J Control Autom Electr Syst (2013) 24:33–45
DOI 10.1007/s40313-013-0009-2

LMI Relaxations for $\mathcal{H}_\infty$ and $\mathcal{H}_2$ Static Output Feedback of Takagi–Sugeno Continuous-Time Fuzzy Systems

Eduardo S. Tognetti · Ricardo C. L. F. Oliveira · Pedro L. D. Peres

Received: 30 April 2012 / Revised: 10 August 2012 / Accepted: 7 September 2012 / Published online: 12 March 2013

Abstract This paper presents new results concerning the problem of static output feedback $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control design for continuous-time Takagi–Sugeno (T–S) fuzzy systems. A fuzzy line integral Lyapunov function with arbitrary polynomial dependence on the premise variables is used to certify closed-loop stability with a bound to the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms, allowing the membership functions to vary arbitrarily (i.e., no bounds on the time-derivative of the membership functions are assumed). The static output feedback fuzzy controller is obtained through a two-step procedure: first, a fuzzy state feedback control gain is determined by means of linear matrix inequalities (LMIs). Then, the state feedback gain matrices are used in the LMI conditions of the second step that, if satisfied, provide the fuzzy static output feedback control law. The proposed approach also allows the output feedback gains to have independent and arbitrary polynomial dependence on some specific premise variables, selected by the designer, with great advantages for practical applications. The efficiency of the proposed strategy is demonstrated by means of numerical examples and time domain simulations.

Keywords Takagi–Sugeno fuzzy systems · $\mathcal{H}_\infty$ control · $\mathcal{H}_2$ control · Static output feedback · Linear matrix inequalities

E. S. Tognetti
Department of Electrical Engineering, University of Brasilia—UnB, Brasilia, DF 70910-900, Brazil
e-mail: estognetti@ene.unb.br

R. C. L. F. Oliveira · P. L. D. Peres (✉)
School of Electrical and Computer Engineering, University of Campinas—UNICAMP, Campinas, SP 13083-852, Brazil
e-mail: peres@dt.fee.unicamp.br

R. C. L. F. Oliveira
e-mail: ricfow@dt.fee.unicamp.br

1 Introduction

The problem of control design for dynamical systems described by Takagi–Sugeno (T–S) fuzzy systems (Takagi and Sugeno 1985) has been largely investigated in the last decades with the use of the Lyapunov theory combined with linear matrix inequalities (LMIs). The initial methods were based on common quadratic Lyapunov functions that yield, in general, conservative results (Tanaka and Wang 2001). Control design approaches based on constant Lyapunov functions can be found in Teixeira and Żak (1999), Teixeira et al. (2000, 2003) and Andrea et al. (2008). Moreover, convergent LMI relaxations for quadratic stability and $\mathcal{H}_\infty$ state feedback control are presented in Sala and Ariño (2007) and Montagner et al. (2009, 2010). In Arrifano et al. (2006) and Tognetti and Oliveira (2010), a class of piecewise quadratic Lyapunov functions is considered.

As an alternative, fuzzy Lyapunov functions (Tanaka et al. 2003), that can be viewed as a blend of multiple quadratic on the state functions, have been used to provide less conservative results. The main drawback of handling fuzzy Lyapunov functions is the presence of the time-derivative of the membership functions in the stability conditions (Tanaka et al. 2003; Mozelli et al. 2009b; Mozelli and Palhares 2011). To circumvent this difficulty, one usual strategy is to consider upper bounds on the time-derivatives in the conditions (Tanaka et al. 2001). However, these bounds are hard to obtain in the control design problem. Another approach, based on line-integral Lyapunov functions (Rhee and Won 2006), avoids the presence of the time-derivative of the membership functions in the stability conditions and, consequently, allows arbitrary variations of the membership functions. In Rhee and Won (2006), this strategy provides sufficient LMI conditions for stability analysis but the structure imposed to the Lyapunov matrix, to assure the line-integral function to be a candidate Lyapunov function, leads to bilinear matrix inequalities (BMIs) in the synthesis conditions. In Mozelli et al. (2009a, 2010) LMI conditions are proposed to design state feedback controllers with the use of slack variables. It is worth to say that line-integral Lyapunov functions can be used for stability analysis of T–S fuzzy systems only if the premise variables are the states of the system, but this can represent a large class of T–S models obtained from nonlinear systems (Tanaka and Wang 2001). Moreover, to implement the parallel distributed compensation (PDC) for the control law, all premise variables have to be measured in real time (Wang et al. 1995). Therefore, the current approaches based on line-integral Lyapunov functions cannot be direct applied to the output feedback problem.

A strategy developed in Tognetti et al. (2011d) for the state feedback control of T–S fuzzy systems allows the selective use of only the available premise variables for the control law. In this case, the designer could also decide what bounds of the time-derivative of the membership functions to take into account in the LMIs. As it will be shown later, this method can be adapted to cope with the output feedback problem. It is also worth to mention some recent methods that address the state feedback control design problem in terms of local stability for continuous (Klug and Castelan 2011; Guerra et al. 2012) and discrete-time (Klug et al. 2011) T–S fuzzy systems. This issue is relevant in practice because, in general, a T–S fuzzy model represents a nonlinear system to be controlled only locally.

In the context of static output feedback synthesis, due to the non-convex nature of this problem, only a few results can be found in the literature of T–S fuzzy system. Static output feedback controllers are simpler to implement in practical applications when compared to the dynamic and observed based ones (Guerra et al. 2006; Mansouri et al. 2009; Nguang and Shi 2006; Guelton et al. 2009; Klug et al. 2011; Tognetti et al. 2012) since there is no differential equations to be solved in real time (Syrmos et al. 1997). On the other hand, the design of static output feedback controllers is difficult and some conservatism is usually introduced in the problem. See Huang and Nguang (2007); Lee and Kim (2009) and Bouarar et al. (2009) for output feedback control design in the T–S literature.

The main contribution of this paper is to propose new LMI conditions to design $\mathcal{H}_\infty$ and $\mathcal{H}_2$ static output feedback controllers for continuous-time T–S fuzzy systems. To obtain less conservative conditions and allow arbitrary variation of the membership functions, line-integral fuzzy Lyapunov functions are used in a two-step design procedure. The strategy in two steps follows general lines presented in Peaucelle and Arzelier (2001); Arzelier et al. (2003, 2010) and Mehdi et al. (2004). First, a state feedback controller is designed and used as an input parameter in the second step, that provides an output feedback controller and an upper bound for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms of the closed-loop system. All conditions are expressed in terms of LMIs and the Lyapunov matrix and decision variables of the problem are homogeneous polynomial matrices of arbitrary and independent degrees. An interesting feature is that the first stage controller (used only as an intermediate step) is not required to stabilize, a priori, the closed-loop T–S fuzzy system. This property can be used to generate a large class of state feedback gains for the second stage. The membership functions are represented by the Cartesian product of simplexes Baranyi (2004); Tognetti et al. (2010a,b), called multi-simplex, allowing a more general structure for the Lyapunov matrix and slack variables (polynomials with arbitrary degree dependence on the membership functions in each simplex) when compared to Rhee and Won (2006) and Mozelli et al. (2009a, 2010). As a by-product, less conservative conditions for synthesis of state feedback controllers are obtained for this type of line-integral Lyapunov functions. Moreover, like in Tognetti et al. (2011d), this strategy allows the selective use of the premise variables, an important aspect in the context of output feedback problems for T–S fuzzy systems, since the premise variables usually are the states of the system. This paper extends the results presented in Tognetti et al. (2011c) and Tognetti et al. (2011b) in the following aspects: the presence of noise is considered in the measured output (very common in practical cases but almost unexplored in the literature); control design with $\mathcal{H}_2$ performance; more relaxed first stage condition; additional remarks about the path-independent Lyapunov function, detailing the structures of the decision variables in the LMIs, implementation issues and how to retrieve a state feedback controller from the second stage condition. Also, an iterative procedure is proposed to enhance the quality of the controller in terms of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ attenuation bounds. Numerical examples and simulations illustrate the effectiveness of the proposed approach.

2 Preliminaries

Consider a class of continuous-time T–S fuzzy system which can be described by the following ℓth fuzzy rule

$$\mathcal{R}_\ell: \text{ If } x_1(t) \text{ is } M_1^{\alpha_{\ell 1}} \text{ and } \ldots \text{ and } x_n(t) \text{ is } M_n^{\alpha_{\ell n}}$$

$$\text{Then} \begin{cases} \dot{x}(t) = A_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} x(t) + B_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} u(t) \\ \qquad +E_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} w(t) \\ z(t) = C_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} x(t) + D_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} u(t) \\ \qquad +F_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} w(t) \\ y(t) = C_{y_{\alpha_{\ell 1}\cdots\alpha_{\ell n}}} x(t) + F_{y_{\alpha_{\ell 1}\cdots\alpha_{\ell n}}} w(t) \end{cases} \quad (1)$$

for $\ell = 1, \ldots, N$, where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output, $w(t) \in \mathbb{R}^o$ is the disturbance input, $u(t) \in \mathbb{R}^m$ is the control input, $z(t) \in \mathbb{R}^q$ is the controlled output used for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ criteria. The linear subsystem matrices are $A_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in$

$\mathbb{R}^{n\times n}$, $B_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in \mathbb{R}^{n\times m}$, $E_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in \mathbb{R}^{n\times o}$, $C_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in \mathbb{R}^{q\times n}$, $D_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in \mathbb{R}^{q\times m}$, $F_{\alpha_{\ell 1}\cdots\alpha_{\ell n}} \in \mathbb{R}^{q\times o}$, $C_{y_{\alpha_{\ell 1}\cdots\alpha_{\ell n}}} \in \mathbb{R}^{p\times n}$ and $F_{y_{\alpha_{\ell 1}\cdots\alpha_{\ell n}}} \in \mathbb{R}^{p\times o}$. The premise variables are the states and $M_j^{\alpha_{\ell j}}$ denotes an x_j-based fuzzy set used for the ℓth fuzzy rule, where $\alpha_{\ell j}$ specifies which x_j-based fuzzy set is used in the ℓth fuzzy rule. N is total number of fuzzy rules and r_j the number of x_j-based fuzzy sets. For instance, if $\alpha_{11} = \alpha_{21} = k$ then it means that in rules 1 and 2 the same $x_1(t)$-based fuzzy set M_1^k is used.

