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Input and Output Finite-Level Quantized Linear Control Systems: Stability Analysis and Quantizer Design

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Abstract This paper investigates the local stability of input- and output-quantized discrete-time linear time-invariant systems considering static finite-level logarithmic quantizers. The sector bound approach together with a relaxed stability notion is applied to derive an LMI-based method to estimate a set of admissible initial states and its attractor in a neighborhood of the system origin assuming that an output feedback controller and the quantizers are given. In addition, the stability analysis method is tailored to design an input and an output static finite-level logarithmic quantizers when a set of admissible initial states and an upper bound on the volume of its attractor are known. Numerical examples are presented to demonstrate the proposed stability analysis and quantizer design methods.

Keywords Quantized feedback systems · Discrete-time linear time-invariant systems · Finite-logarithmic quantizers · Practical stability

1 Introduction

With the increasing application of communication links to exchange information and control signals between spatially distributed system components, networked control systems (NCS) have recently attracted growing interest of the control community motivated by the fact that NCS bring a new range of control applications (Hespanha et al. 2007; Schenato et al. 2007; You and Xie 2013). Since in many situations quantization errors are unavoidable, their effects cannot be neglected at the cost of an inadequate closed-loop performance and even the lost of stability. As a result, the study of quantized feedback systems is of relevance in networked control systems.

Originally, quantized feedback systems were studied to evaluate and mitigate the quantization errors arising from the digital implementation of feedback systems; see, for instance, Kalman (1956), Slaughter (1964) and Delchamps (1990). Nowadays, with the increasing application of NCS in which control system components (i.e., sensor, controller and actuator) are connected via a shared digital communication network, the problem of limited bandwidth becomes an issue of great interest. In this context, quantized feedback methods can be applied to deal with the bandwidth allocation problem by constraining the number of bits to be transmitted in the communication link (Maestrelli et al. 2014). As a result, an increasing number of works in the last ten or more years has focused on the topic of quantized feedback systems, such as the references Brockett and Liberzon (2000); Ishii and Francis (2003); Fu and Xie (2005); Coutinho et al. (2010); Wei et al. (2014) to cite a few.

Signal quantization may be performed in several ways. For instance, the quantization levels can be uniformly or non-uniformly distributed, and the quantization policy can be divided into static and dynamic constructive laws. In Elia and Mitter (2001), it has been shown that for a quadratically stabilizable system a logarithmic quantizer (i.e., the quantization levels are linear in a logarithmic scale) is the optimal solution in terms of coarse quantization density. In addition,

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logarithmic quantizers can achieve superior dynamic range for a given number of bits (Rasool et al. 2012). However, the optimal solution requires an *ideal logarithmic quantizer*, namely a quantizer with an infinite number of quantization levels, which may be overcome via a finite-level quantizer combined with a dynamic scaling factor (Fu and Xie 2009). In this line, Fu and Xie (2005) have introduced the sector bound approach for static logarithmic quantized feedback systems, giving simple formulae to the stabilization problem via state and output feedback controllers. Since then, the sector bound approach has been applied to solve a variety of problems such as, quantized robust control of linear uncertain systems (Fu and Xie 2010), state estimation with quantized measurements (Fu and de Souza 2009), local stability analysis of control linear systems with a static finite-level quantizer (de Souza et al. 2010), and feedback control of quantized nonlinear systems (Liu et al. 2012).

The aforementioned works assume the presence of a single quantizer in the feedback loop either in the input channel or in the output channel. However, since in NCS the information (control signal and measurements) is generally exchanged through a shared communication channel with limited bandwidth, it is natural to consider that both control and measurement signals are quantized (Zhai et al. 2005). To date, few results have addressed the stability and stabilization problems for input- and output-quantized feedback systems as, for instance, Richter and Misawa (2003); Zhai et al. (2005); Yue et al. (2006); Picasso and Bicchi (2007); Coutinho et al. (2010); Liu et al. (2011); Rasool et al. (2012); Yan et al. (2013). In particular, Coutinho et al. (2010) have extended the sector bound approach to deal with input- and output-quantized discrete-time linear systems assuming static logarithmic quantizers having an infinite number of quantization levels. To the authors' knowledge, the study of local stability properties of input and output finite-level logarithmic quantized linear control systems has not yet been fully addressed in the literature despite some recent results on global stability analysis of input and output finite-level quantized systems (Richter and Misawa 2003; Xia et al. 2013).

The local stability of input- and output-quantized SISO discrete-time linear time-invariant feedback systems considering static finite-level logarithmic quantizers was investigated by Maestrelli et al. (2012), where LMI-based conditions are proposed to estimate a set of admissible initial states and its attractor in a neighborhood of the state-space origin assuming a controller and a quantizer are given *a priori*. This paper expands this earlier result by addressing the quantizer design problem when the set of admissible initial states and an upper bound on the volume of the attractor are known, which is an important issue for bandwidth management with a policy based on limiting the amount of information (Maestrelli et al. 2014). In addition, the main result of (Maestrelli et al. 2012) is revised and a procedure to jointly optimize the estimates of the set of admissible initial states and its attractor is proposed. Numerical examples are presented to demonstrate the potentials of the proposed stability analysis and quantizer design methods.

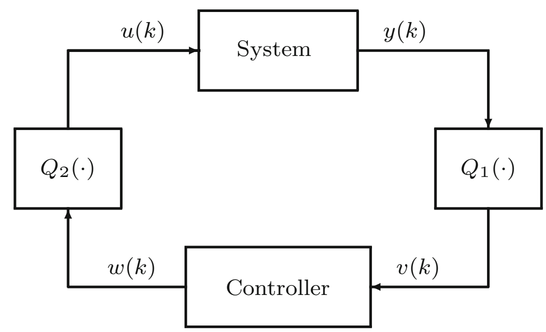

Fig. 1 Feedback control system with input and output quantization

Notation For a real matrix S, S' is its transpose, $\mathrm{diag}\{\cdots\}$ denotes a block-diagonal matrix and $S > 0$ ($S \geq 0$) means that S is symmetric and positive definite (nonnegative definite). For a symmetric block matrix, the symbol $*$ stands for the transpose of the blocks outside the main diagonal block. For two sets $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{B} \subset \mathcal{A}$, $\mathcal{A} \setminus \mathcal{B}$ stand for $\mathcal{A}$ excluded $\mathcal{B}$. Finally, the argument k of $x(k)$ as well as matrix and vector dimensions are often omitted.

2 Problem Statement

Consider the quantized feedback system in Fig. 1, where the system is represented by the following state-space model:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad\; y(k) = Cx(k) \end{cases}, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is the measurement, and A, B, and C are given matrices with appropriate dimensions. This system is to be controlled by an output feedback controller with a state-space representation as follows:

$$\begin{cases} \xi(k+1) = A_c\xi(k) + B_c v(k) \\ \quad\; w(k) = C_c\xi(k) \end{cases}, \tag{2}$$

where $\xi \in \mathbb{R}^{n_c}$ is the controller state, $v \in \mathbb{R}$ is the controller input, $w \in \mathbb{R}$ is the controller output, and A_c, B_c and C_c are given matrices with appropriate dimensions.

