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New Algorithms for Solving the Split Equality Problems in Hilbert Spaces

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Abstract
We introduce a new approach by using unconstrained optimization to find a solution to the system of the split equality problems in real Hilbert spaces. Our new algorithms do not depend on the norm of the transfer mappings. We also give the relaxed iterative algorithms corresponding to the proposed algorithms. Finally, we present some numerical experiments to demonstrate the performance of the main results.

Keywords Hilbert space · Metric projection · Split equality problem

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1 Introduction

Let H_1, H_2 and H be three real Hilbert spaces, let $C \subseteq H_1$ and $Q \subseteq H_2$ be two nonempty, closed and convex sets, let $A : H_1 \to H$ and $B : H_2 \to H$ be two bounded linear operators. The *split equality problem* (SEP, for short) was first introduced and studied by Moudafi et al. in 2013 (see, e.g., [11, 12]). The SEP is stated as follows:

$$\text{Find } v \in C, w \in Q \text{ such that } A(v) = B(w).$$

The SEP links closely to many different important problems. For instance, in game theory, in decomposition methods for PDE's, in decision sciences and inertial Nash equilibration processes (see, e.g., [1, 2]), and the split feasibility problem which was later approached for inversion problems in intensity modulated radiation therapy (see, e.g., [4, 5]).

To find a solution to the SEP, in [11], Moudafi considered the constrained optimization problem:

$$\min_{v\in C, w\in Q} \frac{1}{2}\|Av - Bw\|_H^2.$$

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From observing and writing down the optimal conditions, he obtained the following fixed point formulation:

$$\begin{cases} v = P_C(v - \gamma A^*(Av - Bw)), \\ w = P_Q(w + \gamma B^*(Av - Bw)), \end{cases}$$

where A^* and B^* are the adjoint operators of A and B, respectively. This equation suggests the possibility of iterating, and thus he considered and established the alternating CQ-algorithm for solving the SEP, that is

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma B^*(Ax_n - By_n)). \end{cases} \tag{1.1}$$

Under some suitable conditions [11, Theorem 2.1], he proved that the iterative sequence generated by (1.1) converges weakly to a solution of the SEP.

Due to their tremendous utility and wide applicability, many algorithms have been set up to solve the SEP or its modified version in different forms. For more details, see, for instance, [6–10, 14–16, 19–22, 24–26] and the references therein.

Very recently, in [23], Tuyen introduced and studied the more general problem, which is said to be the system of split equality problems (SSEP, for short). Namely, suppose that

(D1) H_1, H_2 and H are three real Hilbert spaces; C_i and Q_i $(i = 1, 2, 3, \ldots, N)$ are nonempty closed convex subsets of H_1 and H_2, respectively.
(D2) $A_i : H_1 \to H$ and $B_i : H_2 \to H$ $(i = 1, 2, 3, \ldots, N)$ are bounded linear operators.
(D3) b_i $(i = 1, 2, 3, \ldots, N)$ are given elements in H.
(D4) $\Omega = \{(v, w) \in \cap_{i=1}^N (C_i \times Q_i) : A_i(v) - B_i(w) = b_i, i = 1, 2, 3, \ldots, N\} \neq \emptyset$.

The SSEP is stated as follows:

$$\text{Find an element } p_* \in \Omega.$$

Using the Tikhonov regularization method, he proposed implicit and explicit iterative algorithms [23, Theorems 3.1 and 3.5] for solving the Problem SSEP. But, in the first algorithm, we have to solve an implicit equation. For the second algorithm, one of the control parameters requires computing or at least estimating the Lipschitz constant and the norm of the objective operators. In general, they are not easy work to perform in practice.

We also note that if $A_i \equiv A$, $B_i \equiv B$ and $b_i = 0$ for all $i = 1, 2, 3, \ldots, N$, then the SSEP becomes the multiple-sets split equality problem (MSSEP, for short) which has been studied by Tian et al. in [22]. They also have established a weak convergence algorithm with the split self-adaptive step size for solving the MSSEP.

In this paper, motivated and inspired by the above works, we will focus on and establish several new algorithms for solving the Problem SSEP with another approach. To begin this, for each $x = (v, w) \in \mathbb{H} := H_1 \times H_2$, we define the mapping $U : \mathbb{H} \to \mathbb{R}$ as follows:

$$U(x) = \frac{\sum_{i=1}^N \left[\|A_i(v) - B_i(w) - b_i\|_H^2 + \|v - P_{C_i}(v)\|_{H_1}^2 + \|w - P_{Q_i}(w)\|_{H_2}^2 \right]}{2}.$$

We now consider the unconstrained optimization problem:

$$\min_{x \in \mathbb{H}} U(x). \tag{1.2}$$

It is easy to see that U is a convex function and Problem SSEP is equivalent to Problem (1.2). Thus, $p_* = (v_*, w_*)$ is a solution of Problem SSEP if and only if $\nabla U(p_*) = 0$, in which

$\nabla U(x) = (U_1(x), U_2(x))$ with

$$U_1(x) = \sum_{i=1}^{N} \left(\left(I^{H_1} - P_{C_i}^{H_1} \right)(v) + A_i^* \left(A_i(v) - B_i(w) - b_i \right) \right),$$
$$U_2(x) = \sum_{i=1}^{N} \left(\left(I^{H_2} - P_{Q_i}^{H_2} \right)(w) - B_i^* \left(A_i(v) - B_i(w) - b_i \right) \right).$$

Moreover, we observe that $\nabla U(p_*) = 0$ is equivalent to the problem of finding a fixed point p_* of $I - \gamma \nabla U$, that is, $p_* = (I - \gamma \nabla U)(p_*)$ for some $\gamma > 0$. Hence, in the present paper, we will introduce and study the convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = x_n - \gamma_n \nabla U(x_n),$$

where $\gamma_n > 0$ (see more detail in Algorithm 1). We first establish the weak convergence of Algorithm 1. Next, to obtain a strong convergence, we give a modification of Algorithm 1 by using the viscosity approximation method (see Algorithm 2). Some corollaries for solving the system of split feasibility problems are introduced in Section 4. Two relaxed iterative algorithms corresponding to Algorithms 1 and 2 are presented and studied in Section 5. Three numerical examples are discussed in Section 6 to examine the performance of the proposed algorithms.

2 Preliminaries

In this section, we denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$ the inner product and the induced norm in a real Hilbert space $\mathcal{H}$. The symbols $\to$ and $\rightharpoonup$ are indicated the strong and weak convergence, respectively.

If H_1 and H_2 are real Hilbert spaces then $\mathbb{H} := H_1 \times H_2$ is also a Hilbert space (see, e.g., [14, Proposition 2.4] and [17, Proposition 2.2]) with the inner product

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\mathbb{H}} = \langle x_1, x_2 \rangle_{H_1} + \langle y_1, y_2 \rangle_{H_2}, \quad \forall (x_1, y_1), (x_2, y_2) \in \mathbb{H},$$

and the norm on $\mathbb{H}$ is defined by

$$\|(x, y)\|_{\mathbb{H}}^2 = \|x\|_{H_1}^2 + \|y\|_{H_2}^2, \quad \forall (x, y) \in \mathbb{H}.$$

The following lemmas are used in the sequel in the proofs of the main results.

Lemma 2.1 *Let $P_C^{\mathcal{H}}$ be a metric projection from a real Hilbert space $\mathcal{H}$ into a nonempty, closed and convex subset C of $\mathcal{H}$. Then the following hold:*

(i) ([3, Theorem 3.14]) $\langle x - P_C^{\mathcal{H}}(x), y - P_C^{\mathcal{H}}(x) \rangle_{\mathcal{H}} \le 0, \quad \forall x \in \mathcal{H}, y \in C.$

(ii) ([23, Lemma 2.1])

$$\langle x - y, P_C^{\mathcal{H}}(x) - P_C^{\mathcal{H}}(y) \rangle_{\mathcal{H}} \ge \|P_C^{\mathcal{H}}(x) - P_C^{\mathcal{H}}(y)\|_{\mathcal{H}}^2, \quad \forall x, y \in \mathcal{H}.$$

(iii) ([23, Lemma 2.1])

$$\langle x - y, (I^{\mathcal{H}} - P_C^{\mathcal{H}})(x) - (I^{\mathcal{H}} - P_C^{\mathcal{H}})(y) \rangle_{\mathcal{H}} \ge \|(I^{\mathcal{H}} - P_C^{\mathcal{H}})(x) - (I^{\mathcal{H}} - P_C^{\mathcal{H}})(y)\|_{\mathcal{H}}^2$$

for all $x, y \in \mathcal{H}$. It also follows that $I^{\mathcal{H}} - P_C^{\mathcal{H}}$ is a nonexpansive mapping.

