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Robust H_∞ Filtering for 2-D Discrete Roesser Systems

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Abstract This paper tackles the H_∞ filtering problem for 2-D discrete systems. The approach is based on the Roesser model. The objective is to propose a new design with sufficient condition via LMI formulations. Less conservative results are obtained by introducing additional free parameters by using the Finsler's Lemma. This method provides extra degree of freedom in optimization of the H_∞ performance. The efficiency of the proposed approach is shown by several examples.

Keywords 2-D discrete systems · Roesser models · H_∞ filtering · Uncertain systems · Linear matrix inequalities (LMIs) · Slack matrices

1 Introduction

Two-dimensional (2-D) system theory has attracted considerable attention due to its extensive applications of many physical systems, such as those in state-space digital filter, image data processing and transmission, thermal processes, biomedical imaging, gas absorbtion, water stream heating, etc. So they are being extensively studied. To mention a few of the results obtained so far, modeling has been studied in Fornasini and Marchisini (1976, 1978), Roesser (1975) and Takagi (1985); the stability has been investigated in Xia and Jia (2002), Hmamed et al. (2008), Dey et al. (2012) and Kokil et al. (2012); H_∞ stabilization and control were solved in Du et al. (2001, 2002), Benhayoun et al. (2013), Hmamed et al. (2010), Xu et al. (2008), Wang et al. (2015), Qiu et al. (2015a, b, c); H_∞ filtering for 2-D linear, delayed and Takagi–Sugeno systems have been studied, respectively, in Gao and Li (2014), Ying and Rui (2011), Gao et al. (2008), El-Kasri et al. (2012, 2013a, b), Du et al. (2000), Xu et al. (2005), Wu et al. (2008), Boukili et al. (2014b), Qiu et al. (2013), Hmamed et al. (2013), Gao and Wang (2004), Chen and Fong (2006), Boukili et al. (2013, 2014a) and Meng and Chen (2014); finally, Li and Gao (2012), Gao and Li (2011) and Li et al. (2012) has addressed the finite frequency H_∞ filtering for 2-D systems.

This paper concentrates on filtering, as it is an important problem in signal processing. More precisely, this paper concentrates on H_∞ filtering: H_∞ filtering for 2-D systems with parameter uncertainties has been studied in Xu et al. (2005), Hmamed et al. (2013), El-Kasri et al. (2013a, b), Boukili et al. (2013), Chen and Fong (2006) and Wu et al. (2008). These previous results on robust H_∞ filtering are mostly based on quadratic stability conditions and are hence inevitably conservative, as the same Lyapunov function is used for the entire uncertainty domain.

To overcome this conservatism, this paper considers parameter-dependent Lyapunov functions, to reduce the overdesign inherent to a quadratic framework Gao and Li (2014), Ying and Rui (2011), Gao et al. (2008) and El-Kasri et al. (2012). In addition, in order to decouple the product terms between the Lyapunov matrix and system matrices and provide extra degrees of freedom, slack matrices are introduced. The key in our approach is then the use of four independent slack matrices and some homogenous polyno-

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mially parameter-dependent matrices of arbitrary degree: as their degree grows, increasing precision is obtained, providing less conservative filter designs. The filters are designed for systems with parameter uncertainties that belong to a polytope, where only the vertices are known. The proposed condition include as special cases the previous quadratic formulations, and also the linearly parameter-dependent approaches (that use linear convex combinations of matrices).

It must be emphasized that the theoretical results are given in the form of linear matrix inequalities (LMIs), which can be solved by standard numerical software, thus providing a simple methodology. An example shows the effectiveness of the proposed approach.

The organization of this paper is as follows: Sect. 2 states the problem to be solved and present some preliminary results. Then the analysis of robust asymptotical stability with H_∞ performance is given in Sect. 3. The H_∞ filter design scheme is then developed in Sect. 4, followed by an example to illustrate the effectiveness of the proposed approach. Finally, some conclusions are given.

Notations: The notation used throughout the paper is standard. The superscript T stands for matrix transposition. $P > 0$ means that the matrix P is real symmetric and positive definite. I is the identity matrix with appropriate dimension. In symmetric block matrices or long matrix expressions, we use an asterisk $*$ to represent terms induced by symmetry. diag$\{\ldots\}$ stands for a block-diagonal matrix. The l_2 norm for a 2-D signal $w(i, j)$ is given by

$$\| w \|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w^{\mathrm{T}}(i, j)w(i, j)}$$

where $w(i, j)$ is said to be in the space $l_2\{[0, \infty), [0, \infty)\}$ or l_2, for simplicity, if $\| w \|_2 < \infty$. A 2-D signal $w(i, j)$ in the l_2 space is an energy-bounded signal.

2 Problem Description

Consider a 2-D discrete system described by the following Roesser model:

$$\begin{aligned}
\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= A_\tau \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + B_\tau w(i, j) \\
y(i, j) &= C_\tau \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + D_\tau w(i, j) \\
z(i, j) &= H_\tau \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \\
x^h(0, k) &= \varphi(k), x^v(0, k) = \phi(k), \quad \forall k,
\end{aligned} \tag{1}$$

where $x^h(i, j) \in R^{n_1}$ is the state vector in the horizontal direction, $x^v(i, j) \in R^{n_2}$ the state vector in the vertical direction, $y(i, j) \in R^m$ is the measured signal vector, $z(i, j) \in R^v$ the signal to be estimated, and $w(i, j) \in R^q$ is the disturbance signal vector. It is assumed that $w(i, j)$ belongs to $L_2\{[0, \infty), [0, \infty)\}$. The system matrices are decomposed in blocks as follows:

$$\begin{aligned}
A_\tau &= \begin{bmatrix} A_{11\tau} & A_{12\tau} \\ A_{21\tau} & A_{22\tau} \end{bmatrix}, \quad B_\tau = \begin{bmatrix} B_{1\tau} \\ B_{2\tau} \end{bmatrix}, \\
C_\tau &= \begin{bmatrix} C_{1\tau} & C_{2\tau} \end{bmatrix}, \quad H_\tau = \begin{bmatrix} H_{1\tau} & H_{2\tau} \end{bmatrix}
\end{aligned} \tag{2}$$

where the dimensions of each block are compatible with the vectors.

The system matrices are assumed to be uncertain and bounded in a polyhedral domain

$$\Omega_\tau \triangleq (A_\tau, B_\tau, C_\tau, D_\tau, H_\tau) \in \mathcal{R} \tag{3}$$

where $\mathcal{R}$ denotes a polytope defined as

$$\mathcal{R} \triangleq \left\{ \Omega_\tau | \Omega_\tau = \sum_{i=1}^{s} \tau_i \Omega_i; \tau \in \Gamma \right\} \tag{4}$$

with $\Omega_i \triangleq (A_i, B_i, C_i, D_i, H_i)$ denoting the vertices of $\mathcal{R}$ and

$$\Gamma \triangleq \left\{ (\tau_1, \tau_2, \ldots, \tau_s) : \sum_{i=1}^{s} \tau_i = 1, \tau_i > 0 \right\} \tag{5}$$

is the unit simplex.

