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Vanishing and Non-negativity of the First Normal Hilbert Coefficient

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Dedicated to Professor Ngo Viet Trung on the occasion of his 70th birthday

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Abstract
Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced. We prove that, when $\widehat{R}$ is S_2, if there exists a parameter ideal $Q \subseteq R$ such that $\bar{e}_1(Q) = 0$, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$. This leads to an affirmative answer to a problem raised by Goto-Hong-Mandal [Goto, S., Hong, J., Mandal, M.: The positivity of the first coefficients of normal Hilbert polynomials. Proc. Amer. Math. Soc. **139**(7), 2399–2406 (2011)]. We also give an alternative proof (in fact a strengthening) of their main result. In particular, we show that if $\widehat{R}$ is equidimensional, then $\bar{e}_1(Q) \geq 0$ for all parameter ideals $Q \subseteq R$, and in characteristic $p > 0$, we actually have $e_1^*(Q) \geq 0$. Our proofs rely on the existence of big Cohen-Macaulay algebras.

Keywords Normal hilbert polynomial · Normal hilbert coefficients · Tight hilbert coefficients · Regular local rings · Big cohen-macaulay algebras

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1 Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d such that $\widehat{R}$ is reduced and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then for $n \gg 0$, $\ell(R/\overline{I^{n+1}})$ agrees with a polynomial in n of degree d, and we have integers $\overline{e}_0(I), \ldots, \overline{e}_d(I)$ such that

$$\ell(R/\overline{I^{n+1}}) = \overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{e}_d(I).$$

These integers $\overline{e}_i(I)$ are called the normal Hilbert coefficients of I.

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It is well-known that $\overline{e}_0(I)$ is the Hilbert-Samuel multiplicity of I, which is always a positive integer. In this paper, we are interested in the first coefficient $\overline{e}_1(I)$. It was proved by Goto-Hong-Mandal [10] that when $\widehat{R}$ is unmixed, $\overline{e}_1(I) \geq 0$ for all $\mathfrak{m}$-primary ideals $I \subseteq R$ (which answers a question posed by Vasconcelos [30]). They proposed a further problem in [10, Section 3] regarding the vanishing of $\overline{e}_1(I)$ and the regularity of the normalization of R. Since any $\mathfrak{m}$-primary ideal I is integral over a parameter ideal when the residue field is infinite, to study $\overline{e}_1(I)$ we may assume that $I = Q$ is a parameter ideal (i.e., it is generated by a system of parameters). In this paper, we prove the following main result which will lead to an affirmative answer to the question proposed in [10]. This theorem is also a generalization of the main result of [27].

Theorem 1.1 (Theorem 3.7) *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced and S_2. If $\overline{e}_1(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is regular and $\nu(\mathfrak{m}/Q) \leq 1$.*

In [7], it was shown that when R has characteristic $p > 0$, for $n \gg 0$, $\ell(R/(I^{n+1})^*)$ also agrees with a polynomial of degree d and one can define the tight Hilbert coefficients $e_0^*(I), \ldots, e_d^*(I)$ in a similar way (see Section 2 for more details). It is easy to see that $\overline{e}_1(I) \geq e_1^*(I)$. We strengthen the main result of [10] in characteristic $p > 0$ by showing that $e_1^*(Q) \geq 0$ for any parameter ideal $Q \subseteq R$ under mild assumptions.

Theorem 1.2 (Corollary 3.3) *Let $(R, \mathfrak{m})$ be an excellent local ring of characteristic $p > 0$ such that $\widehat{R}$ is reduced and equidimensional. Then we have $e_1^*(Q) \geq 0$ for all parameter ideals $Q \subseteq R$.*

Our proofs of both theorems rely on the existence of big Cohen-Macaulay algebras. In fact, we show that the tight Hilbert coefficients $e_i^*(I)$ is a special case of what we call the BCM Hilbert coefficients $e_i^B(I)$ associated to a big Cohen-Macaulay algebra B, and the latter can be defined in arbitrary characteristic. In this context, we will show in Theorem 3.1 that $\overline{e}_1(Q) \geq e_1^B(Q) \geq 0$ for all parameter ideals $Q \subseteq R$ when B satisfies some mild assumptions. This recovers and extends the main result of [10] in arbitrary characteristic.

Throughout this article, all rings are commutative with multiplicative identity 1. We will use $(R, \mathfrak{m})$ to denote a Noetherian local ring with unique maximal ideal $\mathfrak{m}$. We refer the reader to [4, Chapter 1-4] for basic notions such as Cohen-Macaulay rings, regular sequence, Euler characteristic, integral closure, and the Hilbert-Samuel multiplicity. We refer the reader to [29, Section 07QS] for the definition and basic properties of excellent rings. The paper is organized as follows. In Section 2 we collect the definitions and some basic results on big Cohen-Macaulay algebras and variants of Hilbert coefficients. In Section 3 we prove our main results and we propose some further questions.

2 Preliminaries

Recall that an element x in a ring R is integral over an ideal $I \subseteq R$ if it satisfies an equation of the form $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, where $a_k \in I^k$. The set of all elements integral over I forms an ideal and is denoted by $\overline{I}$, called the integral closure of I. An ideal $I \subseteq R$ is called integrally closed if $I = \overline{I}$. It is well-known that an element $x \in R$ is integral over I if and only if the image of x in $R/\mathfrak{p}$ is integral over $I(R/\mathfrak{p})$ for all minimal primes $\mathfrak{p}$, see [21, Proposition 1.1.5].

Suppose that R is a Noetherian ring of prime characteristic $p > 0$. The tight closure of an ideal $I \subseteq R$, introduced by Hochster–Huneke, is defined as follows:

$$I^* := \{x \in R \mid \text{there exists } c \in R - \cup_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p} \text{ such that } cx^{p^e} \in I^{[p^e]} \text{ for all } e \gg 0\}.$$

An ideal $I \subseteq R$ is called tightly closed if $I = I^*$. In general, tight closure is always contained in the integral closure, that is, $I^* \subseteq \overline{I}$ (see [15, Proposition on page 58]). Similar to integral closure, an element $x \in R$ is in the tight closure of I if and only if the image of x in $R/\mathfrak{p}$ is in the tight closure of $I(R/\mathfrak{p})$ for all minimal primes $\mathfrak{p}$, see [15, Theorem on page 49].