The system in (1) and the above controller are interconnected via the following relations:

$$v(k) = Q_1(y(k)), \quad u(k) = Q_2(w(k)), \tag{3}$$

where $Q_1(\cdot)$ and $Q_2(\cdot)$ are static finite-level logarithmic quantizers with quantization levels given by the sets as follows:

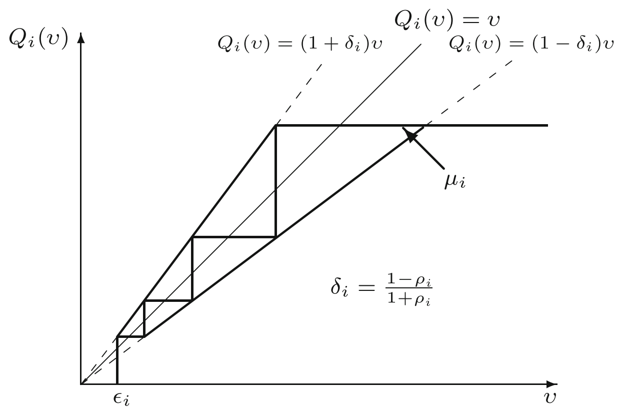

Fig. 2 Logarithmic quantizer with a finite number of quantization levels

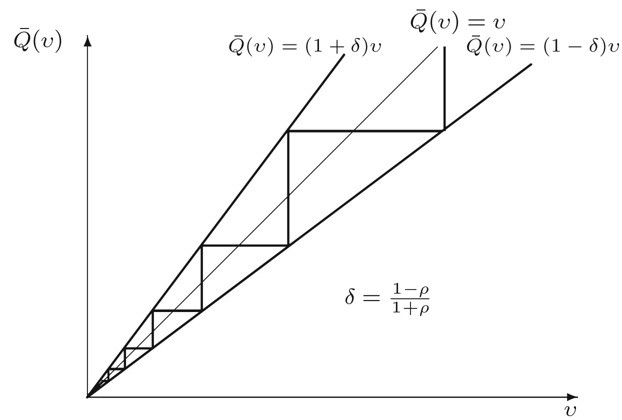

Fig. 3 Logarithmic quantizer with an infinite number of quantization levels

$$\begin{aligned}\mathcal{V}_i = \left\{ \pm m_{i,j} : m_{i,j} = \rho_i^j \mu_i,\ j = 0, 1, \ldots, N_i - 1 \right\} \\ \cup \{0\},\ \rho_i \in (0,1),\ i = 1, 2\end{aligned}, \tag{4}$$

where N_i is the number of positive quantization levels and $\mu_i > 0$ is the largest admissible level. Note that a small (large) ρ_i implies coarse (dense) quantization. As an abuse of terminology, ρ_i will be referred to as *quantization density*.

This paper is concerned with investigating the stability of the closed-loop system of (1)–(3), where $Q_1(\cdot)$ and $Q_2(\cdot)$ are quantizers with finite alphabets obeying the following constructive law (see Fig. 2):

$$Q_i(\upsilon) = \begin{cases} \mu_i, & \text{if } \ \upsilon > \frac{\mu_i}{(1-\delta_i)}, \ \ \mu_i > 0 \\ \rho_i^j \mu_i, & \text{if } \ \frac{\rho_i^j \mu_i}{(1+\delta_i)} < \upsilon \le \frac{\rho_i^j \mu_i}{(1-\delta_i)}, \\ & \quad j = 0, 1, \ldots, N_i - 1 \\ 0, & \text{if } \ 0 \le \upsilon \le \epsilon_i \\ -Q(-\upsilon), & \text{if } \ \upsilon < 0 \end{cases}, \tag{5}$$

where

$$\delta_i = \frac{1-\rho_i}{1+\rho_i}, \quad \epsilon_i = \frac{\rho_i^{N_i - 1} \mu_i}{(1+\delta_i)}. \tag{6}$$

It is assumed that the input and output quantizers are independent and possibly different, i.e., they can have different parameters ρ_i, μ_i, and N_i, $i = 1, 2$.

3 Preliminaries

This section reviews a result derived in Coutinho et al. (2010) on quadratic stability of input- and output-quantized feedback linear SISO systems with ideal static logarithmic quantizers. To this end, consider feedback system in (1)–(3) under the assumption that $Q_1(\cdot)$ and $Q_2(\cdot)$ are ideal static logarithmic quantizers. Note that an ideal static logarithmic quantizer $\bar{Q}(\cdot)$ is defined as follows (see Fig. 3):

$$\bar{Q}(\upsilon) = \begin{cases} \rho^j \mu, & \text{if } \ \frac{\rho^j \mu}{(1+\delta)} < \upsilon \le \frac{\rho^j \mu}{(1-\delta)}, \\ & \quad j = 0, \pm 1, \pm 2, \ldots \\ 0, & \text{if } \ \upsilon = 0 \\ -\bar{Q}(-\upsilon), & \text{if } \ \upsilon < 0 \end{cases} \tag{7}$$

Theorem 1 *Coutinho et al. (2010) The closed-loop system of (1)–(3), where $Q_1(\cdot)$ and $Q_2(\cdot)$ are static infinite-level logarithmic quantizers, is quadratically stable if and only if there exists a matrix $P > 0$ such that*

$$\begin{aligned}&\bar{A}(\Delta_1, \Delta_2)' P \bar{A}(\Delta_1, \Delta_2) - P < 0, \\ &\forall\, \Delta_1, \Delta_2 \in \mathbb{R} : |\Delta_1| \le \delta_1,\ |\Delta_2| \le \delta_2,\end{aligned} \tag{8}$$

where

$$\bar{A}(\Delta_1, \Delta_2) = \begin{bmatrix} A & B(1+\Delta_2)C_c \\ B_c(1+\Delta_1)C & A_c \end{bmatrix}. \tag{9}$$

Theorem 1 establishes that the quadratic stabilization problem for an input–output-quantized feedback system with infinite-level logarithmic quantizers can be transformed, with no conservatism, into a standard robust control problem. Specifically, the system in (1) with given static infinite-level logarithmic quantizers $Q_1(\cdot)$ and $Q_2(\cdot)$ is quadratically stabilizable via an output feedback controller in (2) satisfying the interconnection relations in (3) if and only if the following uncertain system:

$$\begin{cases} x(k+1) = Ax(k) + B(1+\Delta_2)w(k) \\ \quad\ \ v(k) = (1+\Delta_1)Cx(k) \end{cases}, \tag{10}$$

where Δ_1 and Δ_2 are uncertain parameters satisfying $|\Delta_i| \le \delta_i$, $i = 1, 2$, is quadratically stabilizable via the controller in (2).

The result of Theorem 1 is strong in the sense that the quadratic stability of the uncertain system $\bar{x}(k+1) =$

$\bar{A}(\Delta_1, \Delta_2)\bar{x}(k)$, with uncertain parameters Δ_1 and Δ_2 satisfying $|\Delta_i| \leq \delta_i$, $i=1, 2$, is a necessary and sufficient condition for the quadratic stability of the quantized closed-loop system. In other words, the sector bound condition

$$[\, Q_i(\upsilon) - (1-\delta_i)\upsilon)\,]\,[\, Q_i(\upsilon) - (1+\delta_i)\upsilon\,] \leq 0 \qquad (11)$$

is non-conservative to model infinite-level logarithmic quantizers in the sense of quadratic stability.

