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The v-Number of Binomial Edge Ideals

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Abstract
The invariant v-number was introduced very recently in the study of Reed-Muller-type codes. Jaramillo and Villarreal (J. Combin. Theory Ser. A 177:105310, 2021) initiated the study of the v-number of edge ideals. Inspired by their work, we take the initiation to study the v-number of binomial edge ideals in this paper. We discuss some properties and bounds of the v-number of binomial edge ideals. We explicitly find the v-number of binomial edge ideals locally at the associated prime corresponding to the cutset ∅. We show that the v-number of Knutson binomial edge ideals is less than or equal to the v-number of their initial ideals. Also, we classify all binomial edge ideals whose v-number is 1. Moreover, we try to relate the v-number with the Castelnuovo-Mumford regularity of binomial edge ideals and give a conjecture in this direction.

Keywords v-number · Binomial edge ideals · Castelnuovo-Mumford regularity · Initial ideals · Completion set

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1 Introduction

Let $R = K[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} R_d$ denote the polynomial ring in n variables over a field K with the standard grading. For a graded ideal I of R, the set of *associated prime* ideals of I, denoted by $\mathrm{Ass}(I)$ or $\mathrm{Ass}(R/I)$, is the collection of prime ideals of R of the form $(I : f)$ for some $f \in R_d$. In 2020, Cooper et al. introduced a new invariant, called v-number, for graded ideals of R during the study of Reed-Muller-type codes [7].

Definition 1.1 ([7, Definition 4.1]) Let I be a proper graded ideal of R. The v-*number* of I, denoted by $\mathrm{v}(I)$, is defined by

$$\mathrm{v}(I) := \min\{d \geq 0 \mid \exists\, f \in R_d \text{ and } \mathfrak{p} \in \mathrm{Ass}(I) \text{ with } (I : f) = \mathfrak{p}\}.$$

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For each $\mathfrak{p} \in \text{Ass}(I)$, we can locally define v-number as

$$\text{v}_{\mathfrak{p}}(I) := \min\{d \geq 0 \mid \exists\, f \in R_d \text{ with } (I : f) = \mathfrak{p}\}.$$

Then $\text{v}(I) = \min\{\text{v}_{\mathfrak{p}}(I) \mid \mathfrak{p} \in \text{Ass}(I)\}$.

This invariant of I helps us understand the asymptotic behaviour of the minimum distance function δ_I of projective Reed-Muller-type codes (see [7]). So far, very little is known about the v-number, and [3, 11, 16, 32] are the only papers written in this direction. The first paper entirely devoted to the v-number was written by Jaramillo and Villarreal [16], where they studied the v-number of edge ideals. Motivated by their work in Jaramillo and Villarreal [16], we take the initiation to study the v-number of binomial edge ideals.

Definition 1.2 Let G be a simple graph on the vertex set $V(G) = [n] = \{1, \ldots, n\}$ with the edge set $E(G)$. Consider the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ over a field K. Then the *binomial edge ideal* of G, denoted by J_G, is an ideal of S defined as

$$J_G = \langle f_{ij} = x_i y_j - x_j y_i \mid \{i, j\} \in E(G) \text{ with } i < j \rangle.$$

The study of binomial edge ideals was started in 2010 independently through the articles [13] and [26]. Since then, this has become an intensive research topic in combinatorial commutative algebra. A binomial edge ideal has a natural determinantal structure in the sense that it can be seen as an ideal generated by a set of 2×2-minors of a $2 \times n$ matrix X of indeterminates. One of the primary motivations behind studying these ideals is their connection to algebraic statistics, particularly their appearance in the study of conditional independence statements [13, Section 4]. Moreover, it is proved in Conca et al. [4] that binomial edge ideals belong to the class of *Cartwright-Sturmfels* ideals, which was introduced in Conca et al. [5] inspired by the work of Cartwright and Sturmfels [2] and has many nice properties.

Generally, people study algebraic properties and invariants of binomial edge ideals by investigating the underlying graphs' combinatorics. So far, lots of studies have been done on binomial edge ideals in several directions (see the survey paper [29] and the references therein). In this paper, we give a new direction in the study of binomial edge ideals by investigating their v-number. We discuss some properties of the v-number of binomial edge ideals and their initial ideals. There are many papers ([9, 18, 19, 21, 28]) on the upper bound of (*Castelnuovo-Mumford*) regularity of binomial edge ideals, but the general lower bound of the regularity of binomial edge ideals is only given by Matsuda and Murai [24]. We try to establish a new lower bound on the regularity of binomial edge ideals using the v-number and give a conjecture on the relation between v-number and regularity of binomial edge ideals. The paper is organized in the following manner:

In Section 2, we discuss the necessary prerequisites. In Section 3, we study some properties of the v-number of binomial edge ideals. For a vertex v of a graph G, we denote by G_v, the following graph:

$$V(G_v) = V(G) \text{ and } E(G_v) = E(G) \cup \{\{i, j\} \mid i, j \in \mathcal{N}_G(v), i \neq j\}.$$

Corresponding to a vertex v of a graph G, we have the following exact sequence (see [8, Proof of Theorem 1.1]):

$$0 \longrightarrow S/J_G \longrightarrow S/J_{G_v} \oplus S/\langle J_{G\setminus\{v\}}, x_v, y_v \rangle \longrightarrow S/\langle J_{G_v\setminus\{v\}}, x_v, y_v \rangle \longrightarrow 0.$$

The graphs $G \setminus \{v\}, G_v, G_v \setminus \{v\}$ play an important role in the study of binomial edge ideals. These graphs and the above exact sequence help inductively to study the invariants (like regularity, depth, etc.) and properties (like unmixed, Cohen-Macaulay, etc.) of J_G. In Proposition 3.3, we show how $\mathrm{v}(J_G)$ is related to $\mathrm{v}(J_{G_v})$ and $\mathrm{v}(J_{G\setminus\{v\}})$. Then we define completion set of G (Definition 3.4) with minimum and maximum completion number of G (denoted by min-comp(G) and max-comp(G), respectively) to find some bounds on $\mathrm{v}(J_G)$. We explicitly find $\mathrm{v}_\emptyset(G)$, the v-number of J_G locally at $P_\emptyset(G)$, and get a combinatorial upper bound of $\mathrm{v}(J_G)$ in the following theorem:

Theorem 3.6 *Let G be a simple graph. Then* $\mathrm{v}_\emptyset(J_G) = \text{min-comp}(G)$. *In particular, we have* $\mathrm{v}(J_G) \leq \text{min-comp}(G)$.

As a corollary of Theorem 3.6, we get $\gamma(G) \leq \mathrm{v}_\emptyset(G)$ in Corollary 3.9, where G is a connected non-complete graph and $\gamma(G)$ denotes the domination number of G. In Theorem 3.11, we prove the additivity of v-number for some radical ideals, and as an application of Theorem 3.11, we get the additivity of v-number of binomial edge ideals as follows:

Corollary 3.12 *Let* $G = G_1 \sqcup G_2$ *be a graph. Then* $\mathrm{v}(J_G) = \mathrm{v}(J_{G_1}) + \mathrm{v}(J_{G_2})$.

Next, we try to establish a relation between the v-number of binomial edge ideals and their initial ideals. We show that under some circumstances, $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ in Theorem 3.15 and as an application we get in Corollary 3.17 that $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ for Knutson binomial edge ideals. In Example 3.18, we show that the v-number of initial ideals of binomial edge ideals depends on the labelling of vertices. Finally, we end up this section by classifying all binomial edge ideals with v-number 1 as follows:

Theorem 3.20 *Let G be a simple connected graph. Then* $\mathrm{v}(J_G) = 1$ *if and only if* $G = \mathrm{cone}(v, H)$ *for some non-complete graph H*.

Section 4 of this paper is devoted to study the relation between regularity and v-number of binomial edge ideals. In [6, Corollary 2.7], Conca and Varbaro showed that for a graded ideal I in a polynomial ring R with a square-free initial ideal $\mathrm{in}_<(I)$ for some term order $<$, we have $\mathrm{reg}(R/I) = \mathrm{reg}(R/\mathrm{in}_<(I))$. Using this fact and looking at the initial ideal, we try to give a relation between the v-number and regularity of binomial edge ideals. One of the main results of this section is the following:

Theorem 4.5 *Let G be a chordal graph. Then* $\text{max-comp}(G) \leq \mathrm{reg}(S/J_G)$. *In particular, we have* $\mathrm{v}(J_G) \leq \mathrm{v}_\emptyset(J_G) \leq \text{max-comp}(G) \leq \mathrm{reg}(S/J_G)$.

Also, we show that $\mathrm{v}_\emptyset(J_G) \leq \mathrm{reg}(S/J_G)$ in Theorem 4.6 for some classes of graphs, including whisker graphs (see Corollary 4.8). In Example 4.9, we show that $\mathrm{v}_\emptyset(J_G)$ could be a better lower bound for $\mathrm{reg}(S/J_G)$ than the lower bound given by Matsuda and Murai in [24]. Also, we show in Example 4.10 that this lower bound is tight by providing a graph G, which satisfies $\mathrm{v}(J_G) = \mathrm{v}_\emptyset(J_G) = \mathrm{reg}(S/J_G)$. At the end, we show that for a given $n \in \mathbb{N}$, there exists a graph G satisfying $\mathrm{reg}(S/J_G) - \mathrm{v}(J_G) = n$ (see Theorem 4.11), i.e. the difference between v-number and regularity of J_G can be arbitrarily large. In the last section (Section 5), we put some open problems to give a future direction on the study of v-number of binomial edge ideals. Also, due to Theorems 4.5, 4.6, and some evidence from our computations, we conjecture the following:

Conjecture 5.3 *Let G be a simple graph. Then* $\mathrm{v}_\emptyset(J_G) \leq \mathrm{reg}(S/J_G)$. *In particular, we have* $\mathrm{v}(J_G) \leq \mathrm{reg}(S/J_G)$.