Let $\vartheta_j^{\alpha_{\ell j}}(x_j(t))$ be the membership function of $M_j^{\alpha_{\ell j}}$. The normalized membership function for each $\alpha_{\ell j} = 1, \ldots, r_j = i$, is

$$\mu_{ji}\left(x_j(t)\right) = \frac{\vartheta_j^i\left(x_j(t)\right)}{\sum_{i=1}^{r_j}\vartheta_j^i\left(x_j(t)\right)}, \quad j = 1, \ldots, n, \quad i = 1, \ldots, r_j,$$

$$0 \le \mu_{ji}\left(x_j(t)\right) \le 1, \qquad \sum_{i=1}^{r_j}\mu_{ji}\left(x_j(t)\right) = 1.$$

Each $\mu_j = (\mu_{j1}, \ldots, \mu_{jr_j})$, $j = 1, \ldots, n$, belongs to the unit simplex

$$\mathcal{U}_{r_j} = \left\{(\lambda_1, \ldots, \lambda_{r_j}) \in \mathbb{R}^{r_j} : \sum_{i=1}^{r_j}\lambda_i = 1,\ \lambda_i \ge 0\right\}.$$

In the adopted modeling technique each membership function $\mu_i(x_i(t))$ depends on only one premise variable. This yields great flexibility over the techniques presented in Rhee and Won (2006) and Mozelli et al. (2009a, 2010) as will be demonstrated in the following sections. The definition of the set where all membership functions lie, called multi-simplex, is the same as in Tognetti et al. (2011d).

Using the multi-simplex structure, the T–S fuzzy system (1) can be rewritten as following[1]

$$\begin{cases} \dot{x}(t) = A(\mu)x(t) + B(\mu)u(t) + E(\mu)w(t) \\ z(t) = C(\mu)x(t) + D(\mu)u(t) + F(\mu)w(t) \\ y(t) = C_y(\mu)x(t) + F_y(\mu)w(t) \end{cases} \tag{2}$$

where

$$\left(A, B, E, C, D, F, C_y, F_y\right)(\mu) = \sum_{i_1=1}^{r_1}\cdots\sum_{i_n=1}^{r_n} \mu_{1i_1}(x_1)\cdots\mu_{ni_n}(x_n)\left(A_{i_1\cdots i_n}, B_{i_1\cdots i_n}, E_{i_1\cdots i_n}, C_{i_1\cdots i_n}, D_{i_1\cdots i_n}, F_{i_1\cdots i_n}, C_{y_{i_1\cdots i_n}}, F_{y_{i_1\cdots i_n}}\right)$$

$$\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathcal{U} = \mathcal{U}_{r_1} \times \mathcal{U}_{r_2} \times \cdots \times \mathcal{U}_{r_n}. \tag{3}$$

Polynomial combinations of arbitrary degree of the membership functions are modeled through the multi-simplex structure, with great advantages to the output feedback problem. Hence, the non-PDC output feedback control law given by

$$u(t) = L(\mu)y(t), \quad \mu \in \mathcal{U} \tag{4}$$

may assume independent degrees in each simplex. The closed-loop T–S fuzzy system is given as

$$\begin{cases} \dot{x}(t) = A_{cl}(\mu)x(t) + E_{cl}(\mu)w(t) \\ z(t) = C_{cl}(\mu)x(t) + F_{cl}(\mu)w(t), \end{cases} \tag{5}$$

with

$$\begin{aligned} A_{cl}(\mu) &\triangleq A(\mu) + B(\mu)L(\mu)C_y(\mu) \\ C_{cl}(\mu) &\triangleq C(\mu) + D(\mu)L(\mu)C_y(\mu) \\ E_{cl}(\mu) &\triangleq E(\mu) + B(\mu)L(\mu)F_y(\mu) \\ F_{cl}(\mu) &\triangleq F(\mu) + D(\mu)L(\mu)F_y(\mu) \end{aligned} \tag{6}$$

for all $\mu \in \mathcal{U}$.

In general, to represent a nonlinear systems through linear T–S fuzzy models, the premise variables are the states of the system. However, in the output feedback problem the states of the system may not be available for measurement. Therefore, some premise variables cannot be used by the control law. The present strategy allows the designer to select only the available premise variables for control design, circumventing the main difficulty in the output feedback control of T–S fuzzy systems.

To cope with arbitrarily fast variations of the premise variables, a fuzzy line-integral Lyapunov function (Rhee and Won 2006) is used for system (2),

$$V(x) = 2\int_{\rho(0,x)} f(\psi)\cdot d\psi \tag{7}$$

where $\rho(0, x)$ is a path from the origin to the present state, $(\cdot)$ stands for the inner product of vectors, ψ is a vector for the integral and $d\psi$ is an infinitesimal displacement vector. The fuzzy vector $f(x)$ is parameterized as

$$f(x) = P_g(\mu)x, \tag{8}$$

$$P_g(\mu) \triangleq \begin{bmatrix} d_{11_{g_1}}(\mu_1) & p_{12} & \cdots & p_{1n} \\ p_{12} & d_{22_{g_2}}(\mu_2) & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & d_{nn_{g_n}}(\mu_n) \end{bmatrix}. \tag{9}$$

The indexes $g = (g_1, g_2, \cdots, g_n)$ denote the degrees of the polynomial membership functions on $\mu_1, \mu_2, \ldots, \mu_n$ of $P_g(\mu)$. The degree $g_i = 0$ stands for a constant element, when x_i is not a premise variable for the T–S model. Note that the off diagonal elements are constants and the structure (9), that allows polynomials of degree g_i for each diagonal element $d_{ii_{g_i}}(\mu_i)$, generalizes the ones used in Rhee and Won (2006) and Mozelli et al. (2009a, 2010) that consider only affine dependence ($g_i = 1,\ i = 1, \ldots, n$).

[1] For simplicity of notation, the dependence of $\mu(x(t))$ on $x(t)$ is omitted hereafter.

Remark 1 To be a Lyapunov function candidate, $V(x)$ has to satisfy the following conditions (Khalil 2002): (i) continuous differentiability; (ii) positive definiteness; (iii) radial unboundedness. Condition (i) can be demonstrated following the lines given in Rhee and Won (2006). To assure that both (ii) and (iii) hold, $V(x)$ given by (7) must be path-independent.[2] This is guaranteed through the structure given by (9), that verifies

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i}, \quad i \neq j = 1, \ldots, n. \tag{10}$$

In fact, condition (10) above holds because $f(x) = [f_1(x), \ldots, f_n(x)]'$ is defined as

$$f_i(x) = d_{ii_{g_i}}(\mu_i(x_i))\, x_i + \sum_{k \neq i}^{n} p_{ik} x_k, \qquad i = 1, \ldots, n,$$

and the partial time-derivative of $f_i(x)$ with respect to the variable x_j, for $j \neq i$, is

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial d_{ii_{g_i}}(\mu_i(x_i))}{\partial x_j} x_i + p_{ij} = p_{ij}, \qquad i = 1, \ldots, n,$$

Similarly,

$$\frac{\partial f_j(x)}{\partial x_i} = p_{ji}$$

and, since $p_{ij} = p_{ji}$, $f(x)$ satisfies the condition in (10).

Before proceeding to the main results, the Elimination Lemma is presented.

Lemma 1 (Elimination Lemma (Skelton et al. 1998)) *Let $\mathcal{U} \in \mathbb{R}^{n \times m}$, $\mathcal{V} \in \mathbb{R}^{k \times m}$ and $\Phi = \Phi' \in \mathbb{R}^{n \times n}$ be given matrices. The following conditions are equivalents:*

(i) There exist a matrix $\mathcal{X} \in \mathbb{R}^{m \times k}$ that satisfies

$$\Phi + \mathcal{V}\mathcal{X}\mathcal{U} + (\mathcal{V}\mathcal{X}\mathcal{U})' < 0$$

(ii) Both conditions

$$\mathcal{N}_v \Phi \mathcal{N}_v' < 0 \text{ or } \mathcal{V}\mathcal{V}' > 0$$
$$\mathcal{N}_u' \Phi \mathcal{N}_u < 0 \text{ or } \mathcal{U}'\mathcal{U} > 0$$

must be verified, where $\mathcal{N}_v$ and $\mathcal{N}_u'$ are respectively the orthogonal complement of $\mathcal{V}$ and $\mathcal{U}'$, that is,

$$\mathcal{N}_v \mathcal{V} = 0, \quad \mathcal{N}_u' \mathcal{U}' = 0.$$

The definitions of the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance criteria used in the paper are given below.

[2] Observe that condition (iii) in Remark 1 always holds in T–S fuzzy systems that, in general, represent nonlinear systems exactly in a subset of $\mathbb{R}^n$ (i.e., the states are constrained).

Definition 1 Suppose that the closed-loop continuous-time T–S fuzzy system (5) is exponentially stable. Then, its $\mathcal{H}_\infty$ performance is defined by (see Green and Limebeer 1995)

$$\gamma \triangleq \sup_{\|w(t)\|_2 \neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}$$

with $w(t) \in \mathcal{L}_2[0, \infty)$ and $z(t) \in \mathcal{L}_2[0, \infty)$, where $\mathcal{L}_2[0, \infty)$ denotes the space of square integrable continuous-time signals over the interval $[0, \infty)$.