Lemma 2.2 [13, Lemma 3] *Let $\mathcal{H}$ be a real Hilbert space and let $\{x_n\}$ be a sequence in $\mathcal{H}$ such that $x_n \rightharpoonup z$ when $n \to \infty$. Then we have*

$$\liminf_{n\to\infty} \|x_n - z\|_{\mathcal{H}} < \liminf_{n\to\infty} \|x_n - x\|_{\mathcal{H}}$$

for all $x \in \mathcal{H}$ and $x \neq z$.

Lemma 2.3 [3, Theorem 4.17] *Let C be a nonempty closed convex bounded subset of a Hilbert space $\mathcal{H}$ and $T : C \to \mathcal{H}$ a nonexpansive mapping. Then the mapping $I^{\mathcal{H}} - T$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C which satisfies $x_n \rightharpoonup x \in C$ and $x_n - T(x_n) \to y \in \mathcal{H}$, it follows that $x - T(x) = y$.*

Lemma 2.4 *For every $x, y \in \mathcal{H}$, we have*

$$\|x + y\|^2_{\mathcal{H}} \leq \|x\|^2_{\mathcal{H}} + 2\langle y, x + y\rangle_{\mathcal{H}}.$$

Lemma 2.5 [18, Lemma 2.6] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{b_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{n=1}^{\infty} b_n = \infty$ and $\{c_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \quad \forall n \geq 1.$$

If

$$\limsup_{k\to\infty} c_{n_k} \leq 0$$

for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying

$$\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

3 Main Results

In this section, we always suppose all conditions from (D1) to (D4) are held. From now on, we consistently denote $\mathbb{H} := H_1 \times H_2$. To solve Problem SSEP, we first introduce the following algorithm.

Algorithm 1 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = x_n - \gamma_n \nabla U(x_n), \tag{3.1}$$

with the parameter $\{\gamma_n\}$ is defined by

$$\gamma_n = \rho_n \frac{D_n}{E_n + F_n + \zeta_n}, \tag{3.2}$$

where $\rho_n \in [a, b] \subset (0, 2)$, $\{\zeta_n\}$ is a sequence of positive real numbers which is upper bounded by ζ, and

$$D_n := \sum_{i=1}^{N} \|(I^{H_1} - P^{H_1}_{C_i})(v_n)\|^2_{H_1} + \sum_{i=1}^{N} \|(I^{H_2} - P^{H_2}_{Q_i})(w_n)\|^2_{H_2} + \sum_{i=1}^{N} \|A_i(v_n) - B_i(w_n) - b_i\|^2_{H},$$

$$E_n := \left\| \sum_{i=1}^{N} \Big((I^{H_1} - P_{C_i}^{H_1})(v_n) + A_i^*(A_i(v_n) - B_i(w_n) - b_i) \Big) \right\|_{H_1}^2,$$

$$F_n := \left\| \sum_{i=1}^{N} \Big((I^{H_2} - P_{Q_i}^{H_2})(w_n) - B_i^*(A_i(v_n) - B_i(w_n) - b_i) \Big) \right\|_{H_2}^2.$$

Step 3. Set $n \leftarrow n+1$, and go to Step 2.

We have the following theorem.

Theorem 3.1 *The sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a solution of Problem SSEP.*

Proof The proof is split into several steps. We take any point $p = (v, w) \in \Omega$.

Claim 1 The sequence $\{x_n\}$ is bounded.

It takes from (3.1) that

$$\begin{aligned} \|x_{n+1} - p\|_{\mathbb{H}}^2 &= \|x_n - \gamma_n \nabla U(x_n) - p\|_{\mathbb{H}}^2 \\ &= \|x_n - p\|_{\mathbb{H}}^2 - 2\gamma_n \langle \nabla U(x_n), x_n - p\rangle_{\mathbb{H}} + \gamma_n^2 \|\nabla U(x_n)\|_{\mathbb{H}}^2. \end{aligned} \tag{3.3}$$

We observe that

$$\begin{aligned} \langle \nabla U(x_n), x_n - p\rangle_{\mathbb{H}} &= \sum_{i=1}^{N} \Big\langle (I^{H_1} - P_{C_i}^{H_1})(v_n), v_n - v \Big\rangle_{H_1} \\ &\quad + \sum_{i=1}^{N} \Big\langle (I^{H_2} - P_{Q_i}^{H_2})(w_n), w_n - w \Big\rangle_{H_2} \\ &\quad + \sum_{i=1}^{N} \langle A_i^*(A_i(v_n) - B_i(w_n) - b_i), v_n - v\rangle_{H_1} \\ &\quad - \sum_{i=1}^{N} \langle B_i^*(A_i(v_n) - B_i(w_n) - b_i), w_n - w\rangle_{H_2} \\ &= \sum_{i=1}^{N} \Big\langle (I^{H_1} - P_{C_i}^{H_1})(v_n) - (I^{H_1} - P_{C_i}^{H_1})(v), v_n - v \Big\rangle_{H_1} \\ &\quad + \sum_{i=1}^{N} \Big\langle (I^{H_2} - P_{Q_i}^{H_2})(w_n) - (I^{H_2} - P_{Q_i}^{H_2})(w), w_n - w \Big\rangle_{H_2} \\ &\quad + \sum_{i=1}^{N} \langle A_i(v_n) - B_i(w_n) - b_i, A_i(v_n) - A_i(v)\rangle_H \\ &\quad - \sum_{i=1}^{N} \langle A_i(v_n) - B_i(w_n) - b_i, B_i(w_n) - B_i(w)\rangle_H \end{aligned}$$

$$= \sum_{i=1}^{N} \Big\langle (I^{H_1} - P_{C_i}^{H_1})(v_n) - (I^{H_1} - P_{C_i}^{H_1})(v), v_n - v \Big\rangle_{H_1}$$
$$+ \sum_{i=1}^{N} \Big\langle (I^{H_2} - P_{Q_i}^{H_2})(w_n) - (I^{H_2} - P_{Q_i}^{H_2})(w), w_n - w \Big\rangle_{H_2}$$
$$+ \sum_{i=1}^{N} \|A_i(v_n) - B_i(w_n) - b_i\|_H^2. \tag{3.4}$$

In view of Lemma 2.1 (iii) and (3.4), we can find that

$$\langle \nabla U(x_n), x_n - p \rangle_{\mathbb{H}} \geq \sum_{i=1}^{N} \Big\| (I^{H_1} - P_{C_i}^{H_1})(v_n) - (I^{H_1} - P_{C_i}^{H_1})(v) \Big\|_{H_1}^2$$
$$+ \sum_{i=1}^{N} \Big\| (I^{H_2} - P_{Q_i}^{H_2})(w_n) - (I^{H_2} - P_{Q_i}^{H_2})(w) \Big\|_{H_2}^2$$
$$+ \sum_{i=1}^{N} \|A_i(v_n) - B_i(w_n) - b_i\|_H^2$$
$$= \sum_{i=1}^{N} \Big\| (I^{H_1} - P_{C_i}^{H_1})(v_n) \Big\|_{H_1}^2 + \sum_{i=1}^{N} \Big\| (I^{H_2} - P_{Q_i}^{H_2})(w_n) \Big\|_{H_2}^2$$
$$+ \sum_{i=1}^{N} \|A_i(v_n) - B_i(w_n) - b_i\|_H^2.$$

This implies that

$$\langle \nabla U(x_n), x_n - p \rangle_{\mathbb{H}} \geq D_n. \tag{3.5}$$

Besides, we also note that

$$\|\nabla U(x_n)\|_{\mathbb{H}}^2 = \Big\| \sum_{i=1}^{N} \Big((I^{H_1} - P_{C_i}^{H_1})(v_n) + A_i^*(A_i(v_n) - B_i(w_n) - b_i) \Big) \Big\|_{H_1}^2$$
$$+ \Big\| \sum_{i=1}^{N} \Big((I^{H_2} - P_{Q_i}^{H_2})(w_n) - B_i^*(A_i(v_n) - B_i(w_n) - b_i) \Big) \Big\|_{H_2}^2$$
$$= E_n + F_n. \tag{3.6}$$

Thus, it follows from (3.2), (3.3), (3.5) and (3.6) that

$$\|x_{n+1} - p\|_{\mathbb{H}}^2 \leq \|x_n - p\|_{\mathbb{H}}^2 - 2\gamma_n D_n + \gamma_n^2 (E_n + F_n)$$
$$= \|x_n - p\|_{\mathbb{H}}^2 - 2\rho_n \frac{D_n^2}{E_n + F_n + \zeta_n} + \rho_n^2 \frac{D_n^2 (E_n + F_n)}{(E_n + F_n + \zeta_n)^2}$$
$$\leq \|x_n - p\|_{\mathbb{H}}^2 - 2\rho_n \frac{D_n^2}{E_n + F_n + \zeta_n} + \rho_n^2 \frac{D_n^2}{E_n + F_n + \zeta_n}$$
$$= \|x_n - p\|_{\mathbb{H}}^2 - \rho_n (2 - \rho_n) \frac{D_n^2}{E_n + F_n + \zeta_n}. \tag{3.7}$$

From the condition $\rho_n \in [a, b] \subset (0, 2)$ and (3.7), we obtain

$$\|x_{n+1} - p\|_{\mathbb{H}}^2 \leq \|x_n - p\|_{\mathbb{H}}^2. \tag{3.8}$$

By employing mathematical induction, we find that the sequence $\{x_n\}$ is bounded.