N.B. Some of the results presented in this article bear resemblance to those found in Jaramillo-Velez and Seccia [17], it's worth noting that our approaches were distinct and undertaken independently.

2 Preliminaries

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, with the standard gradation. An ideal I of R is said to be a *monomial ideal* if I is generated by a set of monomials and the unique minimal generating set of I is denoted by $G(I)$. A monomial ideal I is said to be *square-free* if $G(I)$ consists of only square-free monomials. It is a well-known fact that square-free monomial ideals are radical ideals and their associated prime ideals are generated by a set of variables. Let $<$ be a monomial order on R. For a graded ideal $I \subseteq R$, we denote the *initial ideal* of I with respect to $<$ by $\text{in}_<(I)$. For a monomial $m \in R$, the *support* of m, denoted by $\text{supp}(m)$, is defined as $\text{supp}(m) := \{x_i \mid x_i \text{ divides } m\}$. In this article, every ideal is assumed to be graded.

In this paper, we assume all graphs are simple and whenever applicable connected also. For $T \subseteq V(G)$, we write $G \setminus T$ to denote the induced subgraph of G on the vertex set $V(G) \setminus T$. Again by $G[T]$, we mean the induced subgraph of G on the vertex set T. For a vertex $v \in V(G)$, we say $\mathcal{N}_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$ the *neighbour set* of v in G. We write $\mathcal{N}_G[v] := \mathcal{N}_G(v) \cup \{v\}$. A *path* from u to v of length n in G is a sequence of vertices $u = v_0, \ldots, v_n = v \in V(G)$, such that $\{v_{i-1}, v_i\} \in E(G)$ for each $1 \leq i \leq n$, and $v_i \neq v_j$ if $i \neq j$.

Definition 2.1 A *path* graph on n vertices, denoted by P_n, is a graph whose vertex set can be ordered as $v_1, \ldots, v_n$ such that $E(P_n) = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\}$. The *length* of P_n is the number of edges in P_n, which is $n - 1$. An *induced path* of a graph G is an induced subgraph of G, which is a path graph. We denote by $\ell(G)$ the maximum length of an induced path in G.

Definition 2.2 A *cycle* of length n, denoted by C_n, is a graph with n vertices $v_1, \ldots, v_n$ such that $E(G) = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\} \cup \{\{v_1, v_n\}\}$. A graph is said to be *chordal* if it has no induced cycle C_n for $n \geq 4$.

Definition 2.3 A graph is said to be *complete* if there is an edge between every pair of vertices. We denote a complete graph on n vertices by K_n.

Remark 2.4 A vertex $v \in V(G)$ is said to be a *free* vertex of G if $\mathcal{N}_G(v)$ is a complete graph. It follows from [27, Proposition 2.1], that, v is a free vertex of G if and only if $v \notin T$ for all $T \in \mathscr{C}(G)$. Also, from [8, Proof of Theorem 1.1] it is observed that $T \in \mathscr{C}(G_v)$ if and only if $v \notin T$ and $T \in \mathscr{C}(G)$.

Definition 2.5 Let G be a graph with $V(G) = [n]$. A path $\pi : i = i_0, i_1, \ldots, i_r = j$ from i to j with $i < j$ in G is said to be an *admissible path* if the following hold:

1. $i_k \neq i_l$ for $k \neq l$;
2. For each $k \in \{1, \ldots, r-1\}$, we have either $i_k < i$ or $i_k > j$;
3. The induced subgraph of G on the vertex set $\{i_0, \ldots, i_r\}$ has no induced cycle.

Remark 2.6 Corresponding to an admissible path $\pi : i = i_0, i_1, \ldots, i_r = j$ from i to j with $i < j$ in G, we associate the monomial

$$u_\pi = \Big(\prod_{i_k > j} x_{i_k}\Big)\Big(\prod_{i_l < i} y_{i_l}\Big).$$

Then $\mathcal{G} = \{u_\pi f_{ij} \mid \pi \text{ is an admissible path from } i \text{ to } j \text{ with } i < j\}$ is a reduced Gröbner basis of J_G with respect to $<$ by [13, Theorem 2.1]. Therefore, we have

$$G(\text{in}_<(J_G)) = \{u_\pi x_i y_j \mid \pi \text{ is an admissible path from } i \text{ to } j \text{ with } i < j\}.$$

Primary Decomposition of Binomial Edge Ideals

A vertex $v \in V(G)$ is said to be a *cut vertex* of G, if removal of v from G increases the number of connected components. Let G be a graph on the vertex set $V(G) = [n]$. A set $T \subseteq [n]$ is said to be a *cutset* of G if each $t \in T$ is a cut vertex of $G \setminus (T \setminus \{t\})$. We denote by $\mathscr{C}(G)$ the set of all cutsets of G. For $T \subseteq [n]$, we denote the number of connected components of the graph $G \setminus T$ by $c_G(T)$ (or sometimes by $c(T)$ if the graph is clearly understood from the context). Let $G_1, \ldots, G_{c(T)}$ be the connected components of $G \setminus T$. For each G_i, we denote by $\tilde{G}_i$, the complete graph on the vertex set $V(G_i)$. We set

$$P_T(G) = \left\langle \bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \ldots, J_{\tilde{G}_{c(T)}} \right\rangle.$$

Then $P_T(G)$ is a prime ideal. By [13, Corollary 2.2], J_G is a radical ideal and from [13, Corollary 3.9], the minimal primary decomposition of J_G is

$$J_G = \bigcap_{T \in \mathscr{C}(G)} P_T(G).$$

Note: Instead of writing $\mathrm{v}_{P_T(G)}(J_G)$, the v-number of J_G locally at an associated prime $P_T(G)$, we will denote it by $\mathrm{v}_T(J_G)$.

Remark 2.7 For a prime ideal $\mathfrak{p}$ in R, we have $(\mathfrak{p} : 1) = \mathfrak{p}$ and hence, $\mathrm{v}(\mathfrak{p}) = 0$. Note that for a graph G, $P_\emptyset(G)$ is a disjoint union of binomial edge ideals of complete graphs. Hence, we have $\mathrm{v}(J_{K_n}) = 0$ and if G is a disjoint union of complete graphs, then also $\mathrm{v}(J_G) = 0$.

Remark 2.8 Let I be a radical ideal with $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$ as a primary decomposition. Then we have by [1, Exercise 1.12, Lemma 4.4] that $(I : f) = \mathfrak{p}_i$ if and only if $f \notin \mathfrak{p}_i$ and $f \in \mathfrak{p}_j$ for all $j \neq i$. We will use this fact frequently in our proofs.

3 Properties of the v-Number of Binomial Edge Ideals

In this section, we study some properties of the v-number of binomial edge ideals. We explicitly find $\mathrm{v}_\emptyset(J_G)$ and give a combinatorial upper bound of $\mathrm{v}(J_G)$. We try to establish the relation between $\mathrm{v}(J_G)$ and $\mathrm{v}(\mathrm{in}_<(J_G))$. Finally, we classify all binomial edge ideals with v-number 1.

Proposition 3.1 *Let G be a simple graph on the vertex set $[n]$. Then for any vertex $i \in [n]$, we have $(J_G : x_i) = J_{G_i}$ and $(J_G : y_i) = J_{G_i}$.*

Proof We first prove $J_{G_i} \subseteq (J_G : x_i)$. Note that $J_{G_i} = J_G + \langle f_{pq} \mid p, q \in N_G(i), p < q, \{p, q\} \notin E(G) \rangle$. Since $J_G \subseteq (J_G : x_i)$, it is enough to show that $f_{pq} \in (J_G : x_i)$ for $p, q \in \mathcal{N}_G(i)$ with $p < q$ and $\{p, q\} \notin E(G)$. Now we can write

$$\begin{aligned} x_i f_{pq} &= x_i(x_p y_q - x_q y_p) \\ &= x_i x_p y_q - x_p x_q y_i + x_p x_q y_i - x_i x_q y_p \\ &= x_p f_{iq} + x_q f_{pi}. \end{aligned}$$

Since $\{p, i\}, \{i, q\} \in E(G)$, we get $x_i f_{pq} \in J_G$. Hence, $J_{G_i} \subseteq (J_G : x_i)$.

Now, we will show $(J_G : x_i) \subseteq J_{G_i}$. Let $f \in (J_G : x_i)$. This implies $f x_i \in J_G$. We know that $J_G \subseteq J_{G_i} \subseteq P_T(G_i)$, for all $T \in \mathscr{C}(G_i)$. Thus, we get $f x_i \in P_T(G_i)$, for all $T \in \mathscr{C}(G_i)$. By Remark 2.4, $\mathscr{C}(G_i) = \{T \in \mathscr{C}(G) \mid i \notin T\}$. Thus, $x_i \notin P_T(G_i)$ for all $T \in \mathscr{C}(G_i)$. Therefore, we get $f \in P_T(G_i)$ for all $T \in \mathscr{C}(G_i)$, which implies $f \in \bigcap_{T \in \mathscr{C}(G_i)} P_T(G_i) = J_{G_i}$. Hence, $(J_G : x_i) \subseteq J_{G_i}$.

