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A T–S Fuzzy Approach to the Local Stabilization of Nonlinear Discrete-Time Systems Subject to Energy-Bounded Disturbances

Michael Klug · Eugênio B. Castelan · Daniel Coutinho

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Abstract This paper addresses the local stabilization problem of nonlinear discrete-time systems subject to energy-bounded disturbances by means of T–S fuzzy models. A fuzzy state feedback controller is designed such that the input-to-state stability in the ℓ_2 sense of the original nonlinear system is guaranteed in a bounded region of the state space. Such a region is related to the exactness of the T–S model and describes the domain around the origin where the convexity property remains valid. In addition, an estimate of the closed-loop reachable set is provided for a given class of ℓ_2 disturbances. Three (convex) optimization problems are proposed to either minimize the estimate of the reachable set, improve the disturbance tolerance or minimize the ℓ_2-gain from the disturbance input to the regulated output. Numerical examples are considered to illustrate the approach demonstrating the effectiveness of the proposed technique for the control synthesis of nonlinear discrete-time systems.

Keywords Nonlinear T–S models · Local stability · Energy-bounded disturbances · ℓ_2-gain

1 Introduction

Takagi–Sugeno (T–S) fuzzy models have been extensively investigated over the last decade to develop the so-called fuzzy model-based (FMB) control techniques (see, for instance, Dong et al. 2009, 2010; Ding 2011; Esfahani and Sichani 2011; Campana et al. 2012). Basically, this type of models allows the representation of nonlinear systems in terms of local linear models that are smoothly connected by means of nonlinear fuzzy membership functions (MFs) so that it is possible to apply, for instance, well-established Lyapunov and LMI-based tools for parameter varying control systems (Tanaka and Wang 2001; Mozelli and Palhares 2011). Thus, T–S fuzzy models provide a systematic framework for dealing with fundamental issues in modern control theory for complex nonlinear systems.

However, in order to obtain numerically tractable solutions for the stability analysis and control design of nonlinear systems, the available T–S fuzzy modeling techniques can only locally guarantee the stability properties of the original nonlinear systems. Notice when deriving a T–S fuzzy model that a normalizing step is used in the defuzzification process which requires that the premise variables are bounded in some chosen compact set. In other words, there exists a bounded region $\mathcal{X}$ of state space containing the origin associated with a region $\varXi$ in the normalized membership functions space. Hence, when applying convex methods to solve fuzzy-based stability conditions on the $\varXi$ space, it is required to take into account that the stability conditions hold only if the state trajectory of the original nonlinear system does not leave $\mathcal{X}$. From this reasoning, we refer to the region $\mathcal{X}$ as the *T–S domain of validity*.

Nonetheless, the inherent local characteristic of T–S modeling techniques is often not considered in most of FMB control design results which may lead to poor performance or even instability of the closed-loop system (consisting of the original nonlinear plant and the designed fuzzy controller). The local stability issue in T–S fuzzy models may also be related to the natural existence of constraints in the state

M. Klug (✉)
Federal Institute of Santa Catarina, IFSC, Joinville, SC, Brazil
e-mail: michael.klug@ifsc.edu.br

E. B. Castelan · D. Coutinho
Federal University of Santa Catarina, DAS/UFSC, Florianópolis, SC, Brazil
e-mail: eugenio.castelan@ufsc.br

D. Coutinho
e-mail: daniel.coutinho@ufsc.br

variables of real systems, due, for example, to safe operational conditions, physical limitations or some desired level of energy consumptions, as discussed in Klug et al. (2014), or related to the presence of time derivatives of the MFs in the stability analysis when dealing with continuous-time systems, as in Guerra et al. (2012), Tognetti et al. (2013). Further, in the presence of exogenous disturbances, the input-to-state stability properties as well as input-to-output performance criteria of nonlinear systems may hold only locally (Rapaport and Astolfi 2002). However, most of the available FMB results do not consider performance and stability analysis in a local context (see, e.g., Chang and Yang 2014; Figueredo et al. 2014; Qiu et al. 2013; Chang 2012; Golabi et al. 2012; Su et al. 2012). For instance, the references Figueredo et al. (2014) and Su et al. (2012) deal with the H_∞ cost for systems with time-varying delays without considering the domain of validity $\mathcal{X}$, and therefore, performance and stability cannot be effectively guaranteed.

Most of the FMB control design results have employed a common quadratic Lyapunov function (Tanaka and Wang 2001) because of the simplicity on deriving numerical and tractable conditions. However, a common quadratic Lyapunov function may lead to a considerable conservatism, since the Lyapunov matrix should be found for all T–S local models. Recently, fuzzy Lyapunov functions (FLF) have been used to obtain less conservative design conditions at the cost of extra computations as proposed, for instance, in Guerra and Vermeiren (2004). In this context, the number of local models required for the T–S model representation may turn the FLF–FMB control design problem computationally untractable. To avoid a large number of rules, approximate models as described in Teixeira and Zak (1999) might be employed but adding some model inaccuracy. Alternatively, the number of fuzzy rules can be reduced without compromising the model exactness by applying the nonlinear T–S fuzzy modeling technique as proposed in the references Dong et al. (2009), Dong et al. (2010), Klug et al. (2013). In this approach, some nonlinear terms may explicitly appear in the T–S fuzzy models at the cost of losing the linearity of classical fuzzy models. Nevertheless, when the nonlinear terms (locally) satisfy sector-bounded conditions, the well-established mathematical machinery of absolute stability theory (Liberzon 2006) can be applied to derive FMB controllers (Klug et al. 2014).

In light of the above scenario, this paper addresses the state feedback input-to-state stabilization problem for nonlinear discrete-time systems subject to energy-bounded disturbances by means of (sector bounded) nonlinear T–S fuzzy models and FLF. More precisely, LMI control design conditions are proposed to locally ensure the input-to-state stability (ISS) and a certain input-to-output performance (i.e., an upper bound for the system ℓ_2-gain) of the original nonlinear discrete-time systems subject to a class of ℓ_2 disturbances. In addition, the design conditions provide an estimate of the closed-loop reachable set (that is, a region inside the T–S domain of validity which bounds the state trajectories driven by the admissible class of ℓ_2 disturbances). Three convex optimization problems demonstrate the effectiveness of the proposed approach as a control design tool for nonlinear discrete-time systems subject to energy-bounded disturbances.

The rest of this paper is organized as follows. The problem of interest and some preliminary results are stated in Sect. 2, and the main result is derived in Sect. 3. Section 4 provides three convex optimization problems for computing nonlinear state feedback control laws. Section 5 presents two numerical examples to illustrate the approach, and some concluding remarks are given in Sect. 6.

Notation: let A, B be two symmetric real matrices and v, s be two real vectors. $A > B$ means that $A - B$ is positive definite, $A^{'}$ denotes the transpose of A, $A_{(i)}$ denotes the ith row of A, $v_{(i)}$ is the i-th component of v, and v_k is the vector v at the k-th sample. $v(s) \in S[0, \Omega]$ represents a cone sector conditions, that is, $v^{'}_{(i)}(s)(v_{(i)}(s) - \Omega_{(i)}s) \leq 0$. The componentwise inequality $v \succeq s$ means that $v_{(i)} \geq s_{(i)}$. For symmetric block matrices, $\star$ stands for block matrices deduced by symmetry. I_n denotes an n-dimensional identity matrix. $\mathrm{diag}(A, B)$ is a block diagonal matrix. The ℓ_2-norm of a discrete vector sequence $\{w_k,\ k = 0, 1, 2, \ldots\}$ is defined as $\|w_k\|_{\ell_2} = \left(\sum_{k=0}^{\infty} w^{'}_k w_k\right)^{\frac{1}{2}}$.