An $\mathcal{H}_\infty$ guaranteed cost γ can be characterized in terms of LMIs by the continuous-time version of the bounded real lemma for time-varying systems, that can be found, for instance, in Boyd et al. (1994).

Definition 2 Suppose that the continuous-time T–S fuzzy system (5), with $F_{cl}(\mu) = 0$, is exponentially stable. Then, its $\mathcal{H}_2$ performance is defined by (see Takaba 1998)

$$\eta^2 \triangleq \lim_{T \to \infty} \sup \mathcal{E} \left\{ \frac{1}{T} \sum_{t=0}^{T} z(t)' z(t) \right\}$$

where the system input $w(t)$ is a zero-mean white noise Gaussian process with identity covariance matrix.

LMI conditions relating the $\mathcal{H}_2$ performance bound η with controllability or observability Gramians can be found in Boyd et al. (1994). See also Stoorvogel (1993).

3 Main Results

The following theorem provides less conservative results to the synthesis of stabilizing state feedback controllers associated to the control law $u(t) = K_s(\mu)x(t)$ for the T–S fuzzy system (2) by means of the fuzzy line-integral Lyapunov function (7). Hereafter, both the Lyapunov matrix and the slack variables of the problem that depend on μ are represented as homogeneous polynomial matrices of arbitrary degrees[3] in the multi-simplex, being denoted, for instance, as $Z_s(\mu)$ (degree s).

Theorem 1 *Let $\beta > 0$ a given scalar. If there exist a symmetric positive definite matrix $W_g(\mu) \in \mathbb{R}^{n \times n}$, as in (9), a matrix $G \in \mathbb{R}^{n \times n}$ with appropriate strucuture and a matrix $Z_s(\mu) \in \mathbb{R}^{m \times n}$, such that the following parameter-dependent LMIs are verified[4] for all $\mu \in \mathcal{U}$*

$$\Gamma(\mu) \triangleq \begin{bmatrix} \Lambda(\mu) + \Lambda(\mu)' & \star \\ W_g(\mu) - G' + \beta\Lambda(\mu) & -\beta(G + G') \end{bmatrix} < 0, \tag{11}$$

[3] The index a of a matrix $M_a(\mu)$ stands for the degree of the polynomial representation of the matrix. The degree of the system matrices in (3) (degree one) is omitted for simplicity. For more details about the notation, see Sect. 4.

[4] The symbol $\star$ stands for symmetric blocks.

where

$$\Lambda(\mu) = A(\mu)G + B(\mu)Z_s(\mu) \tag{12}$$

then

$$K_s(\mu) = Z_s(\mu)G^{-1} \tag{13}$$

is a polynomial stabilizing state feedback control gain of degree s for the T–S fuzzy system (2).

Proof Pre and post-multiplying $\Gamma(\mu)$ by T e T', respectively, with $T = \text{diag}((G')^{-1}, (G')^{-1})$, yields

$$\mathcal{Y}(\mu) \triangleq \begin{bmatrix} M\overline{A}(\mu) + \overline{A}(\mu)'M' & \star \\ P_g(\mu) - M' + \beta M\overline{A}(\mu) & -\beta(M + M') \end{bmatrix} \tag{14}$$

where $M \triangleq (G')^{-1}$, $Z_s(\mu) \triangleq K_s(\mu)G$ and

$$\overline{A}(\mu) \triangleq A(\mu) + B(\mu)K_s(\mu), \tag{15}$$

$$P_g(\mu) \triangleq (G')^{-1}W_g(\mu)G^{-1}. \tag{16}$$

Pre- and post-multiplying $\mathcal{Y}(\mu)$ by $[I \quad \overline{A}(\mu)']$ and post-multiplying by its transpose to obtain $\overline{A}(\mu)'P_g(\mu) + P_g(\mu)\overline{A}(\mu) < 0$ and, since $P_g(\mu)$ has the structure (9), one has $\dot{V}(x) < 0$. □

Remark 2 To obtain the structure (9) in $P_g(\mu)$ under the transformation (16), one has to impose the same structure to $W_g(\mu)$ and, moreover, G must also have a special structure. Therefore, the parameter-dependent diagonal terms in $P_g(\mu)$ are placed in the same position in $W_g(\mu)$. Moreover, from (16), for each entry $d_{ii_{g_i}}(\mu_i)$ in the main diagonal of $P_g(\mu)$ that is a function of μ, one has to impose zero to all elements of the corresponding i-th row of G^{-1}, except the entry in the main diagonal. This is accomplished[5] by imposing zeros at the same position for G. As an example, consider a fourth order T–S model where $x_1(t)$ and $x_3(t)$ are the premise variables. Then $g = (g_1, 0, g_3, 0)$ and matrices $W_g(\mu)$ and G must have the following structures

$$W_g(\mu) = \begin{bmatrix} d_{11_{g_1}}(\mu_1) & p_{12} & p_{13} & p_{14} \\ p_{12} & d_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & d_{33_{g_3}}(\mu_3) & p_{34} \\ p_{14} & p_{24} & p_{34} & d_{44} \end{bmatrix}$$

and

$$G = \begin{bmatrix} g_{11} & 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & g_{24} \\ 0 & 0 & g_{33} & 0 \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}.$$

In order to generate a wider class of state feedback controllers as input parameters for Theorem 2, in the sense of improving the chances of finding stabilizing static output feedback controllers with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance at the second stage, the constraints on matrices $W_g(\mu)$ and G can be dropped, as presented in the following corollary. Although the corollary does not guarantee the stability of the closed-loop T–S system for the obtained state feedback controller, if the conditions of Theorem 2 and 3 hold, both the state feedback and the output controllers ensure the stability of the T–S fuzzy system (2) and $\mathcal{H}_\infty$ and $\mathcal{H}_2$ attenuation bounds for the closed-loop system, respectively.

Corollary 1 *Let $\beta > 0$ be a given scalar. If there exist a symmetric positive definite matrix $W_g(\mu) \in \mathbb{R}^{n\times n}$, a matrix $G \in \mathbb{R}^{n\times n}$ and a matrix $Z_s(\mu) \in \mathbb{R}^{m\times n}$ such that, for all $\mu \in \mathcal{U}$, the LMIs (11) are verified, then $K_s(\mu)$ given by (13) is a state feedback controller for the T–S fuzzy system (2), that guarantees the closed-loop T–S system dynamic matrix, for fixed values of the premise variables, to be Hurwitz, i.e., to have eigenvalues with negative real parts for all $\mu \in \mathcal{U}$.*

Note that, for time-varying systems, the fact that the closed-loop dynamic matrix is Hurwitz for all $\mu \in \mathcal{U}$ does not imply stability. Therefore, there is no guarantee of stabilizing properties for the state feedback controller obtained from Corollary 1 for the closed T–S fuzzy system. On the other hand, the next theorem can provide a stabilizing static output feedback controller with a guaranteed $\mathcal{H}_\infty$ attenuation bound using, as an input, the gain provided by Theorem 1 or Corollary 1.

Theorem 2 *Let $K_s(\mu) \in \mathbb{R}^{m\times n}$ be a given polynomial matrix. If there exist a symmetric positive definite matrix $P_g(\mu)$, with structure (9), matrices $S_q(\mu)$, $G_q(\mu)$, $Q_q(\mu)$, $H_v(\mu)$ and $J_v(\mu)$ with appropriate dimensions, and a scalar $\gamma > 0$ such that condition (17) (below) holds for all $\mu \in \mathcal{U}$,*

$$\begin{bmatrix} \overline{A}(\mu)'S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star & \star & \star & \star \\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)' & \star & \star & \star \\ E(\mu)'S_q(\mu)' & E(\mu)'G_q(\mu)' & -\gamma^2 I & \star & \star \\ Q_q(\mu)'(C(\mu) + D(\mu)K_s(\mu)) & 0 & Q_q(\mu)'F(\mu) & I - Q_q(\mu) - Q_q(\mu)' & \star \\ B(\mu)'S_q(\mu)' + J_v(\mu)C_y(\mu) - H_v(\mu)K_s(\mu) & B(\mu)'G_q(\mu)' & J_v(\mu)F_y(\mu) & D(\mu)'Q_q(\mu) & -H_v(\mu) - H_v(\mu)' \end{bmatrix} < 0 \tag{17}$$

[5] Note that G^{-1} is computed in terms of the adjoint matrix of G.

with $\overline{A}(\mu)$ *given by* (15)*, then*

$$L(\mu) = H_v(\mu)^{-1} J_v(\mu) \quad (18)$$

is a stabilizing static output feedback fuzzy controller for the T–S fuzzy system (2) *with* $\mathcal{H}_\infty$ *attenuation bound given by* γ.