4 Local Stability Analysis

The result in Sect. 3 applies to input- and output-quantized feedback systems where the quantizers follow a logarithmic law and have an infinite number of levels. For finite-level quantizers, due to quantizers' dead-zone, the convergence of the state trajectory to the system origin (the equilibrium point under analysis) cannot be in general guaranteed. In such scenario, LMI-based conditions are derived in the sequel to ascertain the state trajectory convergence, in finite time, to a small invariant region in the neighborhood of the system origin.

Firstly, consider the following augmented system, which represents the closed-loop system of (1)–(3):

$$\begin{cases} \zeta(k+1) = A_{\mathrm{a}}\zeta(k) + B_{\mathrm{a}}p(k) \\ \quad q(k) = C_{\mathrm{a}}\zeta(k) \\ \quad p(k) = Q_{\mathrm{a}}(q(k)) \end{cases}, \qquad (12)$$

where

$$\zeta = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \upsilon \\ u \end{bmatrix}, \quad q = \begin{bmatrix} y \\ w \end{bmatrix},$$
$$A_{\mathrm{a}} = \begin{bmatrix} A & 0 \\ 0 & A_{\mathrm{c}} \end{bmatrix}, \quad B_{\mathrm{a}} = \begin{bmatrix} 0 & B \\ B_{\mathrm{c}} & 0 \end{bmatrix}, \qquad (13)$$
$$C_{\mathrm{a}} = \begin{bmatrix} C & 0 \\ 0 & C_{\mathrm{c}} \end{bmatrix}, \quad Q_{\mathrm{a}}(q) = \begin{bmatrix} Q_1(y) & 0 \\ 0 & Q_2(w) \end{bmatrix}.$$

Associated to the augmented system in (12) and the quantizers as in (5), let the following sets:

$$\mathcal{B}_i =: \{\zeta \in \mathbb{R}^{n_\zeta} : |C_{\mathrm{a}_i}\zeta| \leq \mu_i/(1-\delta_i)\}, \; i=1, 2 \qquad (14)$$
$$\mathcal{C}_i =: \{\zeta \in \mathbb{R}^{n_\zeta} : |C_{\mathrm{a}_i}\zeta| \leq \epsilon_i\}, \; i=1, 2, \qquad (15)$$

where $n_\zeta = n+n_c$ and C_{a_i} denotes the i-th row of the matrix C_{a}, namely

$$C_{\mathrm{a}_1} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad C_{\mathrm{a}_2} = \begin{bmatrix} 0 & C_{\mathrm{c}} \end{bmatrix}.$$

The sets $\mathcal{B}_i$ and $\mathcal{C}_i$ are related to, respectively, the largest and smallest quantization levels of the quantizer Q_i, $i = 1, 2$. These sets are symmetric with respect to the origin, are unbounded in the direction of the vectors of an orthogonal basis of the null space of C_{a_i}, and are bounded by two hyperplanes orthogonal to C'_{a_i}. The distance between these hyperplanes is $2\mu_i(1-\delta_i)^{-1}/\sqrt{C_{\mathrm{a}_i}C'_{\mathrm{a}_i}}$ for $\mathcal{B}_i$ and $2\epsilon_i/\sqrt{C_{\mathrm{a}_i}C'_{\mathrm{a}_i}}$ for $\mathcal{C}_i$.

Note that whenever the state ζ of system (12) lies in $\mathcal{C}_1 \cap \mathcal{C}_2$, one has $Q_{\mathrm{a}}(q) = 0$, which leads to a zero input signal p to system (12). Hence, in general, the trajectory of ζ will not converge to the origin and thus quadratic stability will not hold. To tackle this behavior, in the sequel we introduce a notion of stability, which was inspired by the concept of practical stability proposed in Elia and Mitter (2001).

To introduce the stability notion adopted in this paper, let the quadratic functions

$$V(\zeta) = \zeta' P\zeta, \;\; V_{\mathrm{a}}(\zeta) = \zeta' P_{\mathrm{a}}\zeta, \quad P > 0, \;\; P_{\mathrm{a}} > 0, \qquad (16)$$

where ζ is as in (13), and consider the sets

$$\mathcal{D} = \{\zeta \in \mathbb{R}^{n_\zeta} : V(\zeta) \leq 1\}, \qquad (17)$$
$$\mathcal{A} = \{\zeta \in \mathbb{R}^{n_\zeta} : V_{\mathrm{a}}(\zeta) \leq 1\}, \qquad (18)$$
$$\mathcal{C}_p = \{\zeta \in \mathcal{C}_1 \cup \mathcal{C}_2 : DV_{\mathrm{a}}(\zeta) \geq 0\}, \qquad (19)$$

where the notation $Dg(\zeta(k))$, for a real function $g(\cdot)$, is defined by $Dg(\zeta(k)) := g(\zeta(k+1)) - g(\zeta(k))$.

Definition 1 The quantized closed-loop system in (12) is said to be *widely quadratically stable* if there exist quadratic functions $V(\zeta)$ and $V_{\mathrm{a}}(\zeta)$ as in (16) satisfying the following conditions:

$$\mathcal{A} \subset \mathcal{D}, \quad \mathcal{D} \subset \mathcal{B}_i, \; i=1, 2, \qquad (20)$$
$$DV(\zeta) < 0, \quad \forall\, \zeta \in \mathcal{D}\backslash(\mathcal{C}_1 \cup \mathcal{C}_2), \qquad (21)$$
$$DV_{\mathrm{a}}(\zeta) < 0, \quad \forall\, \zeta \in \mathcal{A}\backslash\mathcal{C}_{\mathrm{p}}, \qquad (22)$$
$$\zeta(k+1) \in \mathcal{A} \;\text{ whenever }\; \zeta(k) \in \mathcal{C}_{\mathrm{p}}\,. \qquad (23)$$

Wide quadratic stability ensures that for any $\zeta(0) \in \mathcal{D}$, the trajectory of $\zeta(k)$ will converge to the set $\mathcal{A}$ in finite time. Moreover, $\zeta(k) \in \mathcal{A}, \forall\, k \geq \bar{k}$, for some finite integer $\bar{k} > 0$. In view of that, $\mathcal{A}$ is said to be an *attractor* of $\mathcal{D}$ and $\mathcal{D}$ will be referred to as a *set of admissible initial states*. Observe that Definition 1 is similar to the notion of practical stability proposed in Elia and Mitter (2001), where the admissible initial states and attractor sets are ellipsoidal sets with the same shape. On the other hand, different shapes for $\mathcal{D}$ and $\mathcal{A}$ are allowed in the Definition 1, which may lead to less conservative sets $\mathcal{D}$ and $\mathcal{A}$ due to the shapes of $\mathcal{B}_1$ and $\mathcal{B}_2$. In addition, we can recover the practical stability definition by setting $P_{\mathrm{a}} = \beta P$, with $\beta > 1$, which forces the sets $\mathcal{A}$ and $\mathcal{D}$ to have the same shape.