Let R be a Noetherian complete local domain and let $I \subseteq R$ be an ideal. The solid closure of I, denoted by $I^\bigstar$, consists of those element $x \in R$ such that there exists an R-algebra S such that $\mathrm{Hom}_R(S, R) \neq 0$ and such that $x \in IS$. One can define solid closure of ideals in more general rings, see [16, Definition 1.2], but we will only need this notion for complete local domains. It was shown in [16, Theorem 5.10] that solid closure is contained in the integral closure, i.e., $I^\bigstar \subseteq \overline{I}$. If R has prime characteristic $p > 0$, then solid closure agrees with tight closure $I^\bigstar = I^*$, see [16, Theorem 8.6].

2.1 Big Cohen-Macaulay Algebras

Let $(R, \mathfrak{m})$ be a Noetherian local ring. An R-algebra B, not necessarily Noetherian, is called balanced big Cohen-Macaulay over R if every system of parameters of R is a regular sequence on B and $\mathfrak{m}B \neq B$. Balanced big Cohen-Macaulay algebras exist, in equal characteristic, this is due to Hochster-Huneke [18], and in mixed characteristic, this is proved by André [1] (see also [2, 3, 12]). In this article, we need to compare the closure operation induced by a balanced big Cohen-Macaulay algebra with integral closure. We begin with the following result.

In what follows, when $R \to S$ is a (not necessarily injective) homomorphism of rings, $IS \cap R$ should be interpreted as the contraction of IS to R. That is, those elements of R whose image in S are contained in IS.

Lemma 2.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then the following conditions are equivalent:*

(1) *$\widehat{R}$ is equidimensional.*
(2) *There exists a balanced big Cohen-Macaulay R-algebra B such that*

$$I^B := IB \cap R \subseteq \overline{I} \text{ for all } \mathfrak{m}\text{-primary ideals } I \subseteq R. \qquad (\dagger)$$

(3) *There exists a balanced big Cohen-Macaulay R-algebra B such that $I^B \subseteq \overline{I}$ for all $I \subseteq R$.*

Proof Since (3) $\Rightarrow$ (2) is obvious, we only need to show (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1). Suppose that $\widehat{R}$ is equidimensional and let $P_1, \ldots, P_n$ be the minimal primes of $\widehat{R}$. Let B_i be any balanced big Cohen-Macaulay algebra over $\widehat{R}/P_i$. Since $\widehat{R}$ is equidimensional, each system of parameters of $\widehat{R}$ is also a system of parameters of $\widehat{R}/P_i$ and thus B_i is a balanced big Cohen-Macaulay algebra over $\widehat{R}$. It follows that $B := \prod_{i=1}^n B_i$ is a balanced big Cohen-Macaulay algebra over $\widehat{R}$.

Claim 2.2 $(I\widehat{R})^B = IB \cap \widehat{R} \subseteq \overline{I\widehat{R}}$.

Proof of the Claim Since integral closure can be checked after modulo each minimal prime, it suffices to show that $(I\widehat{R})^B \cdot (\widehat{R}/P_i) \subseteq \overline{I(\widehat{R}/P_i)}$. It is easy to see (by our construction of B) that

$$(I\widehat{R})^B \cdot (\widehat{R}/P_i) = (I(\widehat{R}/P_i))^{B_i}.$$

Since B_i is a solid algebra over the complete local domain $\widehat{R}/P_i$ by [16, Corollary 2.4], we have

$$(I(\widehat{R}/P_i))^{B_i} \subseteq (I(\widehat{R}/P_i))^{\bigstar} \subseteq \overline{I(\widehat{R}/P_i)},$$

where the second inclusion follows from [16, Theorem 5.10]. □

By the claim above, we have

$$I^B \subseteq (I\widehat{R})^B \cap R \subseteq \overline{I\widehat{R}} \cap R = \overline{I},$$

where the last equality follows from [21, Proposition 1.6.2].

We next assume there exists a balanced big Cohen-Macaulay R-algebra B that satisfies (†). We first note that $\widehat{B}$ (the $\mathfrak{m}$-adic completion of B) is still a balanced big Cohen-Macaulay algebra over $\widehat{R}$ by [4, Corollary 8.5.3]. If I is an $\mathfrak{m}$-primary ideal, then we have $R/I \cong \widehat{R}/I\widehat{R}$ and $B/IB \cong \widehat{B}/I\widehat{B}$ (see [29, Tag 05GG]). It follows that $(I\widehat{R})^{\widehat{B}} = (I^B)\widehat{R} \subseteq \overline{I}\widehat{R} = \overline{I\widehat{R}}$ (where the last equality follows from [21, Lemma 9.1.1]). Thus without loss of generality, we may replace R by $\widehat{R}$ and B by $\widehat{B}$ to assume that R is complete. Suppose that R is not equidimensional. Let $P_1, \dots, P_n$ be all the minimal primes of R such that $\dim(R/P_i) = \dim(R)$, and $Q_1, \dots, Q_m$ be all the minimal primes of R such that $\dim(R/Q_j) < d$. We pick $y \in Q_1 \cap \cdots \cap Q_m \setminus P_1 \cup \cdots \cup P_n$. Then y is a parameter element in R, and thus y is a nonzerodivisor on B, since B is balanced big Cohen-Macaulay. Since $y \cdot (P_1 \cap \cdots \cap P_n) \subseteq \sqrt{0}$, there exists t such that $y^t \cdot (P_1 \cap \cdots \cap P_n)^t = 0$. It follows that $(P_1 \cap \cdots \cap P_n)^t B = 0$. Hence

$$(P_1 \cap \cdots \cap P_n)^t \subseteq \mathfrak{m}^k B \cap R \subseteq \overline{\mathfrak{m}^k}$$

for all k by (†). Thus $(P_1 \cap \cdots \cap P_n)^t \subseteq \cap_k \overline{\mathfrak{m}^k} = \sqrt{0}$ by [21, Exercise 5.14], which is a contradiction. □

Remark 2.3 In the proof of Lemma 2.1, we have proved the fact that when $(R, \mathfrak{m})$ is a Noetherian complete local domain, then every balanced big Cohen-Macaulay algebra B satisfies (†). We suspect that when $(R, \mathfrak{m})$ is Noetherian, complete, reduced and equidimensional, then every balanced big Cohen-Macaulay algebra B such that $\mathrm{Supp}(\widehat{B}) = \mathrm{Spec}(R)$ satisfies (†).