The boundary condition of the system fulfills

$$\lim_{n \mapsto \infty} \sum_{k=1}^{n} (|x^h(0, k)|^2 + |x^v(0, k)|^2) < \infty \tag{6}$$

In this paper, we consider a 2-D filter represented by the following Roesser model:

$$\begin{aligned}
\begin{bmatrix} \hat{x}^h(i+1, j) \\ \hat{x}^v(i, j+1) \end{bmatrix} &= A_f \begin{bmatrix} \hat{x}^h(i, j) \\ \hat{x}^v(i, j) \end{bmatrix} + B_f y(i, j) \\
\hat{z}(i, j) &= C_f \begin{bmatrix} \hat{x}^h(i, j) \\ \hat{x}^v(i, j) \end{bmatrix} \\
\hat{x}^h(0, k) &= 0, \quad \hat{x}^v(0, k) = 0, \ \forall k,
\end{aligned} \tag{7}$$

where $\hat{x}^h(i, j) \in R^{n_1}$ is the filter state vector in the horizontal direction, $\hat{x}^v(i, j) \in R^{n_2}$ is the filter state vector in the vertical direction and $\hat{z}(i, j) \in R^p$ is the estimation of $z(i, j)$. The matrices are real valued and are decomposed in the following block form

$$A_f = \begin{bmatrix} A_{f11} & A_{f12} \\ A_{f21} & A_{f22} \end{bmatrix}, \quad B_f = \begin{bmatrix} B_{f1} \\ B_{f2} \end{bmatrix},$$
$$C_f = \begin{bmatrix} C_{f1} & C_{f2} \end{bmatrix}. \tag{8}$$

Defining the augmented state vectors

$$\zeta^h(i,j) = \begin{bmatrix} x^h(i,j)^{\mathrm{T}} & \hat{x}^h(i,j)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$
$$\zeta^v(i,j) = \begin{bmatrix} x^v(i,j)^{\mathrm{T}} & \hat{x}^v(i,j)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{9}$$

and the estimation error

$$e(i,j) = z(i,j) - \hat{z}(i,j) \tag{10}$$

gives the following filtering error system:

$$\begin{bmatrix} \zeta^h(i+1,j) \\ \zeta^v(i,j+1) \end{bmatrix} = \bar{A}_\tau \begin{bmatrix} \zeta^h(i,j) \\ \zeta^v(i,j) \end{bmatrix} + \bar{B}_\tau w(i,j)$$
$$e(i,j) = \bar{C}_\tau \begin{bmatrix} \zeta^h(i,j) \\ \zeta^v(i,j) \end{bmatrix} \tag{11}$$

where

$$\bar{A}_\tau = \Upsilon^{\mathrm{T}} \begin{bmatrix} A_\tau & 0 \\ B_f C_\tau & A_f \end{bmatrix} \Upsilon, \quad \bar{B}_\tau = \Upsilon^{\mathrm{T}} \begin{bmatrix} B_\tau \\ B_f D_\tau \end{bmatrix},$$
$$\bar{C}_\tau = \begin{bmatrix} H_\tau & -C_f \end{bmatrix} \Upsilon, \tag{12}$$
$$\Upsilon = \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix} = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}$$

The transfer function of the filtering error system is then

$$T_{ew}(z_1, z_2, \tau) = \bar{C}_\tau [\mathrm{diag}\{z_1 I_{2\times n_1}, z_2 I_{2\times n_2}\} - \bar{A}_\tau]^{-1} \bar{B}_\tau \tag{13}$$

Thus, the robust H_∞ filtering error problem can be stated as follows:

Problem description: Given the Roesser system (1) with parameter uncertainty (3), find a filter (7), such that the filter error system (11) is robustly asymptotically stable for all $\tau \in \Gamma$ and satisfies the following robust H_∞ performance:

$$\|T_{ew}(z_1, z_2, \tau)\|_\infty < \gamma, \quad \forall \tau \in \Gamma \tag{14}$$

where γ is a given positive scalar.

Remark 2.1 The parameter uncertainties considered in this paper are assumed to be of polytopic type, entering into all the matrices of the system model. This description has been widely used for robust control and filtering (see, Gao and Wang 2004; Xia and Jia 2002), as many practical systems present parameter uncertainties which can be exactly modeled by a polytopic uncertainty, or at least bounded.

To derive our main results, we use Finsler's Lemma:

Lemma 2.2 *(Lacerda et al. 2011) Let $\zeta \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{R}^{n\times n}$ and $\mathcal{B} \in \mathbb{R}^{m\times n}$ with rank $(\mathcal{B}) = r < n$ and $\mathcal{B}^\perp \in \mathbb{R}^{n\times(n-r)}$ be full-column-rank matrix satisfying $\mathcal{B}\mathcal{B}^\perp = 0$. Then, the following conditions are equivalent:*

1. $\zeta^{\mathrm{T}} \mathcal{Q} \zeta < 0, \forall \zeta \neq 0 : \mathcal{B}\zeta = 0$
2. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^\perp < 0$
3. $\exists \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^{\mathrm{T}} \mathcal{B} < 0$
4. $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^{\mathrm{T}} \mathcal{X}^{\mathrm{T}} < 0$

3 H_∞ Filtering Analysis

In this section, the filtering analysis problem is considered. More specifically, we assume that the filter matrices in (8) are known, and we will study the condition under which the filtering error system (11) is asymptotically stable with H_∞-norm bounded γ. To solve the robust H_∞ filtering problem, we first recall the following result (Gao et al. 2008; Du et al. 2002).

Lemma 3.1 *Given a positive scalar γ, if $(A_\tau, B_\tau, C_\tau, D_\tau, H_\tau) \in \Omega$ are arbitrary but fixed, then the filtering error system (11) is asymptotically stable and satisfies the H_∞ performance γ for any fixed $\tau \in \Gamma$, if there exists a block-diagonal matrix $P_\tau = diag\{P_\tau^h, P_\tau^v\} > 0$, where $P_\tau^h \in \mathbb{R}^{(2\times n_1)\times(2\times n_1)}$ and $P_\tau^v \in \mathbb{R}^{(2\times n_2)\times(2\times n_2)}$, such that*

$$\begin{bmatrix} -P_\tau & P_\tau \bar{A}_\tau & P_\tau \bar{B}_\tau & 0 \\ \bar{A}_\tau^{\mathrm{T}} P_\tau & -P_\tau & 0 & \bar{C}_\tau^{\mathrm{T}} \\ \bar{B}_\tau^{\mathrm{T}} P_\tau & 0 & -\gamma^2 I & 0 \\ 0 & \bar{C}_\tau & 0 & -I \end{bmatrix} < 0 \tag{15}$$