Claim 2 For every $i = 1, 2, 3, \ldots, N$, we have

$$\left\|(I^{H_1} - P_{C_i}^{H_1})(v_n))\right\|_{H_1}^2 \to 0, \tag{3.9}$$

$$\left\|(I^{H_2} - P_{Q_i}^{H_2})(w_n)\right\|_{H_2}^2 \to 0, \tag{3.10}$$

$$\|A_i(v_n) - B_i(w_n) - b_i\|_H^2 \to 0. \tag{3.11}$$

From (3.7), we have

$$\rho_n(2 - \rho_n)\frac{D_n^2}{E_n + F_n + \zeta_n} \leq \|x_n - p\|_{\mathbb{H}}^2 - \|x_{n+1} - p\|_{\mathbb{H}}^2.$$

On the other hand, it takes from (3.8) that the finite limit of the sequence $\{\|x_n - p\|_{\mathbb{H}}^2\}$ exists. Thus, from the conditions $\rho_n \in [a, b] \subset (0, 2)$, $0 < \zeta_n \leq \zeta$ and the above inequality, we can infer that

$$\frac{D_n^2}{E_n + F_n + \zeta} \to 0.$$

This leads to

$$D_n \to 0. \tag{3.12}$$

From the definition of D_n and (3.12), we obtain the limitations (3.9), (3.10) and (3.11), as claimed.

Claim 3 The sequence $\{x_n\}$ converges weakly to $p_* \in \Omega$.

Since $\{x_n\}$ is a bounded sequence, there exists the subsequence $\{x_{n_k}\} := \{(v_{n_k}, w_{n_k})\}$ of $\{x_n\}$ which converges weakly to some $z = (v_*, w_*) \in \mathbb{H}$, that is,

$$v_{n_k} \rightharpoonup v_*, \quad w_{n_k} \rightharpoonup w_*.$$

In view of Lemma 2.3, (3.9) and (3.10), we get $(v_*, w_*) \in C_i \times Q_i$ for all $i = 1, 2, \ldots, N$. On the other hand, since A_i and B_i are bounded linear operators, we have

$$A_i(v_{n_k}) - B_i(w_{n_k}) - b_i \rightharpoonup A_i(v_*) - B_i(w_*) - b_i, \quad \forall i = 1, 2, \ldots, N.$$

Combining with (3.11), we can infer that

$$A_i(v_*) - B_i(w_*) - b_i = 0, \quad \forall i = 1, 2, \ldots, N.$$

Therefore, we have $z \in \Omega$.

Finally, we shall demonstrate that $x_n \rightharpoonup z$. Suppose that there is another subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $x_{n_m} \rightharpoonup \bar{z}$ with $\bar{z} \neq z$. Using an argument similar to the one used above, we again also get that $\bar{z} \in \Omega$. It follows from Lemma 2.2 and the existence of the finite limit of

$\{\|x_n - z\|_{\mathbb{H}}\}$ that

$$\begin{aligned}\liminf_{k\to\infty} \|x_{n_k} - z\|_{\mathbb{H}} &< \liminf_{k\to\infty} \|x_{n_k} - \bar{z}\|_{\mathbb{H}} \\ &= \liminf_{m\to\infty} \|x_{n_m} - \bar{z}\|_{\mathbb{H}} \\ &< \liminf_{m\to\infty} \|x_{n_m} - z\|_{\mathbb{H}} \\ &= \liminf_{k\to\infty} \|x_{n_k} - z\|_{\mathbb{H}}.\end{aligned}$$

This is a contradiction. It implies that $x_{n_m} \rightharpoonup z$. Therefore, we obtain that $x_n \rightharpoonup z$.

This completes the proof. □

To obtain the strong convergence theorem, we now combine Algorithm 1 with the viscosity approximation method. The second algorithm is established as follows:

Algorithm 2 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)(x_n - \gamma_n \nabla U(x_n)), \tag{3.13}$$

where $h : \mathbb{H} \to \mathbb{H}$ is a contraction mapping with constant $\delta \in [0, 1)$, $\{\gamma_n\}$ is defined as in (3.2) and $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Step 3. Set $n \leftarrow n + 1$, and go to Step 2.

Theorem 3.2 *The sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to $p_* = P_\Omega(h(p_*))$.*

Proof The proof is divided into several steps. We first put $y_n = x_n - \gamma_n \nabla U(x_n)$ and take any $p \in \Omega$.

Claim 1 The sequence $\{x_n\}$ is bounded.

It follows from (3.13) that

$$\begin{aligned}\|x_{n+1} - p\|_{\mathbb{H}} &= \|\alpha_n h(x_n) + (1 - \alpha_n)(x_n - \gamma_n \nabla U(x_n)) - p\|_{\mathbb{H}} \\ &= \|\alpha_n (h(x_n) - p) + (1 - \alpha_n)(y_n - p)\|_{\mathbb{H}}.\end{aligned}$$

By the convexity of $\|\cdot\|_{\mathbb{H}}$ and h is a contraction mapping with constant $\delta \in [0, 1)$, we can find that

$$\begin{aligned}\|x_{n+1} - p\|_{\mathbb{H}} &\le \alpha_n \|h(x_n) - p\|_{\mathbb{H}} + (1 - \alpha_n)\|y_n - p\|_{\mathbb{H}} \\ &\le \alpha_n [\|h(x_n) - h(p)\|_{\mathbb{H}} + \|h(p) - p\|_{\mathbb{H}}] + (1 - \alpha_n)\|y_n - p\|_{\mathbb{H}} \\ &\le \alpha_n [\delta\|x_n - p\|_{\mathbb{H}} + \|h(p) - p\|_{\mathbb{H}}] + (1 - \alpha_n)\|y_n - p\|_{\mathbb{H}}. \qquad (3.14)\end{aligned}$$

By an argument similar as in Claim 1 of Theorem 3.1, we can find that

$$\begin{aligned}\|y_n - p\|_{\mathbb{H}}^2 &= \|x_n - \gamma_n \nabla U(x_n) - p\|_{\mathbb{H}}^2 \\ &\le \|x_n - p\|_{\mathbb{H}}^2 - \rho_n(2 - \rho_n)\frac{D_n^2}{E_n + F_n + \zeta_n} \qquad (3.15) \\ &\le \|x_n - p\|_{\mathbb{H}}^2. \qquad (3.16)\end{aligned}$$

From (3.14) and (3.16), we can infer that

$$\begin{aligned}\|x_{n+1}-p\|_{\mathbb{H}} &\le \alpha_n[\delta\|x_n-p\|_{\mathbb{H}}+\|h(p)-p\|_{\mathbb{H}}]+(1-\alpha_n)\|x_n-p\|_{\mathbb{H}}\\ &=(1-\alpha_n(1-\delta))\|x_n-p\|_{\mathbb{H}}+\alpha_n\|h(p)-p\|_{\mathbb{H}}\\ &=(1-\alpha_n(1-\delta))\|x_n-p\|_{\mathbb{H}}+\alpha_n(1-\delta)\frac{\|h(p)-p\|_{\mathbb{H}}}{1-\delta}\\ &\le \max\left\{\|x_n-p\|_{\mathbb{H}},\frac{\|h(p)-p\|_{\mathbb{H}}}{1-\delta}\right\}.\end{aligned}$$

By employing mathematical induction, we find that the sequence $\{x_n\}$ is bounded. Hence, the sequences $\{y_n\}$ and $\{h(x_n)\}$ are also bounded.

Claim 2 We have

$$\rho_n(2-\rho_n)\frac{D_n^2}{E_n+F_n+\zeta_n}\le\|x_n-p\|_{\mathbb{H}}^2-\|x_{n+1}-p\|_{\mathbb{H}}^2+\alpha_n M_1, \tag{3.17}$$

where $M_1=\sup_n\{\|h(x_n)-p\|_{\mathbb{H}}^2\}<\infty$.