2 Problem Statement

Consider the following class of nonlinear systems:

$$\begin{aligned} x_{k+1} &= f(x_k) + g(x_k)u_k + h(x_k)w_k \\ z_k &= f_z(x_k) + g_z(x_k)u_k + h_z(x_k)w_k \end{aligned} \tag{1}$$

where $x_k \in \Re^{n_x}$, $u_k \in \Re^{n_u}$, $w_k \in \mathcal{W} \subset \Re^{n_w}$ and $z_k \in \Re^{n_z}$ are the state, the control input, the exogenous disturbance vector and the regulated output, respectively. The functions $f(\cdot) : \Re^{n_x} \to \Re^{n_x}$, with $f(0) = 0$, $f_z(\cdot) : \Re^{n_x} \to \Re^{n_z}$, with $f_z(0) = 0$, $h(\cdot) : \Re^{n_x} \to \Re^{n_x \times n_w}$, $h_z(\cdot) : \Re^{n_x} \to \Re^{n_z \times n_w}$, $g(\cdot) : \Re^{n_x} \to \Re^{n_x \times n_u}$ and $g_z(\cdot) : \Re^{n_x} \to \Re^{n_z \times n_u}$ are continuous and bounded for all x_k. The disturbance input vector w_k is assumed to lie inside the following class of square summable sequences:

$$\mathcal{W} := \{w_k : \|w_k\|^2_{\ell_2} \leq \delta^{-1}\}. \tag{2}$$

where δ is a positive scalar defining the size of $\mathcal{W}$ (i.e., the energy bound of w_k). The set $\mathcal{W}$ will be often referred as the class of admissible disturbances.

In order to design a state feedback control law $u_k = \kappa(x_k)$, the nonlinear system (1) will be represented by means of a nonlinear T–S fuzzy model (which we refer as the N-fuzzy model) having $\mathcal{R}_i$, $i = 1, \dots, n_r$, fuzzy rules defined by (Dong et al. 2010):

$$\mathcal{R}_i : \begin{cases} \text{IF} \quad \nu_{k(1)} \text{ is } M_1^i, \nu_{k(2)} \text{ is } M_2^i, \dots, \nu_{k(n_s)} \text{ is } M_{n_s}^i \\ \text{THEN} \quad x_{k+1} = A_i x_k + B_i u_k + B_{wi} w_k + G_i \varphi_k \\ \qquad\qquad z_k \;\; = C_{zi} x_k + B_{zi} u_k + B_{zwi} w_k + G_{zi} \varphi_k \end{cases} \tag{3}$$

with M_j^i, $j = 1, \dots, n_s$, representing the fuzzy sets, $\nu_k := [\nu_{k(1)}, \nu_{k(2)}, \dots, \nu_{k(n_s)}]$ the premise variables, and $(A_i, B_i, B_{wi}, G_i, C_{zi}, B_{zi}, B_{zwi}, G_{zi})$ the matrices defining the fuzzy local models. Furthermore, the number of fuzzy rules n_r is associated with the number of premise variables n_s by the relation $n_r = 2^{n_s}$ to derive a precise representation of nonlinear system (1). The vector function $\varphi_k = \varphi(Lx_k) \in \Re^{n_\varphi}$, with $\varphi(0) = 0$ and $L \in \Re^{n_\varphi \times n_x}$, is a known nonlinear function of x_k satisfying a (local) cone sector condition $\varphi(\cdot) \in S[0, \Omega]$ for all $x_k \in \mathcal{X} \subset \Re^{n_x}$ with $\mathcal{X}$ to be defined later. Thus, as in Jungers and Castelan (2011), Klug et al. (2014), consider the existence of a free positive diagonal matrix $\Delta \in \Re^{n_\varphi \times n_\varphi}$ such that

$$\varphi_k' \Delta^{-1}[\varphi_k - \Omega L x_k] \leq 0 \,, \; \forall\, x_k \in \mathcal{X}. \tag{4}$$

From the definition of Δ, we see that if (4) is verified, then $\jmath$ independent classical conditions, $\varphi_{k(\jmath)}[\varphi_k - \Omega x_k]_{(\jmath)} \leq 0$, are also assured. Therefore, Δ represents a degree of freedom for the purpose of design and optimization.

The above N-fuzzy model is based on the representation proposed in Dong et al. (2009) and Dong et al. (2010). Notice if $\varphi_k = 0$ that the rules $\mathcal{R}_1, \dots, \mathcal{R}_{n_r}$ recover the classical definition of T–S fuzzy models (Takagi and Sugeno 1985).

Let $\alpha_k \equiv \alpha(x_k) \in \Xi$ be the vector of normalized grades of membership functions with simplex structure (Feng 2010) with Ξ defined as follows:

$$\Xi = \left\{ \alpha_k \in \Re^{n_r} : \sum_{i=1}^{n_r} \alpha_{k(i)} = 1,\; \alpha_{k(i)} \geq 0,\; i = 1, \dots, n_r \right\}. \tag{5}$$

Hence, the N-fuzzy model (3) can be rewritten as the following nonlinear fuzzy system:

$$\begin{aligned} x_{k+1} &= A(\alpha_k)x_k + B(\alpha_k)u_k + B_w(\alpha_k)w_k + G(\alpha_k)\varphi_k \\ z_k &= C_z(\alpha_k)x_k + B_z(\alpha_k)u_k + B_{zw}(\alpha_k)w_k + G_z(\alpha_k)\varphi_k \end{aligned} \tag{6}$$

where

$$\begin{bmatrix} A(\alpha_k) & B(\alpha_k) & B_w(\alpha_k) & G(\alpha_k) \\ C_z(\alpha_k) & B_z(\alpha_k) & B_{zw}(\alpha_k) & G_z(\alpha_k) \end{bmatrix} = \sum_{i=1}^{n_r} \alpha_{k(i)} \begin{bmatrix} A_i & B_i & B_{wi} & G_i \\ C_{zi} & B_{zi} & B_{zwi} & G_{zi} \end{bmatrix}.$$

Remark 1 It is important to emphasize that constraining the membership function α_k to a polytope, i.e., $\alpha(x_k) \in \Xi$, is an essential step for solving the control algorithms (thanks to convexity properties). However, when deriving the membership function of the T–S fuzzy model, a normalizing step is used in the defuzzification process which requires that premise variables are bounded in some chosen compact set. As a result, there exists a related region of state space, containing the origin, $0 \subset \mathcal{X} \subset \Re^{n_x}$, where the convexity of (6) is guaranteed. In other words, $x_k \in \mathcal{X} \Rightarrow \alpha_k \in \Xi$, as illustrated in the numerical examples later in this paper.

Assuming that the normalized membership functions α_k can be computed in real time, a nonlinear controller can be proposed with the same fuzzy rules as the nonlinear T–S model in (6). In this case, the following nonlinear state feedback control law is proposed:

$$u_k = \kappa(x_k) = K(\alpha_k)x_k + \Gamma(\alpha_k)\varphi_k \tag{7}$$

where $K(\alpha_k) = \sum_{i=1}^{n_r} \alpha_{k(i)} K_i$, $K_i \in \Re^{n_u \times n_x}$, and $\Gamma(\alpha_k) = \sum_{i=1}^{n_r} \alpha_{k(i)} \Gamma_i$, $\Gamma_i \in \Re^{n_u \times n_\varphi}$.