Proof The LMI (17) can be rewritten as the condition (i) of Lemma 1, that is,

$$\Omega(\mu) = \Phi + \mathcal{V}\mathcal{X}\mathcal{U} + (\mathcal{V}\mathcal{X}\mathcal{U})' < 0,$$

with $\mathcal{X} = H(\mu)$ and

$$\mathcal{U} = \begin{bmatrix} Y(\mu) & 0 & \overline{Y}(\mu) & 0 & -I \end{bmatrix},$$

$$\mathcal{V}' = \begin{bmatrix} 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where

$$Y(\mu) = H_v(\mu)^{-1} J_v(\mu) C_y(\mu) - K_s(\mu) \quad (19)$$

$$\overline{Y}(\mu) = H_v(\mu)^{-1} J_v(\mu) F_y(\mu) \quad (20)$$

and

$$\Phi = \begin{bmatrix} \overline{A}(\mu)' S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star & \star & \star & \star \\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)' & \star & \star & \star \\ E(\mu)' S_q(\mu)' & E(\mu)' G_q(\mu)' & -\gamma^2 I & \star & \star \\ Q_q(\mu)'\overline{C}(\mu) & 0 & Q_q(\mu)' F(\mu) & I - Q_q(\mu) - Q_q(\mu)' & \star \\ B(\mu)' S_q(\mu)' & B(\mu)' G_q(\mu)' & 0 & D(\mu)' Q_q(\mu) & 0 \end{bmatrix}$$

with $\overline{A}(\mu)$ as in (15) and

$$\overline{C}(\mu) = C(\mu) + D(\mu) K_s(\mu). \quad (21)$$

Defining $\mathcal{N}_v$ and $\mathcal{N}_v$ as

$$\mathcal{N}_v = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{N}_u = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ Y(\mu) & 0 & \overline{Y}(\mu) & 0 \end{bmatrix},$$

such that $\mathcal{N}_v \mathcal{V} = 0$ and $\mathcal{N}_u' \mathcal{U}' = 0$, the inequalities of condition (ii) of Lemma 1 yield

$$\mathcal{N}_v \Phi \mathcal{N}_v' = \begin{bmatrix} \overline{A}(\mu)' S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star \\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)' \end{bmatrix} < 0 \quad (22)$$

and

$$\begin{bmatrix} S_q(\mu) A_{cl}(\mu) + A_{cl}(\mu)' S_q(\mu)' & \Delta(\mu) & S_q(\mu) E_{cl}(\mu) & C_{cl}(\mu)' Q_q(\mu) \\ \star & -G_q(\mu) - G_q(\mu)' & G_q(\mu) E_{cl}(\mu) & 0 \\ \star & \star & -\gamma^2 I & F_{cl}(\mu)' Q_q(\mu) \\ \star & \star & \star & -Q_q(\mu)' Q_q(\mu) \end{bmatrix} \leq \mathcal{N}_u' \Phi \mathcal{N}_u < 0 \quad (23)$$

with

$$\Delta(\mu) \triangleq P_g(\mu) - S_q(\mu) + A_{cl}(\mu)' G_q(\mu)' \quad (24)$$

and $A_{cl}(\mu)$, $C_{cl}(\mu)$, $E_{cl}(\mu)$ and $F_{cl}(\mu)$ given by (6). Note that

$$(I - Q_q(\mu))'(I - Q_q(\mu)) \geq 0 \Rightarrow$$
$$-Q_q(\mu)' Q_q(\mu) \leq I - Q_q(\mu) - Q_q(\mu)'. \quad (25)$$

The LMI (22) is a stability condition for $A(\mu) + B(\mu) K_s(\mu)$, ensuring that $K_s(\mu)$ is a stabilizing controller for the T–S system (2).

The multiplication of (23) on the right by T_3 and on the left by T_3', with

$$T_3 = \begin{bmatrix} I & 0 & 0 \\ A_{cl}(\mu) & E_{cl}(\mu) & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_q(\mu)^{-1} \end{bmatrix},$$

yields

$$\begin{bmatrix} A_{cl}(\mu)' P_g(\mu) + P_g(\mu) A_{cl}(\mu) & \star & \star \\ E_{cl}(\mu)' P_g(\mu) & -\gamma^2 I & \star \\ C_{cl}(\mu) & F_{cl}(\mu) & -I \end{bmatrix} < 0,$$

that is, the *bounded real lemma* (Boyd et al. 1994) with $P_g(\mu)$ as in (9), implying that $\dot{V}(x) + y'y - \gamma^2 w'w < 0$. Thus, the static output feedback controller given in (18) stabilizes the T–S fuzzy system (2) and provides an $\mathcal{H}_\infty$ attenuation bound given by γ. □

The next theorem provides sufficient conditions for the existence of a static output feedback controller with a guaranteed $\mathcal{H}_2$ attenuation bound. In $\mathcal{H}_2$ control, $F(\mu) = 0$ and $F_y(\mu) = 0$ are considered in the T–S fuzzy system (2).

Theorem 3 *Let* $K_s(\mu) \in \mathbb{R}^{m \times n}$ *be a given polynomial matrix. If there exist a symmetric positive definite matrix* $P_g(\mu)$*, with structure* (9)*, matrices* $X_q(\mu) = X_q(\mu)'$, $S_q(\mu)$, $G_q(\mu)$, $Q_q(\mu)$, $H_v(\mu)$ *and* $J_v(\mu)$ *with appropriate dimensions, and a scalar* $\eta > 0$ *such that the conditions*

$$\min_{\eta} Tr(X_q(\mu)) < \eta^2 \quad (26)$$

$$E(\mu)' P_g(\mu) E(\mu) - X_q(\mu) < 0 \quad (27)$$

and condition (28) *(top of next page) hold for all* $\mu \in \mathcal{U}$,

$$\begin{bmatrix} \overline{A}(\mu)'S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star & \star & \star \\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)' & \star & \star \\ Q_q(\mu)'(C(\mu) + D(\mu)K_s(\mu)) & 0 & I - Q_q(\mu) - Q_q(\mu)' & \star \\ B(\mu)'S_q(\mu)' + J_v(\mu)C_y(\mu) - H_v(\mu)K_s(\mu) & B(\mu)'G_q(\mu)' & D(\mu)'Q_q(\mu) & -H_v(\mu) - H_v(\mu)' \end{bmatrix} < 0 \quad (28)$$

with $\overline{A}(\mu)$ given by (15), then (18) is a stabilizing static output feedback fuzzy controller for the T–S fuzzy system (2), assuring an $\mathcal{H}_2$ attenuation bound given by η.

Proof Following the same steps as in the proof of Theorem 2, LMI (28) can be rewritten as condition (i) of Lemma 1, that is,

$$\Omega(\mu) = \Phi + \mathcal{V}\mathcal{X}\mathcal{U} + (\mathcal{V}\mathcal{X}\mathcal{U})' < 0,$$

with $\mathcal{X} = H(\mu)$ and

$$\mathcal{U} = \begin{bmatrix} Y(\mu) \; 0 \; 0 \; -I \end{bmatrix},$$
$$\mathcal{V}' = \begin{bmatrix} 0 \; 0 \; 0 \; I \end{bmatrix},$$

where $Y(\mu)$ as in (19) and

$$\Phi = \begin{bmatrix} \overline{A}(\mu)'S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star & \star & \star \\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)' & \star & \star \\ Q_q(\mu)'\overline{C}(\mu) & 0 & I - Q_q(\mu) - Q_q(\mu)' & \star \\ B(\mu)'S_q(\mu)' & B(\mu)'G_q(\mu)' & D(\mu)'Q_q(\mu) & 0 \end{bmatrix}$$

with $\overline{A}(\mu)$ as in (15) and $\overline{C}(\mu)$ as in (21).

Defining $\mathcal{N}_v$ and $\mathcal{N}_v$ as

$$\mathcal{N}_v = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \; \mathcal{N}_u = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ Y(\mu) & 0 & 0 \end{bmatrix},$$

such that $\mathcal{N}_v\mathcal{V} = 0$ and $\mathcal{N}_u'\mathcal{U}' = 0$, the inequalities of condition (ii) of Lemma 1 yield (22) and

$$\begin{bmatrix} S_q(\mu)A_{cl}(\mu) + A_{cl}(\mu)'S_q(\mu)' & \Delta(\mu) & C_{cl}(\mu)'Q_q(\mu) \\ \star & -G_q(\mu) - G_q(\mu)' & 0 \\ \star & \star & -Q_q(\mu)'Q_q(\mu) \end{bmatrix} \leq \mathcal{N}_u'\Phi\mathcal{N}_u < 0, \quad (29)$$

with $\Delta(\mu)$ as in (24), and $A_{cl}(\mu)$, $C_{cl}(\mu)$, $E_{cl}(\mu)$ and $F_{cl}(\mu)$ given by (6), also considering (25).

The multiplication of (29) on the right by T_3 and on the left by T_3', with

$$T_3 = \begin{bmatrix} I & 0 \\ A_{cl}(\mu) & 0 \\ 0 & Q_q(\mu)^{-1} \end{bmatrix},$$

yields

$$\begin{bmatrix} A_{cl}(\mu)'P_g(\mu) + P_g(\mu)A_{cl}(\mu) & C_{cl}(\mu)' \\ C_{cl}(\mu) & -I \end{bmatrix} < 0$$

which, by Schur complement (Boyd et al. 1994), is equivalent to

$$A_{cl}(\mu)'P_g(\mu) + P_g(\mu)A_{cl}(\mu) + C_{cl}(\mu)'C_{cl}(\mu) < 0. \quad (30)$$

Therefore, LMIs (26), (27) and (30) are well-known conditions (Boyd et al. 1994) that provide an $\mathcal{H}_2$ attenuation bound given by η for the closed-loop system (5) with a static output feedback gain given by (18). □

In the conditions presented in Theorems 1, 2 and 3 and Corollary 1, the matrices that compose the control law are dissociated from the Lyapunov matrix. Therefore, structural constraints can be imposed independently. Moreover, any state feedback gain could be used as input data in the second stage conditions, even with more complex structures than (13). In the conditions of the first step, the scalar β can assume any positive value, constituting an extra degree of freedom to be explored. A heuristic search can be done or a given set of values for β can be tested with the aim of obtaining the smallest values for the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ attenuation bounds in the second step.

Remark 3 Choosing $C_y(\mu) = I$, Theorems 2 and 3 can also provide stabilizing state feedback controllers for the T–S fuzzy system (2) with $\mathcal{H}_\infty$ and $\mathcal{H}_2$ attenuation bounds, respectively. This strategy has been explored in Tognetti et al. (2011a), where less conservative results than Rhee and Won (2006), and Mozelli et al. (2009a, 2010) are presented.

4 Implementation Issues and LMI Relaxations

The proposed LMIs depend on the membership functions, represented in terms of multi-simplexes, and cannot be treated numerically since they are of infinite dimension. To construct numerical tractable conditions it is necessary to represent the homogeneous polynomials in terms of their coefficients and to homogenize the polynomial matrices in the LMIs to the same degree in each simplex.