Indeed, from (3.13), (3.15) and Lemma 2.4, we have

$$\begin{aligned}\|x_{n+1}-p\|_{\mathbb{H}}^2 &= \|\alpha_n(h(x_n)-p)+(1-\alpha_n)(y_n-p)\|_{\mathbb{H}}^2\\ &\le \alpha_n\|h(x_n)-p\|_{\mathbb{H}}^2+(1-\alpha_n)\|y_n-p\|_{\mathbb{H}}^2\\ &\le M_1\alpha_n+\|y_n-p\|_{\mathbb{H}}^2\\ &\le \|x_n-p\|_{\mathbb{H}}^2+\alpha_n M_1-\rho_n(2-\rho_n)\frac{D_n^2}{E_n+F_n+\zeta_n}.\end{aligned}$$

It is easy to see that the last inequality can be rewritten in the form (3.17), as claimed.

Claim 3 We have the following inequality:

$$a_{n+1}\le(1-b_n)a_n+b_nc_n,\quad \forall n\ge 1, \tag{3.18}$$

where

$$\begin{aligned}a_n &:= \|x_n-p\|_{\mathbb{H}}^2,\\ b_n &:= \alpha_n(1-\delta),\\ c_n &:= \frac{2\langle h(p)-p,x_{n+1}-p\rangle_{\mathbb{H}}}{1-\delta}.\end{aligned}$$

Indeed, one again, from (3.13), (3.16) and Lemma 2.4, we can see that

$$\begin{aligned}\|x_{n+1}-p\|_{\mathbb{H}}^2 =&\|\alpha_n(h(x_n)-p)+(1-\alpha_n)(y_n-p)\|_{\mathbb{H}}^2\\ =&\|\alpha_n(h(x_n)-h(p))+(1-\alpha_n)(y_n-p)+\alpha_n(h(p)-p)\|_{\mathbb{H}}^2\\ \le&\|\alpha_n(h(x_n)-h(p))+(1-\alpha_n)(y_n-p)\|_{\mathbb{H}}^2\\ &+2\alpha_n\langle h(p)-p,x_{n+1}-p\rangle_{\mathbb{H}}\\ \le&\alpha_n\|h(x_n)-h(p)\|_{\mathbb{H}}^2+(1-\alpha_n)\|y_n-p\|_{\mathbb{H}}^2\\ &+2\alpha_n\langle h(p)-p,x_{n+1}-p\rangle_{\mathbb{H}}\\ \le&\alpha_n\delta\|x_n-p\|_{\mathbb{H}}^2+(1-\alpha_n)\|x_n-p\|_{\mathbb{H}}^2\\ &+2\alpha_n\langle h(p)-p,x_{n+1}-p\rangle_{\mathbb{H}}\\ =&(1-\alpha_n(1-\delta))\|x_n-p\|_{\mathbb{H}}^2+\alpha_n(1-\delta)\frac{2\langle h(p)-p,x_{n+1}-p\rangle_{\mathbb{H}}}{1-\delta}.\end{aligned}$$

It is not difficult to see that the above inequality can be rewritten in the form (3.18), as claimed.

Claim 4 The sequence $\{x_n\}$ converges strongly to $p_* = P_\Omega(h(p_*))$.

Suppose that $\{\|x_{n_m} - p_*\|^2\}$ is an arbitrary subsequence of $\{\|x_n - p_*\|^2\}$ such that

$$\liminf_{k\to\infty}(\|x_{n_m+1} - p_*\|^2 - \|x_{n_m} - p_*\|^2) \geq 0.$$

It follows from Claim 2, $\alpha_{n_m} \to 0$ and $\rho_{n_m} \in [a, b] \subset (0, 2)$ that

$$\frac{D_{n_m}^2}{E_{n_m} + F_{n_m} + \zeta_{n_m}} \to 0.$$

Since $\zeta_{n_m} \leq \zeta$, we have

$$\frac{D_{n_m}^2}{E_{n_m} + F_{n_m} + \zeta} \to 0,$$

which implies that $D_{n_m} \to 0$. Hence, we can find that

$$\left\|(I^{H_1} - P_{C_i}^{H_1})(v_{n_m})\right\|_{H_1}^2 \to 0, \tag{3.19}$$

$$\left\|(I^{H_2} - P_{Q_i}^{H_2})(w_{n_m})\right\|_{H_2}^2 \to 0, \tag{3.20}$$

$$\|A_i(v_{n_m}) - B_i(w_{n_m}) - b_i\|_H^2 \to 0. \tag{3.21}$$

In addition, we also have

$$\begin{aligned}\|y_{n_m} - x_{n_m}\|_{\mathbb{H}}^2 &= \gamma_{n_m}^2(E_{n_m} + F_{n_m})\\ &= \rho_{n_m}^2 \frac{D_{n_m}^2(E_{n_m} + F_{n_m})}{(E_{n_m} + F_{n_m} + \zeta_{n_m})^2}\\ &\leq b^2 \frac{D_{n_m}^2}{E_{n_m} + F_{n_m} + \zeta_{n_m}} \to 0.\end{aligned}$$

This implies that

$$\|y_{n_m} - x_{n_m}\|_{\mathbb{H}} \to 0. \tag{3.22}$$

By the boundedness of $\{x_{n_m}\}$ and $\{h(x_{n_m})\}$, we observe that

$$\begin{aligned}\|x_{n_m+1} - x_{n_m}\|_{\mathbb{H}} &= \|\alpha_{n_m}(h(x_{n_m}) - x_{n_m}) + (1 - \alpha_{n_m})(y_{n_m} - x_{n_m})\|_{\mathbb{H}}\\ &\leq \alpha_{n_m}\|h(x_{n_m}) - x_{n_m}\|_{\mathbb{H}} + (1 - \alpha_{n_m})\|y_{n_m} - x_{n_m}\|_{\mathbb{H}}\\ &\leq \alpha_{n_m} M_2 + (1 - \alpha_{n_m})\|y_{n_m} - x_{n_m}\|_{\mathbb{H}},\end{aligned} \tag{3.23}$$

where $M_2 = \sup_m\{\|h(x_{n_m}) - x_{n_m}\|_{\mathbb{H}}\}$. Thus, it takes from (3.22) and (3.23) that

$$\|x_{n_m+1} - x_{n_m}\|_{\mathbb{H}} \to 0. \tag{3.24}$$

Finally, to apply Lemma 2.5, from Claim 3, it suffices to prove the following inequality

$$\limsup_{m\to\infty} c_{n_m} \leq 0.$$

It is equivalent to show that $\limsup_{m\to\infty}\langle h(p_*)-p_*, x_{n_m+1}-p_*\rangle \le 0$. We first note that

$$\begin{aligned}&\langle h(p_*)-p_*, x_{n_m+1}-p_*\rangle\\ =&\langle h(p_*)-p_*, x_{n_m+1}-x_{n_m}\rangle+\langle h(p_*)-p_*, x_{n_m}-p_*\rangle\\ \le&\|h(p_*)-p_*\|\|x_{n_m+1}-x_{n_m}\|+\langle h(p_*)-p_*, x_{n_m}-p_*\rangle.\end{aligned} \tag{3.25}$$

Since $\{x_{n_m}\}$ is a bounded sequence (Claim 1), there exists a subsequence $\{x_{n_{m_j}}\}$ of $\{x_{n_m}\}$ which converges weakly to some $z\in\mathbb{H}$, such that

$$\begin{aligned}\limsup_{m\to\infty}\langle h(p_*)-p_*, x_{n_m}-p_*\rangle_{\mathbb{H}} &= \lim_{j\to\infty}\langle h(p_*)-p_*, x_{n_{m_j}}-p_*\rangle_{\mathbb{H}}\\ &=\langle h(p_*)-p_*, z-p_*\rangle_{\mathbb{H}}.\end{aligned}$$

Furthermore, from (3.19), (3.20), (3.21) and using an argument similar to the proof of Claim 3 in Theorem 3.1, we obtain that $z\in\Omega$. Besides, from the definition of p_* and Lemma 2.1 (i), we obtain that

$$\limsup_{m\to\infty}\langle h(p_*)-p_*, x_{n_m}-p_*\rangle=\langle h(p_*)-p_*, z-p_*\rangle\le 0. \tag{3.26}$$

Using (3.24), (3.25), (3.26), we find that $\limsup_{m\to\infty} c_{n_m}\le 0$. Hence, it is not difficult to see that all the hypotheses of Lemma 2.5 are satisfied. This guarantees that $\|x_n-p_*\|\to 0$.