Now, taking into account Remark 1, the T–S model domain of validity $\mathcal{X}$ is for convenience defined by means of the following polyhedral set:

$$\mathcal{X} = \{x_k \in \Re^{n_x} : |N x_k| \preceq \phi\}, \tag{8}$$

where $\phi \in \Re^{n_\phi}$ and $N \in \Re^{n_\phi \times n_x}$ are constant and given, with $n_\phi \leq n_x$, representing the constraints which characterize the region $\mathcal{X}$.

Taking (6) and (7) into account, the closed-loop T–S fuzzy model is as follows:

$$\begin{aligned} x_{k+1} &= \mathcal{A}(\alpha_k)x_k + B_w(\alpha_k)w_k + \mathcal{G}(\alpha_k)\varphi_k \\ z_k &= \mathcal{C}(\alpha_k)x_k + B_{zw}(\alpha_k)w_k + \mathcal{F}(\alpha_k)\varphi_k \end{aligned} \tag{9}$$

with $\mathcal{A}(\alpha_k) = A(\alpha_k) + B(\alpha_k)K(\alpha_k)$, $\mathcal{G}(\alpha_k) = G(\alpha_k) + B(\alpha_k)\Gamma(\alpha_k)$, $\mathcal{C}(\alpha_k) = C_z(\alpha_k) + B_z(\alpha_k)K(\alpha_k)$ and $\mathcal{F}(\alpha_k) = G_z(\alpha_k) + B_z(\alpha_k)\Gamma(\alpha_k)$. The closed-loop matrices $\mathcal{A}(\alpha_k)$, $\mathcal{G}(\alpha_k)$, $\mathcal{C}(\alpha_k)$ and $\mathcal{F}(\alpha_k)$ can be generically rewritten, through summation properties, as

$$\mathcal{T}(\alpha_k) = \sum_{i=1}^{n_r} \sum_{j=i}^{n_r} \mu_{ij} \alpha_{k(i)} \alpha_{k(j)} \frac{T_i + X_i Y_j + T_j + X_j Y_i}{2} \tag{10}$$

where the tuple $(\mathcal{T}, T, X, Y)$ represents either $(\mathcal{A}, A, B, K)$, $(\mathcal{G}, G, B, \Gamma)$, $(\mathcal{C}, C_z, B_z, K)$ or $(\mathcal{F}, G_z, B_z, \Gamma)$, and

$$\mu_{ij} = \begin{cases} 2, & i \neq j, \\ 1, & \text{otherwise.} \end{cases} \tag{11}$$

Notice that the ISS stability of system (9) for all $\alpha_k \in \Xi$ implies that the original nonlinear system in (1) with (7) is also ISS stable. This is satisfied if the trajectory x_k of (9) driven by $w_k \in \mathcal{W}$ remains in $\mathcal{X}$ for all $k \geq 0$. In order to obtain LMI constraints guaranteeing the state trajectory boundness inside $\mathcal{X}$, the following problem will be addressed in this paper.

Problem 1 Determine the gain matrices K_i and Γ_i, for $i = 1, \ldots, n_r$, such that the trajectories of system (9) remain bounded in some region $\mathcal{D}$ containing the origin such that $\mathcal{D} \subset \mathcal{X}$ for any $w_k \in \mathcal{W}$ and for all $\alpha_k \in \Xi$. In addition, determine a positive constant γ which bounds the induced ℓ_2-norm from w_k to z_k.

To end this section, we provide some preliminary results which will be instrumental to derive the main contributions of this paper.

Let $V(x_k, \alpha_k)$ be a fuzzy Lyapunov function (FLF)

$$V(x_k, \alpha_k) : \Re^{n_x} \times \Xi \to \Re^+, \ \ V(0, \alpha_k) = 0 \quad \forall \alpha_k \in \Xi \quad (12)$$

and the set $\mathcal{D}$ defined as follows

$$\mathcal{D} \triangleq \{x_k \in \Re^{n_x} : V(x_k, \alpha_k) \leq \delta^{-1}, \forall \alpha_k \in \Xi\}\,, \quad (13)$$

where δ is the positive scalar defining the bound of $\mathcal{W}$ in (2).

In the following, we define the notion of ℓ_2 input-to-state stability for nonlinear discrete-time systems to be considered in this paper.

Definition 1 Consider the system (1), with $x(0) = 0$, and the level set $\mathcal{D}$ as defined in (13) for a given positive scalar δ. The unforced system in (1) is said to be ℓ_2-ISS$_{\mathcal{D}}$ (input-to-state stable with respect to $\mathcal{D}$), if for any $w_k \in \mathcal{W}$ the system state x_k remains bounded in $\mathcal{D}$ for all $k \geq 0$.

Observe that the above definition implies that $\mathcal{D}$ is a positively invariant set. Thus, $x(0) \in \mathcal{D}$ implies that

$$V(x_k, \alpha_k) \leq \delta^{-1}, \ \forall k \geq 0\,, \ \alpha_k \in \Xi. \quad (14)$$

Lemma 1 *The unforced system (1), with $x(0) = 0$, is ℓ_2-ISS$_{\mathcal{D}}$ and there exists an upper bound γ on the ℓ_2-gain from w_k to z_k if the following holds for all $x_k \in \mathcal{D}$, $\alpha_k \in \Xi$ and $w_k \in \mathcal{W}$:*

$$\Delta V_k \triangleq V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) + \gamma^{-2} z_k' z_k - w_k' w_k < 0 \quad (15)$$

Proof Assume that (15) holds $\forall x_k \in \mathcal{D}$, $\alpha_k \in \Xi$, $w_k \in \mathcal{W}$. Then, for any $\bar{k} > 0$, we get:

$$\sum_{k=0}^{\bar{k}-1} \Delta V_k = V(x_{\bar{k}}, \alpha_{\bar{k}}) - V(x_0, \alpha_0) + \gamma^{-2} \sum_{k=0}^{\bar{k}-1} z_k' z_k - \sum_{k=0}^{\bar{k}-1} w_k' w_k < 0 \quad (16)$$

Thus, in view of (2) and (12), the above implies:

(i) ℓ_2 input-to-state stability: note that $V(x_0, \alpha_0) = 0$, since $x(0) = 0$. Then, we have $V(x_{\bar{k}}, \alpha_{\bar{k}}) \leq \|w_k\|_2^2 \leq \delta^{-1}, \forall \bar{k} > 0$. That is, $\mathcal{D}$ is a positive invariant set.

(ii) input-to-output performance: taking $\bar{k} \to \infty$, it follows that $\|z_k\|_2 < \gamma \|w_k\|_2$. That is, γ is an upper bound on the system ℓ_2-gain.

(iii) internal stability: let $\tilde{k}$ be a positive integer. If $w_k = 0$ for all $k \geq \tilde{k}$, then the condition in (15) implies that $V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) < -\gamma^{-2} z_k' z_k < 0$ guaranteeing that $x_k \to 0$ as $k \to \infty$. In other words, $\mathcal{D}$ is a contractive positive invariant set whenever the disturbance w_k vanishes (see, e.g., Klug et al. 2011). □

3 Main Results

In this paper, we consider the N-fuzzy model (9) for designing the control law (7) which ensures that the nonlinear system (1) is locally ℓ_2-ISS$_{\mathcal{D}}$ in closed loop. To this end, let the following FLF:

$$V(x_k, \alpha_k) = x_k' Q^{-1}(\alpha_k) x_k, \ \ Q(\alpha_k) = \sum_{i=1}^{n_r} \alpha_{k(i)} Q_i\,, \quad (17)$$

with $Q_i = Q_i' > 0 \in \Re^{n_x \times n_x}$, $i = 1, \ldots, n_r$, to be determined.