In the following, some notations of multi-simplex polynomial matrices, necessary to construct numerical tractable conditions, are introduced. For more details, see Tognetti et al. (2011d).

For $n \in \mathbb{N}$, $r \in \mathbb{N}^n$ and $g = (g_1, \dots, g_n) \in \mathbb{N}^n$, let $\mathcal{K}_{r_i}(g_i)$, $i = 1, \dots, n$, be the set of r_i-tuples obtained from all possible combinations of r_i nonnegative integers with sum g_i and $\mathbf{K}_r(g)$ defined as the Cartesian product of $\mathcal{K}_r(g_i)$, $i = 1, \dots, n$, that is

$$\mathcal{K}_{r_i}(g_i) = \left\{k_i = (k_{i1}, k_{i2}, \dots, k_{ir_i}) \in \mathbb{N}^{r_i} : \sum_{j=1}^{r_i} k_{ij} = g_i\right\}$$

$$\mathbf{K}_r(g) = \mathcal{K}_{r_1}(g_1) \times \cdots \times \mathcal{K}_{r_n}(g_n).$$

A homogeneous polynomial matrix $M_g(\mu)$ of partial degrees $g = (g_1, g_2, \dots, g_n)$ can be generically represented by

$$M_g(\mu) \triangleq \sum_{k \in \mathbf{K}_r(g)} \mu^k M_k = \sum_{k_1 \in \mathcal{K}_{r_1}(g_1)} \cdots \sum_{k_n \in \mathcal{K}_{r_n}(g_n)} \mu_1^{k_1} \cdots \mu_n^{k_n} M_{k_1 \cdots k_n}, \quad (31)$$

where μ^k are homogeneous monomials of degree g_i in each variable μ_i, i.e.,

$$\mu^k = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n}, \quad \mu_i^{k_i} = \mu_{i1}{}^{k_{i1}} \mu_{i2}{}^{k_{i2}} \cdots \mu_{ir_i}{}^{k_{ir_i}}, \quad (32)$$

$k_i = (k_{i1}, k_{i2}, \dots, k_{ir_i})$ is such that $k_{i1} + k_{i2} + \cdots + k_{ir_i} = g_i$ and M_k are the corresponding matrix-valued coefficients. For instance, a homogeneous polynomial matrix $M_g(\mu)$ of partial degrees $g = (1, 2)$ and $r = (2, 2)$ (polynomial dependence of degree one on $\mu_1 \in \mathcal{U}_2$ and polynomial dependence of degree two on $\mu_2 \in \mathcal{U}_2$), yields $\mathbf{K}_r(g) = \mathcal{K}_2(1) \times \mathcal{K}_2(2) = \{(0, 1), (1, 0)\} \times \{(0, 2), (1, 1), (2, 0)\}$, corresponding to the following matrix-valued polynomial:

$$\begin{aligned} M_g(\mu) = \mu_{11}\Big(&\mu_{21}{}^2 M_{((1,0),(2,0))} + \mu_{21}\mu_{22} M_{((1,0),(1,1))} \\ &+ \mu_{22}{}^2 M_{((1,0),(0,2))}\Big) + \mu_{12}\Big(\mu_{21}{}^2 M_{((0,1),(2,0))} \\ &+ \mu_{21}\mu_{22} M_{((0,1),(1,1))} + \mu_{22}{}^2 M_{((0,1),(0,2))}\Big). \end{aligned} \quad (33)$$

More details about the construction of finite dimensional LMIs from homogeneous polynomials of arbitrary degree represented in the multi-simplex can be found in the Appendix, in Oliveira et al. (2008) and Tognetti et al. (2011d). The codes developed for Theorems 1 and 2 and Corollary 1 are available for download in http://www.lara.unb.br/~eduardo/softwares/sba12.zip.

The conditions presented are only sufficient, but increasingly precise results can be obtained as the degrees g, associated to the Lyapunov function, q, to the slack variables and s, to the first step controller, increase, at the price of a higher computational effort. On the other hand, the choices of v, i.e., the degree of the output control gain, depend on the design purposes. Although the structure (9) assumes that all the states are premise variables (as in the fuzzy rule (1)), some of the states can be discarded from being premise variables by imposing the respective diagonal terms in (9) as constants (degree $g_i = 0$), as in Mozelli et al. (2009a, Example 4) and Rhee and Won (2006, Example 2). A constant gain (not depending on any premise variables) can be obtained by selecting $v = (0, \dots, 0)$. A controller that depends only on a specific premise variable is constructed by choosing a nonzero corresponding degree v_i.

Remark 4 An iterative procedure to obtain smaller values of $\mathcal{H}_\infty$ attenuation bounds γ can be constructed in the following way. For each iteration k, the input matrix

$$K_{v+1}(\mu)^{(k)} = L(\mu)^{(k-1)} C_y(\mu), \quad (34)$$

where $L(\mu)^{(k-1)}$ is obtained from (18) at iteration $k - 1$, is used as new input data for Theorem 2. The iterations stop when the obtained $\mathcal{H}_\infty$ attenuation bound $\gamma^{(k)}$ of the closed-loop system

$$\begin{cases} \dot{x}(t) = A^{(k)}(\mu)x(t) + E^{(k)}(\mu)w(t) \\ z(t) = C^{(k)}(\mu)x(t) + F^{(k)}(\mu)w(t), \end{cases} \quad (35)$$

with

$$\begin{aligned} A^{(k)}(\mu) &= A(\mu) + B(\mu)L(\mu)^{(k-1)} C_y(\mu) \\ E^{(k)}(\mu) &= E(\mu) + B(\mu)L(\mu)^{(k-1)} F_y(\mu) \\ C^{(k)}(\mu) &= C(\mu) + D(\mu)L(\mu)^{(k-1)} C_y(\mu) \\ F^{(k)}(\mu) &= F(\mu) + D(\mu)L(\mu)^{(k-1)} F_y(\mu), \end{aligned}$$

is such that $|\gamma^{(k-1)} - \gamma^{(k)}| < \varepsilon$, where ε is a tolerance defined *a priori*. Numerical experiments have showed that $\gamma^{(k)} \leq \gamma^{(k-1)}$, that is, smaller values of the $\mathcal{H}_\infty$ attenuation bound can be obtained at the price of increasing the computational burden. Note that the iterative procedure can also be applied in Theorem 3 to reduce the $\mathcal{H}_2$ bounds.

5 Numerical Examples

The numerical complexity associated with an optimization problem based on LMIs can be estimated from the number V of scalar variables and the number L of LMI rows. The results were obtained using YALMIP (Löfberg 2004) and SeDuMi (Sturm 1999) within the MATLAB 7.4.0 environment running on a personal computer with a 3.00 GHz Intel Core 2 Duo processor and 2.00 GB of RAM running Windows XP SP3.

In the following examples, the scalar β of Theorem 1 and Corollary 1 has been chosen from the set $\{1, 0.5, 0.1, 0.05, 0.04, 0.001, 10^{-5}\}$. The best result (in terms of smallest $\mathcal{H}_\infty$ or $\mathcal{H}_2$ bounds) has been retained.

Example 1 Consider the T–S fuzzy system given by (2), adapted from Guelton et al. (2009), with the following matrices

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$
$$B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \ E_1 = E_2 = \begin{bmatrix} 0 \\ -0.25 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 2 & -10 \\ 5 & -1 \end{bmatrix}, \ C_2 = \begin{bmatrix} -3 & 20 \\ -7 & -2 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \ D_2 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix},$$
$$F_1' = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}, \ F_2' = \begin{bmatrix} 0.35 & 0.5 \end{bmatrix}$$
$$C_{y_1} = C_{y_2} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ F_{y_1} = F_{y_2} = 0.1,$$

where the premise variable is $x_1(t)$.

Table 1 shows the $\mathcal{H}_\infty$ attenuation bounds obtained with Corollary 1 and Theorem 2 for several values of degrees of the Lyapunov function (g), slack variables (q), the first stage state feedback controller (s) and the static output controller (v). Observe that the values of the $\mathcal{H}_\infty$ attenuation bound decrease as the degrees increase, requiring higher computational efforts, that still remains acceptable. The first two values of γ in Table 1 show that the use of a non-constant Lyapunov function is essential to decrease the $\mathcal{H}_\infty$ attenuation bound and the last two results in the same table illustrate that the presented technique allows to reduce conservatism, keeping the static output controller simple to implement (degree $v = 1$), concomitantly.

Adapting the system to allow a comparison with the conditions proposed in Bouarar et al. (2009) (considering $y(t) = z(t)$, that is, $C_{y_i} = C_i$, $F_{y_i} = F_i$ and $D_i' = [0 \ \ 0]$, $i = 1, 2$), Table 2 shows the values of γ obtained with the conditions of Theorem 2 in Bouarar et al. (2009) (BGM09), that take into account a bound ϕ of the time-derivative of the membership functions and with the conditions of Corollary 1 and Theorem 2 for different degrees. Observe that the approach proposed in Bouarar et al. (2009) can assure $\gamma = 0.71$ only for very slow rates of variation $\dot{\mu}_{11}(x_1)$ (the variation rate of the membership function $\mu_{11}(x_1)$ of the T–S system) whereas the proposed conditions allow arbitrary variation of the membership functions. The conditions of Theorem 1 in Kau et al. (2007), that use a constant Lyapunov matrix, were not able to find a solution (considering also $y(t) = z(t)$, that is, $C_{y_i} = C_i$, $F_{y_i} = F_i = [0 \ \ 0]$ and $D_i' = [0 \ \ 0]$, $i = 1, 2$).