This completes the proof. □

Remark 3.1 It follows from (3.6) that if $E_n+F_n=0$ then $\nabla U(x_n)=0$. In this case, we have that $x_n=(v_n,w_n)$ is a solution to the SSEP and, thus, we can stop the algorithm. If otherwise, we can select the parameter $\zeta_n=0$ which leads to $\gamma_n=\rho_n\frac{D_n}{E_n+F_n}$. On the other hand, we note that

$$\begin{aligned}E_n+F_n\le&2\left(\sum_{i=1}^N\left\|(I^{H_1}-P_{C_i}^{H_1})(v_n)\right\|_{H_1}^2+\sum_{i=1}^N\left\|(I^{H_2}-P_{Q_i}^{H_2})(w_n)\right\|_{H_2}^2\right)\\ &+2\sum_{i=1}^N\|A_i\|^2\|A_i(v_n)-B_i(w_n)-b_i\|_H^2\\ &+2\sum_{i=1}^N\|B_i\|^2\|A_i(v_n)-B_i(w_n)-b_i\|_H^2\le\kappa D_n,\end{aligned}$$

where $\kappa=2\max\{1,\max_{1\le i\le N}\{\|A_i\|^2\}+\max_{1\le i\le N}\{\|B_i\|\}^2\}$. This guarantees that $D_n\to 0$ whenever $\frac{D_n^2}{E_n+F_n}\to 0$.

Hence, the conclusions of Theorems 3.1 and 3.2 are still valued by employing an argument similar to the one used in the proof of these.

4 Corollaries

It is easy to see that if $H\equiv H_2$, B_i is the identity mapping on H and $b_i=0$ for all $i=1,2,3,\ldots,N$, then Problem SSEP becomes the system of split feasibility problems, that is,

$$\text{Find an element } p_*\in\widehat{\Omega}, \tag{4.1}$$

where

$$\widehat{\Omega} = \left\{(v, w) \in \cap_{i=1}^{N}(C_i \times Q_i) : A_i(v) - w = 0,\ i = 1, 2, 3, \dots, N\right\} \neq \emptyset.$$

Furthermore, Problem (4.1) reduces to the split feasibility problem in the case that $N = 1$.

We now denote $\nabla\widehat{U}(x) := (\widehat{U}_1(x), \widehat{U}_2(x))$ for all $x = (v, w) \in \mathbb{H}$ with

$$\widehat{U}_1(x) := \sum_{i=1}^{N}\left((I^{H_1} - P_{C_i}^{H_1})(v) + A_i^*(A_i(v) - w)\right),$$

$$\widehat{U}_2(x) := \sum_{i=1}^{N}\left((I^{H_2} - P_{Q_i}^{H_2})(w) - (A_i(v) - w)\right).$$

From Algorithm 1, we obtain the following algorithm.

Algorithm 3 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = x_n - \widehat{\gamma}_n \nabla\widehat{U}(x_n),$$

with the parameter $\{\gamma_n\}$ is defined by

$$\widehat{\gamma}_n = \rho_n \frac{\widehat{D}_n}{\widehat{E}_n + \widehat{F}_n + \zeta_n},$$

where $\rho_n \in [a, b] \subset (0, 2)$, $\{\zeta_n\}$ is a sequence of positive real numbers which is upper bounded by ζ, and

$$\widehat{D}_n := \sum_{i=1}^{N}\left[\left\|(I^{H_1} - P_{C_i}^{H_1})(v_n)\right\|_{H_1}^2 + \left\|(I^{H_2} - P_{Q_i}^{H_2})(w_n)\right\|_{H_2}^2 + \|A_i(v_n) - w_n\|_{H_2}^2\right],$$

$$\widehat{E}_n := \|\sum_{i=1}^{N}\left((I^{H_1} - P_{C_i}^{H_1})(v_n) + A_i^*(A_i(v_n) - w_n)\right)\|_{H_1}^2,$$

$$\widehat{F}_n := \|\sum_{i=1}^{N}\left((I^{H_2} - P_{Q_i}^{H_2})(w_n) - (A_i(v_n) - w_n)\right)\|_{H_2}^2.$$

Step 3. Set $n \leftarrow n + 1$, and go to Step 2.

Theorem 4.1 *The sequence $\{x_n\}$ generated by Algorithm 3 converges weakly to a solution $p_* = (v_*, w_*)$ to Problem (4.1).*

From Algorithm 2, we obtain Algorithm 4 below.

Algorithm 4 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)(x_n - \widehat{\gamma}_n \nabla\widehat{U}(x_n)),$$

where $h : \mathbb{H} \to \mathbb{H}$ and $\{\alpha_n\}$ are defined as in Step 2 of Algorithm 2 while $\{\widehat{\gamma}_n\}$ is defined as in Step 2 of Algorithm 3.
Step 3. Set $n \leftarrow n + 1$, and go to Step 2.

Theorem 4.2 *The sequence $\{x_n\}$ generated by Algorithm 4 converges strongly to a unique solution (v_*, w_*) to Problem (4.1) such that $p_* = P_{\widehat{\Omega}}(h(p_*))$.*

5 Relaxed Iterative Algorithms

In this section, we consider Problem SSEP when C_i and Q_i are sublevel sets of the lower semicontinuous convex functions $c_i : H_1 \to \mathbb{R}$ and $q_i : H_2 \to \mathbb{R}$ and $i = 1, 2, \ldots, N$, respectively. Namely,

$$C_i = \{v \in H_1 : c_i(v) \leq 0\},$$
$$Q_i = \{w \in H_2 : q_i(w) \leq 0\},$$

where c_i and q_i are respectively subdifferentiable on H_1 and H_2, and that the subdifferentials ∂c_i and ∂q_i are bounded (on bounded sets).

At points $v_n \in H_1$ and $w_n \in H_2$, we define the subsets $C_{i,n}$ and $Q_{i,n}$ as follows:

$$C_{i,n} = \{v \in H_1 : c_i(v_n) \leq \langle v_n - v, \mathfrak{c}_{i,n} \rangle_{H_1}\},$$
$$Q_{i,n} = \{w \in H_2 : q_i(w_n) \leq \langle w_n - w, \mathfrak{q}_{i,n} \rangle_{H_2}\},$$

where $\mathfrak{c}_{i,n} \in \partial c_i(v_n)$ and $\mathfrak{q}_{i,n} \in \partial q_i(w_n)$. It is not hard to find that $C_{i,n}$ and $Q_{i,n}$ are half-spaces of H_1 and H_2. They are respectively called the relaxed sets of C_i and Q_i. Besides, we also have $C_i \subset C_{i,n}$ and $Q_i \subset Q_{i,n}$.

In general, we are not easy to compute the metric projections $P_{C_i}^{H_1}$ and $P_{Q_i}^{H_2}$. It depends on the construct of the sets C_i and Q_i. However, we do have the explicit expression of the metric projections $P_{C_{i,n}}^{H_1}$ and $P_{Q_{i,n}}^{H_2}$, which are

$$P_{C_{i,n}}^{H_1}(v) = v - \max\left\{0, \frac{\langle v - v_n, \mathfrak{c}_{i,n} \rangle_{H_1} + c_i(v_n)}{\|\mathfrak{c}_{i,n}\|_{H_1}^2}\right\} \mathfrak{c}_{i,n},$$
$$P_{Q_{i,n}}^{H_1}(w) = w - \max\left\{0, \frac{\langle w - w_n, \mathfrak{q}_{i,n} \rangle_{H_2} + q_i(w_n)}{\|\mathfrak{q}_{i,n}\|_{H_2}^2}\right\} \mathfrak{q}_{i,n}.$$

Therefore, we obtain relaxed iterative algorithms corresponding to Algorithm 1 and Algorithm 2, where $P_{C_i}^{H_1}$ and $P_{Q_i}^{H_2}$ are respectively replaced by $P_{C_{i,n}}^{H_1}$ and $P_{Q_{i,n}}^{H_2}$.

We denote $\nabla \tilde{U}(x) := (\tilde{U}_1(x), \tilde{U}_2(x))$ for all $x = (v, w) \in \mathbb{H}$, where

$$\tilde{U}_1(x) := \sum_{i=1}^{N} \left((I^{H_1} - P_{C_{i,n}}^{H_1})(v) + A_i^*(A_i(v) - B_i(w) - b_i) \right),$$
$$\tilde{U}_2(x) := \sum_{i=1}^{N} \left((I^{H_2} - P_{Q_{i,n}}^{H_2})(w) - B_i^*(A_i(v) - B_i(w) - b_i) \right).$$

From Algorithm 1, we obtain the following algorithm.