In light of the above, notice that we have to additionally consider $\mathcal{D} \subset \mathcal{X}$ for all $\alpha_k \in \Xi$ in Lemma 1 to guarantee the convexity of the fuzzy model in (9). Furthermore, it can be shown that the level set $\mathcal{D}$ as defined in (13) with (17) will be the intersection of n_r ellipsoidal sets (Hu 2002; Jungers and Castelan 2011). In this paper, we consider the following definition for $\mathcal{D}$:

$$\mathcal{D} \triangleq \bigcap_{i \in \{1, \ldots n_r\}} \mathcal{E}(Q_i^{-1}, \delta^{-1}) \quad (18)$$

where $\mathcal{E}(Q_i^{-1}, \delta^{-1}) = \left\{x_k \in \Re^{n_x} : x_k' Q_i^{-1} x_k \leq \delta^{-1}\right\}$ is the i-th ellipsoidal set.

In the sequel, we present sufficient design conditions based on LMIs to determine the control law (7) which locally stabilizes the nonlinear system (1) in the ℓ_2-ISS$_{\mathcal{D}}$ sense.

Theorem 1 *Suppose there exist symmetric positive definite matrices $Q_i \in \Re^{n_x \times n_x}$, $i = 1, \ldots, n_r$; a diagonal positive definite matrix $\Delta \in \Re^{n_\varphi \times n_\varphi}$; matrices $Y_{1i} \in \Re^{n_u \times n_x}$, $Y_{2i} \in \Re^{n_u \times n_\varphi}$, $i = 1, \ldots, n_r$, and $U \in \Re^{n_x \times n_x}$; and positive scalars δ and γ satisfying the following LMIs:*

$$\begin{bmatrix} -Q_q & \Pi^1_{ij} & \Pi^3_{ij} & \dfrac{B_{wi}+B_{wj}}{2} & 0 \\ \star & \Pi^2_{ij} & U'L'\Omega & 0 & \Pi^4_{ij} \\ \star & \star & -2\Delta & 0 & \Pi^5_{ij} \\ \star & \star & \star & -\gamma^2 I & \dfrac{B'_{zwi}+B'_{zwj}}{2} \\ \star & \star & \star & \star & -I \end{bmatrix} < 0$$

$$\forall\, q, i = 1, \dots, n_r \ \ and \ \ j = i, \dots, n_r \tag{19}$$

$$\begin{bmatrix} -Q_i & Q_i N'_{(l)} \\ \star & -\delta\phi^2_{(l)} \end{bmatrix} \le 0\,,\ \forall\, i = 1, \dots, n_r \ \ and \ \ l = 1, \dots, n_\phi, \tag{20}$$

where

$$\begin{aligned} \Pi^1_{ij} &= 0.5\left(A_i U + B_i Y_{1j} + A_j U + B_j Y_{1i}\right), \\ \Pi^2_{ij} &= 0.5\left(Q_i + Q_j\right) - U - U', \\ \Pi^3_{ij} &= 0.5\left(G_i \Delta + B_i Y_{2j} + G_j \Delta + B_j Y_{2i}\right), \\ \Pi^4_{ij} &= 0.5\left(U' C'_{zi} + Y'_{1i} B'_{zj} + U' C_{zj} + Y'_{1j} B'_{zi}\right), \\ \Pi^5_{ij} &= 0.5\left(\Delta G'_{zi} + Y'_{2i} B'_{zj} + \Delta G'_{zj} + Y'_{2j} B'_{zi}\right). \end{aligned} \tag{21}$$

Let $K_i = Y_{1i} U^{-1}$ and $\Gamma_i = Y_{2i}\Delta^{-1}$, $i = 1, \dots, n_r$. In addition, consider the nonlinear system (1), with (7), and its exact N-fuzzy representation in (9). Then, the following holds for zero initial conditions:

(a) x_k remains bounded in $\mathcal{D}$ for any $w_k \in \mathcal{W}$;
(b) $\|z_k\|_2 \le \gamma\, \|w_k\|_2$ for all $w_k \in \mathcal{W}$;
(c) $x_k \to 0$ as $k \to \infty$ if there exists $\tilde{k} > 0$ such that $w_k = 0$ for all $k \ge \tilde{k}$;
d) $\mathcal{D} \subseteq \mathcal{E}(Q_i^{-1}, \delta^{-1}) \subset \mathcal{X}$, for $i = 1, \dots, n_r$.

Proof Assume that (19) is verified for all $q, i = 1, \dots, n_r$ and $j = i, \dots, n_r$. Replace Y_{1i} and Y_{2i}, respectively, by $K_i U$ and $\Gamma_i \Delta$. Multiply the resulting inequalities successively by $\alpha_{k(i)}$, $\alpha_{k(j)}$, $\alpha_{k+1(q)}$, and sum up on $i, q = 1, \dots, n_r$ and $j = i, \dots, n_r$. Thus, the inequality $\mathcal{M}(\alpha_k) < 0$ holds if $\alpha_k \in \Xi$ with

$$\mathcal{M}(\alpha_k) = \begin{bmatrix} -Q(\alpha_k^+) & \mathcal{A}(\alpha_k)U & \mathcal{G}(\alpha_k)\Delta & B_w(\alpha_k) & 0 \\ \star & U'Q^{-1}(\alpha_k)U & U'L'\Omega & 0 & U'\mathcal{C}(\alpha_k) \\ \star & \star & -2\Delta & 0 & \Delta\mathcal{F}(\alpha_k) \\ \star & \star & \star & -\gamma^2 I & B'_{zw}(\alpha_k) \\ \star & \star & \star & \star & -I \end{bmatrix}$$

and the shorthands $\alpha = \alpha_k$ and $\alpha_k^+ = \alpha_{k+1}$. Note that the matrices $Q(\alpha_k)$, $B_w(\alpha_k)$ and $B_{zw}(\alpha_k)$ can be written as (Silva et al. 2014)

$$\begin{bmatrix} Q(\alpha_k) \\ B_w(\alpha_k) \\ B_{zw}(\alpha_k) \end{bmatrix} = \sum_{i=1}^{n_r}\sum_{j=i}^{n_r} \mu_{ij}\alpha_{k(i)}\alpha_{k(j)} \left(\frac{1}{2} \begin{bmatrix} Q_i + Q_j \\ B_{wi} + B_{wj} \\ B_{zwi} + B_{zwj} \end{bmatrix} \right).$$

and that $U'Q^{-1}(\alpha)U \ge -Q(\alpha) + U' + U$ is verified since U is full rank from the (2, 2) block of the left-hand side of (19).