In order to ensure wide quadratic stability of the closed-loop system in (12), firstly observe that the condition $DV(\zeta) < 0$ is written as

$$\begin{bmatrix}\zeta\\p\end{bmatrix}'\begin{bmatrix}A_a'PA_a-P & *\\ B_a'PA_a & B_a'PB_a\end{bmatrix}\begin{bmatrix}\zeta\\p\end{bmatrix}<0. \quad (24)$$

Moreover, note that for all $\zeta\in(\mathcal{B}_1\cap\mathcal{B}_2)\backslash(\mathcal{C}_1\cup\mathcal{C}_2)$, the input vector $p(k)$ of system (12) satisfies the following multivariable sector-bound condition (Coutinho et al. 2010; Tarbouriech et al. 2011):

$$\left[p-\tilde{\Delta}q\right]'T\left[p-\hat{\Delta}q\right]\le 0, \quad (25)$$

where $T>0$ is a free diagonal matrix to be determined and

$$\tilde{\Delta}=\begin{bmatrix}(1-\delta_1) & 0\\ 0 & (1-\delta_2)\end{bmatrix},\ \hat{\Delta}=\begin{bmatrix}(1+\delta_1) & 0\\ 0 & (1+\delta_2)\end{bmatrix}.$$

Thus, considering that the second inclusion in (20) holds, condition (21) is satisfied if (24) holds subject to (25), which is guaranteed by the inequality

$$\begin{bmatrix}\zeta\\p\end{bmatrix}'\begin{bmatrix}\Upsilon_1 & *\\ \Upsilon_2 & \Upsilon_3\end{bmatrix}\begin{bmatrix}\zeta\\p\end{bmatrix}<0, \quad (26)$$

where

$$\begin{cases}\Upsilon_1=A_a'PA_a-P-C_a'(\tilde{\Delta}T\hat{\Delta}+\hat{\Delta}T\tilde{\Delta})C_a,\\ \Upsilon_2=B_a'PA_a+T(\hat{\Delta}+\tilde{\Delta})C_a,\\ \Upsilon_3=B_a'PB_a-2T.\end{cases} \quad (27)$$

Similarly, to ensure (22), the following condition is considered:

$$\begin{bmatrix}\zeta\\p\end{bmatrix}'\begin{bmatrix}\Upsilon_{a_1} & *\\ \Upsilon_{a_2} & \Upsilon_{a_3}\end{bmatrix}\begin{bmatrix}\zeta\\p\end{bmatrix}<0, \quad (28)$$

where the matrices Υ_{a_1}, Υ_{a_2}, and Υ_{a_3} are similar to Υ_1, Υ_2, and Υ_3, respectively, with the matrices P and T replaced by, respectively, P_a and T_a, where $T_a>0$ is a diagonal matrix to be determined.

The inequalities in (26) and (28) together with (20) and considering the definition of the set $\mathcal{C}_p$ will ensure the feasibility of (22). Further, conditions (22) and (23) ensure that $\mathcal{C}_p$ is bounded and $\mathcal{C}_p\subset\mathcal{A}$, otherwise $\zeta(k)$ could eventually leave $\mathcal{A}$.

In view of the above, we have the following stability result:

Theorem 2 *Consider system (1), a given controller (2) and the feedback law in (3) with finite-level quantizers $Q_1(\cdot)$ and $Q_2(\cdot)$ as defined in (5), where μ_i, ρ_i, and N_i are given. The closed-loop system (12) is widely quadratically stable if there exist matrices P and P_a, diagonal matrices $T>0$ and $T_a>0$, and positive scalars τ, τ_i, $\bar{\tau}_i$, $\hat{\tau}_i$ and $\tilde{\tau}_i$, $i=1,2$ satisfying the following inequalities:*

$$P>0,\quad P_a-P>0, \quad (29)$$

$$P-(1-\delta_i)^2\mu_i^{-2}C_{a_i}'C_{a_i}>0,\quad i=1,2, \quad (30)$$

$$\begin{bmatrix}\Upsilon_1 & *\\ \Upsilon_2 & \Upsilon_3\end{bmatrix}<0,\quad \begin{bmatrix}\Upsilon_{a_1} & *\\ \Upsilon_{a_2} & \Upsilon_{a_3}\end{bmatrix}<0, \quad (31)$$

$$\tau-(\tau_1+\tau_2)\ge 0,\quad \tilde{\tau}_i-\bar{\tau}_i\ge 0,\ i=1,2, \quad (32)$$

$$P_a+\sum_{i=1}^{2}\tau_i\epsilon_i^{-2}C_{a_i}'C_{a_i}-(1+\tau)A_a'P_aA_a\ge 0, \quad (33)$$

$$\begin{bmatrix}U_1(i,j) & *\\ U_2(i,j) & U_3(i,j)\end{bmatrix}\ge 0,\quad i,\ j=1,2,\quad i\ne j, \quad (34)$$

where δ_i and ϵ_i are defined in (6) and

$$\begin{cases}U_1(i,j)=P_a+\hat{\tau}_j(1-\delta_j^2)C_{a_j}'C_{a_j}\\ \qquad\qquad +\bar{\tau}_i\epsilon_i^{-2}C_{a_i}'C_{a_i}-(1+\tilde{\tau}_i)A_a'P_aA_a,\\ U_2(i,j)=-\hat{\tau}_jC_{a_j}-(1+\tilde{\tau}_i)B_{a_j}'P_aA_a,\\ U_3(i,j)=\hat{\tau}_j-(1+\tilde{\tau}_i)B_{a_j}'P_aB_{a_j},\\ B_{a_1}=\left[0\ B_c'\right]',\quad B_{a_2}=\left[B'\ 0\right]'.\end{cases} \quad (35)$$

Moreover, the set of admissible initial states $\mathcal{D}$ and the attractor $\mathcal{A}$ are given by (17) and (18), respectively.

Proof Firstly, in view of (14), (17), and (18), the second inequality of (29) together with (30) ensure that $\mathcal{A}\subset\mathcal{D}$ and $\mathcal{D}\subset\mathcal{B}_i$, $i=1,2$, respectively. Next, the first inequality of (31) ensures the feasibility of (26), which in turns implies (21). Further, the second inequality of (31) guarantees that (28) holds, which together with (20) and the definition of $\mathcal{C}_p$ imply that (22) is satisfied.