2.2 Hilbert Coefficients

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d and let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Then for all $n \gg 0$ we have

$$\ell(R/I^{n+1}) = e_0(I)\binom{n+d}{d} - e_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d(I),$$

where $e_0(I), \dots, e_d(I)$ are all integers, and are called the Hilbert coefficients of I.

Now suppose that $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots$ is module-finite over the Rees algebra $R[It]$. For instance, by a famous result of Rees (see [21, Corollary 9.2.1]), this is the case when $\widehat{R}$ is reduced. Then one can show that for all $n \gg 0$, $\ell(R/\overline{I^{n+1}})$ agrees with a polynomial in n and one can write

$$\ell(R/\overline{I^{n+1}}) = \bar{e}_0(I)\binom{n+d}{d} - \bar{e}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_d(I),$$

where the integers $\overline{e}_0(Q), \ldots, \overline{e}_d(Q)$ are called the normal Hilbert coefficients. It is well-known that $e_0(I) = \overline{e}_0(I)$ agrees with the Hilbert-Samuel multiplicity $e(I, R)$ of I.

We also recall the tight Hilbert coefficients studied in [7]. Again, we suppose that $\widehat{R}$ is reduced and R has characteristic $p > 0$. Then we have

$$\ell(R/(I^{n+1})^*) = e_0^*(I)\binom{n+d}{d} - e_1^*(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^*(I)$$

for all $n \gg 0$, and the integers $e_0^*(I), \ldots, e_d^*(I)$ are called the tight Hilbert coefficients, see [7] for more details.

Now if B is a balanced big Cohen-Macaulay R-algebra that satisfies (†), then we know that $R \oplus I^B t \oplus (I^2)^B t^2 \oplus \cdots$ is an R-algebra that is also module-finite over $R[It]$: the fact that it is an R-algebra follows from the fact that $(I^a)^B(I^b)^B \subseteq (I^{a+b})^B$ for all a, b (i.e., $\{(I^n)^B\}_n$ form a graded family of ideals), and that it is module-finite over $R[It]$ follows because by (†), it is an $R[It]$-submodule of $R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots$, and the latter is module-finite over $R[It]$ (note that $R[It]$ is Noetherian). Based on the discussion above, one can show that for all $n \gg 0$, $\ell(R/(I^{n+1})^B)$ also agrees with a polynomial in n, and we write

$$\ell(R/(I^{n+1})^B) = e_0^B(I)\binom{n+d}{d} - e_1^B(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d^B(I)$$

for all $n \gg 0$ (see [19] for more general results). We call the integers $e_0^B(I), \ldots, e_d^B(I)$ the BCM Hilbert coefficients with respect to B. It is easy to see that $e_0^B(I) = e(I, R)$ is still the Hilbert-Samuel multiplicity of I, and that we always have $\overline{e}_1(I) \geq e_1^B(I) \geq e_1(I)$ by comparing the coefficients of n^{d-1} and noting that $I^n \subseteq (I^n)^B \subseteq \overline{I^n}$ for all n by (†).

Remark 2.4 We point out that when $(R, \mathfrak{m})$ is excellent and $\widehat{R}$ is reduced and equidimensional of characteristic $p > 0$, the tight Hilbert coefficient is a particular case of BCM Hilbert coefficient. This follows from the fact that under these assumptions, there exists a balanced big Cohen-Macaulay algebra B such that $I^* = I^B$ for all $I \subseteq R$ (and any such B will satisfy (†), since tight closure is contained in the integral closure [20, Theorem 1.3]). When R is a complete local domain this is proved in [15, Theorem on page 250]. In general, one can take such a B_i for each complete local domain $\widehat{R}/P_i$, where P_i is a minimal prime of $\widehat{R}$, and let $B = \prod B_i$. Since R is excellent, $I^*\widehat{R} = (I\widehat{R})^*$ (see [20, Proposition 1.5]) and as tight closure can be checked after modulo each minimal prime, it follows that $I^*\widehat{R} = (I\widehat{R})^B$ and thus $I^* = I^B$.

Throughout the rest of this article, we will be mainly working with parameter ideals, i.e., ideals generated by a system of parameters. As we mentioned in the introduction, this will not affect the study of $\overline{e}_1(I)$, since we can often enlarge the residue field and replace I by its minimal reduction.

3 The Main Results

In this section we prove our main results that $e_1^B(Q)$ (and hence $\overline{e}_1(Q)$) is always nonnegative for a parameter ideal Q, and that $\overline{e}_1(Q) = 0$ for some parameter ideal Q implies R is regular.

3.1 Non-negativity of $\bar{e}_1(Q)$ and $e_1^B(Q)$

Theorem 3.1 *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced and equidimensional. Let B be any balanced big Cohen-Macaulay R-algebra that satisfies* (†). Then for all parameter ideals $Q \subseteq R$ we have

$$\bar{e}_1(Q) \geq e_1^B(Q) \geq 0 \geq e_1(Q).$$

Remark 3.2 $\bar{e}_1(Q) \geq 0$ was the main theorem of [10, Theorem 1.1], and $0 \geq e_1(Q)$ was first proved in full generality in [26, Theorem 3.6]. Our method gives alternative proofs, and is inspired by some work of Goto [9] (in fact the proof that $e_1(Q) \leq 0$ via this method is due to Goto [9], see also [13, Theorem 1.1] for a generalization).

Corollary 3.3 *Let $(R, \mathfrak{m})$ be an excellent local ring of characteristic $p > 0$ such that $\widehat{R}$ is reduced and equidimensional. Then we have $e_1^*(Q) \geq 0$ for all parameter ideals $Q \subseteq R$.*

Proof This follows from Theorem 3.1 and Remark 2.4. □

Proof of Theorem 3.1 Let $Q = (x_1, \ldots, x_d) \subseteq R$. Set $S = R[[y_1, \ldots, y_d]]$ and $\mathfrak{q} = (y_1 - x_1, \ldots, y_d - x_d) \subseteq S$. For all $n \geq 0$ we have $y_1, \ldots, y_d$ is a system of parameters on $S/\mathfrak{q}^{n+1}$, and that

$$R/Q^{n+1} = S/(\mathfrak{q}^{n+1} + (y_1, \ldots, y_d)).$$

We next note that

$$\begin{aligned} e_0(Q) = e(Q, R) &= \chi(x_1, \ldots, x_d; R) \\ &= \chi(x_1, \ldots, x_d, y_1, \ldots, y_d; S) \\ &= \chi(y_1, \ldots, y_d, y_1 - x_1, \ldots, y_d - x_d; S) \\ &= \chi(y_1, \ldots, y_d; S/\mathfrak{q}) \\ &= e(y_1, \ldots, y_d; S/\mathfrak{q}), \end{aligned}$$

where the equalities on the second and the fourth lines follow from the fact that $y_1, \ldots, y_d$ and $y_1 - x_1, \ldots, y_d - x_d$ are both regular sequences on S. Now since $S/\mathfrak{q}^{n+1}$ has a filtration by $\binom{n+d}{d}$ copies of $S/\mathfrak{q}$, by the additivity formula for multiplicity (see [21, Theorem 11.2.3]) we have

$$e(y_1, \ldots, y_d; S/\mathfrak{q}^{n+1}) = \binom{n+d}{d} e(y_1, \ldots, y_d; S/\mathfrak{q}).$$