Further, let the congruence transformation $\Pi\mathcal{M}(\alpha)\Pi'$ with

$$\Pi = \begin{bmatrix} 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & (U')^{-1} & 0 \\ 0 & 0 & \Delta^{-1} & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Thus, applying the Schur's complement to $\Pi\mathcal{M}(\alpha)\Pi' < 0$ yields:

$$\mathcal{M}_S(\alpha_k) = \vartheta'_{1,k} Q^{-1}(\alpha_k^+)\vartheta_{1,k} + \begin{bmatrix} \mathcal{C}'(\alpha_k) \\ B'_{zw}(\alpha_k) \\ \mathcal{F}'(\alpha_k) \end{bmatrix} \begin{bmatrix} \mathcal{C}'(\alpha_k) \\ B'_{zw}(\alpha_k) \\ \mathcal{F}'(\alpha_k) \end{bmatrix}' - \begin{bmatrix} Q^{-1}(\alpha_k) & 0 & -L'\Omega\Delta^{-1} \\ 0 & \gamma^2 I & 0 \\ -\Delta^{-1}\Omega L & 0 & 2\Delta^{-1} \end{bmatrix} < 0. \tag{22}$$

with $\vartheta_{1,k} = \left[\mathcal{A}(\alpha_k)\ B_w(\alpha_k)\ \mathcal{G}(\alpha_k)\right]$. Now, let $\vartheta_{2,k} = \left[x'_k\ w'_k\ \varphi'_k\right]'$. Then, we obtain the following in view of (22):

$$\vartheta'_{2,k}\mathcal{M}_S(\alpha_k)\vartheta_{2,k} = \Delta V_k - 2\varphi'_k\Delta^{-1}(\varphi_k - \Omega L x_k) < 0\,, \tag{23}$$

if $\alpha_k \in \Xi$.

Hence, the condition (23) implies that $\Delta V_k < 0$ whenever $\alpha_k \in \Xi$ and the sector condition (4) is verified. Assuming that x_k does not leave $\mathcal{X}$, for all $k \ge 0$, we can infer that condition (16) is also satisfied. In this way, the properties a), b) and c) in Theorem 1 are guaranteed, and consequently i), ii) and iii) from Lemma 1.

Now, we need to show that $x_k \in \mathcal{X}$, for all $k \ge 0$, and consequently $\alpha_k \in \Xi$. To this end, assume that (20) is verified. Then, multiplying (20) by $\alpha_{k(i)}$ and summing up on $i = 1, \dots, n_r$ leads to:

$$\Lambda = \begin{bmatrix} -Q(\alpha_k) & Q(\alpha_k)N'_{(l)} \\ \star & -\delta\phi^2_{(l)} \end{bmatrix} \le 0.$$

Let $\mathcal{F} = \mathrm{diag}\{Q^{-1}(\alpha_k), 1\}$. Hence, the congruence transformation $\mathcal{F}'\Lambda\mathcal{F} = \tilde{\Lambda}$ yields

$$\tilde{\Lambda} = \begin{bmatrix} -Q^{-1}(\alpha_k) & N'_{(l)} \\ \star & -\delta\phi^2_{(l)} \end{bmatrix} \le 0.$$

By applying the Schur's complement to $\tilde{\Lambda}$, we obtain:

$$N'_{(l)}(\delta\phi^2_{(l)})^{-1}N_{(l)} - Q^{-1}(\alpha_k) \le 0.$$

Pre- and post-multiplying the above, respectively, by x'_k and x_k and considering the S-procedure lead to:

$$x_k^{'} N_{(l)}^{'} \phi_{(l)}^{-2} N_{(l)} x_k \leq 1,\ \forall\, x_k \in \mathcal{D},$$
$$\mathcal{D} = \{x : x_k^{'} Q^{-1}(\alpha_k) x_k \leq \delta^{-1}\}$$

That is, $\mathcal{E}(Q_i^{-1}, \delta^{-1}) \subset \mathcal{X}, \forall i = 1, \ldots, n_r$. Recalling from (18) that $\mathcal{D} \subseteq \mathcal{E}(Q_i^{-1}, \delta^{-1})$, we can ensure the property d) and infer from Lemma 1 that $x_k \in \mathcal{D}$, for all $k \geq 0$, which concludes the proof. □

Remark 2 We can apply Theorem 1 to systems represented by classical T–S fuzzy models (without the nonlinear term) by eliminating the third row and column block of the matrix on the left-hand side of (19).

Remark 3 It is interesting to note that the stabilization condition (19) has a reduced number of LMIs when compared to other techniques in the literature. This is due to the use of the property (10) and the definition of the variable μ_{ij} in (11). See, for instance, Theorem 6.6 of Feng (2010), where the resulting LMIs are required to be verified $\forall\ i, j, q = 1, \ldots, n_r$.

4 Design Issues

Now, we propose three extensions of Theorem 1 in order to demonstrate the potential of the proposed approach as a control design tool for nonlinear discrete-time systems subject to energy-bounded disturbances.

4.1 Disturbance Tolerance

The disturbance tolerance criterion consists in maximizing a bound on the disturbance energy for which we can ensure that the system trajectories remain bounded (and inside the domain of validity). This can be accomplished by the following optimization problem.

$$\min_{Q_i, \Delta, Y_{1i}, Y_{2i}, U} \delta \quad \begin{cases} \text{subject to} \\ \text{LMIs (19) and (20).} \end{cases} \tag{24}$$

Notice that the minimization of δ implies in maximizing the set of admissible disturbances $\mathcal{W}$.

4.2 Disturbance Attenuation

For a given disturbance energy level δ^{-1}, the disturbance attenuation criterion consists in minimizing an upper bound on the ℓ_2-gain from w_k to z_k while guaranteeing that $x_k \in \mathcal{X}$, which can be obtained from the solution of the following optimization problem:

$$\min_{Q_i, \Delta, Y_{1i}, Y_{2i}, U} \gamma \quad \begin{cases} \text{subject to} \\ \text{LMIs (19) and (20).} \end{cases} \tag{25}$$

4.3 Reachable Set Estimation

The reachable set estimation criterion consists in minimizing the set $\mathcal{D}$ (an estimate of the reachable set) for a specific disturbance energy level δ^{-1} and a guaranteed bound γ on the system ℓ_2-gain. This objective can be accomplished by considering the inclusion $\mathcal{E}(Q_i^{-1}, \delta^{-1}) \subset \beta\mathcal{X}, \forall i = 1, \ldots, n_r$, $\beta \in (0, 1]$, which is obtained by modifying the condition in (20) as follows:

$$\begin{bmatrix} -Q_i & Q_i N_{(l)}^{'} \\ \star & -\beta^2 \delta \phi_{(l)}^2 \end{bmatrix} \leq 0 \quad \forall i = 1, \ldots, n_r \ \text{ and } \ l = 1, \ldots, \eta \tag{26}$$

Then, the objective is to obtain the lowest value for β which will be useful in practice whenever the effects of the disturbances over the system trajectories are to be minimized. These criteria can be obtained by the following optimization problem:

$$\min_{Q_i, \Delta, Y_{1i}, Y_{2i}, U} \beta \quad \begin{cases} \text{subject to} \\ \text{LMIs (19), (26) and } 0 < \beta \leq 1. \end{cases} \tag{27}$$

5 Numerical Examples

In this section, we present two numerical examples. The first one demonstrates some stability issues that can occur when the domain of validity is not considered. In this sense, a nonlinear plant is modeled by the classical T–S form and the region $\mathcal{X}$, where the model convexity is guaranteed, is not taken into account in the design phase. In the second example, it is shown the effectiveness of the proposed technique considering the optimization problems described in the previous section.