Proposition 3.2 *Given a positive scalar γ, if $(A_\tau, B_\tau, C_\tau, D_\tau, H_\tau) \in \Omega$ are arbitrary but fixed, the filtering error system (11) is asymptotically stable with H_∞-norm bounded γ if there exist parameter-dependent symmetric positive definite matrices $P_\tau = diag\{P_\tau^h, P_\tau^v\}$, and parameter-dependent matrices M_τ, S_τ, R_τ and F_τ such that:*

$$\Theta = \begin{bmatrix} \Gamma_1 & M_\tau^{\mathrm{T}} \bar{A}_\tau - S_\tau & M_\tau^{\mathrm{T}} \bar{B}_\tau - R_\tau & -F_\tau \\ * & \Gamma_2 & \Gamma_3 & \Gamma_4 \\ * & * & \Gamma_5 & \bar{B}_\tau^{\mathrm{T}} F_\tau \\ * & * & * & -I \end{bmatrix} < 0 \tag{16}$$

where

$$\Gamma_1 = P_\tau - M_\tau - M_\tau^{\mathrm{T}}, \quad \Gamma_2 = S_\tau^{\mathrm{T}} \bar{A}_\tau + \bar{A}_\tau^{\mathrm{T}} S_\tau - P_\tau$$
$$\Gamma_3 = S_\tau^{\mathrm{T}} \bar{B}_\tau + \bar{A}_\tau^{\mathrm{T}} R_\tau, \quad \Gamma_4 = \bar{A}_\tau^{\mathrm{T}} F_\tau + \bar{C}_\tau^{\mathrm{T}}$$
$$\Gamma_5 = \bar{B}_\tau^{\mathrm{T}} R_\tau + R_\tau^{\mathrm{T}} \bar{B}_\tau - \gamma^2 I.$$

Proof To prove the theorem above, we consider the following matrices

$$\mathcal{Q} = \begin{bmatrix} P_\tau & 0 & 0 & 0 \\ 0 & -P_\tau & 0 & \bar{C}_\tau^{\mathrm{T}} \\ 0 & 0 & -\gamma^2 I & 0 \\ 0 & \bar{C}_\tau & 0 & -I \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} -I \\ \bar{A}_\tau^{\mathrm{T}} \\ \bar{B}_\tau^{\mathrm{T}} \\ 0 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{B}^{\perp T} = \begin{bmatrix} \bar{A}_\tau^{\mathrm{T}} & I & 0 & 0 \\ \bar{B}_\tau^{\mathrm{T}} & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} M_\tau, & S_\tau, & R_\tau, & F_\tau \end{bmatrix}^{\mathrm{T}}$$

Therefore, the condition (2) of Lemma 2.2 is equivalent to

$$\begin{bmatrix} -P_\tau + \bar{A}_\tau^{\mathrm{T}} P_\tau \bar{A}_\tau & \bar{A}_\tau^{\mathrm{T}} P_\tau \bar{B}_\tau & \bar{C}_\tau^{\mathrm{T}} \\ * & \bar{B}_\tau^{\mathrm{T}} P_\tau \bar{B}_\tau - \gamma^2 I & 0 \\ * & 0 & -I \end{bmatrix} < 0 \tag{17}$$

By Schur complement argument, it can be seen that the inequality (17) is equivalent to condition (15), which completes the proof.

Remark 3.3 M_τ, S_τ, R_τ and F_τ act as slack variables to provide extra degrees of freedom in the solution space of the robust H_∞ filtering problem. By setting $R_\tau = 0$ and $F_\tau = 0$, Proposition 3.2 coincides with the results of Theorem 1 in Ying and Rui (2011). Thanks to lack variable matrices, we obtain an LMI in which the Lyapunov matrix P_τ is not involved in any product with the system matrices. This enables us to derive a robust H_∞ filtering condition that is less conservative than previous results due to the extra degrees of freedom (see the numerical example at the end of the paper).

4 H_∞ Filter Design

In this section, a methodology is established for designing the H_∞ filter (7), that is, to determine the filter matrices (8) such that the filtering error system (11) is asymptotically stable with an H_∞-norm bounded by γ.

Based on Proposition 3.2, we select for variables P_τ and M_τ the following structures (Gao et al. 2008):

$$P_\tau = \begin{bmatrix} P_{1\tau}^h & P_{2\tau}^h & 0 & 0 \\ (P_{2\tau}^h)^{\mathrm{T}} & P_{3\tau}^h & 0 & 0 \\ 0 & 0 & P_{1\tau}^v & P_{2\tau}^v \\ 0 & 0 & (P_{2\tau}^v)^{\mathrm{T}} & P_{3\tau}^v \end{bmatrix},$$

$$M_\tau = \begin{bmatrix} M_{1\tau}^h & M_4^h & 0 & 0 \\ M_{2\tau}^h & M_3^h & 0 & 0 \\ 0 & 0 & M_{1\tau}^v & M_4^v \\ 0 & 0 & M_{2\tau}^v & M_3^v \end{bmatrix} \tag{18}$$

Then, let the slack variables S_τ, F_τ and R_τ take the following structure (Lacerda et al. 2011)

$$S_\tau = \begin{bmatrix} S_{1\tau}^h & \lambda_1 M_4^h & 0 & 0 \\ S_{2\tau}^h & \lambda_2 M_3^h & 0 & 0 \\ 0 & 0 & S_{1\tau}^v & \lambda_3 M_4^v \\ 0 & 0 & S_{2\tau}^v & \lambda_4 M_3^v \end{bmatrix},$$
$$R_\tau = \begin{bmatrix} R_\tau^h, & 0, & R_\tau^v, & 0 \end{bmatrix},$$
$$F_\tau = \begin{bmatrix} F_\tau^h, & 0, & F_\tau^v, & 0 \end{bmatrix} \tag{19}$$

where P_τ, $M_{1\tau}^h$, $M_{2\tau}^h$, $M_{1\tau}^v$, $M_{2\tau}^v$, $S_{1\tau}^h$, $S_{2\tau}^h$, $S_{1\tau}^v$, $S_{2\tau}^v$, R_τ^h, R_τ^v, F_τ^h, F_τ^v depend on the parameter τ, while M_3^h, M_4^h, M_3^v, and M_4^v are fixed for the entire uncertainty domain and, without loss of generality, invertible; the scalar parameters λ_1, λ_2, λ_3 and λ_4 will be used as optimization parameters.

Remark 4.1 The structure of S_τ in (19) is different than the one proposed in Ying and Rui (2011), in which $S = \delta M$, so it depended on M. It is important to note that $S_{1\tau}^h$, $S_{2\tau}^h$, $S_{1\tau}^v$ and $S_{2\tau}^v$ of S in the new structure (19) are free slack variables completely independent of M. This provides extra degrees of freedom in the solution space for the LMI optimization problems derived from Theorem 4.3.