Table 1 $\mathcal{H}_\infty$ attenuation bounds for the system of Example 1 obtained with Corollary 1 and Theorem 2 (C1–T2) for different values of (g, q, s, v) (denoted as C1–T2$_{(g,q,s,v)}$) and β in Corollary 1. V is the number of scalar variables and L of LMI rows

Method	γ	β	L	V	Time (s)
C1–T2$_{(0,1,1,1)}$	3.34	0.1	48	43	0.28
C1–T2$_{(1,1,1,1)}$	0.98	1.0	52	47	0.31
C1–T2$_{(6,6,6,6)}$	0.92	0.04	172	147	0.68
C1–T2$_{(6,6,6,1)}$	0.97	0.05	172	137	0.60

Table 2 $\mathcal{H}_\infty$ attenuation bounds for the adapted system (considering $y(t) = z(t)$) of Example 1 obtained with Corollary 1 and Theorem 2 (C1–T2) for different values of (g, q, s, v) (denoted as C1–T2$_{(g,q,s,v)}$) and with BGM09 for several values of the bound ϕ of the time-derivative $\dot{\mu}_{11}(x_1)$ of the membership function $\mu_{11}(x_1)$

C1–T2			BGM09	
Degree	γ	β	γ	ϕ
(1, 1, 1, 1)	0.89	0.5	3.67	2.5
(2, 2, 2, 2)	0.87	1.0	1.01	1.5
(6, 6, 6, 6)	0.77	0.05	0.85	1.0
(10, 10, 10, 10)	0.71	0.05	0.71	0.1

It can also be observed that the iterative procedure described in Remark 4 is able to further improve the results. Applying Corollary 1 and Theorem 2 with $(g, q, s, v) = (1, 1, 0, 0)$ (constant state and static output controller) and $\beta = 0.05$ in Corollary 1, $\gamma = 3.69$ is obtained at the first iteration, $\gamma = 1.21$ at the second and $\gamma = 1.19$ at the third. It is also worth to say that, as far as the authors know, no condition in the T–S fuzzy literature can deal effectively with noise in the measurement output ($F_y(\mu) \neq 0$) in the static output feedback problem for continuous-time systems.

Example 2 Consider a flexible joint-inverted pendulum device (Liu et al. 2005) described by the dynamic equation

$$\begin{aligned} I_1\ddot{\theta}_1(t) + I_2\ddot{\theta}_2(t) &= mgl\sin\theta_2(t) + u(t) - \alpha w(t) \\ I_2\ddot{\theta}_2(t) &= \beta_d(\dot{\theta}_2(t) - \dot{\theta}_1(t)) - \beta_s(\theta_2(t) - \theta_1(t)) \\ &\quad + mgl\sin\theta_2(t), \end{aligned} \tag{36}$$

where $\theta_1(t)$ and $\theta_2(t)$ denote the angle (rad) of the pendulum and of the rotor from the vertical, $u(t)$ is the control torque (Nm), $w(t)$ is the disturbance torque (Nm), I_1 and I_2 are the moment of inertia (Kg m^2) of the rotor and of the pendulum, m is the mass (Kg) of the pendulum, l is the length (m) from the center of mass of the pendulum round its center of mass, and $g = 9.8\,\text{m/s}^2$ is the gravitational acceleration constant. Suppose the shaft is not rigid, but is modeled as a parallel combination of a linear torsion spring with constant $\beta_s > 0$ and a linear torsion damper with coefficient $\beta_d > 0$. In the numerical simulations, $m = 1\,\text{Kg}$, $l = 1\,\text{m}$, $\beta_s = 2\,\text{Nm}$, $\beta_d = 3\,\text{Nms}$ and $\alpha = 0.5$ were used.

Let $x_1(t) = \theta_2(t)$, $x_2(t) = \dot{\theta}_2(t)$, $x_3(t) = \theta_2(t) - \theta_1(t)$, $x_4(t) = \dot{\theta}_2(t) - \dot{\theta}_1(t)$ and assume $-\pi < x_1(t) < \pi$. Then, the nonlinear system (36) is exactly represented by the following T–S fuzzy model

$$\mathcal{R}_i: \text{ If } x_1(t) \text{ is } M_1^i \text{ then}$$
$$\begin{cases} \dot{x} = A_i x + E_i w + B_i u \\ z = C_i x + F_i w + D_i u \\ y = C_{y_i} x \end{cases} \tag{37}$$

for $i = 1, 2$, with membership functions

$$\mu_{11}(x_1) = \begin{cases} \frac{\sin x_1(t)}{x_1(t)}, & x_1(t) \neq 0 \\ 1, & x_1(t) = 0 \end{cases}$$
$$\mu_{12}(x_1) = 1 - \mu_{11}(x_1).$$

and matrices (as in Chen et al. (2012))

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1.96 & 0 & -0.4 & -0.6 \\ 0 & 0 & 0 & 1 \\ 1.96 & 0 & -2.4 & -3.6 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -0.4 & -0.6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2.4 & -3.6 \end{bmatrix}, \quad E_1 = E_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix},$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix},$$

$$C_{y_1} = C_{y_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad F_1 = F_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The static output controller (4) of degree $v_1 = 1$, composed by the matrices

$$L_1 = \begin{bmatrix} -16.0 & -17.5 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -16.0 & -18.0 \end{bmatrix},$$

has been obtained with Corollary 1 ($\beta = 0.1$) and Theorem 2 with partial degrees $g_1 = s_1 = q_1 = 1$. The $\mathcal{H}_\infty$ attenuation bound is $\gamma = 0.17$. The controller has been implemented and simulations of the closed-loop nonlinear system have been performed. The states of the closed-loop system, the output and control input signal are shown, respectively, in Figs. 1 and 2 for the initial condition $x(0) = [1\,0\,0\,0]'$ and no disturbance ($w(t) = 0$). As expected, the trajectories converge to the origin. The controlled output of the closed-loop system with a static output controller and a state feedback controller (obtained with Corollary 1 and Theorem 2, considering the same values of partial degrees, β and $C_y(\mu) = I$) are showed in Fig. 3, for the disturbance signal $w(t) = e^{-t/3}\sin(\pi t)$, illustrated in Fig. 4. It can be observed that the attenuation level for the static output feedback controller is very close to the one obtained with the state feedback controller. Actually, both controllers provide practically the same upper bound γ for the $\mathcal{H}_\infty$ norm, illustrating the effectiveness of the static output feedback controller obtained with the proposed approach.

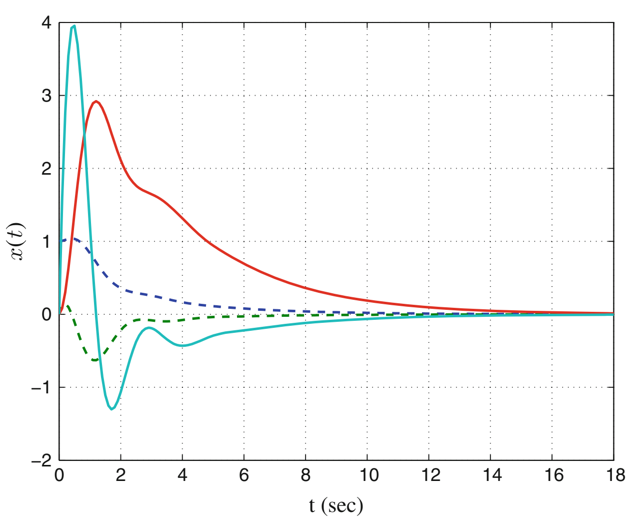

Fig. 1 State trajectories of the closed-loop nonlinear system (36) with static output feedback controller and initial condition $x(0) = [1\,0\,0\,0]'$ and $w(t) = 0$. The measured output signals ($y(t)$) are indicated by *dashed lines*

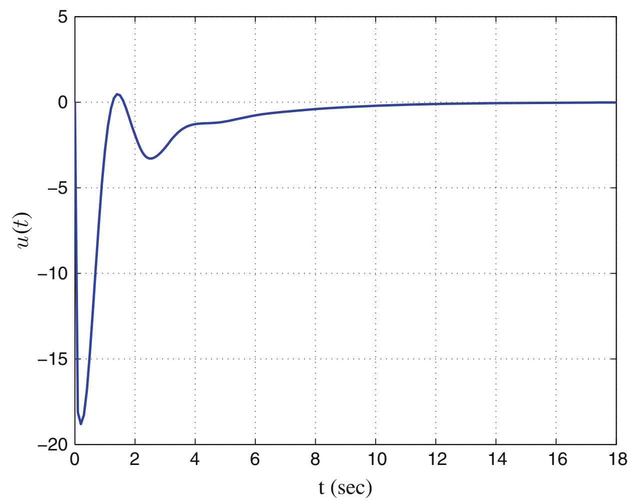

Fig. 2 Control signal generated by the static output feedback controller for the nonlinear system of (36) with initial condition $x(0) = [1\,0\,0\,0]'$ and $w(t) = 0$

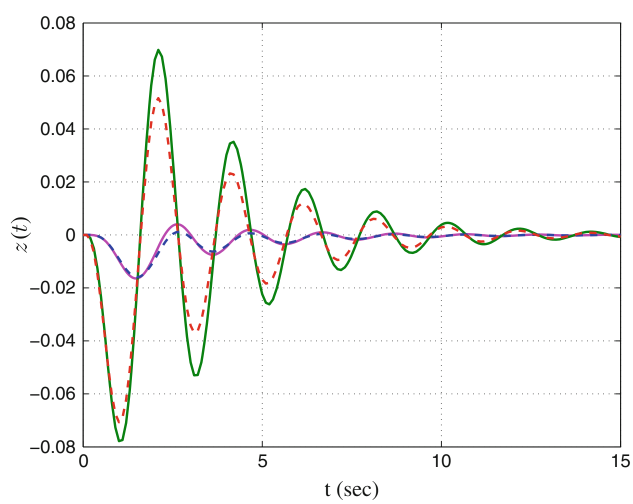

Fig. 3 Controlled output signals for the nonlinear system of (36) with static output (*solid*) and state (*dashed*) feedback controllers for the initial condition $x(0) = [0\,0\,0\,0]'$ and disturbance $w(t) = e^{-t/3}\sin(\pi t)$