Putting these together, we have

$$\binom{n+d}{d} e_0(Q) = \binom{n+d}{d} e(y_1, \ldots, y_d; S/\mathfrak{q}) = e(y_1, \ldots, y_d; S/\mathfrak{q}^{n+1}),$$

and

$$\ell(R/Q^{n+1}) = \ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \ldots, y_d)S/\mathfrak{q}^{n+1}}\right).$$

Since $y_1, \ldots, y_d$ is a system of parameters of $S/\mathfrak{q}^{n+1}$, we have

$$\ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \ldots, y_d)S/\mathfrak{q}^{n+1}}\right) \geq e(y_1, \ldots, y_d; S/\mathfrak{q}^{n+1}).$$

It follows that

$$\ell(R/Q^{n+1}) \geq \binom{n+d}{d} e_0(Q),$$

and thus $e_1(Q) \leq 0$ (note that this does not require any assumption on $\widehat{R}$).

It remains to show that $e_1^B(Q) \geq 0$ (since $\overline{e}_1(Q) \geq e_1^B(Q)$ always holds, see the discussion in Section 2.2). Since $e_0^B(Q) = e_0(Q)$, it is enough to show that

$$\ell(R/(Q^{n+1})^B) \leq \binom{n+d}{d} e_0(Q) \tag{1}$$

for any balanced big Cohen-Macaulay algebra B. Below we will prove a slightly stronger result. Recall that for a parameter ideal $(z_1, \ldots, z_d)$ of R, the limit closure is defined as $(z_1, \ldots, z_d)^{\lim_R} := \bigcup_t (z_1^{t+1}, \ldots, z_d^{t+1}) : (z_1 z_2 \cdots z_d)^t$. The limit closure does not depend on the choice of the elements $z_1, \ldots, z_d$ (i.e., it only depends on the ideal $(z_1, \ldots, z_d)$). This is because $(z_1, \ldots, z_d)^{\lim_R}/(z_1, \ldots, z_d)$ is the kernel of the natural map $R/(z_1, \ldots, z_d) \to H_{\mathfrak{m}}^d(R)$.

Claim 3.4 Set $\Lambda_{n+1} = \{(\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \mid \alpha_i \geq 1 \text{ and } \sum_{i=1}^d \alpha_i = 1 + n\}$ and for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Lambda_{n+1}$, set $Q(\alpha) = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})$. Then we have

$$\ell\left(R/(\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R})\right) \leq \binom{n+d}{d} e_0(Q).$$

Proof of the Claim Recall that we have already proved that

$$\binom{n+d}{d} e_0(Q) = e(y_1, \ldots, y_d; S/\mathfrak{q}^{n+1}).$$

Moreover, we always have (for example, see [23, Theorem 9])

$$e(y_1, \ldots, y_d; S/\mathfrak{q}^{n+1}) \geq \ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \ldots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}}\right).$$

Therefore it is enough to prove that

$$\ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1, \ldots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}}\right) \geq \ell\left(R/(\bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R})\right). \tag{2}$$

Consider $z \in S$ whose image in $S/\mathfrak{q}^{n+1}$ is contained in $(y_1, \ldots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}$. This means there exists some $t \geq 1$ such that

$$\begin{aligned}(y_1 y_2 \cdots y_d)^t z &\in (y_1^{t+1}, \ldots, y_d^{t+1}, (y_1 - x_1, \ldots, y_d - x_d)^{n+1}) \\ &\subseteq (y_1^{t+1}, \ldots, y_d^{t+1}, (y_1 - x_1)^{\alpha_1}, \ldots, (y_d - x_d)^{\alpha_d})\end{aligned}$$

for each $\alpha = (\alpha_1, \ldots, \alpha_d) \in \Lambda_{n+1}$. This implies

$$z \in (y_1, \ldots, y_d, (y_1 - x_1)^{\alpha_1}, \ldots, (y_d - x_d)^{\alpha_d})^{\lim_S} = (y_1, \ldots, y_d, x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})^{\lim_S}.$$

But since $S = R[[y_1, \ldots, y_d]]$, it is straightforward to check that

$$(y_1, \ldots, y_d, x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})^{\lim_S} = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})^{\lim_R} S + (y_1, \ldots, y_d) S.$$

Thus if the image of z is contained in $(y_1, \ldots, y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}$, then after modulo $(y_1, \ldots, y_d)S$, $\overline{z} \in (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d})^{\lim_R}$ for each $(\alpha_1, \ldots, \alpha_d) \in \Lambda_{n+1}$, i.e., $\overline{z} \in \bigcap_{\alpha \in \Lambda_{n+1}} Q(\alpha)^{\lim_R}$. It follows that the natural surjection

$$S/\mathfrak{q}^{n+1} \xrightarrow{\text{mod } (y_1, \ldots, y_d)S} R/Q^{n+1}$$

induces a surjection

$$\frac{S/\mathfrak{q}^{n+1}}{(y_1,\ldots,y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}} \twoheadrightarrow \frac{R}{\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^{\lim_R}}.$$

This clearly establishes (2) and completes the proof of the claim. □

Finally, since $x_1,\ldots,x_d$ is a regular sequence on B, we have $Q(\alpha)^{\lim_R} \subseteq Q(\alpha)^B$ for each α. It follows that $\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^{\lim_R} \subseteq \bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^B$. Now if $x \in \bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^B$, then we have $x \in \left(\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)B\right) \cap R$. But since $x_1,\ldots,x_d$ is a regular sequence on B, it is not hard to check that $\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)B = Q^{n+1}B$ (see [28, Remark 3.3] or [11]) and thus $x \in Q^{n+1}B\cap R = (Q^{n+1})^B$. Therefore we have $\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^B = (Q^{n+1})^B$. Putting these together, we have

$$\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^{\lim_R} \subseteq \bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^B = (Q^{n+1})^B.$$