Define matrices

$$\Pi = \operatorname{diag}\left\{I,\ (M_3^h)^{-T}(M_4^h)^{\mathrm{T}},\ I,\ (M_3^v)^{-T}(M_4^v)^{\mathrm{T}}\right\},$$

$$\bar{P}_\tau = \begin{bmatrix} \bar{P}_{1\tau}^h & \bar{P}_{2\tau}^h & 0 & 0 \\ (\bar{P}_{2\tau}^h)^{\mathrm{T}} & \bar{P}_{3\tau}^h & 0 & 0 \\ 0 & 0 & \bar{P}_{1\tau}^v & \bar{P}_{2\tau}^v \\ 0 & 0 & (\bar{P}_{2\tau}^v)^{\mathrm{T}} & \bar{P}_{3\tau}^v \end{bmatrix} = \Pi^{\mathrm{T}} P_\tau \Pi$$

Applying congruence transformations to (16) by $\operatorname{diag}\{\Pi, \Pi, I, I\}$ we get

$$\Theta = \begin{bmatrix} \Pi^{\mathrm{T}}\Gamma_1\Pi & \Pi^{\mathrm{T}}\Gamma_2\Pi & \Pi^{\mathrm{T}}\Gamma_3 & -\Pi^{\mathrm{T}}F_\tau \\ \Pi^{\mathrm{T}}\Gamma_2^{\mathrm{T}}\Pi & \Pi^{\mathrm{T}}\Gamma_4\Pi & \Pi^{\mathrm{T}}\Gamma_5 & \Pi^{\mathrm{T}}\Gamma_6 \\ \Gamma_3^{\mathrm{T}}\Pi & \Gamma_5^{\mathrm{T}}\Pi & \Gamma_7 & \bar{B}_\tau^{\mathrm{T}}F_\tau \\ -F_\tau^{\mathrm{T}}\Pi & \Gamma_6^{\mathrm{T}}\Pi & F_\tau^{\mathrm{T}}\bar{B}_\tau & -I \end{bmatrix} < 0 \tag{20}$$

where

$$\Gamma_1 = P_\tau - M_\tau - M_\tau^{\mathrm{T}},\quad \Gamma_2 = M_\tau^{\mathrm{T}}\bar{A}_\tau - S_\tau,$$

$\Gamma_3 = M_\tau^{\mathrm{T}}\bar{B}_\tau - R_\tau,\quad \Gamma_4 = S_\tau^{\mathrm{T}}\bar{A}_\tau + \bar{A}_\tau^{\mathrm{T}}S_\tau - P_\tau$
$\Gamma_5 = S_\tau^{\mathrm{T}}\bar{B}_\tau + \bar{A}_\tau^{\mathrm{T}}R_\tau,\quad \Gamma_6 = \bar{A}_\tau^{\mathrm{T}}F_\tau + \bar{C}_\tau^{\mathrm{T}},$
$\Gamma_7 = \bar{B}_\tau^{\mathrm{T}}R_\tau + R_\tau^{\mathrm{T}}\bar{B}_\tau - \gamma^2 I.$

We define

$$
\begin{aligned}
N_\tau &= \begin{bmatrix} N_\tau^h & 0 \\ 0 & N_\tau^v \end{bmatrix} = \begin{bmatrix} M_4^h(M_3^h)^{-1}M_{2\tau}^h & 0 \\ 0 & M_4^v(M_3^v)^{-1}M_{2\tau}^v \end{bmatrix},\\
U &= \begin{bmatrix} U^h & 0 \\ 0 & U^v \end{bmatrix} = \begin{bmatrix} M_4^h(M_3^h)^{-T}(M_4^h)^{\mathrm{T}} & 0 \\ 0 & M_4^v(M_3^v)^{-T}(M_4^v)^{\mathrm{T}} \end{bmatrix},\\
M_\tau &= \begin{bmatrix} M_\tau^h & 0 \\ 0 & M_\tau^v \end{bmatrix} = \begin{bmatrix} M_{1\tau}^h & 0 \\ 0 & M_{1\tau}^v \end{bmatrix}, R_\tau = \begin{bmatrix} R_\tau^h \\ R_\tau^v \end{bmatrix},\\
Q_\tau &= \begin{bmatrix} Q_\tau^h & 0 \\ 0 & Q_\tau^v \end{bmatrix} = \begin{bmatrix} M_4^h(M_3^h)^{-1}S_{2\tau}^h & 0 \\ 0 & M_4^v(M_3^v)^{-1}S_{2\tau}^v \end{bmatrix},\\
S_\tau &= \begin{bmatrix} S_\tau^h & 0 \\ 0 & S_\tau^v \end{bmatrix} = \begin{bmatrix} S_{1\tau}^h & 0 \\ 0 & S_{1\tau}^v \end{bmatrix}, F_\tau = \begin{bmatrix} F_\tau^h \\ F_\tau^v \end{bmatrix},\\
\bar{B}_f &= \begin{bmatrix} \bar{B}_{f1} \\ \bar{B}_{f2} \end{bmatrix} = \begin{bmatrix} (M_4^h)^{\mathrm{T}}B_{f1} \\ (M_4^v)^{\mathrm{T}}B_{f2} \end{bmatrix}\\
\bar{C}_f &= [\,\bar{C}_{f1}\ \bar{C}_{f2}\,] = [\,C_{f1}(M_3^h)^{-T}(M_4^h)^{\mathrm{T}}\ \ C_{f2}(M_3^v)^{-T}(M_4^v)^{\mathrm{T}}\,],\\
\bar{A}_f &= \begin{bmatrix} \bar{A}_{f11} & \bar{A}_{f12} \\ \bar{A}_{f21} & \bar{A}_{f22} \end{bmatrix} = \begin{bmatrix} M_4^h A_{f11}(M_3^h)^{-T}(M_4^h)^{\mathrm{T}} & M_4^h A_{f12}(M_3^v)^{-T}(M_4^v)^{\mathrm{T}} \\ M_4^v A_{f21}(M_3^h)^{-T}(M_4^h)^{\mathrm{T}} & M_4^v A_{f22}(M_3^v)^{-T}(M_4^v)^{\mathrm{T}} \end{bmatrix}
\end{aligned}
$$

$$
\begin{bmatrix} \bar{A}_f & \bar{B}_f \\ \bar{C}_f & 0 \end{bmatrix} = \left[\begin{array}{cc|c} M_4^h & 0 & 0 \\ 0 & M_4^v & 0 \\ \hline 0 & 0 & I \end{array}\right]\left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & 0 \end{array}\right] \times \left[\begin{array}{cc|c} (M_3^h)^{-T}(M_4^h)^{\mathrm{T}} & 0 & 0 \\ 0 & (M_3^v)^{-T}(M_4^v)^{\mathrm{T}} & 0 \\ \hline 0 & 0 & I \end{array}\right] \tag{21}
$$

With a new change of variables in inequality (16) by the above matrices, we obtain the following result.