Algorithm 5 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = x_n - \tilde{\gamma}_n \nabla \tilde{U}(x_n),$$

with the parameter $\{\tilde{\gamma}_n\}$ is defined by

$$\tilde{\gamma}_n = \rho_n \frac{\tilde{D}_n}{\tilde{E}_n + \tilde{F}_n + \zeta_n},$$

where $\rho_n \in [a, b] \subset (0, 2)$, $\{\zeta_n\}$ is a sequence of positive real numbers which is upper bounded by ζ, and

$$\begin{aligned}\tilde{D}_n &:= \sum_{i=1}^{N} \left\|(I^{H_1} - P^{H_1}_{C_{i,n}})(v_n)\right\|^2_{H_1} + \sum_{i=1}^{N} \left\|(I^{H_2} - P^{H_2}_{Q_{i,n}})(w_n)\right\|^2_{H_2} \\ &\quad + \sum_{i=1}^{N} \|A_i(v_n) - B_i(w_n) - b_i\|^2_H, \\ \tilde{E}_n &:= \left\|\sum_{i=1}^{N} \Big((I^{H_1} - P^{H_1}_{C_{i,n}})(v_n) + A_i^*(A_i(v_n) - B_i(w_n) - b_i)\Big)\right\|^2_{H_1}, \\ \tilde{F}_n &:= \left\|\sum_{i=1}^{N} \Big((I^{H_2} - P^{H_2}_{Q_{i,n}})(w_n) - B_i^*(A_i(v_n) - B_i(w_n) - b_i)\Big)\right\|^2_{H_2}.\end{aligned}$$

Step 3. Set $n \leftarrow n + 1$, and go to Step 2.

Theorem 5.1 *The sequence $\{x_n\}$ generated by Algorithm 5 converges weakly to a solution of Problem SSEP.*

Proof In view of the proof of Theorem 3.1, we can infer that the sequence $\{x_n\}$ is bounded and

$$\left\|(I^{H_1} - P^{H_1}_{C_{i,n}})(v_n))\right\|^2_{H_1} \to 0, \tag{5.1}$$

$$\left\|(I^{H_2} - P^{H_2}_{Q_{i,n}})(w_n)\right\|^2_{H_2} \to 0, \tag{5.2}$$

$$\|A_i(v_n) - B_i(w_n) - b_i\|^2_H \to 0. \tag{5.3}$$

We will prove that all weak sequential limits of $\{x_n\}$ belong to Ω. Indeed, since $\{x_n\}$ is a bounded sequence, there exists the subsequence $\{x_{n_k}\} := \{(v_{n_k}, w_{n_k})\}$ of $\{x_n\}$ which converges weakly to some $z = (v_*, w_*) \in \mathbb{H}$. It is equivalent to

$$v_{n_k} \rightharpoonup v_*, \quad w_{n_k} \rightharpoonup w_*.$$

Since the subdifferential ∂c_i is assumed to be bounded on bounded sets and the sequence $\{x_n\}$ is bounded, there exists a positive real number M_3 such that

$$\|\mathfrak{c}_{i,n}\|_{H_1} \leq M_3$$

for all $n \in \mathbb{N}$. It follows from $P^{H_1}_{C_{i,n}}(v_n) \in C_{i,n}$ and the definition of $C_{i,n}$ that

$$\begin{aligned}c_i(v_{n_k}) &\leq \Big\langle (I^{H_1} - P^{H_1}_{C_{i,n_k}})(v_{n_k})), \mathfrak{c}_{i,n_k} \Big\rangle_{H_1} \\ &\leq \left\|(I^{H_1} - P^{H_1}_{C_{i,n_k}})(v_{n_k})\right\|_{H_1} \|\mathfrak{c}_{i,n_k}\|_{H_1} \\ &\leq M_3 \left\|(I^{H_1} - P^{H_1}_{C_{i,n_k}})(v_{n_k})\right\|_{H_1}.\end{aligned} \tag{5.4}$$

From (5.1) and (5.4), we can find that

$$\liminf_{k\to\infty} c_i(v_{n_k}) \leq 0.$$

By the lower semicontinuity of the function c, we have

$$c_i(v_*) \leq \liminf_{k\to\infty} c_i(v_{n_k}) \leq 0.$$

Therefore, we obtain $v_* \in C_i$. By an argument similar to the one above and using (5.2), we also obtain that $w_* \in Q_i$. Furthermore, using (5.3) and repeating the proof of Theorem 3.1 in Claim 3, we can deduce that

$$A_i(v_*) - B_i(w_*) - b_i = 0.$$

Hence, we have $(v_*, w_*) \in \Omega$.

Once again, we use a similar argument to the one employed in the last proof of Theorem 3.1 and can conclude that (v_*, w_*) is the unique weak sequential limit of $\{x_n\}$ and that $v_n \to v_*$ and $w_n \to w_*$.

This completes the proof. □

From Algorithm 2, we obtain the algorithm below.

Algorithm 6 **Step 1.** Choose $x_0 = (v_0, w_0) \in \mathbb{H} := H_1 \times H_2$ arbitrarily and set $n := 0$.
Step 2. Given $x_n = (v_n, w_n)$, compute

$$x_{n+1} = \alpha_n h(x_n) + (1 - \alpha_n)(x_n - \tilde{\gamma}_n \nabla \tilde{U}(x_n)),$$

where $h : \mathbb{H} \to \mathbb{H}$ and $\{\alpha_n\}$ are defined as in Step 2 of Algorithm 2 while $\{\tilde{\gamma}_n\}$ is defined as in Step 2 of Algorithm 5.
Step 3. Set $n \leftarrow n + 1$, and go to Step 2.

By using a line of proof similar to the one in the proof of Theorem 3.2 and combining it with Theorem 5.1, we obtain the following theorem.

Theorem 5.2 *The sequence $\{x_n\}$ generated by Algorithm 6 converges strongly to $p_* = P_\Omega(h(p_*))$.*

6 Numerical Test

Our algorithms are implemented in MATLAB 14a running on the DESKTOP-8LDGIN0, Intel(R) Core(TM) i5-4210U CPU @ 1.70GHz with 2.40 GHz and 4GB RAM.

Example 6.1 We consider the Problem SSEP under the following hypotheses:

(D1) $H_1 = \mathbb{R}^m$, $H_2 = \mathbb{R}^k$ and $H = \mathbb{R}^p$ are three finite Euclidean spaces. For each $i = 1, 2, 3$, the sets C_i and Q_i are defined by

$$C_i = \{x \in \mathbb{R}^m : \|x - \mathfrak{a}_i\|^2 \leq \mathfrak{R}_i^2\},$$
$$Q_i = \{x \in \mathbb{R}^k : \|x - \widehat{\mathfrak{a}}_i\|^2 \leq \widehat{\mathfrak{R}}_i^2\},$$

where the coordinates of the centers $\mathfrak{a}_i$ and $\widehat{\mathfrak{a}}_i$ are randomly generated in the interval $[-2, 2]$, the radii $\mathfrak{R}_i$ and $\widehat{\mathfrak{R}}_i$ are also randomly generated in the intervals $[10, 20]$ and $[20, 30]$, respectively.

(D2) $A_i : \mathbb{R}^m \to \mathbb{R}^p$ and $B_i : \mathbb{R}^k \to \mathbb{R}^p$ $(i = 1, 2, 3)$ are bounded linear operators where the elements of their representing matrices are randomly generated in the closed interval $[-5, 5]$.

(D3) $b_i = 0$ for all $i = 1, 2, 3$.