In the sequel, it will be shown that (32)–(34) guarantee that (23) holds. For that, we partition the set $\mathcal{C}_p$ into three complementary subsets as follows:

$$\mathcal{C}_{p_1}=\left\{\zeta\in\mathcal{C}_1\backslash\mathcal{C}_2: DV_a(\zeta)\ge 0\right\},$$
$$\mathcal{C}_{p_2}=\left\{\zeta\in\mathcal{C}_2\backslash\mathcal{C}_1: DV_a(\zeta)\ge 0\right\},$$
$$\mathcal{C}_{p_3}=\left\{\zeta\in\mathcal{C}_1\cap\mathcal{C}_2: DV_a(\zeta)\ge 0\right\},$$

and consider two cases:

(a) $\zeta(k)\in\mathcal{C}_{p_3}$: Letting $\phi\in\mathbb{R}^{n_\zeta}$ and adding the first inequality of (32) to (33) pre-multiplied by ϕ' and post-multiplied by ϕ, and then dividing both sides by $\tau>0$, we get

$$(1-\phi'A_a'P_aA_a\phi)-\tau^{-1}\phi'(A_a'P_aA_a-P_a)\phi$$
$$-\tau^{-1}\sum_{i=1}^{2}\tau_i(1-\epsilon_i^{-2}\phi'C_{a_i}'C_{a_i}\phi)\ge 0,\ \forall\phi\in\mathbb{R}^{n_\zeta}.$$

Applying the S-procedure, the latter condition yields

$$\phi' A_a' P_a A_a \phi \le 1, \quad \forall\, \phi \in \mathbb{R}^{n_\zeta} : \phi'(A_a' P_a A_a - P_a)\phi \ge 0, \quad \epsilon_i^{-2}\phi' C_{a_i}' C_{a_i}\phi \le 1,\ i=1,2. \tag{36}$$

Now, let $\phi = \zeta(k)$ as defined in (13). Since the last condition in (36) is equivalent to $\zeta(k) \in \mathcal{C}_1 \cap \mathcal{C}_2$, and in such a case the input signal $p(k)$ of (12) is zero, then it holds that $A_a\phi = \zeta(k+1)$. Therefore, (36) leads to

$$\zeta(k+1)' P_a \zeta(k+1) \le 1, \quad \forall\, \zeta(k) \in \mathcal{C}_1 \cap \mathcal{C}_2 : \quad \zeta(k+1)' P_a \zeta(k+1) - \zeta(k)' P_a \zeta(k) \ge 0.$$

This guarantees that $\zeta(k+1) \in \mathcal{A}$ whenever $\zeta(k) \in \mathcal{C}_{p_3}$.
(b) $\zeta(k) \in \mathcal{C}_{p_i}$, $i=1,2$: Let $\phi \in \mathbb{R}^{n_\zeta}$ and $\psi \in \mathbb{R}$. Adding the second inequality of (32) to (34) and pre-multiplied by $[\phi \quad \psi]'$ and post-multiplied by $[\phi' \quad \psi']'$, and then diving both sides by $\tilde{\tau}_i > 0$, we obtain

$$\begin{aligned} &1 - (A_a\phi + B_{a_j}\psi)' P_a (A_a\phi + B_{a_j}\psi) \\ &+ \tilde{\tau}_i^{-1}\bar{\tau}_i(\epsilon_i^{-2}\phi' C_{a_i}' C_{a_i}\phi - 1) \\ &- \tilde{\tau}_i^{-1}\begin{bmatrix}\phi\\ \psi\end{bmatrix}' \begin{bmatrix} A_a' P_a A_a - P_a & * \\ B_{a_j}' P_a A_a & B_{a_j}' P_a B_{a_j}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix} \\ &+ \tilde{\tau}_i^{-1}\hat{\tau}_j[\psi - (1-\delta_j)C_{a_j}\phi]'[\psi - (1+\delta_j)C_{a_j}\phi] \ge 0, \\ &\quad i, j = 1, 2, i \ne j \end{aligned}$$

for all $\phi \in \mathbb{R}^{n_\zeta}$ and $\psi \in \mathbb{R}$. By the S-procedure, the latter inequality implies that for $i, j = 1, 2, i \ne j$

$$(A_a\phi + B_{a_j}\psi)' P_a (A_a\phi + B_{a_j}\psi) \le 1, \quad \forall\, \phi \in \mathbb{R}^{n_\zeta},\ \psi \in \mathbb{R} : \begin{cases} \begin{bmatrix}\phi\\ \psi\end{bmatrix}' \begin{bmatrix} A_a' P_a A_a - P_a & * \\ B_{a_j}' P_a A_a & B_{a_j}' P_a B_{a_j}\end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix} \ge 0, \\ \left[\psi - (1-\delta_j)C_{a_j}\phi\right]' \left[\psi - (1+\delta_j)C_{a_j}\phi\right] \le 0, \\ \epsilon_i^{-2}\phi' C_{a_i}' C_{a_i}\phi \le 1. \end{cases} \tag{37}$$

Note that the last inequality of (37) is equivalent to $\phi \in \mathcal{C}_i$. Let $\phi = \zeta(k)$ and $\psi = p_j(k)$ as in (13). Since for $\zeta(k) \in \mathcal{C}_i$, the input signal $p_i(k)$ of (12) is zero and $p_j(k)$ satisfies the sector-bound inequality in (11), by considering (12) we obtain from (37) that

$$\zeta(k+1)' P_a \zeta(k+1) \le 1, \quad \forall\, \zeta(k) \in \mathcal{C}_i \backslash \mathcal{C}_j : \quad \zeta(k+1)' P_a \zeta(k+1) - \zeta(k)' P_a \zeta(k) \ge 0$$

which ensures $\zeta(k+1) \in \mathcal{A}$ whenever $\zeta(k) \in \mathcal{C}_{p_i}$, $i=1,2$.

In the light of the above, we conclude that system (12) is widely quadratically stable. □

Remark 1 Observe that (33) and (34) are not jointly convex in $\tau, \tilde{\tau}_1, \tilde{\tau}_2$, and P_a. However, the conditions in (29)–(34) turn out to be LMIs when the scalars $\tau, \tilde{\tau}_1, \tilde{\tau}_2$ are given *a priori*. Thus, applying a gridding procedure, we can perform a search on the latter scalars to obtain a feasible solution to the inequalities in (29)–(34). □

In general, we may desire to obtain a maximized set $\mathcal{D}$ (in the sense of its volume), or a minimized set $\mathcal{A}$. As the set $\mathcal{D}$ is an ellipsoid, a way to approximately maximize its size is to minimize trace(P). The reason for this is that for $P \in \mathbb{R}^{n_\zeta \times n_\zeta}$ we have $n_\zeta(\text{trace}(P))^{-1} \le \text{trace}(P^{-1})$ and trace(P^{-1}) is the sum of the squared semi-axis lengths of the ellipsoid $\mathcal{D}$. Specifically, the size of $\mathcal{D}$ in Theorem 2 can be approximately maximized by means of the following optimization problem:

$$\begin{cases} \min\limits_{\gamma_1,\, P,\, P_a,\, T,\, T_a,\, \tau,\, \tau_i,\, \tilde{\tau}_i\, \bar{\tau}_i,\, \hat{\tau}_i\, i=1,2} \gamma_1, \quad \text{subject to} \\ (29) - (34) \text{ and } \gamma_1 - \text{trace}(P) \ge 0\,. \end{cases} \tag{38}$$

Similarly, we can approximately minimize the size of $\mathcal{A}$ by maximizing trace(P_a), which can be achieved via the optimization problem as follows:

$$\begin{cases} \max\limits_{\gamma_2,\, P,\, P_a,\, T,\, T_a,\, \tau,\, \tau_i,\, \tilde{\tau}_i\, \bar{\tau}_i,\, \hat{\tau}_i\, i=1,2} \gamma_2, \quad \text{subject to} \\ (29) - (34) \text{ and } \text{trace}(P_a) - \gamma_2 \ge 0\,. \end{cases} \tag{39}$$