5.1 Example 1

Consider the control problem of backing-up a truck-trailer as studied in Feng and Ma (2001), Lo and Lin (2003), Tanaka and Wang (2001). The state space representation of the system is described by

$$\begin{aligned}
x_{1,k+1} &= x_{1,k} - \frac{vT}{\mathcal{L}} \sin(x_{1,k}) + \frac{vT}{l} u_k \\
x_{2,k+1} &= x_{2,k} + \frac{vT}{\mathcal{L}} \sin(x_{1,k}) + 0.2 w_k \\
x_{3,k+1} &= x_{3,k} + vT \cos(x_{1,k}) \sin\left(x_{2,k} + \frac{vT}{2\mathcal{L}} \sin(x_{1,k})\right) \\
&\quad + 0.1 w_k \\
z_k &= 7x_{1,k} - vT x_{2,k} + 0.03 x_{3,k} - \frac{vT}{l} u_k
\end{aligned} \tag{28}$$

where $x_{1,k}$ represents the angle between the truck and the trailer, $x_{2,k}$ denotes the angle of the trailer, $x_{3,k}$ is the vertical position of the rear, and l and $\mathcal{L}$ represent the length of the vehicle and of the trailer, respectively. T is the sampling time, and v the constant reverse speed. In particular, we consider $l = 2.8\,\text{m}$, $\mathcal{L} = 5.5\,\text{m}$, $v = -1.0\,\text{m/s}$ and $T = 2.0\,\text{s}$. Due to physical limitations and/or to guarantee a safe operation of the system, such as preventing the *jack-knife* effect that occurs when $x_{1,k} = \pm\pi/2$, the considered domain of validity $\mathcal{X}$ in (8) is defined as follows (Lo and Lin 2003)

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \phi = \begin{bmatrix} \frac{\pi}{3} & \frac{170\pi}{180} \end{bmatrix}'.$$

In order to demonstrate the problems that can occur in practice when the model validity domain is not considered (usually assumed in the literature), we consider the classical T–S model of the system (28) with eight linear local rules, given by the following equation (Tanaka and Wang 2001)

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{8} \alpha_{k(i)} \left(A_i x_k + B_i u_k + B_{wi} w_k\right) \\ z_k &= C_z x_k + B_z u_k + B_{zw} w_k \end{aligned} \tag{29}$$

where

$$A_i = A_{jkl} = \begin{bmatrix} 1 - \frac{vT}{\mathcal{L}} b_j & 0 & 0 \\ \frac{vT}{\mathcal{L}} b_j & 1 & 0 \\ \frac{v^2T^2}{2\mathcal{L}} b_j d_k g_l & vT d_k g_l & 1 \end{bmatrix}, \quad \begin{matrix} i = l + 2(k-1) + 4(j-1) \\ j, k, l = \{1, 2\} \end{matrix}$$

$$B_i = B = \begin{bmatrix} \frac{vT}{l} \\ 0 \\ 0 \end{bmatrix}, \quad B_{wi} = B_w = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix},$$

$$C_z = \begin{bmatrix} 7 & -2 & 0.03 \end{bmatrix}, \quad B_z = -\frac{vT}{l}, \quad \text{and } B_{zw} = 0,$$

with $b_1 = 1, b_2 = 0.827, d_1 = 1, d_2 = 0.5, g_1 = 1$ and $g_2 = 10^{-2}/\pi$. The membership functions $\alpha_{k(i)}, i = 1, \ldots, 8$, are the binary product between functions M_j^i, $j = \{1, 2\}$ and $i = \{1, 2, 3\}$, defined as:

$$M_1^1 = \begin{cases} \dfrac{\sin(x_{1,k}) - b_2 x_{1,k}}{x_{1,k}(b_1 - b_2)}, & x_{1,k} \neq 0 \\ 1, & x_{1,k} = 0 \end{cases}, \quad M_2^1 = 1 - M_1^1,$$

$$M_1^2 = \frac{\cos(x_{1,k}) - d_2}{d_1 - d_2}, \quad M_2^2 = 1 - M_1^2 \quad \text{and}$$

$$M_1^3 = \begin{cases} \dfrac{\sin(\rho) - g_2\rho}{\rho(g_1 - g_2)}, & \rho \neq 0 \\ 1, & \rho = 0 \end{cases}, \quad M_2^3 = 1 - M_1^3,$$

with $\rho = x_{2,k} + \frac{vT}{2\mathcal{L}} \sin(x_{1,k})$

For comparison purposes, the following control approaches using classical T–S fuzzy models are taken into account:

- *case 1*: Theorem 6.6 of Feng (2010);
- *case 2*: Remark 2 without the inclusion constraint in (20).

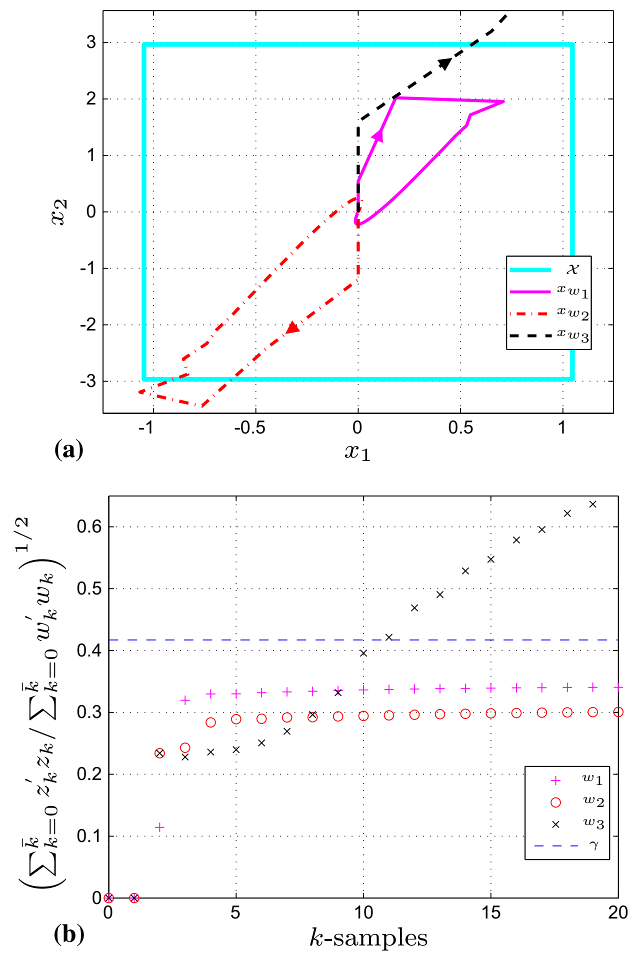

Fig. 1 **a** Domain of validity and trajectories for different disturbances; **b** ℓ_2-gain from w_k to z_k

By solving an optimization problem aiming the minimization of the upper bound on the ℓ_2-gain from w_k to z_k, we have, respectively, obtained the upper bounds $\gamma = 0.4171$ and $\gamma = 0.3430$ for the cases 1 and 2. Notice that the bound obtained in case 2 is less conservative than the one obtained in case 1.

In Fig. 1a, we observe the projections of the closed-loop system trajectories[1] on the plane x_1, x_2 and ℓ_2-gains using the results obtained in case 1, for the following three different disturbance signals with $\|w_{1,k}\|_{\ell_2} = 7.8731$, $\|w_{2,k}\|_{\ell_2} = 10.3923$ and $\|w_{3,k}\|_{\ell_2} = 24$,

$$w_{1,k} = \begin{cases} e^k, & 1 \leq k \leq 2 \\ 0, & k < 1, k > 2 \end{cases}, \quad w_{2,k} = \begin{cases} -6, & 1 \leq k \leq 3 \\ 0, & k < 1, k > 3 \end{cases},$$

$$w_{3,k} = \begin{cases} 8, & 1 \leq k \leq 9 \\ 0, & k < 1, k > 9 \end{cases}$$

[1] The simulations were performed considering the closed-loop system composed of the designed fuzzy controller and the original nonlinear plant.