Therefore by Claim 3.4, we have

$$\ell(R/(Q^{n+1})^B) \le \binom{n+d}{d} e_0(Q)$$

as wanted. □

Remark 3.5 With notation as in Theorem 3.1, we do not know whether we have

$$\ell\left(\frac{S/\mathfrak{q}^{n+1}}{(y_1,\ldots,y_d)^{\lim_{S/\mathfrak{q}^{n+1}}}}\right) = \ell\left(\frac{R}{\bigcap_{\alpha\in\Lambda_{n+1}} Q(\alpha)^{\lim_R}}\right).$$

Remark 3.6 With notation as in Claim 3.4, fix a generating set $(x_1,\ldots,x_d)$ of Q, one may try to define $(Q^n)^{\lim} := \bigcap_{\alpha\in\Lambda_n} Q(\alpha)^{\lim}$ and call this the limit closure of Q^n. However, it is not clear to us whether this is independent of the choice of the generators $x_1,\ldots,x_d$. It is also not clear to us (even when fixing the generators $(x_1,\ldots,x_d)$ of Q) whether $\{(Q^n)^{\lim}\}_n$ form a graded family of ideals, i.e., we do not know whether $(Q^a)^{\lim}(Q^b)^{\lim} \subseteq (Q^{a+b})^{\lim}$ for all a, b.

3.2 Vanishing of $\bar{e}_1(Q)$

In this subsection we prove our main result. Recall that for a finitely generated R-module M, we use the notation $\nu(M)$ to denote its minimal number of generators.

Theorem 3.7 *Let $(R,\mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced and S_2. If $\bar{e}_1(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is regular and $\nu(\mathfrak{m}/Q) \le 1$.*

Proof We first note that if R is Cohen-Macaulay, then by [19, Corollary 4.9], Q is integrally closed.[1] But then by the main result of [8], R is regular and $\nu(\mathfrak{m}/Q) \le 1$.

We may assume that R is complete. We use induction on $d := \dim(R)$. If $d \le 2$, then R is Cohen-Macaulay and we are done by the previous paragraph. Now suppose that $d \ge 3$ and we have established the theorem in dimension $< d$. Let $Q = (x_1,\ldots,x_d)$, $R' = R[t_1,\ldots,t_d]_{\mathfrak{m}R[t_1,\ldots,t_d]}$, and $x = t_1x_1 + \cdots + t_dx_d$.

[1] Using the language of [19], $\bar{e}_1(Q) = 0$ in a Cohen-Macaulay ring implies that the reduction number of the filtration $\{\overline{Q^n}\}_n$ is 0, i.e., a minimal reduction of $\overline{Q}$ is equal to $\overline{Q}$, this is saying that Q is integrally closed.

Claim 3.8 We have $R'' := \widehat{R'}/x\widehat{R'}$ is reduced, equidimensional, and S_2 on the punctured spectrum. Moreover, we have $\overline{e}_1(QR'') = 0$.

Proof This is essentially contained in [10, Proof of Theorem 1.1] under the assumption that R is (complete and) normal. The key ingredient is [22, Theorem 2.1]. Since [22] does not require the normal assumption, the same proof as in [10] works in our setup. For the ease of the reader (and also because the S_2 on the punctured spectrum conclusion is not stated in [10]), we give a complete and self-contained argument here.

First of all, since R' is S_2 and R_0, we know that R'/xR' is S_1 and R_0 (see [25, Lemma 10]), so R'/xR' and thus R'' is reduced (as R'/xR' is excellent). R'' is clearly equidimensional since $\widehat{R'}$ is so and x is a parameter in $\widehat{R'}$. To see R'' is S_2 on the punctured spectrum, it is enough to show R'/xR' is S_2 on the punctured spectrum (as R'/xR' is excellent). Now we use a similar argument as in [25, Lemma 10] (the idea follows from [14]): every non-maximal $P' \in \operatorname{Spec}(R'/xR')$ corresponds to a prime ideal of R' that contracts to a non-maximal $P \in \operatorname{Spec}(R)$, thus $(R'/xR')_{P'}$ is a localization of $R_P[t_1, \dots, t_d]/(t_1x_1 + \cdots + t_dx_d)$, but at least one x_i is invertible in R_P (say x_1 is invertible) so the latter is isomorphic to $R_P[t_2, \dots, t_d]$, which is S_2 as R_P is S_2, thus R'/xR' is S_2 on the punctured spectrum as wanted.

It remains to show that $\overline{e}_1(QR'') = 0$. By [21, Corollary 6.8.13], we have a short exact sequence

$$0 \to R'/\overline{Q^n} \xrightarrow{\cdot x} R'/\overline{Q^{n+1}} \to R'/(x, \overline{Q^{n+1}}) \to 0.$$

Since $\overline{e}_1(Q) = 0$, for $n \gg 0$ we have

$$\ell(R'/\overline{Q^{n+1}}) = \overline{e}_0(Q) \cdot \binom{n+d}{d} + \overline{e}_2(Q) \cdot \binom{n+d-2}{d-2} + o(n^{d-2}),$$

$$\ell(R'/\overline{Q^{n}}) = \overline{e}_0(Q) \cdot \binom{n+d-1}{d} + \overline{e}_2(Q) \cdot \binom{n+d-3}{d-2} + o(n^{d-2}).$$

It follows that

$$\ell(R'/(x, \overline{Q^{n+1}})) = \overline{e}_0(Q) \cdot \binom{n+d-1}{d-1} + o(n^{d-2}). \tag{3}$$

We next show that for all $n \gg 0$, $\overline{Q^n}(R'/xR') = \overline{Q^n(R'/xR')}$. Once this is proved, we will have $\overline{Q^n}R'' = \overline{Q^nR''}$ for all $n \gg 0$ by [21, Lemma 9.1.1] and thus (3) will tell us that

$$\ell(R''/\overline{Q^{n+1}R''}) = \overline{e}_0(Q)\binom{n+d-1}{d-1} + o(n^{d-2}).$$

Since x is a general element of Q, we have $\overline{e}_0(Q) = e(Q, R') = e(QR'', R'') = \overline{e}_0(QR'')$ and so the above equation implies that $\overline{e}_1(QR'') = 0$ as wanted.