Table 1 Numerical results of Algorithm 2 with different choices of α_n

s	TOL	n	err	CPU-Time (s)
0.05	10^{-4}	7	$1.223748605499165e-05$	0.0360
	10^{-6}	9	$2.701731245917301e-07$	0.0369
	10^{-8}	11	$6.843509460101972e-09$	0.0383
0.25	10^{-4}	13	$8.546088255403800e-05$	0.0397
	10^{-6}	20	$6.961546898147826e-07$	0.0448
	10^{-8}	28	$8.508223635581934e-09$	0.0500
0.5	10^{-4}	26	$9.588803526858486e-05$	0.0526
	10^{-6}	49	$9.718300091195404e-07$	0.0726
	10^{-8}	87	$9.319365234169132e-09$	0.0942
0.75	10^{-4}	45	$9.251179259082480e-05$	0.0592
	10^{-6}	143	$9.833815955402861e-07$	0.1251
	10^{-8}	473	$9.925476384362636e-09$	0.3404

(D4) Since $0 = (0, 0) \in \Omega$, we have

$$\Omega = \{(v, w) \in \cap_{i=1}^{3}(C_i \times Q_i) : A_i(v) - B_i(w) = 0, i = 1, 2, 3\} \neq \emptyset.$$

We use Algorithm 2 to $m = 100, k = 200, p = 300$ and $h(x) = 0.05x$ for all $x \in \mathbb{R}^m \times \mathbb{R}^k$. It is not difficult to see that $p_* = (0, 0)$. We take the initial point $x_0 = (v_0, w_0)$ which has the coordinates of v_0 and w_0 randomly generated in the closed interval [20, 40], and select the control parameters as follows:

$$\rho_n = 1.5, \quad \zeta_n = 0.05, \quad \alpha_n = \frac{1}{(n+1)^s} \ (0 < s < 1).$$

We use the stopping rule

$$\text{err} = \|x_n\| = \sqrt{\|v_n\|^2 + \|w_n\|^2} < \text{TOL},$$

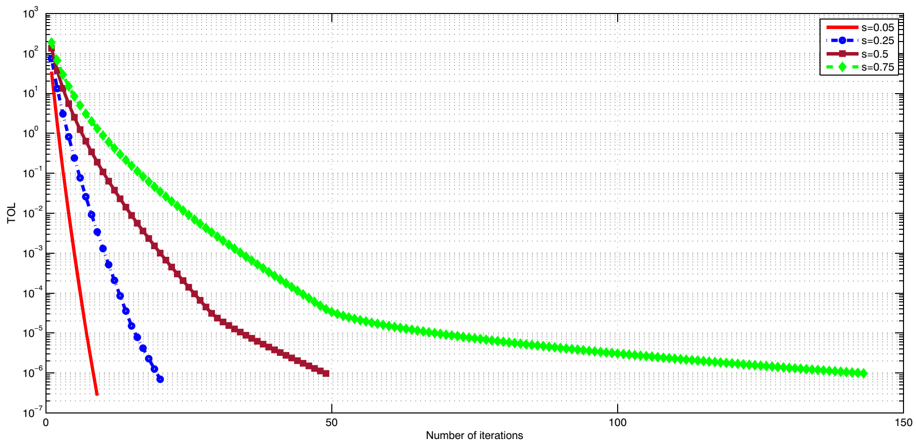

Fig. 1 The behavior of err with TOL=10^{-6}

Table 2 Numerical results of ALGO-T with different choices of α_n

t	TOL	n	err	CPU-Time (s)
0.47	10^{-4}	657742	$9.999945847476398e-05$	459.5096
	10^{-6}	828434	$9.999729512234562e-07$	537.7465
	10^{-8}	991308	$9.999925758077272e-09$	653.8293
0.485	10^{-4}	609990	$9.999951236130542e-05$	501.4462
	10^{-6}	767229	$9.999764897057308e-07$	524.3640
	10^{-8}	917084	$9.999860550931206e-09$	588.7401
0.5	10^{-4}	566197	$9.999722893875010e-05$	383.5082
	10^{-6}	711175	$9.999743125018068e-07$	474.2676
	10^{-8}	849179	$9.999927835113614e-09$	569.5937

where TOL is a given tolerant and $x_n = (v_n, w_n)$. The numerical results are presented in Table 1. The behavior of err is shown in Fig. 1.

We also compare our Algorithm 2 with the algorithm defined by [23, Theorem 3.5] (ALGO-T, for short). The parameters for the ALGO-T are chosen as follows:

$$\alpha_n = \frac{1}{(n+1)^t}, \quad \varepsilon_n = \frac{1.9999}{\alpha_n^{0.495}[(N+\alpha_n)^2 + \gamma_{A,B}^4](1+4N^2)},$$

where $N = 3$ and $\gamma_{A,B} = \max_{1\leq i\leq 3}\{\|A_i\|, \|B_i\|\}$. The numerical results are presented in Table 2.

Example 6.2 We consider Problem (4.1) under the following hypotheses:

(D1) $H_1 = H_2 = H = L^2[0,1]$ with the inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)dt, \quad \forall x = x(t), y = y(t) \in L^2[0,1],$$

and the norm

$$\|x\| = \left(\int_0^1 x^2(t)dt\right)^{\frac{1}{2}}, \quad \forall x = x(t) \in L^2[0,1].$$

The sets C_i and Q_i $(i = 1, 2, 3)$ are given by

$$C_i = \{x \in L^2[0,1] : \|x\| \leq r_i\},$$
$$Q_i = \{x \in L^2[0,1] : \langle \mathfrak{b}_i, x\rangle = \mathfrak{c}_i\},$$

where

$$r_i = i+1, \quad \mathfrak{b}_i = \exp(t) + i, \quad \mathfrak{c}_i = \frac{i+2}{i^2+1}.$$

(D2) For each $i = 1, 2, 3$, the operators $A_i : L^2[0,1] \to L^2[0,1]$ are defined by

$$A_i(x)(t) = \frac{2x(t)}{i^2+1}, \quad \forall x = x(t) \in L^2[0,1].$$

(D3) $b_i = 0$ for all $i = 1, 2, 3$.

Table 3 Numerical results of Algorithm 4 with different choices of ζ_n

ζ_n	TOL	n	err	CPU-Time (s)
10	10^{-4}	11	$5.723109509087199e-05$	0.0014
	10^{-5}	21	$9.904141641479197e-06$	0.0033
	10^{-6}	221	$9.977175102468254e-07$	0.0266
20	10^{-4}	11	$5.248590023519470e-05$	0.0014
	10^{-5}	14	$7.940026272800991e-06$	0.0019
	10^{-6}	110	$9.926253421438724e-07$	0.0164
30	10^{-4}	11	$5.118087290559601e-05$	0.0014
	10^{-5}	13	$7.933972283986941e-06$	0.0018
	10^{-6}	73	$9.892201029412421e-07$	0.0091

(D4) It is easy to find that

$$\Omega = \{(v, w) \in \cap_{i=1}^{3}(C_i \times Q_i) : A_i(v) - w = 0, i = 1, 2, 3\}$$

is a nonempty set because $(t, 2t/(i^2+1)) \in \Omega$.

We use Algorithm 4 with $h(x) = 0.25x$ for all $x \in L^2[0,1] \times L^2[0,1]$, the initial point $x_0 = (\exp(t), \log(t+1))$, and select the parameters as follows:

$$\rho_n = 0.5, \quad \zeta_n = 10i, \quad \alpha_n = \frac{1}{(n+1)^{0.025}}.$$

We use the stopping criterion

$$\text{err} = \|x_{n+1} - x_n\| < \text{TOL},$$

where TOL is a given tolerant. The numerical results are presented in Table 3. The behavior of err is described in Fig. 2.

We also compare our Algorithm 4 with the ALGO-T. The parameters for the ALGO-T are chosen as follows:

$$\alpha_n = \frac{1}{(n+1)^s}, \quad \varepsilon_n = \frac{1.9999}{\alpha_n^{0.75}[(N+\alpha_n)^2 + \gamma_{A,B}^4](1+4N^2)},$$

where $N = 3$ and $\gamma_{A,B} = \max_{1 \le i \le 3}\{\|A_i\|, \|B_i\|\}$. The numerical results are presented in Table 4.

Example 6.3 Let $\mathbb{R}^m$ and $\mathbb{R}^k$ be two finite Euclidean spaces. We consider the signal recovery problem through the following LASSO problem:

$$\min\left\{\frac{1}{2}\|Av - w\|^2 : \; v \in \mathbb{R}^k, \; \|v\|_1 \le p\right\},$$

where $A \in \mathbb{R}^m \times \mathbb{R}^k$, $w \in \mathbb{R}^m$, $p > 0$ and $\|\cdot\|_1$ is l_1-norm. A is a perception matrix, which is generated from a standard normal distribution. The true sparse signal v_* is constructed from the uniform distribution in the interval $[-2, 2]$ with random p nonzero elements. In the sample data $w_* = Av_*$ no noise is assumed.