Similarly, one can show that $(J_G : y_i) = J_{G_i}$. The proof follows same as the above, where for the first part of the proof, we get $f_{pq} y_i = y_q(f_{pi}) + y_p(f_{iq})$. □

Proposition 3.2 *Let G be a simple graph on vertex set $[n]$. Then for $i, j \in [n]$ we get $(G_i)_j = (G_j)_i$.*

Proof It is enough to prove that $J_{(G_i)_j} = J_{(G_j)_i}$. By Proposition 3.1, we can write $J_{(G_i)_j} = (J_{G_i} : x_j) = ((J_G : x_i) : x_j) = (J_G : x_i x_j) = (J_G : x_j x_i) = ((J_G : x_j) : x_i) = (J_{G_j} : x_i) = J_{(G_j)_i}$. □

Proposition 3.3 *Let G be a simple graph. Then the following hold:*

(a) *For any $v \in V(G)$, we have $\mathrm{v}(J_G) \leq \mathrm{v}(J_{G_v}) + 1$.*
(b) *For a vertex v of G, if there exists a cutset T of G such that $v \notin T$ and $\mathrm{v}(J_G) = \mathrm{v}_T(J_G)$, then $\mathrm{v}(J_{G_v}) \leq \mathrm{v}(J_G)$.*
(c) *For a vertex v of G, if there exists a cutset T of G such that $v \in T$ and $\mathrm{v}(J_G) = \mathrm{v}_T(J_G)$, then $\mathrm{v}(J_{G \setminus \{v\}}) \leq \mathrm{v}(J_G)$.*

Proof (a): We know $\mathscr{C}(G_v) = \{T \in \mathscr{C}(G) \mid v \notin T\}$ by Remark 2.4. Let f be a homogeneous polynomial and $T \in \mathscr{C}(G_v)$ such that $(J_{G_v} : f) = P_T(G_v)$ and $\deg(f) = \mathrm{v}(J_{G_v})$. Note that $T \in \mathscr{C}(G)$ and $P_T(G) = P_T(G_v)$ as $v \notin T$. Then $(J_{G_v} : f) = P_T(G_v)$ implies $(J_G : x_v f) = P_T(G)$ by Proposition 3.1. Hence, $\mathrm{v}(J_G) \leq \mathrm{v}(J_{G_v}) + 1$.

(b): Let f be a homogeneous polynomial such that $(J_G : f) = P_T(G)$ and $\mathrm{v}(J_G) = \deg(f)$. Since $v \notin T$, we have $T \in \mathscr{C}(G_v)$ by Remark 2.4. Now $(J_G : f) = P_T(G)$ implies $f \in P_{T'}(G)$ for all $T' \in \mathscr{C}(G)$ with $T' \neq T$ and $f \notin P_T(G)$. Note that for every $T' \in \mathscr{C}(G_v)$, we have $P_{T'}(G_v) = P_{T'}(G)$. Therefore, $f \in P_{T'}(G_v)$ for all $T' \in \mathscr{C}(G_v)$ with $T' \neq T$ and $f \notin P_T(G_v)$. Hence, $(J_{G_v} : f) = P_T(G_v)$ and so, $\mathrm{v}(J_{G_v}) \leq \mathrm{v}(J_G)$.

(c): T is a cutset of G implies every $t \in T$ is a cut vertex of $G \setminus (T \setminus \{t\})$. Then every $t \in T \setminus \{v\}$ is a cut vertex of $G \setminus (T \setminus \{t\}) = (G \setminus \{v\}) \setminus ((T \setminus \{v\}) \setminus \{t\})$, which gives $T \setminus \{v\} \in \mathscr{C}(G \setminus \{v\})$. Since $v \in T$, we have $P_T(G) = \langle x_v, y_v \rangle + P_{T \setminus \{v\}}(G \setminus \{v\})$. Since $x_v, y_v \in P_T(G)$, we can choose a homogeneous polynomial $f \in K[\{x_i, y_i \mid i \in V(G \setminus \{v\})\}]$ such that $(J_G : f) = P_T(G)$ and $\mathrm{v}(J_G) = \deg(f)$. Now $f P_T(G) \subseteq J_G$ implies $f P_{T \setminus \{v\}}(G \setminus \{v\}) \subseteq J_{G \setminus \{v\}}$ as $f \in K[\{x_i, y_i \mid i \in V(G \setminus \{v\})\}]$. Thus, $P_{T \setminus \{v\}}(G \setminus \{v\}) \subseteq (J_{G \setminus \{v\}} : f)$. Let $g \in (J_{G \setminus \{v\}} : f)$. Then $fg \in J_{G \setminus \{v\}} \subseteq P_{T \setminus \{v\}}(G \setminus \{v\})$. Since $(J_G : f) = P_T(G)$, we have $f \notin P_T(G)$ and so, $f \notin P_{T \setminus \{v\}}(G \setminus \{v\})$. Thus, $g \in P_{T \setminus \{v\}}(G \setminus \{v\})$. Therefore, we get $(J_{G \setminus \{v\}} : f) = P_{T \setminus \{v\}}(G \setminus \{v\})$. Hence, $\mathrm{v}(J_{G \setminus \{v\}}) \leq \mathrm{v}(J_G)$. □

Definition 3.4 (Completion set) Let G be a simple graph. For a set $V = \{v_1, \ldots, v_k\} \subseteq V(G)$, we write $G_V = G_{v_1 v_2 \cdots v_k} := (\ldots((G_{v_1})_{v_2})\ldots)_{v_k}$. Due to Proposition 3.2, G_V does not depend on the ordering of the elements of V, and thus, the definition of G_V is well-defined. A set $W \subseteq V(G)$ is said to be a *completion set* of G if G_W is a disjoint union of complete graphs. A completion set W is said to be a *minimal completion set* of G if G_U is not a disjoint union of complete graphs for every $U \subsetneq W$. The minimum (respectively, maximum) cardinality among all the minimal completion sets of G is denoted by min-comp(G) (respectively, max-comp(G)) and we call it the *minimum completion* (respectively, *maximum completion*) number of G.

Lemma 3.5 *Let G be a graph. Let $T_1, \dots, T_k \in \mathscr{C}(G) \setminus \{\emptyset\}$ be some collection of cutsets of G. Write $I_j = \langle x_i, y_i \mid i \in T_j \rangle$ for each $1 \le j \le k$. Then*

$$(I_1 + P_\emptyset(G)) \cap \cdots \cap (I_k + P_\emptyset(G)) = (I_1 \cap \cdots \cap I_k) + P_\emptyset(G).$$

Proof It is enough to consider G is connected. Note that $(I_1 \cap \cdots \cap I_k) + P_\emptyset(G) \subseteq (I_1 + P_\emptyset(G)) \cap \cdots \cap (I_k + P_\emptyset(G))$ is clear. We use induction on k to prove the reverse inclusion. If $k = 1$, then there is nothing to prove. Suppose $(I_1 + P_\emptyset(G)) \cap \cdots \cap (I_{k-1} + P_\emptyset(G)) = (I_1 \cap \cdots \cap I_{k-1}) + P_\emptyset(G)$ holds. Let $f \in (I_1 + P_\emptyset(G)) \cap \cdots \cap (I_k + P_\emptyset(G)) = ((I_1 \cap \cdots \cap I_{k-1}) + P_\emptyset(G)) \cap (I_k + P_\emptyset(G))$. Since $P_\emptyset(G) \subseteq I_k + P_\emptyset(G)$, we have $((I_1 \cap \cdots \cap I_{k-1}) + P_\emptyset(G)) \cap (I_k + P_\emptyset(G)) = (I_1 \cap \cdots \cap I_{k-1}) \cap (I_k + P_\emptyset(G)) + P_\emptyset(G)$. Then we can write $f = g + h$, where $g \in (I_1 \cap \cdots \cap I_{k-1}) \cap (I_k + P_\emptyset(G))$ and $h \in P_\emptyset(G)$. Note that $\{x_i, y_i \mid i \in T_k\} \cup \{f_{pq} \mid p < q \text{ and } p, q \in V(G) \setminus T_k\}$ is a reduced Gröbner basis of $I_k + P_\emptyset(G)$. Therefore, we can write $g = g' + h'$ in the reduced form, such that $g' \in I_k$ and $h' \in P_\emptyset(G)$. Now $g \in I_1 \cap \cdots \cap I_{k-1}$ implies each monomial term of g' and h' belongs to $I_1 \cap \cdots \cap I_{k-1}$ as $I_1 \cap \cdots \cap I_{k-1}$ is a monomial ideal. Thus, $g' \in I_1 \cap \cdots \cap I_k$ and hence, $f = g' + h' + h \in (I_1 \cap \cdots \cap I_k) + P_\emptyset(G)$. □

Theorem 3.6 *Let G be a simple graph. Then* $\mathrm{v}_\emptyset(J_G) = \text{min-comp}(G)$. *In particular, we have* $\mathrm{v}(J_G) \le \text{min-comp}(G)$.

Proof Let $\text{min-comp}(G) = k$ and $\{v_1, \dots, v_k\}$ be a minimal completion set of G. Then $J_G : x_{v_1} \cdots x_{v_k} = P_\emptyset(G)$ due to Proposition 3.1. Therefore, $\mathrm{v}_\emptyset(J_G) \le \text{min-comp}(G)$. For the reverse inequality, let f be a homogeneous polynomial such that $(J_G : f) = P_\emptyset(G)$ and $\mathrm{v}_\emptyset(J_G) = \deg(f)$. Then $f \in \bigcap_{T \in \mathscr{C}(G)\setminus\{\emptyset\}} P_T(G)$ and $f \notin P_\emptyset(G)$. For every $T \in \mathscr{C}(G)$, we can write $P_T(G) = \langle x_i, y_i \mid i \in T \rangle + I_T$, where I_T is a binomial edge ideal of disjoint union of some complete graphs such that $I_T \subseteq P_\emptyset(G)$. Since $f \notin P_\emptyset(G)$, using Lemma 3.5, we can write $f = g + h$ such that $(0 \neq)\, g \in I$ and $h \in P_\emptyset(G)$, where I is the square-free monomial ideal given by

$$I = \bigcap_{T \in \mathscr{C}(G)\setminus\{\emptyset\}} \langle x_i, y_i \mid i \in T \rangle.$$

Thus, $\deg(f) \ge \min\{\deg(m) \mid m \in G(I)\}$. For every $m \in G(I)$, we have

$$m \in \bigcap_{T \in \mathscr{C}(G)\setminus\{\emptyset\}} P_T(G) \text{ and } m \notin P_\emptyset(G).$$

Thus, $(J_G : m) = P_\emptyset(G)$ for every $m \in G(I)$. Therefore, we can choose f such that $f \in G(I)$. Suppose $f = x_{i_1} \cdots x_{i_r} y_{j_1} \cdots y_{j_s}$. Then by Proposition 3.1, we get $\{i_1, \dots, i_r, j_1, \dots, j_s\}$ is a completion set of G. Thus, $\deg(f) = \mathrm{v}_\emptyset(J_G) \ge \text{min-comp}(G)$. Hence, $\mathrm{v}_\emptyset(J_G) = \text{min-comp}(G)$ and $\mathrm{v}(J_G) \le \mathrm{v}_\emptyset(J_G) = \text{min-comp}(G)$. □

Remark 3.7 Let G be a graph with connected components $G_1, \dots, G_k$. It is easy to see from Definition 3.4 that $\text{min-comp}(G) = \text{min-comp}(G_1) + \cdots + \text{min-comp}(G_k)$. Then by Theorem 3.6, we get $\mathrm{v}_\emptyset(J_G) = \mathrm{v}_\emptyset(J_{G_1}) + \cdots + \mathrm{v}_\emptyset(J_{G_k})$.