Proposition 4.2 *Given the 2-D system in (1), for the filter in (7), an any fixed $\tau \in \Gamma$, there exist a matrix $P_\tau = diag\{P_\tau^h, P_\tau^v\}$ and filter matrices A_f, B_f, C_f satisfying (15) if there exist matrices*
$\bar{P}_\tau = diag\left\{\bar{P}_\tau^h\ \ \bar{P}_\tau^v\right\} > 0, M_\tau = diag\left\{M_\tau^h\ \ M_\tau^v\right\},$
$S_\tau = diag\left\{S_\tau^h\ \ S_\tau^v\right\}, N_\tau = diag\left\{N_\tau^h\ \ N_\tau^v\right\},$
$U = diag\left\{U^h\ \ U^v\right\}, Q_\tau = diag\left\{Q_\tau^h\ \ Q_\tau^v\right\},$
$R_\tau = \left[(R_\tau^h)^{\mathrm{T}}\ \ (R_\tau^v)^{\mathrm{T}}\right]^{\mathrm{T}}, F_\tau = \left[(F_\tau^h)^{\mathrm{T}}\ \ (F_\tau^v)^{\mathrm{T}}\right]^{\mathrm{T}},$
$\bar{A}_f, \bar{B}_f, \bar{C}_f, \Lambda_1 = diag\{\lambda_1,\ \lambda_3\}, \Lambda_2 = diag\{\lambda_2,\ \lambda_4\}$ with $\lambda_1, \lambda_2, \lambda_3$ and λ_4 real scalars satisfying:

$$
\Xi_\tau = \begin{bmatrix} \bar{P}_\tau - \Psi_{1\tau} & \Psi_{2\tau} & \Psi_{3\tau} & -\Upsilon_1^{\mathrm{T}}F_\tau^{\mathrm{T}} \\ * & \Psi_{4\tau} & \Psi_{5\tau} & \Psi_{6\tau} \\ * & * & \Psi_{7\tau} & B_\tau^{\mathrm{T}}F_\tau^{\mathrm{T}} \\ * & * & * & -I \end{bmatrix} < 0 \tag{22}
$$

where

$$
\begin{aligned}
\Psi_{1\tau} &= \Upsilon_1^{\mathrm{T}}[M_\tau^{\mathrm{T}} + M_\tau]\Upsilon_1 + \Upsilon_1^{\mathrm{T}}N_\tau^{\mathrm{T}}\Upsilon_2 + \Upsilon_2^{\mathrm{T}}N_\tau\Upsilon_1\\
&\quad + \Upsilon_2^{\mathrm{T}}U^{\mathrm{T}}[\Upsilon_1 + \Upsilon_2] + [\Upsilon_1^{\mathrm{T}} + \Upsilon_2^{\mathrm{T}}]U\Upsilon_2.\\
\Psi_{2\tau} &= \Upsilon_1^{\mathrm{T}}[M_\tau A_\tau + \bar{B}_f C_\tau]\Upsilon_1 + \Upsilon_1^{\mathrm{T}}\bar{A}_f\Upsilon_2 + \Upsilon_2^{\mathrm{T}}\bar{A}_f\Upsilon_2\\
&\quad + \Upsilon_2^{\mathrm{T}}[N_\tau A_\tau + \bar{B}_f C_\tau]\Upsilon_1 - \Upsilon_1^{\mathrm{T}}S_\tau^{\mathrm{T}}\Upsilon_1 - \Upsilon_1^{\mathrm{T}}Q_\tau^{\mathrm{T}}\Upsilon_2\\
&\quad - \Upsilon_2^{\mathrm{T}}[\Lambda_1 U^{\mathrm{T}}\Upsilon_1 + \Lambda_2 U^{\mathrm{T}}\Upsilon_2].\\
\Psi_{3\tau} &= \Upsilon_1^{\mathrm{T}}[M_\tau B_\tau + \bar{B}_f D_\tau] + \Upsilon_2^{\mathrm{T}}[N_\tau B_\tau + \bar{B}_f D_\tau]\\
&\quad - \Upsilon_1^{\mathrm{T}}R_\tau^{\mathrm{T}}.\\
\Psi_{4\tau} &= -\bar{P}_\tau + \Upsilon_1^{\mathrm{T}}[S_\tau A_\tau + A_\tau^{\mathrm{T}}S_\tau^{\mathrm{T}}]\Upsilon_1 + \Upsilon_1^{\mathrm{T}}[\Lambda_1\bar{B}_f C_\tau\\
&\quad + C_\tau^{\mathrm{T}}\bar{B}_f^{\mathrm{T}}\Lambda_1]\Upsilon_1 + \Upsilon_2^{\mathrm{T}}[Q_\tau A_\tau + \Lambda_2\bar{B}_f C_\tau]\Upsilon_1\\
&\quad + [\Upsilon_2^{\mathrm{T}}\Lambda_2 + \Upsilon_1^{\mathrm{T}}\Lambda_1]\bar{A}_f\Upsilon_2 + \Upsilon_1^{\mathrm{T}}[A_\tau^{\mathrm{T}}Q_\tau^{\mathrm{T}}\\
&\quad + C_\tau^{\mathrm{T}}\bar{B}_f^{\mathrm{T}}\Lambda_2]\Upsilon_2 + \Upsilon_2^{\mathrm{T}}\bar{A}_f^{\mathrm{T}}[\Lambda_2\Upsilon_2 + \Lambda_1\Upsilon_1].\\
\Psi_{5\tau} &= \Upsilon_1^{\mathrm{T}}S_\tau B_\tau + [\Upsilon_1^{\mathrm{T}}\Lambda_1 + \Upsilon_2^{\mathrm{T}}\Lambda_2]\bar{B}_f D_\tau + \Upsilon_2^{\mathrm{T}}Q_\tau B_\tau\\
&\quad + \Upsilon_1^{\mathrm{T}}A_\tau^{\mathrm{T}}R_\tau^{\mathrm{T}}.\\
\Psi_{6\tau} &= \Upsilon_1^{\mathrm{T}}H_\tau^{\mathrm{T}} - \Upsilon_2^{\mathrm{T}}\bar{C}_f^{\mathrm{T}} + \Upsilon_1^{\mathrm{T}}A_\tau^{\mathrm{T}}F_\tau^{\mathrm{T}}.\\
\Psi_{7\tau} &= R_\tau B_\tau + B_\tau^{\mathrm{T}}R_\tau^{\mathrm{T}} - \gamma^2 I.
\end{aligned}
$$

Moreover, under the above condition, the matrices for an admissible robust H_∞ filter are given by

$$
\begin{bmatrix} A_f & B_f \\ C_f & 0 \end{bmatrix} = \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \bar{A}_f & \bar{B}_f \\ \bar{C}_f & 0 \end{bmatrix} \tag{23}
$$

Proof We know the transfer function of the filter (7) from $y(i, j)$ to $\bar{z}(i, j)$ is given by

$$
T_{\bar{z}y}(z_1, z_2) = C_f[\mathrm{diag}\{z_1 I_{n_1}, z_2 I_{n_2}\} - A_f]^{-1}B_f \tag{24}
$$

Substituting (21) into this transfer function and considering

$$
U_h = M_4^h(M_3^h)^{-T}(M_4^h)^{\mathrm{T}}, U_v = M_4^v(M_3^v)^{-T}(M_4^v)^{\mathrm{T}}, \tag{25}
$$

we get

$$
T_{\bar{z}y}(z_1, z_2) = \bar{C}_f[\mathrm{diag}\{z_1 I_{n_1}, z_2 I_{n_2}\} - U^{-1}\bar{A}_f]^{-1}S^{-1}\bar{B}_f. \tag{26}
$$

Therefore, the desired filter is given by (23) and the proof is completed.