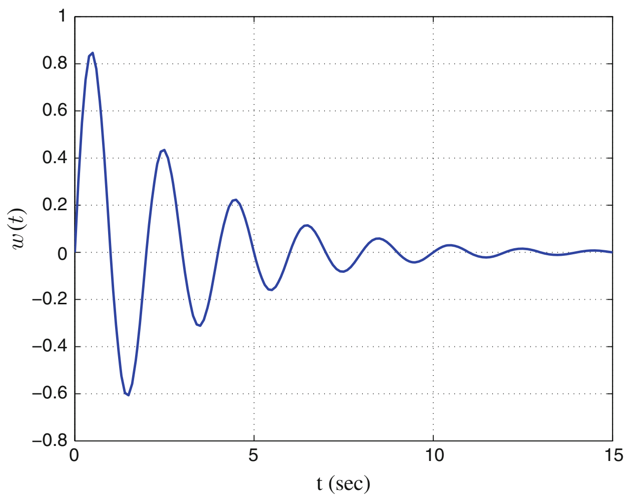

Fig. 4 Disturbance signal $w(t) = e^{-t/3}\sin(\pi t)$ applied to the nonlinear system (36)

6 Conclusion

This paper has proposed parameter-dependent LMI conditions for the synthesis of $\mathcal{H}_\infty$ and $\mathcal{H}_2$ static output feedback controllers for continuous-time T–S fuzzy systems. The method combines a line-integral Lyapunov function and slack variables represented by homogeneous polynomial matrices of arbitrary degree in a two-step approach. Thanks to the multi-simplex representation, the designer has the flexibility to choose only the available premise variables for the control law. The methodology can also be extended to cope with the dynamic output feedback control problem, as well as to deal with the discrete-time case, in the context of T–S fuzzy systems.

Acknowledgments This work was partially supported by the Brazilian agencies FAPESP, CAPES, and CNPq. The authors wish to thank the reviewers and the associate editor for the suggestions and remarks that helped to improve the paper.

7 Appendix

This Appendix illustrates how to construct numerical tractable conditions from infinite dimension LMIs represented by homogeneous polynomials in multi-simplexes. For that, as stated in Sect. 4, the homogeneous polynomials are given in terms of their coefficients.

As a simple example, consider matrix $M_g(\mu)$ given in (33). The inequality $M_g(\mu) > 0$, for all $\mu \in \mathcal{U}$, holds if all terms of (33) are positive definite, that is,

$$M_{((1,0),(2,0))} > 0,\quad M_{((1,0),(1,1))} > 0,\quad M_{((1,0),(0,2))} > 0,$$
$$M_{((0,1),(2,0))} > 0,\quad M_{((0,1),(1,1))} > 0,\quad M_{((0,1),(0,2))} > 0.$$

Using the notations introduced in Sect. 4, one can write

$$M_{k_1k_2} > 0,\quad \forall k_1 \in \mathcal{K}_2(1),\quad \forall k_2 \in \mathcal{K}_2(2)$$

or simply

$$M_k > 0,\quad \forall k \in \mathbf{K}_r(g) = \mathcal{K}_2(1)\times\mathcal{K}_2(2),\quad g=(1,2).$$

When other matrices and variables are involved, all terms of the parameter-dependent LMI matrix must be in the same degree to allow the construction of finite dimension LMIs. The advantage of handling multi-simplexes is that each simplex is homogenized to the same degree independently.

Before presenting details of how to construct numerical tractable conditions, from the multinomial theory (generalization of the binomial theorem to polynomials) define

$$U_g(\mu) \triangleq \prod_{i=1}^{n} \underbrace{\left(\sum_{j=1}^{r_i}\mu_{ij}\right)^{g_i}}_{\sum_{k_i\in\mathcal{K}_{r_i}(g_i)} \frac{g_i!}{k_i!}\mu_i^{k_i}} \triangleq \sum_{k\in\mathbf{K}_r(g)} \frac{g!}{k!}\mu^k. \tag{38}$$

As an example, considering LMI (22), the maximum degree of the polynomial matrix is given by $w = \max\{g, q+\sigma\}$, $\sigma = \mathtt{ones}(1,n)$. Therefore, all terms of (22) are homogenized to the same degree w, that is

$$\begin{aligned}&\begin{bmatrix}\overline{A}(\mu)'S_q(\mu)' + S_q(\mu)\overline{A}(\mu) & \star\\ P_g(\mu) - S_q(\mu)' + G_q(\mu)\overline{A}(\mu) & -G_q(\mu) - G_q(\mu)'\end{bmatrix}\\ &= \begin{bmatrix}\mathcal{M}_{11}(\mu) & \mathcal{M}_{12}(\mu)\\ \star & -\mathcal{M}_{22}(\mu)\end{bmatrix}\\ &= \sum_{k\in\mathbf{K}_r(w)}\mu^k\begin{bmatrix}\mathcal{M}_{11_k} & \mathcal{M}_{12_k}\\ \star & -\mathcal{M}_{22_k}\end{bmatrix} < 0\end{aligned} \tag{39}$$

where

$$\begin{aligned}\mathcal{M}_{11}(\mu) &= U_{w-q-\sigma}(\mu)(\overline{A}(\mu)'S_q(\mu)' + S_q(\mu)\overline{A}(\mu))\\ \mathcal{M}_{12}(\mu) &= U_{w-g}(\mu)P_g(\mu) - U_{w-q}(\mu)S_q(\mu)\\ &\quad + U_{w-q-\sigma}(\mu)G_q(\mu)\overline{A}(\mu)\\ \mathcal{M}_{22}(\mu) &= U_{w-q}(\mu)(G_q(\mu) + G_q(\mu)')\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_{11_k} = &\sum_{\substack{\tilde{k}\in\mathcal{K}_r(w-s-\sigma)\\ \tilde{k}\preceq k}}\ \sum_{\substack{\hat{k}\in\mathcal{K}_r(\sigma)\\ \tilde{k}+\hat{k}\preceq k}} \frac{(w-s-\sigma)!}{\tilde{k}!}\ \overline{A}'_{\hat{k}}S'_{k-\tilde{k}-\hat{k}}\\ &+ S_{k-\tilde{k}-\hat{k}}\overline{A}_{\hat{k}}\end{aligned}$$

$$\begin{aligned}\mathcal{M}_{12_k} = &\sum_{\substack{\tilde{k}\in\mathcal{K}_r(w-g)\\ \tilde{k}\preceq k}} \frac{(w-g)!}{\tilde{k}!}P_{k-\tilde{k}} - \sum_{\substack{\tilde{k}\in\mathcal{K}_r(w-q)\\ \tilde{k}\preceq k}} \frac{(w-q)!}{\tilde{k}!}S_{k-\tilde{k}}\\ &+ \sum_{\substack{\tilde{k}\in\mathcal{K}_r(w-s-\sigma)\\ \tilde{k}\preceq k}}\ \sum_{\substack{\hat{k}\in\mathcal{K}_r(\sigma)\\ \tilde{k}+\hat{k}\preceq k}} \frac{(w-s-\sigma)!}{\tilde{k}!}G_{k-\tilde{k}-\hat{k}}\overline{A}_{\hat{k}}\end{aligned}$$

$$\mathcal{M}_{22_k} = \sum_{\substack{\tilde{k}\in\mathcal{K}_r(w-q)\\ \tilde{k}\preceq k}} \frac{(w-q)!}{\tilde{k}!} G_{k-\tilde{k}} + G'_{k-\tilde{k}}.$$

As can be noted, the last term of (39) is written as a homogeneous polynomial with LMI coefficients. If all the coefficients are imposed to be negative definite (numerically tractable conditions), the feasibility of (22) is assured.

References

Andrea, C. Q., Pinto, J. O. P., Assunção, E., Teixeira, M. C. M., & Junior, L. G. (2008). Controle ótimo $\mathcal{H}_\infty$ de sistemas não-lineares com modelos fuzzy Takagi-Sugeno. *SBA: Controle & Automação*, *19*(3), 256–269.

Arrifano, N. S. D., Oliveira, V. A., & Cossi, L. V. (2003). Synthesis of an LMI-based fuzzy control system with guaranteed cost performance: A piecewise Lyapunov approach, *SBA: Controle & Automação*, *17*(2), 213–225.

Arzelier, D., Gryazina, E. N., Peaucelle, D. & Polyak, B. T. (2010). *Mixed LMI/Randomized methods for static output feedback control design: Proceedings of the 2010 American Control Conference* (pp. 4683–4688). Baltimore, MD, USA.

Arzelier, D., Peaucelle, D. & Salhi, S. (2003). *Robust static output feedback stabilization for polytopic uncertain systems: Improving the guaranteed performance bound: Proceedings of the 4th IFAC Symposium on Robust Control Design (ROCOND 2003)* (pp. 425–430). Milan, Italy.

Baranyi, P. (2004). TP model transformation as a way to LMI-based controller design. *IEEE Transactions on Industrial Electronics*, *51*(2), 387–400.

Bouarar, T., Guelton, K. & Manamanni, N. (2009). *Static output feedback controller design for Takagi-Sugeno systems—A fuzzy Lyapunov LMI approach: Proceedings of the 48th IEEE Conference on Decision and Control—28th Chinese Control Conference* (pp. 4150–4155) Shanghai, P. R. China

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics.

Chen, J., Sun, Y., Min, H., Sun, F., & Zhang, Y. (2012). New results on static output feedback $\mathcal{H}_\infty$ control for fuzzy singularly perturbed systems: A linear matrix inequality approach. *International Journal of Robust and Nonlinear Control*. doi:10.1002/rnc.2787.