To show $\overline{Q^n}(R'/xR') = \overline{Q^n(R'/xR')}$ for $n \gg 0$, let $\mathcal{R}'$ denote the integral closure of $R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]$. Concretely, $\mathcal{R}'$ is the $\mathbb{Z}$-graded ring such that $\mathcal{R}_n = \overline{Q^n}t^n$ for $n > 0$ and $\mathcal{R}'_n = R't^n$ for $n \le 0$. Consider the map

$$\mathcal{R}'/(xt)\mathcal{R}' \to R'[t, t^{-1}]/(xt)R'[t, t^{-1}].$$

If we localize at any prime ideal $\mathcal{P}$ of $R'[Qt, t^{-1}]$ that does not contain (Qt, t^{-1}), then we note that $(\mathcal{R}'/(xt)\mathcal{R}')_{\mathcal{P}}$ is integrally closed inside $(R'[t, t^{-1}]/(xt)R'[t, t^{-1}])_{\mathcal{P}}$. To see this, one can "unlocalize" the ring R', and consider the integral closure of $R[t_1, \dots, t_d][Qt, t^{-1}]$ inside $R[t_1, \dots, t_d][t, t^{-1}]$, call this ring $\mathcal{R}$. If one localizes the map

$$\mathcal{R}/(xt)\mathcal{R} \to R[t_1, \dots, t_d][t, t^{-1}]/(xt)R[t_1, \dots, t_d][t, t^{-1}]$$

at any prime ideal that does not contain (Qt, t^{-1}) (say it does not contain x_1t), then the resulting map is a localization of $R[t_2, \ldots, t_d][\overline{Q^n t^n}, t^{-1}][\frac{1}{x_1t}] \to R[t_2, \ldots, t_d][t, t^{-1}][\frac{1}{x_1t}]$, and the former is already integrally closed in the latter.

Since the radical of (Qt, t^{-1}) is the unique homogeneous maximal ideal of $R[Qt, t^{-1}]$, it follows that $\mathcal{R}'/(xt)\mathcal{R}'$ and the integral closure of $R'[Qt, t^{-1}]/(xt)R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]/(xt)R'[t, t^{-1}]$ agree in large degree. But note that for $n > 0$,

$$[\mathcal{R}'/(xt)\mathcal{R}']_n \cong \frac{\overline{Q^n}}{x\overline{Q^{n-1}}} \cdot t^n \cong \frac{\overline{Q^n}}{x(\overline{Q^n} : x)} \cdot t^n \cong \frac{\overline{Q^n}}{(xR') \cap \overline{Q^n}} \cdot t^n \cong \overline{Q^n}(R'/xR') \cdot t^n,$$

where we have used [21, Corollary 6.8.13] again, while the degree n part of the integral closure of $R'[Qt, t^{-1}]/(xt)R'[Qt, t^{-1}]$ inside $R'[t, t^{-1}]/(xt)R'[t, t^{-1}]$ is $\overline{Q^n(R'/xR')} \cdot t^n$. Thus the fact that they agree in degree $n \gg 0$ is precisely saying that $\overline{Q^n}(R'/xR') = \overline{Q^n(R'/xR')}$ for $n \gg 0$. $\square$

Now we come back to the proof of the theorem. Let S be the S_2-ification of R''. We have a short exact sequence

$$0 \to R'' \to S \to S/R'' \to 0$$

such that S/R'' has finite length (since R'' is S_2 on the punctured spectrum). Also note that $(S, \mathfrak{n})$ is (complete) local by [17, Proposition (3.9)] and that S is reduced (since S is a subring of the total quotient ring of R''). Since $R'' \to S$ is an integral extension, we have $\overline{IS} \cap R'' = \overline{I}$ for every ideal $I \subseteq R''$ by [21, Proposition 1.6.1]. It follows that $\ell_{R''}(S/\overline{Q^nS}) \geq \ell_{R''}(R''/\overline{Q^nR''})$ for all $n \geq 0$. Thus for $n \gg 0$ we have

$$\begin{aligned} &\overline{e}_0(QR'')\binom{n+d}{d} - \overline{e}_1(QR'')\binom{n+d-1}{d-1} + o(n^{d-1}) \\ &= \ell_{R''}(R''/\overline{Q^{n+1}R''}) \\ &\leq \ell_{R''}(S/\overline{Q^{n+1}S}) \\ &= [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \ell_S(S/\overline{Q^{n+1}S}) \\ &= [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \left(\overline{e}_0(QS)\binom{n+d}{d} - \overline{e}_1(QS)\binom{n+d-1}{d-1} + o(n^{d-1})\right). \end{aligned}$$

Since S is a rank one module over R'', we also know that

$$\overline{e}_0(QR'') = e(QR'', R'') = [S/\mathfrak{n} : R/\mathfrak{m}] \cdot e(QS, S) = [S/\mathfrak{n} : R/\mathfrak{m}] \cdot \overline{e}_0(QS),$$

where the second equality is the projection formula for the Hilbert-Samuel multiplicity (which can be seen by combining [21, Theorems 11.2.4 and 11.2.7]). Putting these together we have

$$[S/\mathfrak{n} : R/\mathfrak{m}] \cdot \overline{e}_1(QS) \leq \overline{e}_1(QR'') = 0.$$

But since $\overline{e}_1(QS) \geq 0$ by [10, Theorem 1.1] (see Theorem 3.1), we must have $\overline{e}_1(QS) = 0$. Now $(S, \mathfrak{n})$ is a reduced complete local ring that is S_2 and $\dim(S) = d - 1$, such that $\overline{e}_1(QS) = 0$. By our inductive hypothesis, we know that S is regular. But since S/R'' has finite length, by the long exact sequence of local cohomology induced by $0 \to R'' \to S \to S/R'' \to 0$, we obtain that

$$H^i_\mathfrak{m}(R'') = 0 \text{ for all } i < \dim(R'') \text{ and } i \neq 1, \text{ and } H^1_\mathfrak{m}(R'') \cong S/R''.$$

At this point, we consider the long exact sequence of local cohomology induced by $0 \to \widehat{R'} \xrightarrow{\cdot x} \widehat{R'} \to R'' \to 0$, we get

$$0 = H^1_\mathfrak{m}(R') \to H^1_\mathfrak{m}(R'') \to H^2_\mathfrak{m}(R') \xrightarrow{\cdot x} H^2_\mathfrak{m}(R') \to H^2_\mathfrak{m}(R'') \to \cdots.$$

If $d \geq 4$, then $\dim(R'') \geq 3$ and thus $H^2_{\mathfrak{m}}(R'') = 0$. Since $\widehat{R'}$ is S_2, $H^2_{\mathfrak{m}}(R')$ has finite length and the above exact sequence tells us that $H^2_{\mathfrak{m}}(R') = 0$ by Nakayama's lemma. But then by the above exact sequence again, we have $H^1_{\mathfrak{m}}(R'') = 0$ and hence $S/R'' = 0$. Thus $R'' \cong S$ is regular. But then $\widehat{R'}$ and hence R is regular as wanted.