Before presenting the formulation of Proposition 4.2 using homogeneous polynomially parameter-dependent matrices, some definitions and preliminaries are needed to represent and handle products and sums of homogeneous polynomials. First, we define the homogeneous polynomially parameter-dependent matrices of degree g by

$$\bar{P}_\tau = \sum_{j=1}^{J(g)} \tau_1^{k_1}\tau_2^{k_2}\dots\tau_s^{k_s}\bar{P}_{k_j(g)}, \quad [k_1,k_2,\dots,k_s] = K_j(g) \tag{27}$$

Similarly, matrices M_τ, N_τ, R_τ, Q_τ, S_τ and F_τ take the same form.

The notations in the above are explained as follows. Define $K(g)$ as the set of s-tuples obtained as all possible combination of $[k_1,k_2,\dots,k_s]$, with k_i being nonnegative integers, such that $k_1+k_2+\dots+k_s = g$. $K_j(g)$ is the j-th s-tuples of $K(g)$ which is lexically ordered, $j = 1,\dots,J(g)$. Since the number of vertices in the polytope Γ is equal to s, the number of elements in $K(g)$ as given by $J(g) = \frac{(s+g-1)!}{g!(s-1)!}$. These elements define the subscripts $k_1,k_2,\dots,k_s$ of the constant matrices
$\bar{P}_{k_1,k_2,\dots k_s} \triangleq \bar{P}_{k_j(g)}$, $M_{k_1,k_2,\dots k_s} \triangleq M_{k_j(g)}$,
$N_{k_1,k_2,\dots k_s} \triangleq N_{k_j(g)}$, $R_{k_1,k_2,\dots k_s} \triangleq R_{k_j(g)}$,
$Q_{k_1,k_2,\dots k_s} \triangleq Q_{k_j(g)}$, $S_{k_1,k_2,\dots k_s} \triangleq S_{k_j(g)}$,
$F_{k_1,k_2,\dots k_s} \triangleq F_{k_j(g)}$, which are used to construct the homogeneous polynomial-dependent matrices $\bar{P}_\tau$, M_τ, N_τ, $R_\tau Q_\tau$, S_τ, F_τ in (27).

For each set $K(g)$, define also the set $I(g)$ with elements $I_j(g)$ given by subsets of i, $i \in \{1,2,\dots,s\}$, associated with s-tuples $K_j(g)$ whose k_i's are nonzero. For each i, i=1,ldots,s, define the s-tuples $K_j^i(g)$ as being equal to $K_j(g)$ but with $k_i > 0$ replaced by $k_i - 1$. Note that the s-tuples $K_j^i(g)$ are defined only in the cases where the corresponding k_i is positive. Note also that, when applied to the elements of $K(g+1)$, the s-tuples $K_j^i(g+1)$ define subscripts $k_1,k_2,\dots,k_s$ of matrices $\bar{P}_{k_1,k_2,\dots,k_s}$, $M_{k_1,k_2,\dots,k_s}$, $N_{k_1,k_2,\dots,k_s}$, $Q_{k_1,k_2,\dots,k_s}$, $S_{k_1,k_2,\dots,k_s}$, $F_{k_1,k_2,\dots,k_s}$, $R_{k_1,k_2,\dots,k_s}$, associated with homogeneous polynomially parameter-dependent matrices of degree g. Finally, define the scalar constant coefficients $\beta_j^i(g+1) = \frac{g!}{(k_1!k_2!\dots k_s!)}$, with $[k_1,k_2,\dots,k_s] \in K_j^i(g+1)$.

The main result in this section is given in the following Theorem 4.3.

Theorem 4.3 *Given a stable 2-D system (1) and a scalar $\gamma > 0$, a filter (7) exists such that the filtering error system (11) is robustly asymptotically stable and satisfies (14). If there exist matrices*

$$\bar{P}_{K_j(g)} = \mathrm{diag}\{\bar{P}^h_{K_j(g)},\ \ \bar{P}^v_{K_j(g)}\} > 0,$$
$$M_{K_j(g)} = \mathrm{diag}\{M^h_{K_j(g)},\ \ M^v_{K_j(g)}\},$$
$$R_{K_j(g)} = \mathrm{diag}\{R^h_{K_j(g)},\ \ R^v_{K_j(g)}\},$$
$$N_{K_j(g)} = \mathrm{diag}\{N^h_{K_j(g)},\ \ N^v_{K_j(g)}\},$$
$$Q_{K_j(g)} = \mathrm{diag}\{Q^h_{K_j(g)},\ \ Q^v_{K_j(g)}\},$$
$$S_{K_j(g)} = \left[(S^h_{K_j(g)})^{\mathrm{T}}\ (S^v_{K_j(g)})^{\mathrm{T}}\right]^{\mathrm{T}},$$
$$F_{K_j(g)} = \left[(F^h_{K_j(g)})^{\mathrm{T}}\ (F^v_{K_j(g)})^{\mathrm{T}}\right]^{\mathrm{T}},$$
$$K_j(g) \in K(g),\ j = 1,2,ldots,J(g),$$

$\bar{A}_f$, $\bar{B}_f$, $\bar{C}_f$, $\Lambda_1 = \mathrm{diag}\{\lambda_1,\ \lambda_3\}$, $\Lambda_2 = \mathrm{diag}\{\lambda_2,\ \lambda_4\}$ with λ_1, λ_2, λ_3 and λ_4 real scalars such that the following LMIs hold for all $K_l(g+1) \in K(g+1)$, l=1,ldots,J(g+1):

$$\Xi_k = \sum_{i \in I_l(g+1)} \begin{bmatrix} \Psi_1 & \Psi_2 & \Psi_3 & -\Upsilon_1^{\mathrm{T}} F^{\mathrm{T}}_{K_j(g)} \\ * & \Psi_4 & \Psi_5 & \Psi_6 \\ * & * & \Psi_7 & B_i^{\mathrm{T}} F^{\mathrm{T}}_{K_j(g)} \\ * & * & * & -\beta_j^i(g+1)I \end{bmatrix} < 0 \tag{28}$$