Green, M., & Limebeer, D. J. N. (1995). *Linear Robust Control*. Englewood Cliffs, NJ: Prentice-Hall.

Guelton, K., Bouarar, T., & Manamanni, N. (2009). Robust dynamic output feedback fuzzy Lyapunov stabilization of Takagi–Sugeno systems - A descriptor redundancy approach. *Fuzzy Sets and Systems*, *160*(19), 2796–2811.

Guerra, T. M., Bernal, M., Guelton, K., & Labiod, S. (2012). Non-quadratic local stabilization for continuous-time Takagi–Sugeno models. *Fuzzy Sets and Systems*, *201*, 40–54.

Guerra, T. M., Kruszewski, A., Vermeiren, L., & Tirmant, H. (2006). Conditions of output stabilization for nonlinear models in the Takagi–Sugeno's form. *Fuzzy Sets and Systems*, *157*(17), 1248–1259.

Huang, D., & Nguang, S. K. (2007). Static output feedback controller design for fuzzy systems: An ILMI approach. *Information Sciences*, *177*(14), 3005–3015.

Kau, S.-W., Lee, H.-J., Yang, C.-M., Lee, C.-H., Hong, L., & Fang, C.-H. (2007). Robust $\mathcal{H}_\infty$ fuzzy static output feedback control of T-S fuzzy systems with parametric uncertainties. *Fuzzy Sets and Systems*, *158*(2), 135–146.

Khalil, H. K. (2002). *Nonlinear systems* (3rd ed.). Upper Saddle River, NJ: Prentice Hall.

Klug, M., & Castelan, E. B. (2011). *Redução de regras e compensação robusta para sistemas Takagi-Sugeno com utilização de modelos não lineares locais* (pp. 909–914). São João del-Rei, MG: Anais do X Congresso Brasileiro de Automação Inteligente.

Klug, M., Castelan, E. B. & Leite, V. J. S. (2011). *A dynamic compensator for parameter varying systems subject to actuator limitations applied to a T-S fuzzy system: Proceedings of the 18th IFAC World Congress* (pp. 14495–14500). Milano, Italy.

Lee, H. J., & Kim, D. W. (2009). Fuzzy static output feedback may be possible in LMI framework. *IEEE Transactions on Fuzzy Systems*, *17*(5), 1229–1230.

Liu, H., Sun, F., & Hu, Y. (2005). $\mathcal{H}_\infty$ control for fuzzy singularly perturbed systems. *Fuzzy Sets and Systems*, *155*(2), 272–291.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, Taipei, Taiwan, pp. 284–289. http://control.ee.ethz.ch/~joloef/yalmip.php.

Mansouri, B., Manamanni, N., Guelton, K., Kruszewski, A., & Guerra, T. M. (2009). Output feedback LMI tracking control conditions with $\mathcal{H}_\infty$ criterion for uncertain and disturbed T-S models. *Information Sciences*, *179*(4), 446–457.

Mehdi, D., Boukas, E. K., & Bachelier, O. (2004). Static output feedback design for uncertain linear discrete time systems. *IMA Journal of Mathematical Control and Information*, *21*(1), 1–13.

Montagner, V. F., Oliveira, R. C. L. F., & Peres, P. L. D. (2009). Convergent LMI relaxations for quadratic stabilizability and $\mathcal{H}_\infty$ control for Takagi-Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, *17*(4), 863–873.

Montagner, V. F., Oliveira, R. C. L. F., & Peres, P. L. D. (2010). Relaxações convexas de convergência garantida para o projeto de controladores para sistemas nebulosos de Takagi-Sugeno. *SBA: Controle & Automação*, *21*(1), 82–95.

Mozelli, L., de Avellar, G. S. C., & Palhares, R. M. (2010). Condições LMIs alternativas para sistemas Takagi-Sugeno via função de Lyapunov fuzzy. *SBA: Controle & Automação*, *21*(1), 96–107.

Mozelli, L. A., & Palhares, R. M. (2011). Stability analysis of Takagi-Sugeno fuzzy systems via LMI: Methodologies based on a new fuzzy Lyapunov function. *SBA: Controle & Automação*, *22*(6), 664–676.

Mozelli, L. A., Palhares, R. M., & Avellar, G. S. C. (2009a). A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems. *Information Sciences*, *179*(8), 1149–1162.

Mozelli, L. A., Palhares, R. M., Souza, F. O., & Mendes, E. M. A. M. (2009b). Reducing conservativeness in recent stability conditions of TS fuzzy systems. *Automatica*, *45*(6), 1580–1583.

Nguang, S. K., & Shi, P. (2006). Robust $\mathcal{H}_\infty$ output feedback control design for fuzzy dynamic systems with quadratic $\mathcal{D}$ stability constraints: An LMI approach. *Information Sciences*, *176*(15), 2161–2191.

Oliveira, R. C. L. F., Bliman, P.-A. & Peres, P. L. D. (2008). *Robust LMIs with parameters in multi-simplex: Existence of solutions and applications: Proceedings of the 47th IEEE Conference on Decision and Control* (pp. 2226–2231). Cancun, Mexico.

Peaucelle, D. & Arzelier, D. (2001). *An efficient numerical solution for $\mathcal{H}_2$ static output feedback synthesis: Proceedings of the 2001 European Control Conference* (pp. 3800–3805). Porto, Portugal.

Rhee, B.-J., & Won, S. (2006). A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design. *Fuzzy Sets and Systems*, *157*(9), 1211–1228.

Sala, A., & Ariño, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, *158*(24), 2671–2686.

Skelton, R. E., Iwasaki, T., & Grigoriadis, K. (1998). *A Unified Algebraic Approach to Linear Control Design*. Bristol, PA: Taylor & Francis.

Stoorvogel, A. A. (1993). The robust $\mathcal{H}_2$ control problem: a worst-case design. *IEEE Transactions on Automatic Control*, *38*(9), 1358–1370.

Sturm, J. F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software, 11*(1–4), 625–653.

Syrmos, V. L., Abdallah, C. T., Dorato, P., & Grigoriadis, K. (1997). Static output feedback—A survey. *Automatica*, *33*(2), 125–137.

Takaba, K. (1998). Robust $\mathcal{H}_2$ control of descriptor system with time-varying uncertainty. *International Journal of Control*, *71*(4), 559–579.

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, *SMC–15*(1), 116–132.

Tanaka, K., Hori, T. & Wang, H. O. (2001). *A fuzzy Lyapunov approach to fuzzy control system design: Proceedings of the 2001 American Control Conference* (pp. 4790–4795). Arlington, VA.

Tanaka, K., Hori, T., & Wang, H. O. (2003). A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, *11*(4), 582–589.

Tanaka, K., & Wang, H. (2001). *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York: John Wiley & Sons.

Teixeira, M. C. M., Assunção, E., & Avellar, R. G. (2003). On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *IEEE Transactions on Fuzzy Systems*, *11*(5), 613–623.

Teixeira, M. C. M., Pietrobom, H. C., & Assunção, E. (2000). Novos resultados sobre a estabilidade e controle de sistemas não-lineares utilizando modelos fuzzy e LMI. *SBA: Controle & Automação*, *11*(2), 37–48.

Teixeira, M. C. M., & Żak, S. H. (1999). Stabilizing controller design for uncertain nonlinear systems using fuzzy models. *IEEE Transactions on Fuzzy Systems*, *7*(2), 133–142.

Tognetti, E. S., Oliveira, R. C. L. F., & Peres, P. L. D. (2010a). *Controle seletivo com critério $\mathcal{H}_2$ de sistemas nebulosos Takagi-Sugeno* (pp. 4118–4125). Bonito, MS: Anais do XVIII Congresso Brasileiro de Automática.

Tognetti, E. S., Oliveira, R. C. L. F. & Peres, P. L. D. (2010b). *Selective stabilization of Takagi-Sugeno fuzzy systems: Proceedings of the 2010 IEEE International Conference on Fuzzy Systems* (pp. 2772–2779). Barcelona, Spain.

Tognetti, E. S., Oliveira, R. C. L. F. & Peres, P. L. D. (2011a). *Improved stabilization conditions for Takagi-Sugeno fuzzy systems via fuzzy integral Lyapunov functions: Proceedings of the 2011 American Control Conference* (pp. 4970–4975). San Francisco, CA.

Tognetti, E. S., Oliveira, R. C. L. F. & Peres, P. L. D. (2011b). *An LMI-based approach to static output feedback stabilization of T-S fuzzy systems: Proceedings of the 18th IFAC World Congress* (pp. 12593–12598). Milano, Italy.

Tognetti, E. S., Oliveira, R. C. L. F., & Peres, P. L. D. (2011c). *Relaxações LMIs para realimentação de saída $\mathcal{H}_\infty$ de sistemas nebulosos Takagi-Sugeno contínuos no tempo*. São João del-Rei, MG: Anais do X Congresso Brasileiro de Automação Inteligente.

Tognetti, E. S., Oliveira, R. C. L. F., & Peres, P. L. D. (2011d). Selective $\mathcal{H}_2$ and $\mathcal{H}_\infty$ stabilization of Takagi-Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, *19*(5), 890–900.

Tognetti, E. S., Oliveira, R. C. L. F., & Peres, P. L. D. (2012). Reduced-order dynamic output feedback control of continuous-time T-S fuzzy systems. *Fuzzy Sets and Systems*, *207*(16), 27–44.

Tognetti, E. S., & Oliveira, V. A. (2010). Fuzzy pole placement based on piecewise Lyapunov functions. *International Journal of Robust and Nonlinear Control*, *20*(5), 571–578.

Wang, H. O., Tanaka, K. & Griffin, M. F. (1995). *Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model: Proceedings of the 4th IEEE International Conference on Fuzzy Systems and The 2nd International Fuzzy Engineering Symposium* (pp. 531–538). Yokohama, Japan.