Definition 3.8 A *dominating set* of a graph G is a set $D \subseteq V(G)$ such that every vertex not in D has a neighbour in D. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a dominating set with minimum vertices.

Corollary 3.9 *Let G be a connected non-complete graph. Then $\gamma(G) \le \mathrm{v}_\emptyset(G)$.*

Proof By Theorem 3.6, we have $\mathrm{v}_\emptyset(G) = \text{min-comp}(G)$. Thus, it is enough to show that any minimal completion set of G is a dominating set of G. Let $V = \{v_1, \dots, v_k\}$ be a minimal completion set of G. Note that V is non-empty as G is non-complete. Suppose there is a vertex $u \in V(G) \setminus V$ such that $u \notin \mathcal{N}_G(v_i)$ for each $1 \le i \le k$. Then $u \notin \mathcal{N}_{G_V}(v_k)$, which gives a contradiction as G_V is a complete graph. Thus, V is a dominating set of G and hence, $\gamma(G) \le \mathrm{v}_\emptyset(G)$. □

Lemma 3.10 *Let $I_1 \subseteq R_1 = K[x_1, \dots, x_n]$ and $I_2 \subseteq R_2 = K[y_1, \dots, y_m]$ be two radical ideals. Suppose $P_1 \in \mathrm{Ass}(I_1)$ and $I_1 + I_2 := I_1R + I_2R$, where $R = R_1 \otimes_K R_2$, is a radical ideal with $\mathrm{Ass}(I_1 + I_2) = \{Q_1 + Q_2 \mid Q_1 \in \mathrm{Ass}(I_1), Q_2 \in \mathrm{Ass}(I_2)\}$. Then $((I_1 + I_2) : P_1) = (I_1 : P_1) + I_2$.*

Proof We write $J = I_1 + I_2$. Let $f \in (I_1 : P_1) + I_2$. Then $f = g + h$ for some $g \in (I_1 : P_1)$ and $h \in I_2$. Since $gP_1 \subseteq I_1$ and $h \in I_2$, we have $fP_1 = gP_1 + hP_1 \subseteq I_1 + I_2 = J$. Therefore, $f \in (J : P_1)$ and $(I_1 : P_1) + I_2 \subseteq (J : P_1)$. Let $f' \in (J : P_1)$. Then $f'P_1 \subseteq J$. Since $G(P_1) \subseteq R_1$ and I_1 is radical, we have $P_1 \nsubseteq Q_1 + Q_2$ for every $Q_1 \in \mathrm{Ass}(I_1) \setminus \{P_1\}$ and $Q_2 \in \mathrm{Ass}(I_2)$. Therefore, $f' \in I_1' + I_2$, where $I_1' = \bigcap_{Q_1 \in \mathrm{Ass}(I_1)\setminus\{P_1\}} Q_1$. Then we can write $f' = g' + h'$ such that $g' \in I_1'$ and $h' \in I_2$. If $g' \in P_1$, then $g \in I_1$ and hence, $f' \in J \subseteq ((I_1 : P_1) + I_2)$. If $g' \notin P_1$, then $I_1 : g' = P_1$ as $g' \in I'$. In this case, we get $f' = g' + h' \in (I_1 : P_1) + I_2$, and hence, $(J : P_1) = (I_1 : P_1) + I_2$. □

Theorem 3.11 (The v-number is additive) *Let $I_1 \subseteq R_1 = K[x_1, \dots, x_n]$ and $I_2 \subseteq R_2 = K[y_1, \dots, y_m]$ be two radical ideals. Suppose $I_1 + I_2 := I_1R + I_2R$, where $R = R_1 \otimes_K R_2$, is a radical ideal with $\mathrm{Ass}(I_1 + I_2) = \{P_1 + P_2 \mid P_1 \in \mathrm{Ass}(I_1), P_2 \in \mathrm{Ass}(I_2)\}$. Then*

$$\mathrm{v}(I_1 + I_2) = \mathrm{v}(I_1) + \mathrm{v}(I_2).$$

Proof Let $f_i \in R_i$ be a homogeneous polynomial and $P_i \in \mathrm{Ass}(I_i)$ such that $I_i : f_i = P_i$ and $\mathrm{v}(I_i) = \deg(f_i)$, where $i \in \{1, 2\}$. Then $f_i \in P$ for all $P \in \mathrm{Ass}(I_i) \setminus \{P_i\}$ and $f_i \notin P_i$ for $i = 1, 2$. It is easy to observe that $f_1 f_2 \in Q$ for all $Q \in \mathrm{Ass}(I_1 + I_2) \setminus \{P_1 + P_2\}$ and $f_1 f_2 \notin P_1 + P_2$. Therefore, $((I_1 + I_2) : f_1 f_2) = P_1 + P_2$ and so, $\mathrm{v}(I_1 + I_2) \le \mathrm{v}(I_1) + \mathrm{v}(I_2)$. For the reverse inequality, we will use [11, Theorem 10]. Let $\mathrm{v}(I_1 + I_2) = \mathrm{v}_{P_1+P_2}(I_1 + I_2)$ for some $P_1 \in \mathrm{Ass}(I_1)$ and $P_2 \in \mathrm{Ass}(I_2)$. Let $J = I_1 + I_2$, $(I_1 : P_1)/I_1 = \langle g_1 + I_1, \dots, g_r + I_1 \rangle$ and $(I_2 : P_2)/I_2 = \langle h_1 + I_2, \dots, h_s + I_2 \rangle$. Consider $\phi : R \mapsto R/J$ and we write $\phi(x) = \overline{x}$ for any $x \in R$. Then we have

$$\begin{aligned}
&\big((I_1 + I_2) : (P_1 + P_2)\big)/J \\
=&((I_1 + I_2) : P_1) \cap ((I_1 + I_2) : P_2)/J \\
=&((I_1 : P_1) + I_2) \cap ((I_2 : P_2) + I_1)/J \quad \text{(by Lemma 3.10)} \\
=&((I_1 : P_1) + I_2)/J \cap ((I_2 : P_2) + I_1)/J \quad \text{(by [15, Lemma 3.2])} \\
=&((I_1 : P_1) + J)/J \cap ((I_2 : P_2) + J)/J \\
=&((I_1 : P_1) \cap (I_2 : P_2) + J)/J \quad \text{(by [15, Lemma 3.2])} \\
=&((I_1 : P_1)(I_2 : P_2) + J)/J \quad \text{(by [12, Lemma 3.1])} \\
=&(((I_1 : P_1) + J)/J)(((I_1 : P_1) + J)/J) \\
=&\langle \overline{g_1}, \dots, \overline{g_r} \rangle \langle \overline{h_1}, \dots, \overline{h_s} \rangle \\
=&\langle \overline{g_i h_j} \mid 1 \le i \le r, 1 \le j \le s \rangle.
\end{aligned}$$

By [11, Theorem 3.2], we have

$$\mathrm{v}_{P_1+P_2}(J) = \min\{\deg(g_i h_j) \mid 1 \le i \le r, 1 \le j \le s \text{ and } (J : g_i h_j) = P_1 + P_2\}.$$

Suppose $\mathrm{v}_{P_1+P_2}(J) = \deg(g_i h_j) = \deg(g_i) + \deg(h_j)$ for some g_i and h_j. Since J, I_1, I_2 are radical, $J : g_i h_j = P_1 + P_2$ implies $I_1 : g_j = P_1$ and $I_2 : h_j = P_2$. Thus,

$$\mathrm{v}(I_1) + \mathrm{v}(I_2) \le \mathrm{v}_{P_1}(I_1) + \mathrm{v}_{P_2}(I_2) \le \deg(g_i) + \deg(h_j) = \mathrm{v}_{P_1+P_2}(I_1 + I_2).$$

Since $\mathrm{v}(I_1 + I_2) = \mathrm{v}_{P_1+P_2}(I_1 + I_2)$, we have $\mathrm{v}(I_1 + I_2) = \mathrm{v}(I_1) + \mathrm{v}(I_2)$. □

Corollary 3.12 *Let $G = G_1 \sqcup G_2$ be a graph. Then $\mathrm{v}(J_G) = \mathrm{v}(J_{G_1}) + \mathrm{v}(J_{G_2})$.*

Proof Since binomial edge ideals are radical ideals and $\mathrm{Ass}(J_G) = \{P_{T_1}(G_1) + P_{T_2}(G_2) \mid T_1 \in \mathscr{C}(G_1), T_2 \in \mathscr{C}(G_2)\}$, we have $\mathrm{v}(J_G) = \mathrm{v}(J_{G_1}) + \mathrm{v}(J_{G_2})$ by Theorem 3.11. □

Now, we will establish the relation between $\mathrm{v}(J_G)$ and $\mathrm{v}(\mathrm{in}_<(J_G))$ for certain class of graphs. For that, let us first discuss the primary decomposition of $\mathrm{v}(\mathrm{in}_<(J_G))$.