Finally, suppose $d = 3$. Let B be a balanced big Cohen-Macaulay algebra of $\widehat{R'}$ that is $\mathfrak{m}$-adic complete, then B/xB is a balanced big Cohen-Macaulay algebra of R''. It follows that the canonical map $R'' \to B/xB$ factors through S.

Claim 3.9 B/xB is a balanced big Cohen-Macaulay algebra over S.

Proof of the Claim It is clear that some system of parameters of S (namely those coming from R'') are regular sequences on B/xB. To see that every system of parameters of S is a regular sequence on B/xB, we first note that B/xB is $\mathfrak{m}$-adically complete: since B is $\mathfrak{m}$-adic complete, B/xB is derived $\mathfrak{m}$-complete by [29, Tag 091U], take (y, z) that is a system of parameters of R'', then as y, z is a regular sequence on B/xB, the derived completion with respect to (y, z), which is B/xB itself, agrees with the usual completion with respect to (y, z) by [29, Tag 0920] (equivalently, with respect to $\mathfrak{m}$ as $\sqrt{(y, z)} = \mathfrak{m}$). Hence by [4, Corollary 8.5.3], every system of parameters of S is a regular sequence on $\widehat{B/xB} \cong B/xB$. □

Note that $\dim(R'') = \dim(S) = 2$ and S is regular, thus the long exact sequence of local cohomology induced by $0 \to R'' \to S \to S/R'' \to 0$ implies that $H^2_{\mathfrak{m}}(R'') \cong H^2_{\mathfrak{m}}(S)$. Hence we have the following commutative algebra:

$$\begin{tikzcd}
 & & H^2_{\mathfrak{m}}(S) \arrow[d, equal] & & & \\
H^2_{\mathfrak{m}}(R') \arrow[r, "\cdot x"] & H^2_{\mathfrak{m}}(R') \arrow[r] \arrow[d] & H^2_{\mathfrak{m}}(R'') \arrow[r] \arrow[d, hook, "\phi"] & H^3_{\mathfrak{m}}(R') \arrow[r] \arrow[d] & H^3_{\mathfrak{m}}(R') \arrow[r] \arrow[d] & 0 \\
 & 0 = H^2_{\mathfrak{m}}(B) \arrow[r] & H^2_{\mathfrak{m}}(B/xB) \arrow[r] & H^3_{\mathfrak{m}}(B) \arrow[r] & H^3_{\mathfrak{m}}(B) \arrow[r] & 0
\end{tikzcd}$$

where the injectivity of ϕ follows from the fact that B/xB is a balanced big Cohen-Macaulay algebra over S and thus faithfully flat over S (as S is regular). Chasing this diagram we find that the map $H^2_{\mathfrak{m}}(R') \xrightarrow{\cdot x} H^2_{\mathfrak{m}}(R')$ is surjective. But since $\widehat{R'}$ is S_2, $H^2_{\mathfrak{m}}(R')$ has finite length, thus $H^2_{\mathfrak{m}}(R') = 0$ by Nakayama's lemma. Hence $\widehat{R'}$ is Cohen-Macaulay and thus R'' is also Cohen-Macaulay. But then $R'' \cong S$ and so R'' is regular and thus $\widehat{R'}$ is regular. Thus R is regular as wanted.

Now we have established that R is regular, we can repeat the argument in the first paragraph of the proof to show that $\nu(\mathfrak{m}/Q) \leq 1$ (essentially, this follows from the main result of [8]). □

As a consequence, we answer the problem raised in [10, Section 3] for excellent rings.

Corollary 3.10 *Let R be an excellent local ring such that $\widehat{R}$ is reduced and equidimensional. Suppose that $I \subseteq R$ is an $\mathfrak{m}$-primary ideal such that $\overline{e}_1(I) = 0$. Then R^{N}, the normalization of R, is regular and IR^{N} is normal (i.e., all powers of IR^{N} are integrally closed in R^{N}).*

Proof Replacing R by $R[t]_{\mathfrak{m}R[t]}$, we may assume that the residue field of R is infinite (we leave it to the readers to check that the hypotheses and conclusions are stable under such a base change). Let S be the S_2-ification of R. We will show that the $\mathfrak{m}$-adic completion of $\widehat{S}$ is regular. Since R is excellent, $\widehat{S}$ agrees with the S_2-ification of $\widehat{R}$ by [17, Proposition 3.8]. Thus $\widehat{S}$ is semilocal, reduced, and S_2. Since $\overline{J\widehat{S}} \cap \widehat{R} = \overline{J}$ for every $\mathfrak{m}$-primary ideal $J \subseteq \widehat{R}$ by [21, Proposition 1.6.1], we have $\ell_{\widehat{R}}(\widehat{R}/\overline{J}) \leq \ell_{\widehat{R}}(\widehat{S}/\overline{J\widehat{S}})$.