Very often, it is desired to jointly optimize the size of $\mathcal{D}$ and $\mathcal{A}$, which is generally a difficult problem to solve. A possible way to approximately jointly maximize $\mathcal{D}$ and minimize $\mathcal{A}$ is obtained by minimizing the scalar $\gamma := \gamma_1/\gamma_2$, where γ_1 and γ_2 are as in (38) and (39), respectively. More specifically, this optimization problem can be formulated as follows. First, define

$$\begin{cases} \kappa = \gamma_2^{-1}, \\ X = \kappa P,\ X_a = \kappa P_a, \\ W = \kappa T,\ W_a = \kappa T_a, \\ \alpha_i = \kappa\tau_i,\ \bar{\alpha}_i = \kappa\bar{\tau}_i,\ \hat{\alpha}_i = \kappa\hat{\tau}_i,\ i=1,2 \end{cases} \tag{40}$$

Note that (29)–(34) can be written as

$$X > 0, \quad X_a - X > 0, \quad \kappa > 0, \tag{41}$$

$$\alpha_i > 0, \quad \bar{\alpha}_i > 0, \quad \hat{\alpha}_i > 0, \quad i = 1, 2, \tag{42}$$

$$X - \kappa(1-\delta_i)^2\mu_i^{-2} C_{a_i}' C_{a_i} > 0, \quad i = 1, 2, \tag{43}$$

$$\begin{bmatrix} \widehat{\Upsilon}_1 & * \\ \widehat{\Upsilon}_2 & \widehat{\Upsilon}_3 \end{bmatrix} < 0, \quad \begin{bmatrix} \widehat{\Upsilon}_{a_1} & * \\ \widehat{\Upsilon}_{a_2} & \widehat{\Upsilon}_{a_3} \end{bmatrix} < 0, \tag{44}$$

$$\kappa\tau-(\alpha_1+\alpha_2)\geq 0,\ \kappa\tilde{\tau}_i-\bar{\alpha}_i\geq 0,\quad i=1,2, \tag{45}$$

$$X_{\rm a}+\sum_{i=1}^{2}\alpha_i\epsilon_i^{-2}C'_{a_i}C_{a_i}-(1+\tau)A_{\rm a'}X_{\rm a}A_{\rm a}\geq 0, \tag{46}$$

$$\begin{bmatrix}\widehat{U}_1(i,j) & *\\ \widehat{U}_2(i,j) & \widehat{U}_3(i,j)\end{bmatrix}\geq 0,\ i,j=1,2,\ i\neq j, \tag{47}$$

where $\widehat{\Upsilon}_k$, $\widehat{\Upsilon}_{a_k}$ and $\widehat{U}_k$, $k=1,2,3$, are similar to, respectively, Υ_k, Υ_{a_k} and U_k, $k=1,2,3$ as in Theorem 2 with P, $P_{\rm a}$, T, T_a, τ_i, $\bar{\tau}_i$ and $\hat{\tau}_i$ replaced by, respectively, X, $X_{\rm a}$, W, $W_{\rm a}$, α_i, $\bar{\alpha}_i$, and $\hat{\alpha}_i$, $i=1,2$.

Considering (38) and (39), the minimization of γ can be achieved via the optimization problem

$$\begin{cases}\min\limits_{\gamma,\kappa,X,X_{\rm a},W,W_{\rm a},\tau,\tilde{\tau}_i,\alpha_i,\bar{\alpha}_i,\hat{\alpha}_i,i=1,2}\ \gamma,\\ \text{subject to (41)}-\text{(47)},\quad \gamma-\mathrm{trace}(X)\geq 0,\\ \text{and}\ \ \mathrm{trace}(X_{\rm a})-1\geq 0\,.\end{cases} \tag{48}$$

5 Quantizer Design

Theorem 2 provides a method for deriving a set $\mathcal{D}$ of admissible initial states and its attractor $\mathcal{A}$ for the closed-loop system of (1)–(3) considering finite-time logarithmic quantizers as in (5) with given maximum quantization levels μ_1 and μ_2, and zero-level errors ϵ_1 and ϵ_2. In this section, we apply Theorem 2 for designing the quantizer parameters μ_i and ϵ_i, $i=1,2$. To this end, let $\mathcal{D}_0=\{\zeta\in\mathbb{R}^{n_\zeta}:\zeta' P_0\zeta\leq 1\}$, $P_0>0$, be a given set of admissible initial states and ϑ be an upper bound on the volume of the set $\mathcal{A}=\{\zeta\in\mathbb{R}^{n_\zeta}:\zeta' P_{\rm a}\zeta\leq 1\}$,[1] which will be the attractor of $\mathcal{D}_0$, where $P_{\rm a}>0$ is to be determined. Assuming there exists an output feedback quadratically stabilizing controller for system (1), the following procedure can be applied to design static finite-level logarithmic quantizers $Q_1(\cdot)$ and $Q_2(\cdot)$ such that the wide quadratic stability of the closed-loop system in (1)–(3) is guaranteed:

Step 1: Assuming logarithmic quantizers with an infinity number of levels, determine a controller and the quantization densities ρ_1 and ρ_2 ensuring the closed-loop quadratic stability by applying the methodology proposed in Coutinho et al. (2010) and let $\delta_i=(1-\rho_i)/(1+\rho_i)$, $i=1,2$.

Step 2: Determine matrices P and P_a and positive scalars $\eta_i,\sigma_i,\beta_i,\bar{\beta}_i,\tilde{\tau}_i,\hat{\tau}_i, i=1,2$, and τ satisfying the inequalities in (31) and the following conditions:

$$P>0,\quad P_0-P>0,\quad P_{\rm a}-P_0>0, \tag{49}$$

$$P-(1-\delta_i)^2\eta_i C'_{a_i}C_{\rm a_i}>0,\quad i=1,2, \tag{50}$$

$$P_{\rm a}+\beta_1 C'_{\rm a_1}C_{\rm a_1}+\beta_2 C'_{\rm a_2}C_{a_2} -(1+\tau)A_{\rm a'}P_{\rm a}A_{\rm a}\geq 0, \tag{51}$$

$$\tau\sigma_2-(\beta_1+\beta_2)\geq 0,\ \sigma_1-\sigma_2\geq 0,\quad \sigma_i\tilde{\tau}_i-\bar{\beta}_i\geq 0,\quad i=1,2, \tag{52}$$

$$\begin{bmatrix}\widetilde{U}_1(i,j) & *\\ \widetilde{U}_2(i,j) & \widetilde{U}_3(i,j)\end{bmatrix}\geq 0,\quad i,j=1,2,\quad i\neq j, \tag{53}$$

$$\vartheta^{\frac{2}{n_\zeta}}P_{\rm a}-c^{\frac{2}{n_\zeta}}I\geq 0, \tag{54}$$

where $\widetilde{U}_k(i,j)$ is similar to $U_k(i,j)$, $k=1,2,3$ as defined in (35) with $\bar{\tau}_i\epsilon_i^{-2}$ replaced by $\bar{\beta}_i$ and c is the constant related to the volume of $\mathcal{A}$. Note that (49)–(53) imply that the conditions in (29)–(34) of Theorem 2 hold with $\mu_i=\eta_i^{-2}$, $\epsilon_i=\sigma_i^{-2}$, $\tau_i=\epsilon_i^2\beta_i$, and $\bar{\tau}_i=\epsilon_i^2\bar{\beta}_i$, $i=1,2$, whereas (54) ensures that ϑ is an upper bound for the volume of $\mathcal{A}$.