Let G be a simple graph. Let $T \in \mathscr{C}(G)$ and $G_1, \ldots, G_{c(T)}$ be the connected components of $G \setminus T$. For $\mathbf{v} = (v_1, \ldots, v_{c(T)}) \in V(G_1) \times \cdots \times V(G_{c(T)})$, we consider the following prime ideal:

$$P_T(\mathbf{v}) = \langle x_i, y_i \mid i \in T \rangle + \sum_{k=1}^{c(T)} \langle \{x_i, y_j \mid i, j \in V(G_k), i < v_k, j > v_k\} \rangle.$$

By [22, Lemma 1], we get

$$\mathrm{in}_<(P_T(G)) = \bigcap_{\mathbf{v} \in V(G_1) \times \cdots \times V(G_{c(T)})} P_T(\mathbf{v}).$$

Since $\mathrm{in}_<(J_G)$ is radical, by [4, Corollary 1.12], we have

$$\mathrm{in}_<(J_G) = \bigcap_{T \in \mathscr{C}(G)} \mathrm{in}_<(P_T(G)).$$

Proposition 3.13 ([31, Proposition 3.3]) *Let G be a simple graph. Then we have*

$$\mathrm{Ass}(\mathrm{in}_<(J_G)) = \{P_T(\mathbf{v}) \mid T \in \mathscr{C}(G) \text{ and } \mathbf{v} \in V(G_1) \times \cdots \times V(G_{c(T)})\}.$$

Lemma 3.14 *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be graded prime ideals of R. If $\mathrm{in}_<(\bigcap_{i=1}^k \mathfrak{p}_i)$ is a square-free monomial ideal, then*

$$\mathrm{in}_<\left(\bigcap_{i=1}^k \mathfrak{p}_i\right) = \bigcap_{i=1}^k \left(\mathrm{in}_<(\mathfrak{p}_i)\right).$$

Proof We have $\mathrm{in}_<(\bigcap_{i=1}^k \mathfrak{p}_i) \subseteq \bigcap_{i=1}^k (\mathrm{in}_<(\mathfrak{p}_i))$. Let $m \in \bigcap_{i=1}^k (\mathrm{in}_<(\mathfrak{p}_i))$ be a monomial. Then there exists $f_i \in \mathfrak{p}_i$ such that $\mathrm{in}_<(f_i) = m$ for each $i \in \{1, \ldots, k\}$. Consider the polynomial $f = f_1 \cdots f_k$. Note that $f \in \bigcap_{i=1}^k \mathfrak{p}_i$ and thus, $\mathrm{in}_<(f) = m^k \in \mathrm{in}_<(\bigcap_{i=1}^k \mathfrak{p}_i)$. It is given that $\mathrm{in}_<(\bigcap_{i=1}^k \mathfrak{p}_i)$ is square-free, i.e., a radical ideal. Therefore, $m \in \mathrm{in}_<(\bigcap_{i=1}^k \mathfrak{p}_i)$ and the equality follows. □

Theorem 3.15 *Let G be a graph. If there exists $T' \in \mathscr{C}(G)$ such that $\mathrm{v}(\mathrm{in}_<(J_G))$ is attained for some $P_{T'}(\mathbf{v}')$ and $\mathrm{in}_<(\bigcap_{T \in \mathscr{C}(G) \setminus \{T'\}} P_T(G))$ is radical, then*

$$\mathrm{v}(J_G) \le \mathrm{v}(\mathrm{in}_<(J_G)).$$

Proof Let $G'_1, \ldots, G'_{c(T')}$ be the connected components of $G \setminus T'$. By the given hypothesis, there exists a square-free monomial m such that $(\mathrm{in}_<(J_G) : m) = P_{T'}(\mathbf{v}')$ for some $\mathbf{v}' \in V(G'_1) \times \cdots \times V(G'_{c(T')})$ and $\mathrm{v}(\mathrm{in}_<(J_G)) = \deg(m)$. Then

$$
\begin{aligned}
&\big(\mathrm{in}_<(J_G) : m\big) = P_{T'}(\mathbf{v}') \\
\implies &\Big(\bigcap_{T \in \mathcal{C}(G)} \mathrm{in}_<(P_T(G)) : m\Big) = P_{T'}(\mathbf{v}') \\
\implies &\bigcap_{T \in \mathcal{C}(G)} \big(\mathrm{in}_<(P_T(G)) : m\big) = P_{T'}(\mathbf{v}') \\
\implies &\bigcap_{T \in \mathcal{C}(G)} \Big(\Big(\bigcap_{\mathbf{v} \in V(G_1) \times \cdots \times V(G_{\mathcal{C}(T)})} P_T(\mathbf{v})\Big) : m\Big) = P_{T'}(\mathbf{v}') \\
\implies &\bigcap_{\substack{T \in \mathcal{C}(G) \\ \mathbf{v} \in V(G_1) \times \cdots \times V(G_{\mathcal{C}(T)})}} (P_T(\mathbf{v}) : m) = P_{T'}(\mathbf{v}').
\end{aligned}
$$

Therefore, by Proposition 3.13, we get $m \in P_T(\mathbf{v})$ for all $P_T(\mathbf{v}) \in \mathrm{Ass}(\mathrm{in}_<(J_G)) \setminus \{P_{T'}(\mathbf{v}')\}$ and $m \notin P_{T'}(\mathbf{v}')$. This gives us that $m \in \bigcap_{T \in \mathcal{A}} \mathrm{in}_<(P_T(G))$, where $\mathcal{A} = \mathscr{C}(G) \setminus \{T'\}$. Since $\mathrm{in}_<(\bigcap_{T \in \mathcal{A}} P_T(G))$ is radical, it follows from Lemma 3.14 that

$$
\bigcap_{T \in \mathcal{A}} \mathrm{in}_<(P_T(G)) = \mathrm{in}_<\Big(\bigcap_{T \in \mathcal{A}} P_T(G)\Big).
$$

Thus, we have $m \in \mathrm{in}_<(\bigcap_{T \in \mathcal{A}} P_T(G))$ and so, there exists $f \in \bigcap_{T \in \mathcal{A}} P_T(G)$ such that $\mathrm{in}_<(f) = m$. Suppose $f \in P_{T'}(G)$. Then $\mathrm{in}_<(f) = m \in \mathrm{in}_<(P_{T'}(G))$, which implies $m \in P_{T'}(\mathbf{v}')$ and this gives a contradiction as $m \notin P_{T'}(\mathbf{v}')$. Therefore, $f \notin P_{T'}(G)$ and we get $J_G : f = P_{T'}(G)$. Hence $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$. □

Conca and Varbaro in [6] introduced the notion of Knutson ideals inspired by the work of Allen Knutson [20] on compatibly split ideals and degeneration.

Definition 3.16 (Knutson ideals) Let $f \in R = K[x_1, \ldots, x_n]$ be a polynomial such that its leading term $\mathrm{in}_<(f)$ is a square-free monomial for some term order $<$. Define C_f to be the smallest set of ideals satisfying the following conditions:

1. $\langle f \rangle \in C_f$;
2. If $I \in C_f$, then $I : J \in C_f$ for every ideal $J \subseteq R$;
3. If I and J are in C_f, then also $I + J$ and $I \cap J$ must be in C_f.

If I is an ideal in C_f, we say that I is a *Knutson ideal* associated with f. More generally, we say that I is a *Knutson ideal* if $I \in C_f$ for some f.

Matsuda introduced the notion of weakly closed graphs [23, Definition 2.1] as a generalization of closed graphs and studied the F-purity of binomial edge ideals of weakly closed graphs. Later, Seccia [33, Theorem 4.1] proved that a graph G is weakly closed if and only if J_G is a Knutson ideal. As an application of Theorem 3.15, we give the following corollary regarding the v-number of binomial edge ideals of weakly closed graphs.

Corollary 3.17 *Let G be a weakly closed graph. Then* $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$.

Proof Since G is weakly closed, J_G is a Knutson ideal. Then by the definition of Knutson ideals, it follows that any associated prime of J_G is Knutson. Let $\mathrm{v}(J_G) = \mathrm{v}_{T'}(J_G)$ for some $T' \in \mathscr{C}(G)$. Then $\bigcap_{T \in \mathscr{C}(G)\setminus\{T'\}} P_T(G)$ is Knutson by condition (3) of Definition 3.16. It has been proved in Knutson [20] that the initial ideal of any Knutson ideal is radical. Thus, G satisfies the hypothesis of Theorem 3.15 and hence, $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$. □

Example 3.18 Consider the graph G in Fig. 1 with the labelling shown in (a). Then using the reduced Gröner basis of J_G discussed in Remark 2.6, we get

$$\mathrm{in}_<(J_G) = \langle x_4y_5, x_3y_6, x_3y_4, x_2y_4, x_2y_3, x_1y_2, x_4y_3y_6, x_5y_3y_4y_6 \rangle.$$

By [11, Procedure A1] and Macaulay2 [10], we get $\mathrm{v}(J_G) = 3$ and $\mathrm{v}(\mathrm{in}_<(J_G)) = 4$. Therefore, in this case, we have $\mathrm{v}(J_G) < \mathrm{v}(\mathrm{in}_<(J_G))$.

Similarly, considering the same graph G in Fig. 1 with the labelling given in (b), we get $\mathrm{v}(\mathrm{in}_<(J_G)) = 3$. Thus, $\mathrm{v}(J_G) = \mathrm{v}(\mathrm{in}_<(J_G))$ in this case.