Let $\mathfrak{n}_1, \ldots, \mathfrak{n}_s$ be the maximal ideals of $\widehat{S}$ and let $S_i := (\widehat{S})_{\mathfrak{n}_i}$ (in fact, since $\widehat{S}$ is complete, we have $\widehat{S} \cong \prod_{i=1}^s S_i$, and each S_i is complete local, reduced, and S_2). Then we have

$$\begin{aligned}
&\overline{e}_0(I)\binom{n+d}{d} - \overline{e}_1(I)\binom{n+d-1}{d-1} + o(n^{d-1})\\
&= \ell_R(R/\overline{I^{n+1}}) = \ell_{\widehat{R}}(\widehat{R}/\overline{I^{n+1}\widehat{R}})\\
&\leq \ell_{\widehat{R}}(\widehat{S}/\overline{I^{n+1}\widehat{S}})\\
&= \sum_{i=1}^{s}[S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \ell_{S_i}(S_i/\overline{I^{n+1}S_i})\\
&= \sum_{i=1}^{s}[S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \left(\overline{e}_0(IS_i)\binom{n+d}{d} - \overline{e}_1(IS_i)\binom{n+d-1}{d-1} + o(n^{d-1})\right),
\end{aligned}$$

where we have used [21, Lemma 9.1.1] for the equality in the second line. Since $\widehat{S}$ is a rank one module over $\widehat{R}$, we also know that

$$\overline{e}_0(I) = e(I\widehat{R}, \widehat{R}) = \sum_{i=1}^{s}[S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot e(IS_i, S_i) = \sum_{i=1}^{s}[S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \overline{e}_0(IS_i),$$

where we have used the projection formula for the Hilbert-Samuel multiplicity (see [21, Theorems 11.2.4 and 11.2.7]). The above inequality implies that

$$\sum_{i=1}^{s}[S_i/\mathfrak{n}_i : R/\mathfrak{m}] \cdot \overline{e}_1(IS_i) \leq \overline{e}_1(I) = 0.$$

But since $\overline{e}_1(IS_i) \geq 0$ by [10, Theorem 1.1], we must have $\overline{e}_1(IS_i) = 0$ for all i. Let Q be a minimal reduction of I (note that Q is a parameter ideal of R, since we have reduced to the case that R has an infinite residue field). It follows that $\overline{e}_1(QS_i) = 0$ and thus by Theorem 3.7, S_i is regular and $\nu(\mathfrak{n}_i/Q) \leq 1$. But then QS_i is normal in S_i. It follows that $\widehat{S} \cong \prod_{i=1}^s S_i$ is regular, $Q\widehat{S}$ is normal in $\widehat{S}$ and in particular, $\overline{Q\widehat{S}} = \overline{I\widehat{S}}$.

Since $S \to \widehat{S} \cong \widehat{R} \otimes_R S$ is faithfully flat with geometrically regular fibers (as R is excellent). We have S is regular and $QS = IS$ is normal in S by [21, Theorem 19.2.1]. Finally, since S is regular, S agrees with the normalization R^{N} of R.

Remark 3.11 The condition $\widehat{R}$ is S_2 cannot be dropped in Theorem 3.7. This was already observed in [10, Section 3]. We give a different example that is a complete local domain. Let $R = k[[x, xy, y^2, y^3]]$, where k is a field. Then the S_2-ification of R is $S = k[[x, y]]$ and we have $0 \to R \to S \to S/R \cong k \cdot \overline{y} \to 0$. Let $Q = (x, y^2) \subseteq R$ and we claim that $\overline{e}_1(Q) = 0$. To see this, note that $QS = (x, y^2) \subseteq S$ is normal and $\ell(S/Q^{n+1}S) = 2 \cdot \binom{n+2}{2}$. It follows from the short exact sequence

$$0 \to R/\overline{Q^{n+1}} \to S/Q^{n+1}S \to k \to 0$$

that $\ell(R/\overline{Q^{n+1}}) = 2 \cdot \binom{n+2}{2} - 1$. In particular, $\overline{e}_1(Q) = 0$.

Recall that a Noetherian local ring $(R, \mathfrak{m})$ of prime characteristic $p > 0$ is called F-rational if every ideal generated by a system of parameters is tightly closed. It was mentioned in [5] that Huneke asked that when $\widehat{R}$ is reduced and equidimensional of prime characteristic $p > 0$, whether $e_1^*(Q) = 0$ for some system of parameters $Q \subseteq R$ implies R is F-rational. In general, counter-examples to the question were constructed in [5, Examples 5.4 and 5.5] (in fact, the example in Remark 3.11 is a counter-example that is a complete local domain). However, all these examples do not satisfy Serre's S_2 condition.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let B be a big Cohen-Macaulay R-algebra. Recall that R is called BCM_B-rational if R is Cohen-Macaulay and the natural map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ is injective, where $d = \dim(R)$. If R is an excellent local ring of prime characteristic $p > 0$, then R is F-rational if and only if R is BCM_B-rational for all big Cohen-Macaulay algebra B, see [24, Proposition 3.5].

We propose the following conjecture relating the vanishing of $e_1^B(Q)$ and BCM_B-rational singularities, which modifies Huneke's question and makes sense in all characteristics.

Conjecture 3.12 *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced and S_2. Let B be a balanced big Cohen-Macaulay R-algebra that satisfies* (†). If $e_1^B(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is BCM_B-rational.

In particular, if R is excellent and has characteristic $p > 0$ (such that $\widehat{R}$ is reduced and S_2), and $e_1^*(Q) = 0$ for some parameter ideal $Q \subseteq R$, then R is F-rational.

We have the following partial result towards Conjecture 3.12, which is an analog of the main result of [27].

Proposition 3.13 *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\widehat{R}$ is reduced and equidimensional. Let B be a balanced big Cohen-Macaulay R-algebra that satisfies* (†). If $e_1^B(Q) = e_1(Q)$ for some parameter ideal $Q \subseteq R$, then R is BCM_B-rational.

In particular, if R is excellent and has characteristic $p > 0$, and $e_1^*(Q) = e_1(Q)$ for some parameter ideal $Q \subseteq R$, then R is F-rational.

Proof By Theorem 3.1, we know that $e_1^B(Q) = e_1(Q) = 0$. By the main result of [6], $e_1(Q) = 0$ implies that R is Cohen-Macaulay. By [19, Corollary 4.9], we have $Q^B = Q$. Now we consider the commutative diagram:

$$\begin{tikzcd} R/Q \arrow[r, hook] \arrow[d, hook] & B/QB \arrow[d, hook] \\ H^d_{\mathfrak{m}}(R) \arrow[r] & H^d_{\mathfrak{m}}(B) \end{tikzcd}$$

where the injectivity of the top row follows from $Q^B = Q$, the injectivity of the left column is because R is Cohen-Macaulay, and the injectivity of the right column is because B is balanced big Cohen-Macaulay. Since R is Cohen-Macaulay, we know that $\mathrm{Soc}(R/Q) \cong \mathrm{Soc}(H^d_{\mathfrak{m}}(R))$. Chasing the commutative diagram we find that the map $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B)$ is injective. Therefore, R is BCM_B-rational. □

Remark 3.14 It is clear from the proof of Proposition 3.13 that Conjecture 3.12 holds when R is Cohen-Macaulay, and this essentially follows from [19, Corollary 4.9].

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