 2 Consider system (8) in Example 1 with the same subsystems and another subsystem:

$$A_3 = \begin{bmatrix} 0.048 & 0 \\ 0 & 0.048 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.16 \\ 0.32 \end{bmatrix}$$

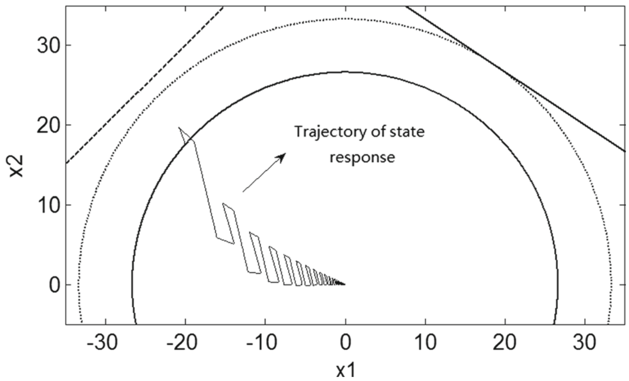

Fig. 2 State responses of Example 1 with shorter MDADT

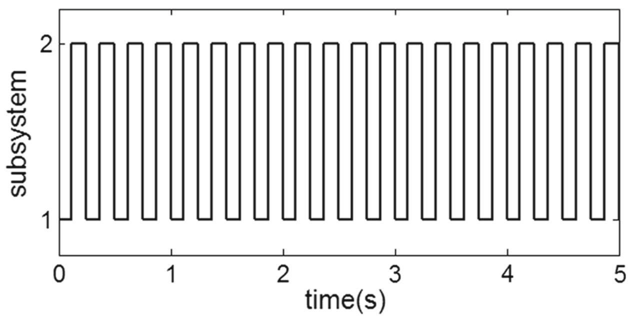

Fig. 3 Switching signal of Example 1 with shorter MDADT

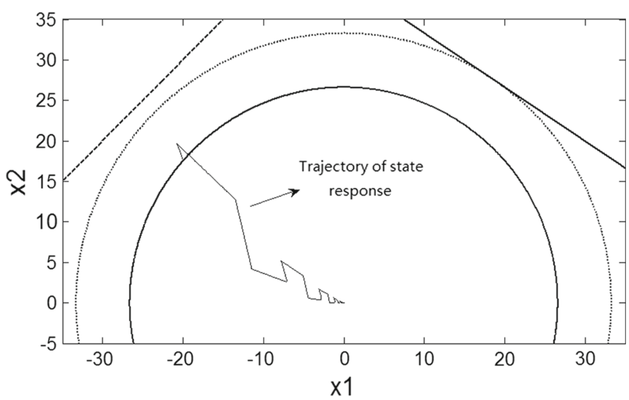

Fig. 4 State responses of Example 1 with longer MDADT

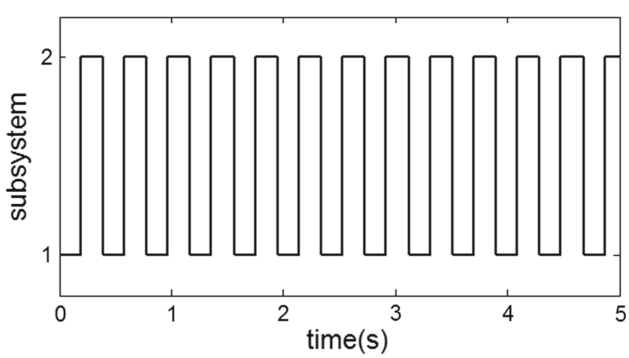

Fig. 5 Switching signal of Example 1 with longer MDADT

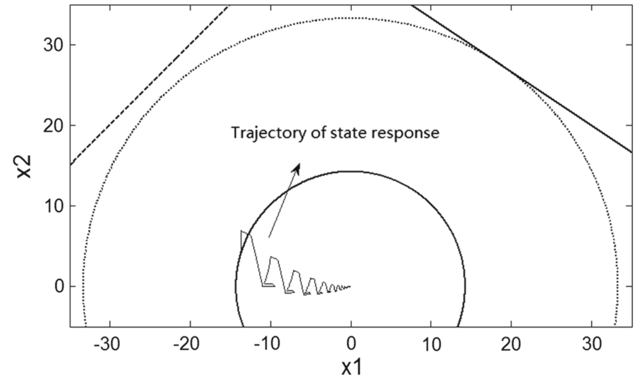

Fig. 6 State responses of Example 2 with shorter MDADT

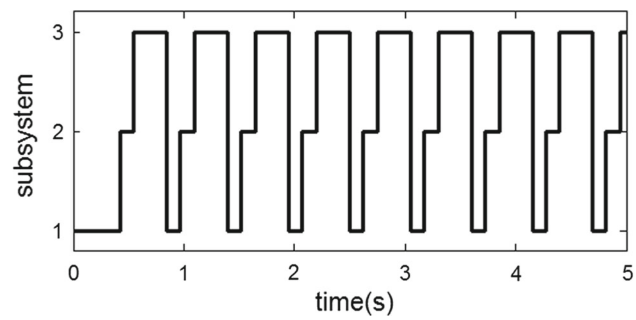

Fig. 7 Switching signal of Example 2 with shorter MDADT

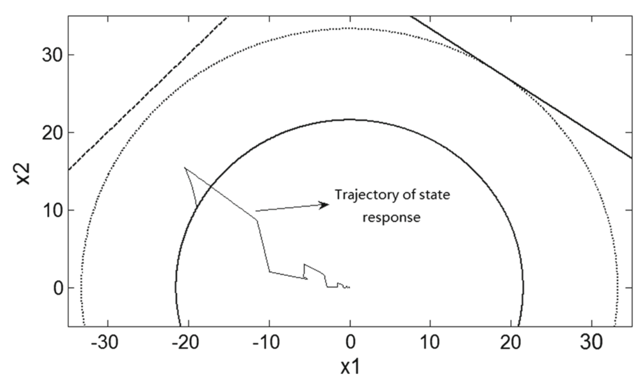

Fig. 8 State responses of Example 2 with longer MDADT

Suppose subsystem 3 cannot be stabilized and $N_{03} = 0$, $T_3 = 0.1\,\mathrm{s}$. By solving the LMIs in Theorem 2 we can get $\alpha_1 = 9.5999, \beta_1 = 0.224, \mu_1 = 1, \alpha_2 = 9.6838, \beta_2 = 0.96$, $\mu_2 = 1$, $\gamma = 3.8398$. $\mu^u = 0.207$. According to (32), we can get $\tau_{a1} > 0.102\,\mathrm{s}$, $\tau_{a2} > 0.110\,\mathrm{s}$, $\tau_{a3} < 0.400\,\mathrm{s}$. Then we choose $\tau_{a1} = 0.12\,\mathrm{s}$, $\tau_{a2} = 0.13\,\mathrm{s}$, $\tau_{a3} = 0.3\,\mathrm{s}$, then $\psi_2 = 5.446$, we choose $x(t_0) = [-14\ 2.7184]$. Figures 6 and 7 illustrate the