Since the primary decomposition of J_G does not depend on the labelling of $V(G)$, $\mathrm{v}(J_G)$ remains the same for any labelling of a given graph. But, $\mathrm{v}(\mathrm{in}_<(J_G))$ may not remain the same with different labelling of $V(G)$.

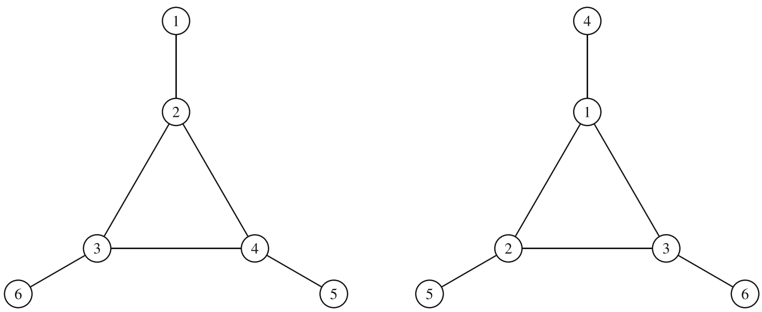
We will now classify those graphs whose binomial edge ideals have v-number 1.

Definition 3.19 Let G be a graph and $v \notin V(G)$ be a vertex. The *cone* of v on G, denoted by $\mathrm{cone}(v, G)$, is the graph with vertex set $V(G) \cup \{v\}$ and edge set $E(G) \cup \{\{u, v\} \mid u \in V(G)\}$.

Theorem 3.20 *Let G be a simple connected graph. Then $\mathrm{v}(J_G) = 1$ if and only if $G = \mathrm{cone}(v, H)$ for some non-complete graph H.*

Proof Suppose $\mathrm{v}(J_G) = 1$. Then there exists a homogeneous linear polynomial f such that $(J_G : f) = P_T(G)$ for some $T \in \mathscr{C}(G)$. Suppose $T \neq \emptyset$. Then there exists $i \in T$ and so, $x_i, y_i \in P_T(G)$. Therefore, $fx_i \in J_G \subseteq P_\emptyset(G)$. But, $P_\emptyset(G)$ cannot contain any linear polynomial, and this gives a contradiction. Therefore, the only possibility is $T = \emptyset$, i.e., $J_G : f = P_\emptyset(G)$. Since $J_G = \cap_{T \in \mathscr{C}(G)} P_T(G)$, we have $f \in P_T(G)$ for all $T \in \mathscr{C}(G)$ with $T \neq \emptyset$ and $f \notin P_\emptyset(G)$. Now each $P_T(G)$ can be written as $P_T(G) = \langle x_i, y_i \mid i \in T \rangle + I_T$, where I_T is an ideal generated by degree two homogeneous binomials. Since f is linear and I_T is a homogeneous binomial ideal of degree two, we have

$$f \in \bigcap_{T \in \mathscr{C}(G)\setminus\{\emptyset\}} \langle x_i, y_i \mid i \in T \rangle.$$



(a) Graph G with $\mathrm{v}(\mathrm{in}_<(J_G)) = 4$ (b) Graph G with $\mathrm{v}(\mathrm{in}_<(J_G)) = 3$

Fig. 1 Graph G with different labelling and different $\mathrm{v}(\mathrm{in}_<(J_G))$

Now, f belongs to a square-free monomial ideal and degree of f is 1 together imply there exists $v \in V(G)$ such that $v \in T$ for all $T \in \mathscr{C}(G) \setminus \{\emptyset\}$. Suppose there exists $u \in V(G)$ such that $u \notin \mathcal{N}_G(v)$. Take $T \subseteq \mathcal{N}_G(v)$ such that there is no path from u to v in $G \setminus T$ and T is minimal with such property. Then it is clear that $T \in \mathscr{C}(G)$ and $T \neq \emptyset$ as G is connected. But, $v \notin T$ and T is non-empty give a contradiction. Hence, $\mathcal{N}_G[v] = V(G)$, i.e., $G = \text{cone}(v, H)$, where H is the induced subgraph of G on $\mathcal{N}_G(v)$. Suppose H is complete. Then G is complete and in this case, $\text{v}(J_G) = 0$ as J_G is prime. Therefore, H is non-complete as $\text{v}(J_G) = 1$.

Conversely, let $G = \text{cone}(v, H)$ for some non-complete graph H. By Proposition 3.1, we get $J_G : x_v = J_{G_v}$. Since $G = \text{cone}(v, H)$, G_v is complete and $J_{G_v} = P_{\emptyset}(G)$. Thus, $\text{v}(J_G) \leq \deg(x_v) = 1$. Now H is non-complete, G is also non-complete, and so, J_G is not a prime ideal. Therefore $\text{v}(J_G) \geq 1$, which gives $\text{v}(J_G) = 1$. □

4 The v-Number and Castelnuovo-Mumford Regularity

In this section, we try to establish a relation between the v-number and Castelnuovo-Mumford regularity of binomial edge ideals. For certain classes of graphs, we show that the v-number is less than or equal to the regularity of binomial edge ideals. Our main technique is to observe the induced matchings of the hypergraphs corresponding to the initial ideals of binomial edge ideals.

Definition 4.1 A *simple hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H})$ is a set of finite elements, known as the *vertex set* of $\mathcal{H}$ and $E(\mathcal{H})$ is a collection of subsets of $V(\mathcal{H})$ such that no two elements of $E(\mathcal{H})$ contain each other, called the *edge set* of $\mathcal{H}$. Elements of $V(\mathcal{H})$ are called vertices of $\mathcal{H}$ and elements of $E(\mathcal{H})$ are called edges of $\mathcal{H}$.

Let $\mathcal{H}$ be a simple hypergraph on the vertex set $V(\mathcal{H}) = \{x_1, \ldots, x_n\}$. For $A \subseteq V(\mathcal{C})$, we consider $X_A := \prod_{x_i \in A} x_i$ as a square-free monomial in the polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. The *edge ideal* of the hypergraph $\mathcal{H}$, denoted by $I(\mathcal{H})$, is an ideal of R defined by

$$I(\mathcal{H}) = \langle X_e \mid e \in E(\mathcal{H}) \rangle.$$

In this sense, the family of square-free monomial ideals are in one to one correspondence with the family of simple hypergraphs. For a square-free monomial ideal I of R, we denote by $\mathcal{H}(I)$ the corresponding simple hypergraph.

Definition 4.2 An *induced matching* in a simple hypergraph $\mathcal{H}$ is a set of pairwise disjoint edges $e_1, \ldots, e_r$ such that the only edges of $\mathcal{H}$ contained in $\bigcup_{i=1}^{r} e_i$ are $e_1, \ldots, e_r$.

Proposition 4.3 ([25, Corollary 3.9]) *Let $\mathcal{H}$ be a simple graph and $M = \{e_1, \ldots, e_r\}$ be an induced matching in $\mathcal{H}$. Then $\sum_{i=1}^{r}(|e_i| - 1) = (\sum_{i=1}^{r} |e_i|) - r \leq \text{reg}(R/I(\mathcal{H}))$.*

Lemma 4.4 *Let $\{v_1, \ldots, v_k\}$ form a minimal completion set of a connected graph G such that $v_i \in \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1})$ for all $2 \leq i \leq k$. Then for each $2 \leq i \leq k$, there exists $u_i \in \mathcal{N}_G(v_i)$ such that $u_i \notin \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1})$ and there is no path from u_i to v_1 in $G[\{v_1, \ldots, \widehat{v_i}, \ldots, v_k, u_i\}]$.*

Proof If $\mathcal{N}_G(v_i) \subseteq \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1})$, then it is clear that $G_{v_1 \cdots \widehat{v_i} \cdots v_k}$ is complete, and this gives a contradiction to the fact that $\{v_1, \ldots, v_k\}$ is a minimal completion set of G. Therefore, $\mathcal{N}_G(v_i) \nsubseteq \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1})$. Let $\mathcal{N}_G(v_i) \setminus (\mathcal{N}_G(v_1) \cup \cdots \cup$

$\mathcal{N}_G(v_{i-1})) = \{u_{i_1}, \ldots, u_{i_r}\}$. Suppose for each $1 \leq j \leq r$, there is a path from u_{i_j} to v_1 in $G[\{v_1, \ldots, \widehat{v_i}, \ldots, v_k, u_{i_j}\}]$. Then $u_{i_j} \in \mathcal{N}_G(v_{s_j})$ for some $s_j \in \{i+1, \ldots, k\}$ and there is a path from v_{s_j} to v_1 in $G[\{v_1, \ldots, \widehat{v_i}, \ldots, v_k\}]$. Let $i' = \max\{s_1, \ldots, s_r\}$ (s_i's need not be distinct). Then $\{u_{i_1}, \ldots, u_{i_r}\} \subseteq \mathcal{N}_{G_{v_1 \cdots \widehat{v_i} \cdots v_{i'}}}(v_{i'})$. Since there exists a path from $v_{i'}$ to v_1 in $G[\{v_1, \ldots, \widehat{v_i}, \ldots, v_k\}]$, we have $G_{v_1 \cdots \widehat{v_i} \cdots v_k}$ is complete, which contradicts the minimality of $\{v_1, \ldots, v_k\}$. This completes the proof. □

Theorem 4.5 *Let G be a chordal graph. Then* $\text{max-comp}(G) \leq \text{reg}(S/J_G)$. *In particular, we have* $\text{v}(J_G) \leq \text{v}_\emptyset(J_G) \leq \text{max-comp}(G) \leq \text{reg}(S/J_G)$.