Step 3: Suitable quantizers parameters μ_i and ϵ_i are given by $\mu_i=1/\sqrt{\eta_i}$ and $\epsilon_i=1/\sqrt{\sigma_i}$, $i=1,2$. Moreover, the numbers of positive quantization levels of quantizers $Q_1(\cdot)$ and $Q_2(\cdot)$ are the smallest integers N_1 and N_2, respectively, satisfying

$$N_i\geq 1+\log_{\rho_i}(\epsilon_i(1+\delta_i)/\mu_i),\ i=1,2. \tag{55}$$

Generally, we are interested in determining the smallest number of quantization levels guaranteeing the wide quadratic stability, which can be achieved by jointly minimizing μ_i and maximizing $\epsilon_i, i=1,2$. To this end, the inequality constraints (31) and (49)–(54) of Step 2 are embedded in the following optimization problem:

$$\begin{cases}\min\limits_{P,P_{\rm a},T,T_{\rm a},\tau,\tilde{\tau}_i,\hat{\tau}_i,\beta_i,\bar{\beta}_i,\sigma_i,\eta_i,i=1,2}\ \theta,\\ \text{subject to (31),(49)}-\text{(54)}\end{cases} \tag{56}$$

where $\theta=\sigma_1+\sigma_2-\eta_1-\eta_2$.

6 Examples

Example 1 Consider the discrete-time system of Example 3.1 in Fu and Xie (2005), which deals with a non-minimum phase open-loop unstable system with a transfer function $G(z)=\frac{z-3}{z(z-2)}$. This system can be represented by the following state-space realization:

$$\begin{cases}x_1(k+1)=x_2(k)\\ x_2(k+1)=2x_2(x)+u(k)\\ y(k)=-3x_1(k)+x_2(k)\end{cases} \tag{57}$$

Firstly, using the methodology proposed in Coutinho et al. (2010) the following quadratically stabilizing output feedback controller is designed to maximize δ_1 and δ_2 assuming the quantizers to be ideal:

[1] The volume of an ellipsoid $\mathcal{A}=\{\zeta\in\mathbb{R}^{n_\zeta}:\zeta' P_{\rm a}\zeta\leq 1,\ P_{\rm a}>0\}$ is given by $c/\sqrt{\det(P_{\rm a})}$, where c is a constant that depends on n_ζ (see, e.g., Bernstein (2009)).

$$\begin{cases} \xi(k+1) = \begin{bmatrix} 0 & 1 \\ -10 & -1.667 \end{bmatrix} \xi(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(k) \\ \quad w(k) = \begin{bmatrix} 0 & 6.667 \end{bmatrix} \xi(k) \end{cases} \tag{58}$$

Next, we assume that the finite-level quantizers $Q_1(\cdot)$ and $Q_2(\cdot)$ have the following parameters:

$$\delta_1 = 10^{-2}, \quad \mu_1 = 18, \quad N_1 = 512,$$
$$\delta_2 = 3 \times 10^{-2}, \quad \mu_2 = 12, \quad N_2 = 128,$$

and according to (6) we have $\epsilon_1 = 6.4910 \times 10^{-4}$ and $\epsilon_2 = 5.702 \times 10^{-3}$. Since the number of bits, N_{b_i}, required for the quantizer $Q_i(\cdot),\ i = 1, 2$ is given by $N_{b_i} = \log_2(2N_i)$, we have that $Q_1(\cdot)$ and $Q_2(\cdot)$ use 10 bits and 8 bits, respectively.

The optimization procedure in (48) together with a griding search procedure on the parameters τ, $\tilde{\tau}_1$, and $\tilde{\tau}_2$ have been applied to approximately jointly optimize the set $\mathcal{D}$ of admissible initial state and its attractor $\mathcal{A}$. Notice that in view of (46) and (47), the later parameters are typically small. For this example, the LMIs in (41)–(47) are feasible if $0.1 \leq \tau, \tilde{\tau}_1, \tilde{\tau}_2 \leq 0.2$. Furthermore, to simplify the griding search, we have considered $0.1 \leq \tau = \tilde{\tau}_1 = \tilde{\tau}_2 \leq 0.2$ leading to $\tau = \tilde{\tau}_1 = \tilde{\tau}_2 = 0.155$ and

$$P = \begin{bmatrix} 22.9607 & -11.1670 & 76.5358 & 1.0331 \\ -11.1670 & 5.5835 & -37.2253 & 0.0000 \\ 76.5358 & -37.2253 & 255.4276 & 3.4410 \\ 1.0331 & 0.0000 & 3.4410 & 2.2467 \end{bmatrix},$$

$$P_{\mathrm{a}} = 10^5 \begin{bmatrix} 0.1681 & -0.0824 & 0.5602 & 0.0057 \\ -0.0824 & 0.0412 & -0.2745 & -0.0000 \\ 0.5602 & -0.2745 & 1.8675 & 0.0188 \\ 0.0057 & -0.0000 & 0.0188 & 0.0120 \end{bmatrix},$$

Figure 4 displays a slice of the set $\mathcal{D}$ for $\xi = 0$ together with the projection of a stable and an unstable state trajectories on the plane defined by $[\, x_1 \ \ x_2 \ \ 0 \ \ 0\,]'$. To evaluate the conservatism of the achieved results, Fig. 4 also shows a zoomed view of the starting sequence of the two state trajectories and the slice of $\mathcal{D}$.

As in Fig. 4 the attractor is too small to be visible, in Fig. 5 we display a detailed view of a slice of $\mathcal{A}$ and the projection of the stable state trajectory on the plane defined by $[\, x_1 \ \ x_2 \ \ 0 \ \ 0\,]'$.