where

$$\begin{aligned}
\Psi_1 &= \bar{P}_{K_j(g)} - \Upsilon_1^{\mathrm{T}}[M^{\mathrm{T}}_{K_j(g)} + M_{K_j(g)}]\Upsilon_1 - \Upsilon_1^{\mathrm{T}} N^{\mathrm{T}}_{K_j(g)}\Upsilon_2 \\
&\quad - \Upsilon_2^{\mathrm{T}} N_{K_j(g)}\Upsilon_1 - \beta_j^i(g+1)\Upsilon_2^{\mathrm{T}} U^{\mathrm{T}}[\Upsilon_1 + \Upsilon_2] \\
&\quad - \beta_j^i(g+1)[\Upsilon_1^{\mathrm{T}} + \Upsilon_2^{\mathrm{T}}]U\Upsilon_2. \\
\Psi_2 &= \Upsilon_1^{\mathrm{T}}[M_{K_j(g)}A_i + \beta_j^i(g+1)\bar{B}_f C_i]\Upsilon_1 - \Upsilon_1^{\mathrm{T}} S^{\mathrm{T}}_{K_j(g)}\Upsilon_1 \\
&\quad - \Upsilon_1^{\mathrm{T}} Q^{\mathrm{T}}_{K_j(g)}\Upsilon_2 + \beta_j^i(g+1)[\Upsilon_1^{\mathrm{T}}\bar{A}_f\Upsilon_2 + \Upsilon_2^{\mathrm{T}}\bar{A}_f\Upsilon_2] \\
&\quad + \Upsilon_2^{\mathrm{T}}[N_{K_j(g)}A_i + \beta_j^i(g+1)\bar{B}_f C_i]\Upsilon_1 \\
&\quad - \beta_j^i(g+1)\Upsilon_2^{\mathrm{T}}[\Lambda_1 U^{\mathrm{T}}\Upsilon_1 + \Lambda_2 U^{\mathrm{T}}\Upsilon_2]. \\
\Psi_3 &= \Upsilon_1^{\mathrm{T}}[M_{K_j(g)}B_i + \beta_j^i(g+1)\bar{B}_f D_i] + \Upsilon_2^{\mathrm{T}}[N_{K_j(g)}B_i \\
&\quad + \beta_j^i(g+1)\bar{B}_f D_i] - \Upsilon_1^{\mathrm{T}} R^{\mathrm{T}}_{K_j(g)}. \\
\Psi_4 &= -\bar{P}_{K_j(g)} + \Upsilon_1^{\mathrm{T}}[S_{K_j(g)}A_i + A_i^{\mathrm{T}} S^{\mathrm{T}}_{K_j(g)}]\Upsilon_1 \\
&\quad + \beta_j^i(g+1)\Upsilon_1^{\mathrm{T}}[\Lambda_1\bar{B}_f C_i + C_i^{\mathrm{T}}\bar{B}_f^{\mathrm{T}}\Lambda_1]\Upsilon_1 \\
&\quad + \Upsilon_2^{\mathrm{T}}[Q_{K_j(g)}A_i + \beta_j^i(g+1)\Lambda_2\bar{B}_f C_i]\Upsilon_1 \\
&\quad + \beta_j^i(g+1)[\Upsilon_2^{\mathrm{T}}\Lambda_2 + \Upsilon_1^{\mathrm{T}}\Lambda_1]\bar{A}_f\Upsilon_2 + \Upsilon_1^{\mathrm{T}}[A_i^{\mathrm{T}} Q^{\mathrm{T}}_{K_j(g)} \\
&\quad + \beta_j^i(g+1)C_i^{\mathrm{T}}\bar{B}_f^{\mathrm{T}}\Lambda_2]\Upsilon_2 + \beta_j^i(g+1)\Upsilon_2^{\mathrm{T}}\bar{A}_f^{\mathrm{T}}[\Lambda_2\Upsilon_2 \\
&\quad + \Lambda_1\Upsilon_1]. \\
\Psi_5 &= \Upsilon_1^{\mathrm{T}} S_{K_j(g)}B_i + \beta_j^i(g+1)[\Upsilon_1^{\mathrm{T}}\Lambda_1 + \Upsilon_2^{\mathrm{T}}\Lambda_2]\bar{B}_f D_i \\
&\quad + \Upsilon_2^{\mathrm{T}} Q_{K_j(g)}B_i + \Upsilon_1^{\mathrm{T}} A_i^{\mathrm{T}} R^{\mathrm{T}}_{K_j(g)}. \\
\Psi_6 &= \beta_j^i(g+1)[\Upsilon_1^{\mathrm{T}} H_i^{\mathrm{T}} - \Upsilon_2^{\mathrm{T}}\bar{C}_f^{\mathrm{T}}] + \Upsilon_1^{\mathrm{T}} A_i^{\mathrm{T}} F^{\mathrm{T}}_{K_j(g)}. \\
\Psi_7 &= R_{K_j(g)}B_i + B_i^{\mathrm{T}} R^{\mathrm{T}}_{K_j(g)} - \beta_j^i(g+1)\gamma^2 I.
\end{aligned}$$

Then, the homogeneous polynomial matrices $\bar{P}_\tau$, M_τ, N_τ, R_τ, Q_τ, S_τ and F_τ assure (22) for all $\tau \in \Gamma$.

Moreover, if the LMIs of (28) are fulfilled for a given degree $\bar{g}$, then the LMIs corresponding to any degree $g > \bar{g}$ are also satisfied.

Proof **First part**: Since $\bar{P}_{K_j(g)} > 0$, $K_{j(g)} \in K(g)$, $j = 1, ldots, J(g)$, we know that $\bar{P}_\tau$ defined in (27) is positive definite for all $\tau \in \Gamma$. Now, note that $\Xi \in$ (22) for

$(A_\tau, B_\tau, C_\tau, D_\tau, H_\tau) \in \Omega$ and $P_\tau, M_\tau, N_\tau, T_\tau, R_\tau$ and S_τ given by (27) are homogeneous polynomial matrix equations of degree $g+1$ that can be written as

$$\Xi(\tau) = \sum_{l=1}^{J(g+1)} \tau_1^{k_1} \tau_2^{k_2} \ldots \tau_s^{k_s} \; \Xi_k \tag{29}$$

Condition (28) imposed for all $l, l = 1, ldots, J(g+1)$ assure that $\Xi_\tau < 0$ for all $\tau \in \Gamma$, and thus the first part is proved.

Second part: Suppose that (28) are fulfilled for a certain degree $\hat{g}$, that is, there exit $J(\hat{g})$ symmetric positive definite matrix $\bar{P}_{K_{j(\hat{g})}}$ and matrices $M_{K_{j(\hat{g})}}, N_{K_{j(\hat{g})}}, Q_{K_{j(\hat{g})}}, S_{K_{j(\hat{g})}}, R_{K_{j(\hat{g})}}, F_{K_{j(\hat{g})}}, j = 1, ldots, J(\hat{g})$ such that $\bar{P}_\tau$, M_τ, N_τ, Q_τ, S_τ, F_τ and R_τ defined in (27) are homogeneous polynomially parameter-dependent Lyapunov matrices assuring $\Xi_\tau < 0$. Then, the terms of the polynomial matrices $\tilde{\bar{P}}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)\bar{P}_\tau$, $\tilde{M}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)M_\tau$, $\tilde{N}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)N_\tau$, $\tilde{Q}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)Q_\tau$, $\tilde{S}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)S_\tau$, $\tilde{F}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)F_\tau$ and $\tilde{R}_\tau = (\tau_1, \tau_2, \ldots, \tau_s)R_\tau$ also satisfy the inequalities of Theorem 4.3 corresponding to the degree $\hat{g}+1$, which can be obtained in this case by linear combination of the inequalities of Theorem 4.3 for $\hat{g}$

Remark 4.4 When the scalars $\lambda_1, \lambda_2, \lambda_3$ and λ_4 of Theorem 4.3 are fixed to be constants, then (28) is an LMI which is effectively linear in the variables. To select values for these scalars, optimization can be used (for example fminsearch in MATLAB) to optimize some performance measure (for example γ, the disturbance attenuation level).