Figure 1b shows that the closed-loop state trajectory may either remain bounded or diverge to infinity. Precisely, the trajectories driven by $w_{1,k}$ and $w_{2,k}$ are bounded and by $w_{3,k}$ goes to infinity. However, it is important to emphasize that although the trajectory imposed by $w_{2,k}$ is bounded and verifies the required performance, it reaches an impracticable *jack-knife* condition for the system. The instability and *jack-knife* condition problems stem from the fact that the fuzzy model domain of validity $\mathcal{X}$ has not been considered in the design phase leading to undesirable system behaviors. Notice by applying optimization problem (25) that we are able to determine control laws which ensure that the state trajectory is confined to the domain of validity while guaranteeing upper bounds on the ℓ_2-gain for the exogenous signals $w_{1,k}$, $w_{2,k}$ and $w_{3,k}$.

5.2 Example 2

Consider the following discrete-time nonlinear system (Klug et al. 2013):

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} -\frac{13}{20} & \frac{11}{20} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} x_k + \begin{bmatrix} 0 \\ \frac{5}{4} \end{bmatrix} u_k + \begin{bmatrix} 0 \\ \frac{51}{100} \end{bmatrix} w_k \\ &+ \begin{bmatrix} \frac{9}{40}x_{1,k}^2 + \frac{3}{40}x_{1,k}x_{2,k} + \frac{3}{10}x_{2,k}(1+\sin(x_{2,k})) \\ \frac{1}{5}x_{1,k}^2 + \frac{1}{20}x_{1,k}x_{2,k} + \frac{1}{40}x_{1,k}u_k + \frac{39}{200}x_{1,k}w_k \end{bmatrix} \\ z_k &= x_{1,k} + \frac{23}{20}u_k + \frac{7}{40}x_{1,k}u_k \end{aligned} \tag{30}$$

where $x_k = \begin{bmatrix} x_{(1,k)} & x_{(2,k)} \end{bmatrix}'$.

Assume that the domain of validity $\mathcal{X}$ as given in (8) is defined by means of $N = I_2$ and $\phi = \begin{bmatrix} 2 & 1.5 \end{bmatrix}'$. Additionally, defining the premise variable $\nu_k = x_{1,k}$, and the sector nonlinearity $\varphi_k = \varphi(Lx_k) = \frac{3}{10}x_{(2,k)}(1+\sin(x_{(2,k)}))$, with $L = \begin{bmatrix} 0 & 1 \end{bmatrix}$, the system dynamics in (30) can be cast as follows:

$$\begin{aligned} x_{k+1} &= \left\{ \begin{bmatrix} -\frac{13}{20} & \frac{11}{20} \\ \frac{1}{5} & \frac{6}{5} \end{bmatrix} + \nu_k \begin{bmatrix} \frac{9}{40} & \frac{3}{40} \\ \frac{1}{5} & \frac{1}{20} \end{bmatrix} \right\} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varphi_k \\ &+ \left\{ \begin{bmatrix} 0 \\ \frac{5}{4} \end{bmatrix} + \nu_k \begin{bmatrix} 0 \\ \frac{1}{40} \end{bmatrix} \right\} u_k \\ &+ \left\{ \begin{bmatrix} 0 \\ \frac{51}{100} \end{bmatrix} + \nu_k \begin{bmatrix} 0 \\ \frac{39}{200} \end{bmatrix} \right\} w_k \\ z_k &= x_{1,k} + \left(\frac{23}{20} + \frac{7}{40}\nu_k \right) u_k \end{aligned} \tag{31}$$

Note that the nonlinearity φ_k can be globally encompassed into a sector bounded nonlinearity, i.e., $\varphi_k \in S[0\ ,\ 0.7]$, as well as $\nu_k \in [d_1, d_2]$, with $d_1 = -2$ and $d_2 = 2$ being the extreme points of ν_k. Thus, the system in (31) can be exactly described by the following N-fuzzy model:

Table 1 Disturbance tolerance

γ	1.5	2	2.5	3
δ	0.4530	0.1860	0.1037	0.0671

Table 2 Disturbance attenuation

δ	0.1	0.2	0.5	1
γ	2.5367	1.9487	1.4595	1.3273

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{2} \alpha_{k(i)} \left\{ A_i x_k + B_i u_k + B_{wi} w_k + G_i \varphi_k \right\} \\ z_k &= \sum_{i=1}^{2} \alpha_{k(i)} \left\{ C_{zi} x_k + B_{zi} u_k + B_{zwi} w_k + G_{zi} \varphi_k \right\} \end{aligned} \tag{32}$$

with $\alpha_{k(1)} = \dfrac{d_2 - \nu_k}{d_2 - d_1}$, $\alpha_{k(2)} = \dfrac{\nu_k - d_1}{d_2 - d_1}$, and the following system matrices according to (3), for $i = 1, 2$:

$$A_i = \begin{bmatrix} -\frac{13}{20} + \frac{9}{40}d_i & \frac{11}{20} + \frac{3}{40}d_i \\ \frac{1}{5} + \frac{1}{5}d_i & \frac{6}{5} + \frac{1}{20}d_i \end{bmatrix},\ B_i = \begin{bmatrix} 0 \\ \frac{5}{4} + \frac{1}{40}d_i \end{bmatrix},$$

$$B_{wi} = \begin{bmatrix} 0 \\ \frac{51}{100} + \frac{39}{200}d_i \end{bmatrix},\ G_i = G = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_{zi} = C_z = \begin{bmatrix} 1 & 0 \end{bmatrix},\ B_{zi} = \frac{23}{20} + \frac{7}{40}d_i,$$

$B_{zwi} = 0$ and $G_{zi} = 0$.

Firstly, applying the disturbance tolerance problem in (24) for a given set of ℓ_2-gain values, we obtain the results shown in Table 1. Notice that smaller is the upper bound γ on the system ℓ_2-gain, larger is the value of δ (i.e., the set of admissible disturbances is smaller).

On the other hand, considering that the bound on the admissible disturbances is known *a priori*, we apply the disturbance attenuation optimization problem in (25). The results are described in Table 2 showing that larger values of δ will lead to smaller upper bounds on the ℓ_2-gain.