state responses and switching signal, respectively. Next, we choose $\tau_{a1} = 0.22\,\mathrm{s}$, $\tau_{a2} = 0.23\,\mathrm{s}$, $\tau_{a3} = 0.2\,\mathrm{s}$, then $\psi_2 = 2.386$, we choose $x(t_0) = [-19\ 10.1601]$. Figures 8 and 9 illustrate the state responses and switching signal, respectively.

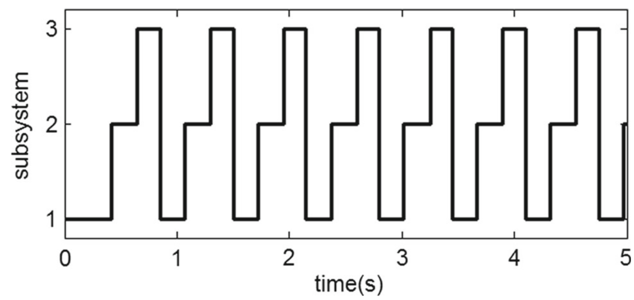

Fig. 9 Switching signal of Example 2 with longer MDADT

From the two examples, it can be noticed that by choosing the appropriate initial states and MDADT, the asynchronous switched system is exponentially stable. Besides, although there may be unstable subsystem, the trajectory of state responses can also remain in the feasible region all the time. The example for Theorem 3 is omitted. The advantage of Theorem 3 is that unstable subsystems can be activated one by one. But the total running time of stable and unstable subsystems must be calculated.

5 Conclusion

This paper provides three theorems which can be used to stabilize asynchronous switched systems with constrained inputs and unstable subsystems. The mode-dependent average dwell time can guarantee that the system is exponentially stable. Even if there are unstable subsystems, the restriction of the running time of unstable subsystems can also guarantee the system is exponentially stable. Furthermore, choosing the initial states within a specific Euclidean ball makes the control inputs admissible all the time.

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