Remark 4.5 As the degree g of the polynomial increases, the conditions become less conservative since new free variables are added to the LMIs. Although the number of LMIs is also increased, each LMI becomes easier to be fulfilled due to the extra degrees of freedom provided by the new free variables; as a consequence, better H_∞ guaranteed costs can be obtained.

5 Numerical Example

Consider the following 2-D static field model described by differential equation (El-Kasri et al. 2012):

$$\begin{aligned}\eta(i+1, j+1) &= \tau_1\eta(i, j+1) + \tau_2\eta(i+1, j)\\ &\quad - \tau_1\tau_2\eta(i, j) + \omega_1(i, j)\end{aligned} \tag{30}$$

where $\eta(i, j)$ is the state of the random field of spacial coordinate (i, j), $\omega_1(i, j)$ is a noise input, τ_1,τ_2 are the vertical and horizontal correlative coefficients of the random field, respectively, satisfying $\tau_1^2 < 1$ and $\tau_2^2 < 1$. The output is then:

$$\begin{aligned}y(i, j) &= \tau_1\eta(i, j+1) + (1-\tau_1\tau_2)\eta(i+1, j)\\ &\quad + \omega_2(i, j)\end{aligned} \tag{31}$$

where $\omega_2(i, j)$ is the measurement noise. The signal to be estimated is

$$z(i, j) = \eta_{i,j} \tag{32}$$

As in Du et al. (2002), define $x^h(i, j) = \eta(i, j+1) - \tau_2\eta(i, j)$, $x^v(i, j) = \eta(i, j)$ and $\omega(i, j) = [\omega_1^{\mathrm{T}}(i, j)\ \omega_2(i, j)^{\mathrm{T}}]^{\mathrm{T}}$. It is easy to see (30)–(32) can be converted into a 2D Roesser model as follows:

$$\begin{aligned}\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= \begin{bmatrix} \tau_1 & 0 \\ 1 & \tau_2 \end{bmatrix}\begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w(i, j)\\ y(i, j) &= \begin{bmatrix} \tau_1 & 1 \end{bmatrix}\begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} w(i, j)\\ z(i, j) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}\end{aligned} \tag{33}$$

where $0.15 \le \tau_1 \le 0.45$, $0.35 \le \tau_2 \le 0.85$. The uncertain 2-D system corresponds to a four vertex polytopic system.

The LMIs (28) were solved using Yalmip and SeDuMi in MATLAB 7.6, for increasing values of the degree g:
For degree $g = 0$, the proposed optimization gives $\lambda_1 = -0.1064$, $\lambda_2 = 0.0228$, $\lambda_3 = 0.0027$, and $\lambda_4 - 0.0002$. For these scalars $\gamma = 2.4342$ and the corresponding filter matrices are:

$$A_f = \begin{bmatrix} 0.3614 & 0.0000 \\ 0.1538 & -0.0000 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.0920 \\ -0.8213 \end{bmatrix},$$
$$C_f = \begin{bmatrix} -0.0605 & -0.9868 \end{bmatrix}$$

The H_∞ norms obtained with this filter at the vertices of the uncertainties are given in Table 1.

Table 1 H_∞ norms at the vertices ($g = 0$)

τ_1	0.15	0.15	0.45	0.45
τ_2	0.35	0.85	0.35	0.85
γ	1.6622	1.3850	1.9632	1.4972

On the other hand, when $g = 1$, $\lambda_1 = -20.3646$, $\lambda_2 = 0.0662$, $\lambda_3 = 0.6369$ and $\lambda_4 - 0.1645$, the disturbance attenuation obtained is $\gamma = 1.8043$ and the corresponding filter matrices are:

$$A_f = \begin{bmatrix} 0.6476 & 1.9416 \\ -0.0151 & 0.2548 \end{bmatrix}, \quad B_f = \begin{bmatrix} 4.0789 \\ -1.2683 \end{bmatrix},$$

Table 2 H_∞ norms at the vertices ($g = 1$)

τ_1	0.15	0.15	0.45	0.45
τ_2	0.35	0.85	0.35	0.85
γ	1.5099	1.5147	1.7851	1.6459

Table 3 H_∞ norms at the vertices (g = 2)

τ_1	0.15	0.15	0.45	0.45
τ_2	0.35	0.85	0.35	0.85
γ	1.5096	1.5108	1.7838	1.6481

Table 4 Comparison with previous results

Degree g	Theorem 4.3	Algo 21 (Gao and Li 2014)	Th 1 (Ying and Rui 2011)	Th 3 (Gao et al. 2008)
0	2.4342	2.4356	2.4360	2.4373
1	1.8043	1.8586	1.8621	1.8627
2	1.8042	1.8295	1.8505	1.8290

$$C_f = \begin{bmatrix} 0.0045 & -0.4699 \end{bmatrix}.$$

The H_∞ norms at the vertices are now given in Table 2.

For degree $g = 2$, $\lambda_1 = -3.3940$, $\lambda_2 = 0.0696$, $\lambda_3 = 0.6291$ and $\lambda_4 = -0.1640$, $\gamma = 1.8042$ and the corresponding filter matrices are:

$$A_f = \begin{bmatrix} 0.6360 & 0.2615 \\ -0.1152 & 0.2544 \end{bmatrix}, \quad B_f = \begin{bmatrix} 0.5458 \\ -1.2617 \end{bmatrix},$$
$$C_f = \begin{bmatrix} 0.0338 & -0.4725 \end{bmatrix}$$

For the filter designed with $g = 2$, the actual H_∞ norms calculated at the four extreme plants are presented in Table 3: it can be seen that all of them are below the guaranteed bound 1.8042.

In summary, it has been shown that less conservative filter designs are achieved as g grows by applying the polynomially parameter-dependent method proposed here.

A comparison with the results obtained with the techniques proposed in Gao and Li (2014), Ying and Rui (2011) and Gao et al. (2008) is presented in Table 4, showing the improvement obtained with the methodology proposed in this paper.

The number of LMIs, the number of scalar variables and the cpu time to solve the LMIs are compared in the following Table 5. It must be pointed out that for this example, increasing the polynomial order to $g > 2$ does not improve the noise reduction properties.

Table 5 The numerical complexity obtained by Theorem 4.3 with **L** is the number of LMIs rows, **V** is the number of scalar variables, **N** is number of LMIs and **Time(s)** is the cpu time to solve the LMIs

Degree g	**N**	**V**	**L**	**Time(s)**
0	4	31	49	0.3420
1	10	91	127	0.4920
2	20	211	261	0.8080

6 Conclusion

This article has investigated the H_∞ filtering problem for 2-D discrete systems described by uncertain Roesser models. A new condition for H_∞ performance analysis has been proposed in the LMI framework, by using a Lyapunov function approach and adding some slack matrix variables with specific structures that make possible to reduce the conservatism of previous works. A numerical example illustrate the effectiveness of the proposed method. As future research, we will use this technique in nonlinear systems based on fuzzy dynamic model and the SOS (Sum Of Square) Technique in this systems.

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