Proof It is enough to assume G is connected. Let $V = \{v_1, \ldots, v_k\}$ be a minimal completion set of G. Since G is connected, after a suitable relabelling of vertices in V we get an order $v_1, \ldots, v_k$ such that

$$v_i \in \mathcal{N}_{G_{v_1 \cdots v_{i-1}}}(v_{i-1}) = \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1}) \tag{1}$$

for each $i = 2, \ldots, k$. From the Gröbner basis of J_G, it is clear that $\text{in}_<(J_G)$ changes with the labelling of $V(G)$. Our aim is to find a labelling of G for which there exists an induced matching M in $\mathcal{H} = \mathcal{H}(\text{in}_<(J_G))$ such that $\sum_{e \in M}(|e| - 1) \geq k$. We will find such suitable labelling of $V(G)$ and corresponding induced matching of $\mathcal{H}$ in a particular algorithmic technique. Let us start.

Step-1: (**Case-1A**) If there exists $u_1 \in \mathcal{N}_G(v_1)$ such that $u_1 \notin \mathcal{N}_G[v_2] \cup \cdots \cup \mathcal{N}_G[v_k]$, then label $v_1 = t_1 = 1$ and $u_1 = t_1 + 1 = 2$. In this case, consider the set $M = \{e_1\}$, where $e_1 = \{x_1, y_2\}$.

(**Case-1B**) If $\mathcal{N}_G(v_1) \subseteq \mathcal{N}_G[v_2] \cup \cdots \cup \mathcal{N}_G[v_k]$, then take u_1 as v_2, and label $v_1 = t_1 = 1$ and $u_1 = v_2 = t_2 = t_1 + 1 = 2$. In this case, consider the set $M = \{e_1\}$, where $e_1 = \{x_{t_1}, y_{t_1+1}\} = \{x_1, y_2\}$.

Step-2: By Lemma 4.4, there exists $u_2 \in \mathcal{N}_G(v_2)$ such that $u_2 \notin \mathcal{N}_G(v_1)$ and there is no path from u_2 to v_1 in $G[\{v_1, \widehat{v_2}, \ldots, v_k, u_2\}]$

(**Case-2A**) If there exists $u_2 \in \mathcal{N}_G(v_2)$ such that $u_2 \notin \mathcal{N}_G[v_1] \cup \widehat{\mathcal{N}_G[v_2]} \cup \cdots \cup \mathcal{N}_G[v_k]$, then we choose such u_2. In this situation, we label the vertices in the following fashion: If $v_2 \neq u_1$, then label $v_2 = t_2 = t_1 + 2 = 3$ and label $u_2 = t_2 + 1 = 4$. In this case, update M as $M = \{e_1, e_2\}$, where $e_2 = \{x_{t_2}, y_{t_2+1}\} = \{x_3, y_4\}$. If $v_2 = u_1$, then we have from the previous case $u_1 = v_2 = t_2 = t_1 + 1$ and label $u_2 = t_2 + 1$. In this case, update M as $M = \{e_1, e_2\}$, where $e_2 = \{x_{t_2}, y_{t_2+1}\}$.

(**Case-2B**) Suppose $\mathcal{N}_G(v_2) \subseteq \mathcal{N}_G[v_1] \cup \widehat{\mathcal{N}_G[v_2]} \cup \cdots \cup \mathcal{N}_G[v_k]$. Then there exists an $u_2 \in \{v_3, \ldots, v_k\}$ satisfying the condition of Lemma 4.4, otherwise $\{v_1, \widehat{v_2}, \ldots, v_k\}$ will be a minimal completion set of G. Let $u_2 = v_{2+j}$ such that j is the smallest. In this case, we will relabel V as follows:

$$\begin{aligned} \text{Label } v_{2+j} &\text{ as } v_3, \\ v_3 &\text{ as } v_4, \\ &\vdots \\ v_{2+j-1} &\text{ as } v_{2+j}, \end{aligned}$$

and others will remain the same. Note that the new ordering of vertices in V also satisfies the property (1). So we can continue with this ordering. Now, label

$$v_2 = t_2 = \begin{cases} t_1 + 1 \text{ if } u_1 = v_2 \text{ in step-1} \\ t_1 + 2 \text{ if } u_1 \neq v_2 \text{ in step-1} \end{cases}$$

and $u_2 = v_3 = t_3 = t_2 + 1$. In this case, we update the set M as $M = \{e_1, e_2\}$, where $e_2 = \{x_{t_2}, y_{t_2+1}\}$.

Continue this process with the following i-th step:

Step-i: By Lemma 4.4, there exists $u_i \in \mathcal{N}_G(v_i)$ such that $u_i \notin \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{i-1})$ and there is no path from u_i to v_1 in $G[\{v_1, \ldots, \widehat{v_i}, \ldots, v_k, u_i\}]$.

(**Case-iA**) If there exists $u_i \in \mathcal{N}_G(v_i)$ such that $u_i \notin \mathcal{N}_G[v_1] \cup \cdots \cup \widehat{\mathcal{N}_G[v_i]} \cup \cdots \cup \mathcal{N}_G[v_k]$, then we choose such u_i. In this situation, we label the vertices in the following fashion: If $v_i \neq u_{i-1}$, then label $v_i = t_i = t_{i-1} + 2$ and label $u_i = t_i + 1$. In this case, update M as $M \cup \{e_i\}$, where $e_i = \{x_{t_i}, y_{t_i+1}\}$. Now if $v_i = u_{i-1}$, then we have from the previous case $u_{i-1} = v_i = t_i = t_{i-1} + 1$ and label $u_i = t_i + 1$. In this case, update M as $M \cup \{e_i\}$, where $e_i = \{x_{t_i}, y_{t_i+1}\}$.

(**Case-iB**) Suppose $\mathcal{N}_G(v_i) \subseteq \mathcal{N}_G[v_1] \cup \cdots \cup \widehat{\mathcal{N}_G[v_i]} \cup \cdots \cup \mathcal{N}_G[v_k]$. Then there exists a $u_i \in \{v_{i+1}, \ldots, v_k\}$ satisfying the condition of Lemma 4.4, otherwise $\{v_1, \ldots, \widehat{v_i}, \ldots, v_k\}$ will be a minimal completion set of G. Let $u_i = v_{i+j}$ such that j is the smallest. In this case, we will relabel V as follows:

$$\begin{aligned} \text{Label } v_{i+j} &\text{ as } v_{i+1}, \\ v_{i+1} &\text{ as } v_{i+2}, \\ &\vdots \\ v_{i+j-1} &\text{ as } v_{i+j}, \end{aligned}$$

and others will remain the same. Note that the new ordering of vertices in V also satisfies the property (1). So we can continue with this ordering. Now, label

$$v_i = t_i = \begin{cases} t_{i-1} + 1 \text{ if } u_{i-1} = v_i \text{ in step-(i-1)} \\ t_{i-1} + 2 \text{ if } u_{i-1} \neq v_i \text{ in step-(i-1)} \end{cases}$$

and $u_i = v_{i+1} = t_{i+1} = t_i + 1$. In this case, we update the set M as $M \cup \{e_i\}$, where $e_i = \{x_{t_i}, y_{t_i+1}\}$.

After completing k steps, we get a set M consisting of k edges $e_1, \ldots, e_k$ of $\mathcal{H}$, where $e_i = \{x_{t_i}, y_{t_i+1}\}$, such that

$$\sum_{i=1}^{k} (|e_i| - 1) = k.$$

Claim: The set M forms an induced matching in $\mathcal{H}$.

Proof of claim. Let $\mathcal{S} = \bigcup_{e \in M} e$. Then $\mathcal{S} = \{x_{t_1}, \ldots, x_{t_k}, y_{t_1+1}, \ldots, y_{t_k+1}\}$. By our choice and labelling of vertices, it is clear that no two elements of M intersect each other. Now we will show that the only edges of $\mathcal{H}$ contained in $\mathcal{S}$ are the edges that belong to M. Suppose $\{x_{t_i}, y_{t_j+1}\} \in E(\mathcal{H})$ for some $i \neq j$ and $x_{t_i}, y_{t_j+1} \in \mathcal{S}$. Then $t_j + 1 > t_i + 1$ and $\{t_i, t_j + 1\} \in E(G)$. We have chosen u_j such that $u_j \notin \mathcal{N}_G(v_1) \cup \cdots \cup \mathcal{N}_G(v_{j-1})$ and so, $t_j + 1 \notin \mathcal{N}_G(t_i)$ as $t_i < t_j$, which is a contradiction. Thus, $\{x_{t_i}, y_{t_j+1}\} \notin E(\mathcal{H})$ when $i \neq j$ and $x_{t_i}, y_{t_j+1} \in \mathcal{S}$. Hence the only edges of $\mathcal{H}$ with cardinality two contained in $\mathcal{S}$ are the edges that belong to

M. Suppose $e \in E(\mathcal{H})$ such that $e \notin M$ and $e \subseteq \mathcal{S}$. Then $|e| > 2$. Corresponding to e there exists an admissible path $\pi : t_i = \alpha_0, \ldots, \alpha_l$ in G such that $\text{supp}(u_\pi x_{t_i} y_{\alpha_l}) \subseteq \mathcal{S}$. If one of $\alpha_1, \ldots, \alpha_l$ (say α_r) is $t_p + 1$ such that $t_p + 1 \notin \{t_1, \ldots, t_k\}$, then either α_{r-1} or α_{r+1} cannot belong to $\{t_1, \ldots, t_k\}$ by our choice of u_i's. Because, if $t_p + 1 \notin \{t_1, \ldots, t_k\}$, then $t_p + 1$ is adjacent to only t_p among $\{t_1, \ldots, t_k\}$. Suppose $\alpha_{r-1} \notin \{t_1, \ldots, t_k\}$ and $\alpha_{r-1} = t_q + 1$ for some $t_q \in \{t_1, \ldots, t_k\}$. In this situation, we will get an induced cycle of length greater than 3 containing the vertices $\{t_p, t_p+1, t_q+1, t_q\}$ and some of $\{t_1, \ldots, t_k\}$, which is a contradiction to the fact that G is chordal. Similarly, we will get a contradiction if $\alpha_{r+1} \notin \{t_1, \ldots, t_k\}$. Therefore, we should have $\{\alpha_0, \ldots, \alpha_l\} \subseteq \{t_1, \ldots, t_k\}$. Now $\alpha_l = u_j$ for some j. Then $v_j \notin \{\alpha_0, \ldots, \alpha_l\}$ as $t_i <$ label of $v_j <$ label of u_j as per our choice labelling. Thus, there will be a path from u_j to v_1 in $G[\{v_1, \ldots, \widehat{v_j}, \ldots, v_k, u_j\}]$, which gives a contradiction due to Lemma 4.4. Hence, M forms an induced matching in $\mathcal{H}$.