Example 2 This example is aimed at illustrating the method of quantizers design of Sect. 5. To this end, consider the following discrete-time linear approximation of an inverted pendulum system with null damping factor taken from de Souza et al. (2010):

$$\begin{cases} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.000 & 0.036 \\ 0.036 & 1.000 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.000 \\ 0.036 \end{bmatrix} u(k), \\ y(k) = x_2, \end{cases}$$

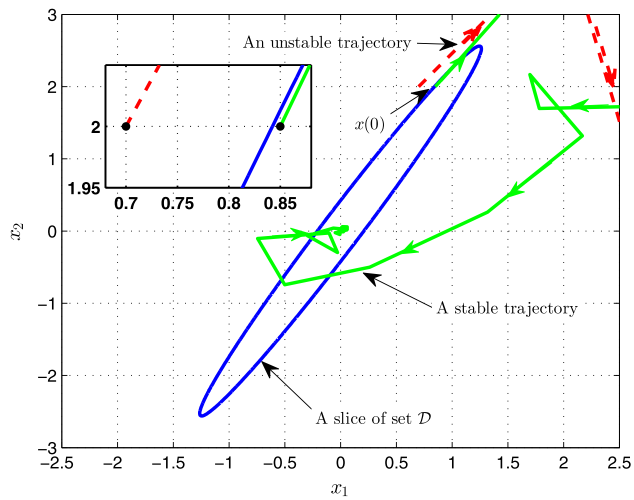

Fig. 4 A slice of the set $\mathcal{D}$ (with $\xi = 0$), and a stable and an unstable state trajectories

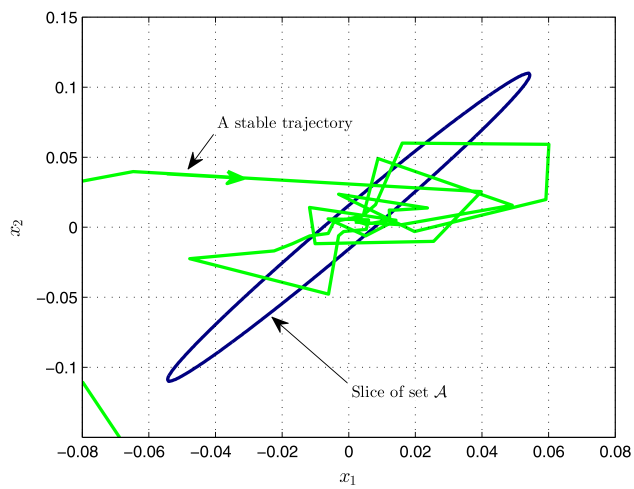

Fig. 5 A slice of the set $\mathcal{A}$ (with $\xi = 0$) and a stable state trajectory

and the following quadratically stabilizing output feedback controller obtained via the method of Coutinho et al. (2010) considering that the quantizers are ideal with the constraint $\delta_1 = \delta_2 = \delta$ and δ being maximized:

$$\begin{cases} \xi(k+1) = \begin{bmatrix} 1.2090 & -0.8758 \\ 0.3825 & -0.6904 \end{bmatrix} \xi(k) + \begin{bmatrix} 87.5314 \\ 146.4775 \end{bmatrix} v(k), \\ w(k) = \begin{bmatrix} 0.0040 & 0.0947 \end{bmatrix} \xi(k)\,. \end{cases}$$

To design the input and output finite-level logarithmic quantizers, we consider the following set $\mathcal{D}_0$ of admissible initial states:

$$\mathcal{D}_0 := \left\{ x \in \mathbb{R}^4 : x' P_0 x \leq 1 \right\}, \ P_0 = \mathrm{diag}\{1.5, 3, 0.1, 0.1\}$$

and the quantization densities $\rho_1 = \rho_2 = 0.7$. Moreover, it is assumed that the maximal volume of the attractor $\mathcal{A}$ is 10% of that of $\mathcal{D}_0$, i.e., $\vartheta = 2.3$ (the volume of $\mathcal{A}$ is $\pi^2/\sqrt{4\det(P_0)}$).

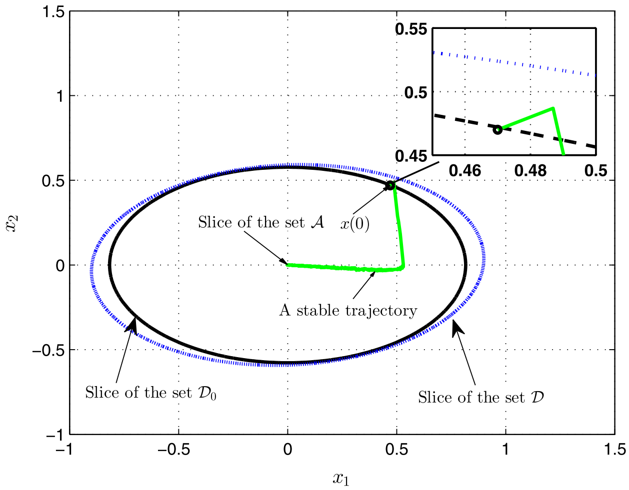

Fig. 6 Slices of the sets $\mathcal{D}$ and $\mathcal{D}_0$ (with $\xi = 0$) and a stable state trajectory

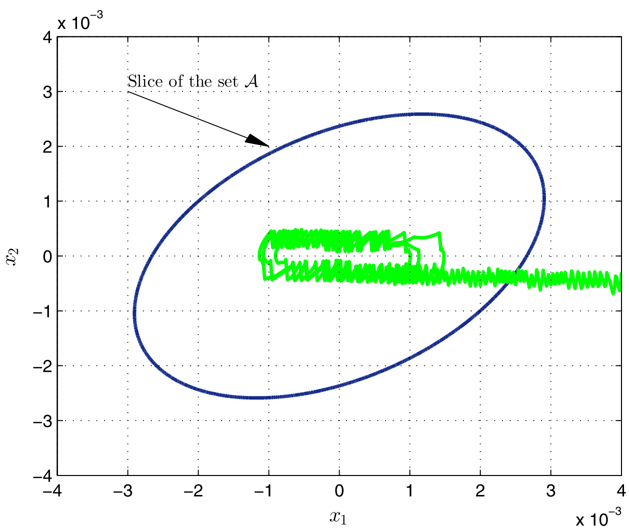

Fig. 7 A slice of the set $\mathcal{A}$ (with $\xi = 0$) and a stable state trajectory

Applying the optimization problem (56) and performing a griding search over $\tilde{\tau}_1$, $\tilde{\tau}_2$, and τ similar as in Example 1, yields:

$$\mu_1 = 0.74, \quad \epsilon_1 = 4.42 \times 10^{-4}, \quad N_1 = 22,$$
$$\mu_2 = 18.51, \quad \epsilon_2 = 4.42 \times 10^{-4}, \quad N_2 = 31,$$

for $\tau = \tilde{\tau}_1 = \tilde{\tau}_2 = 0.0068$.

Figure 6 shows slices of the sets $\mathcal{D}$ and $\mathcal{D}_0$ with $\xi = 0$ along with the projection of a stable state trajectory on the plane defined by $\left[x_1 \ x_2 \ 0 \ 0\right]'$. A zoomed view of the starting sequence of the state trajectory and the slices of $\mathcal{D}$ and $\mathcal{D}_0$ are also displayed in Fig. 6. Furthermore, Fig. 7 gives a detailed view of the slice of $\mathcal{A}$ and the projection of the stable state trajectory.

7 Concluding Remarks

This paper has extended the results of Maestrelli et al. (2012) on local stability analysis of SISO discrete-time linear time-invariant systems with input and output static finite-level logarithmic quantizers. Firstly, an optimization problem with LMI constraints has been proposed to jointly optimize the estimates of the set of admissible initial states and the associated invariant attractor set in a neighborhood of the state-space origin, based on a relaxed stability notion, referred to as wide quadratic stability. Then, assuming these sets are given, we have also provided a method to design the input and output quantizers ensuring wide quadratic stability. Numerical examples have demonstrated the potentials of the proposed approach. Future research is concentrated on extending these results to nonlinear quadratic systems.

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