These two experiments clearly demonstrate that the disturbance attenuation properties of the original nonlinear system are state dependent contrasting with standard T–S fuzzy approaches which assume a constant ℓ_2-gain regardless the disturbance energy. To emphasize this point, Fig. 2a, b show the estimates of the reachable set (given by $\mathcal{D}$) and the state trajectory evolution of the closed-loop system considering controllers derived from (24) and (25), respectively, for the pairs $\{\gamma\ ,\ \delta\} = \{2.5\ ,\ 0.1037\}$ and $\{\delta\ ,\ \gamma\} = \{0.1\ ,\ 2.5367\}$. The disturbance signals are, respectively, similar to the $w_{2,k}$ and $w_{1,k}$ signals considered in Example 1, but with a reduced amplitude to achieve the desired energy level, where x_{w_1} and x_{w_2} means state trajectories driven, respectively, by w_1 and w_2. Notice in both cases that i) the state trajectories remains bounded in $\mathcal{X}$ for all samples; ii) a certain duality between

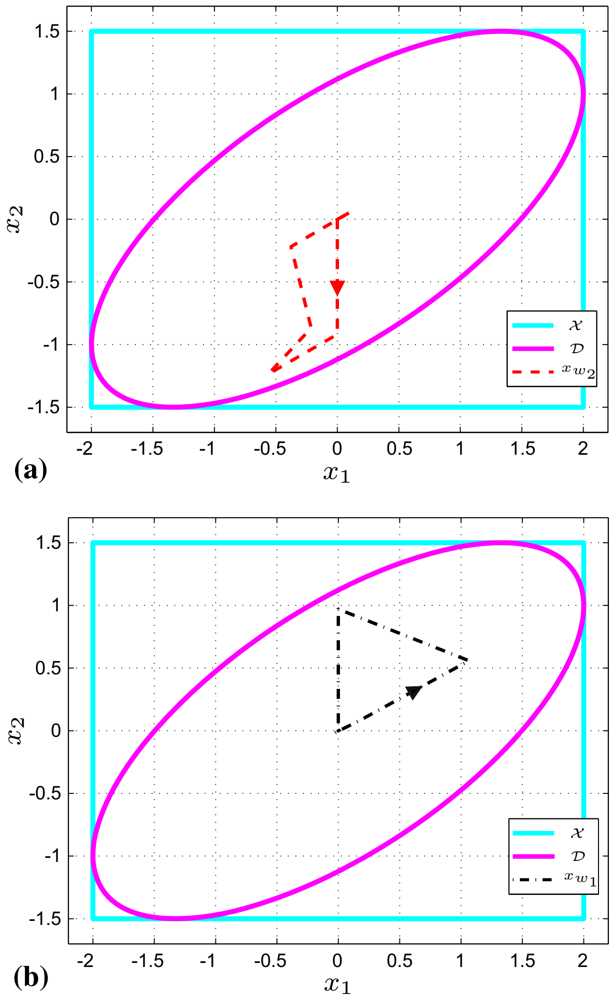

Fig. 2 **a** Regions for disturbance tolerance algorithm; **b** regions for disturbance attenuation algorithm

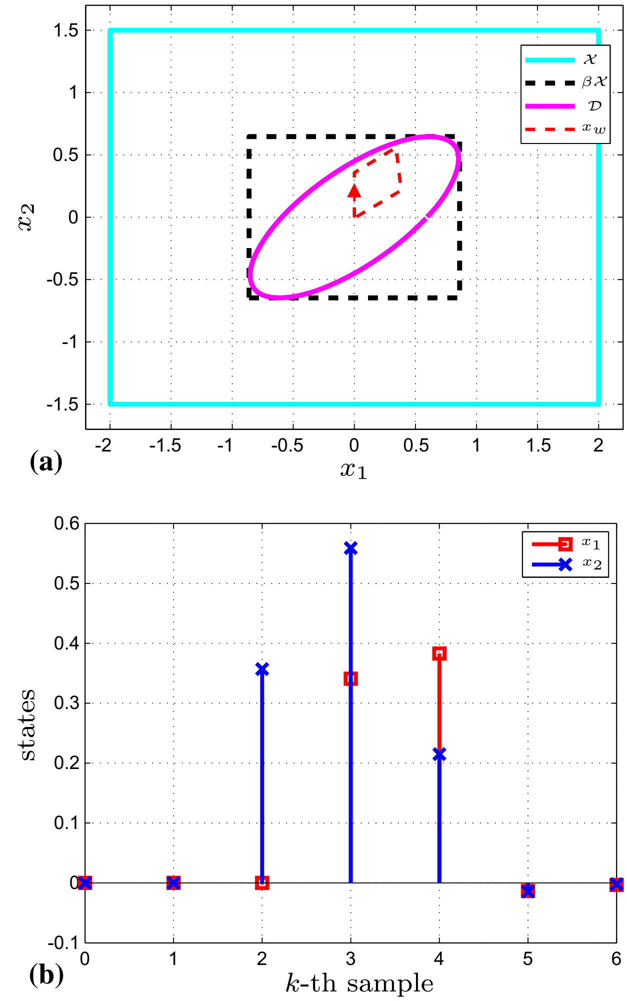

Fig. 3 **a** Regions for disturbance attenuation algorithm; **b** state trajectories

the optimization problems (24) and (25) since they led to similar estimates of the reachable set.

Finally, consider $\delta = 1$, $\gamma = 2$ and the reachable set estimation algorithm in (27). Thus, the following controller matrices are obtained:

$$K_1 = \begin{bmatrix} -0.4215 & -0.6371 \end{bmatrix}, \; K_2 = \begin{bmatrix} -0.5453 & -0.7070 \end{bmatrix},$$
$$\Gamma_1 = 0.3949, \text{ and } \Gamma_2 = 0.4186.$$

In Fig. 3a, we observe the region estimated for this particular case with an optimal $\beta = 0.4313 \leq 1$. To evaluate the method conservativeness, the state trajectory driven by the following signal (which respects the energy bound)

$$w_k = \begin{cases} 0.7, & 1 \leq k \leq 2 \\ 0, & \text{elsewhere} \end{cases}$$

is also plotted in Fig. 3a, demonstrating that the reachable set estimate is tight. For illustrative purposes, the time response of the state trajectories and the control effort are also shown in Figs. 3b and 4, respectively.

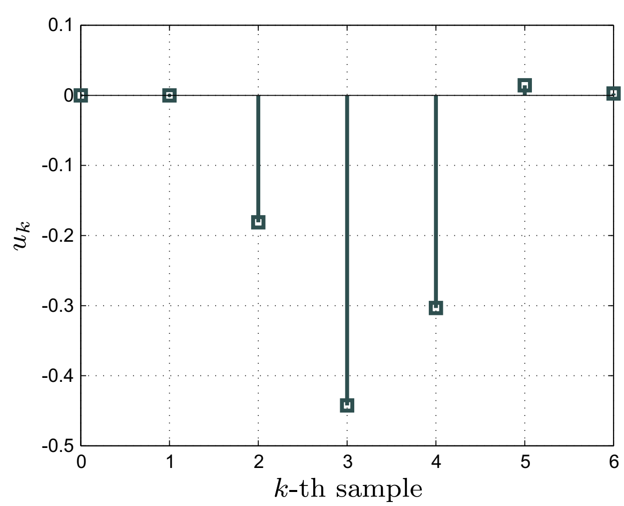

Fig. 4 Control effort

6 Concluding Remarks

We have proposed a convex approach for the design of fuzzy controllers that locally stabilizes nonlinear discrete-time systems considering either classical or N-fuzzy T–S models. Considering fuzzy Lyapunov functions, three optimization problems in terms of LMI constraints are proposed to design a nonlinear state feedback control law which is a function of the membership fuzzy functions and cone sector nonlinearities. It turns out that the proposed approach locally ensures the ℓ_2-ISS of the original nonlinear system while guaranteeing a certain input-to-output performance for a given class of disturbance signals. Numerical examples have demonstrated the potentialities of the proposed technique as a tool for the control design of nonlinear discrete-time systems. As in Klug et al. (2014), the proposed approach can be extended to deal with the local ℓ_2-ISS stabilization problem in the presence of control saturation (Tarbouriech et al. 2011). Our future research is concentrated in extending the proposed framework for considering persistent disturbances.

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