By [6, Corollary 2.7], we have $\text{reg}(S/J_G) = \text{reg}(S/\text{in}_<(J_G))$. Again by Proposition 4.3, $k \leq \text{reg}(R/\text{in}_<(J_G))$ as M is an induced matching in $\mathcal{H}$ with $\sum_{e \in M}(|e| - 1) = k$. The completion set V is chosen arbitrarily and hence, $\text{max-comp}(G) \leq \text{reg}(S/J_G)$. □

Theorem 4.6 *If a graph G has a minimal completion set $\{v_1, \ldots, v_k\}$ such that for each $1 \leq i \leq k$ there exists $u_i \in \mathcal{N}_G(v_i)$ such that $u_i \notin \mathcal{N}_G[v_1] \sqcup \cdots \sqcup \widehat{\mathcal{N}_G[v_i]} \sqcup \cdots \sqcup \mathcal{N}_G[v_k]$, then $\text{v}_\emptyset(J_G) \leq k \leq \text{reg}(S/J_G)$.*

Proof Let us label the vertex v_i as i and the vertex u_i as $k+i$ for each $1 \leq i \leq k$. The remaining vertices can be labelled arbitrarily. Let $\mathcal{H}$ be the corresponding hypergraph of the $\text{in}_<(J_G)$ with respect to our choice of labelling. Now consider the set $M = \{e_1, \ldots, e_k\} \subseteq E(\mathcal{H})$, where $e_i = \{x_i, y_{k+i}\} \in E(\mathcal{H})$ for each $i = 1, \ldots, k$. Looking at the Gröbner basis of J_G, it is easy to see that M forms an induced matching in $\mathcal{H}$ as $k+i \in \mathcal{N}_G(i)$, but $k+i \notin \mathcal{N}_G[1] \sqcup \cdots \sqcup \widehat{\mathcal{N}_G[i]} \sqcup \cdots \sqcup \mathcal{N}_G[k]$. Thus, $\text{reg}(S/I(\mathcal{H})) \geq \sum_{i=1}^{k}(|e_i| - 1) = k$ by Proposition 4.3. Hence, by [6, Corollary 2.7] and Theorem 3.6, we get $\text{v}_\emptyset(J_G) \leq k \leq \text{reg}(S/I(\mathcal{H})) = \text{reg}(S/J_G)$. □

Definition 4.7 Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. The *whisker graph* of G, denoted by W_G, is the graph attaching n new vertices $\{u_1, \ldots, u_n\}$ to G as follows:

- $V(W_G) = \{v_1, \ldots, v_n, u_1, \ldots, u_n\}$,
- $E(W_G) = E(G) \cup \{\{v_i, u_i\} \mid i = 1, \ldots, n\}$.

Corollary 4.8 *Let G be a graph with $V(G) = [n]$. Then $\text{v}_\emptyset(W_G) = n \leq \text{reg}(S/J_{W_G})$.*

Proof Note that $\{1, \ldots, n\}$ is contained in every completion set of W_G. But, $\{1, \ldots, n\}$ itself is a minimal completion set of W_G. Thus, $\{1, \ldots, n\}$ is the only minimal completion set of W_G. Hence, we get the desired result by Theorems 3.6 and 4.6. □

Example 4.9 Let G be the graph given in Fig. 2 and $S = \mathbb{Q}[x_1, \ldots, x_{10}, y_1, \ldots, y_{10}]$. Using Macaulay2, we get $\text{reg}(S/J_G) = 6$. Also, we see that $\ell(G) =$ length of the longest induced path in $G = 4$. By Corollary 4.8, we get $\text{v}_\emptyset(J_G) = 5$. Thus, we have $\ell(G) < \text{v}_\emptyset(J_G) < \text{reg}(S/J_G)$. This example shows that $\text{v}_\emptyset(J_G)$ can be a better lower bound for regularity of binomial edge ideals than the lower bound given by Matsuda and Murai [24].

Example 4.10 Let G be the graph shown in Fig. 3 and $S = \mathbb{Q}[x_1, \ldots, x_8, y_1, \ldots, y_8]$. Using [11, Procedure A1] and Macaulay2, we get $\text{v}(J_G) = 4$ and $\text{reg}(S/J_G) = 4$. On the other hand, $\{1, 2, 3, 4\}$ is a completion set of G. Therefore, $\text{v}_\emptyset(J_G) = 4$ by Theorem 3.6. In this example, we get $\text{v}(J_G) = \text{v}_\emptyset(J_G) = \text{reg}(S/J_G)$. Hence, our given bound in Theorem 4.6 is sharp.

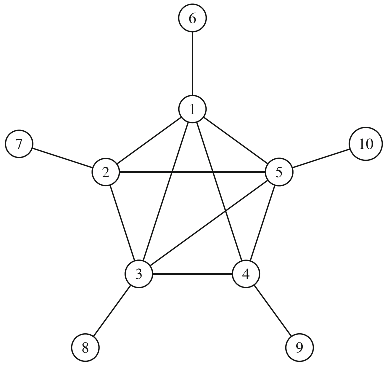


Fig. 2 Graph G with $\ell(G) < \mathrm{v}_{\emptyset}(J_G) < \mathrm{reg}(S/J_G)$

Theorem 4.11 *For every $n \in \mathbb{N}$, there exists a graph G such that $\mathrm{reg}(S/J_G) - \mathrm{v}(J_G) = n$. Moreover, for every $n \in \mathbb{N}$, there exists a graph G such that $\mathrm{reg}(S/J_G) - \mathrm{v}_{\emptyset}(J_G) = n$.*

Proof For $n = 0$, the result follows from Example 4.10. Let $n \in \mathbb{N}^+$. Consider the graph $H = P_{n+2}$, a path graph on $n+2$ vertices. Then by [14, Corollary 7.35], we have $\mathrm{reg}(S/J_H) = n+1$. Now, let $G = \mathrm{cone}(v, H)$, where $v \notin V(H)$. Then $\mathrm{v}(J_G) = \mathrm{v}_{\emptyset}(G) = 1$ by Theorem 3.20. Also, using [30, Theorem 2.1], we get $\mathrm{reg}(S/J_G) = n+1$. Thus, we get $\mathrm{reg}(S/J_G) - \mathrm{v}(J_G) = (n+1) - 1 = n$. In this case, $\mathrm{v}(J_G) = \mathrm{v}_{\emptyset}(J_G)$ and the further hypothesis follows. □

5 Some Open Problems on $\mathrm{v}(J_G)$

In Section 3, we discuss some properties of v-number of binomial edge ideals and give a combinatorial bound. In [16], the authors managed to give a combinatorial description of v-number for edge ideals of graphs. We ask the following question on the combinatorial aspects of the v-number of binomial edge ideals.

Question 5.1 Let G be a simple graph. Can we find some homogeneous polynomial f just using the combinatorics of the graph G such that $\mathrm{v}(J_G) = \deg(f)$? Equivalently, does there exist any graph invariant of G which is equal to $\mathrm{v}(J_G)$?

In Theorem 3.15, we prove that $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ for some classes of binomial edge ideals and as an application, we get in Corollary 3.17 that $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ hold for

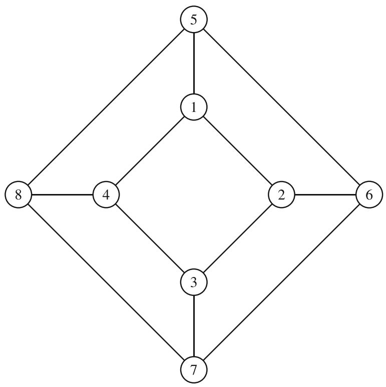


Fig. 3 Graph G with $\mathrm{v}(J_G) = \mathrm{v}_{\emptyset}(J_G) = \mathrm{reg}(S/J_G)$

weakly closed graphs. Also, we see in Example 3.18 that $\mathrm{v}(\mathrm{in}_<(J_G))$ depends on the labelling of vertices and $\mathrm{v}(J_G)$ can be strictly less than $\mathrm{v}(\mathrm{in}_<(J_G))$. With the virtue of these results and our computation, we put the following question.

Question 5.2 Is it true that $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ for all graphs G with all possible labelling of $V(G)$? If not, then can we say that for a graph G, there exists a labelling of $V(G)$ for which $\mathrm{v}(J_G) \leq \mathrm{v}(\mathrm{in}_<(J_G))$ holds?

In Section 4, we try to relate the v-number with (Castelnuovo-Mumford) regularity of binomial edge ideals. In Theorems 4.5 and 4.6, we show that $\mathrm{v}_\emptyset(J_G) \leq \mathrm{reg}(S/J_G)$ for some large classes of graphs including chordal and whisker graphs. Using [11, Procedure A1] and Macaulay2 [10], we investigate many graphs from several classes and witness that $\mathrm{v}_\emptyset(J_G) \leq \mathrm{reg}(S/J_G)$ hold for all of those graphs. Our strong intuition forces us to give the following conjecture.

Conjecture 5.3 Let G be a simple graph. Then $\mathrm{v}_\emptyset(J_G) \leq \mathrm{reg}(S/J_G)$. In particular, we have $\mathrm{v}(J_G) \leq \mathrm{reg}(S/J_G)$.

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Declarations

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