In relation with the Problem (4.1) and the $N = 1$ case, we define

$$C = \{v \in \mathbb{R}^k : \|v\|_1 \le p\}, \quad Q = \{w_*\}.$$

Table 4 Numerical results of ALGO-T with different choices of α_n

s	TOL	n	err	CPU-Time (s)
0.025	10^{-4}	1217	$9.955781066302840e-05$	0.2601
	10^{-5}	1737	$9.976146968701992e-06$	0.3232
	10^{-6}	7964	$9.998917402238431e-07$	1.3724
0.05	10^{-4}	1144	$9.981918927409253e-05$	0.2373
	10^{-5}	1927	$9.993330212057856e-06$	0.4393
	10^{-6}	16046	$9.999703190770469e-07$	2.7482
0.1	10^{-4}	1052	$9.965899513177035e-05$	0.2196
	10^{-5}	3291	$9.997812913924868e-06$	0.6199
	10^{-6}	30645	$9.999874607650179e-07$	5.0932

Thus, we define a convex function

$$c(v) = \|v\|_1 - p$$

and denote the relaxed set C_n by

$$C_n = \{v \in \mathbb{R}^k : c(v_n) \leq \langle v_n - v, \mathfrak{c}_n \rangle\},$$

where $\mathfrak{c}_n \in \partial c(v_n)$. The subdiffrential ∂c at $v_n \in \mathbb{R}^k$ is defined by

$$[\partial c(v_n)]_j = \text{sign}((v_n)_j), \quad j = 1, 2, \ldots, k.$$

We use Algorithm 5 and Algorithm 6 with the initial point $x_0 = (v_0, w_0)$, where v_0 and w_0 are the original points of $\mathbb{R}^k$ and $\mathbb{R}^m$, and select the parameters as follows:

$$\rho_n = 1, \quad \zeta_n = 1, \quad \alpha_n = \frac{1}{10^4 n}.$$

In Algorithm 6, the mapping $h : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^k$ is given by $h(x) = 0.5x$ for all $x \in \mathbb{R}^m \times \mathbb{R}^k$.

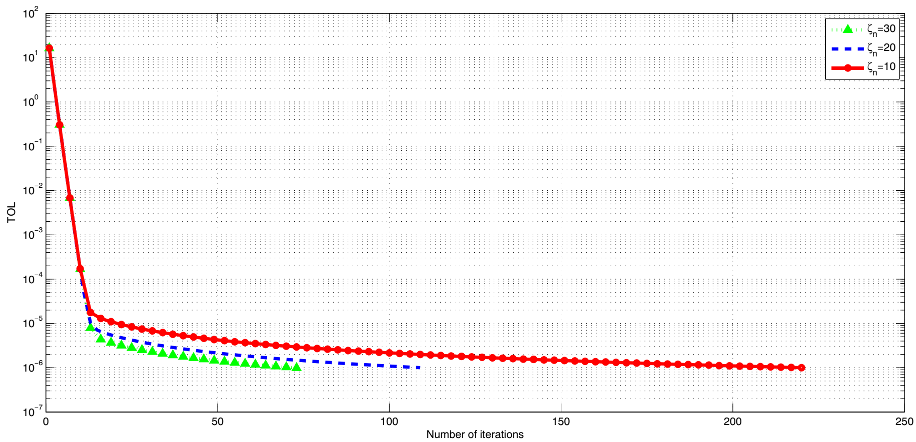

Fig. 2 The behavior of err with TOL=10^{-6}

Table 5 Table of numerical results and comparisons among algorithms

	$k = 1024,\ m = 2048,\ p = 128$			$k = 1024,\ m = 2048,\ p = 256$		
	err	n	CPU-Time (s)	err	n	CPU-Time (s)
TOL=10^{-4}						
Algorithm 5	$8.9639e-05$	31	0.0392	$9.0156e-05$	34	0.0470
Algorithm 6	$9.0195e-05$	31	0.0398	$9.0948e-05$	34	0.0477
Algorithm A	$9.4987e-05$	73	1.0631	$9.3423e-05$	80	1.1726
Algorithm B	$9.5222e-05$	64	0.9101	$9.9153e-05$	102	1.4686
Algorithm C	$9.9863e-05$	73	0.0551	$9.8711e-05$	80	0.0647
TOL=10^{-5}						
Algorithm 5	$9.8757e-06$	47	0.0551	$9.7017e-06$	50	0.0636
Algorithm 6	$8.9280e-06$	48	0.0603	$8.9587e-06$	51	0.0665
Algorithm A	$9.8132e-06$	95	1.4282	$9.6389e-06$	102	1.4870
Algorithm B	$9.9517e-06$	111	1.5908	$9.6227e-06$	158	2.2261
Algorithm C	$9.4523e-06$	96	0.0726	$9.3330e-06$	103	0.0794
TOL=10^{-6}						
Algorithm 5	$9.8952e-07$	70	0.0784	$9.9119e-07$	79	0.0895
Algorithm 6	$9.5004e-07$	73	0.0877	$9.9963e-07$	92	0.0935
Algorithm A	$9.1353e-07$	118	1.7117	$9.9407e-07$	124	1.7968
Algorithm B	$9.8015e-07$	169	2.4079	$9.8293e-07$	217	3.0598
Algorithm C	$9.9037e-07$	118	0.0951	$9.7732e-07$	125	0.0948

The following method of mean square error is used for measuring the recovery accuracy:

$$\text{MSE} = \frac{\|v_n - v_*\|^2}{k},$$

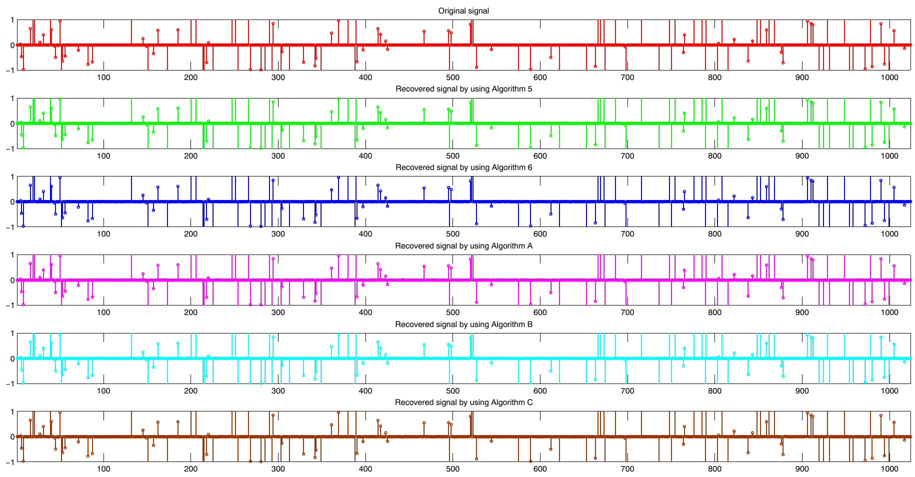

Fig. 3 Original signal and recovered signal with $p = 128$ and TOL=10^{-5}

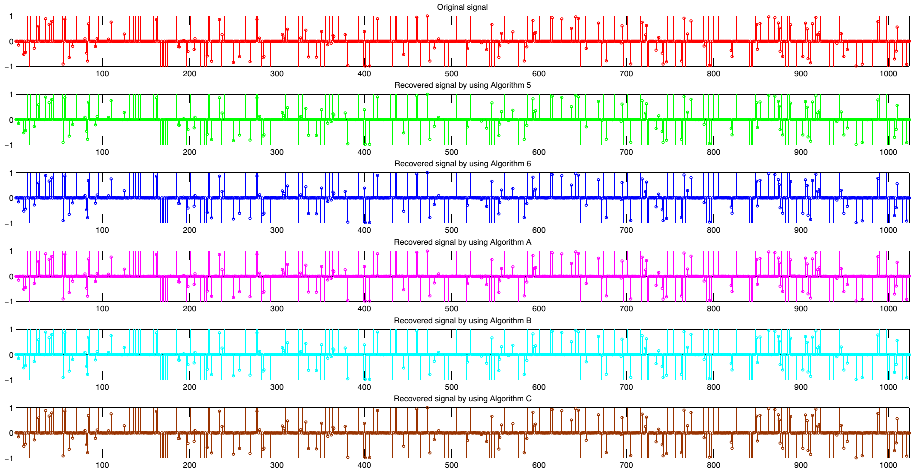

Fig. 4 Original signal and recovered signal with $p = 256$ and TOL=10^{-5}

which is required to be smaller than the given tolerant TOL.

We also compare our Algorithms with some previous algorithms ([24, Algorithm 2-SEF], [25, Algorithm (2.3)] and [10, Algorithm 4.1]). The parameters for each algorithm are chosen as follows:

- Algorithm A ([24, Algorithm 2-SEF]): $\rho_n = 3.9$.
- Algorithm B ([25, Algorithm (2.3)]): $\gamma = \dfrac{1}{2\|A\|^2}$.
- Algorithm C ([10, Algorithm 4.1]): $\rho_n = 3.9$.

The numerical results that we have obtained are shown in Table 5. In Figs. 3 and 4, we present the illustration of the original signal and recovered signal by using the above algorithms.

Remark 6.1 The numerical experiments above show that our new algorithms outperform several previous algorithms proposed in [10, 23–25] concerning the number of iterations and the CPU time.

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Declarations

Competing interests The authors declare that they have no conflict of interest.

Ethical Approval Not applicable.

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