Acta Mathematica Vietnamica (2024) 49:3–18
https://doi.org/10.1007/s40306-023-00515-3

On the Infinitely Generated Locus of Frobenius Algebras of Rings of Prime Characteristic

Alberto F. Boix[1] · **Danny A.J. Gómez–Ramírez**[2] · **Santiago Zarzuela**[3]

To our friend Ngo Viet Trung for his 70th birthday

Received: 26 April 2023 / Accepted: 16 August 2023 / Published online: 5 December 2023

Abstract
Let R be a commutative Noetherian ring of prime characteristic p. The main goal of this paper is to study in some detail when

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \ : \ \mathcal{F}^{E_{\mathfrak{p}}} \text{ is finitely generated as a ring over its degree zero piece}\}$$

is an open set in the Zariski topology, where $\mathcal{F}^{E_{\mathfrak{p}}}$ denotes the Frobenius algebra attached to the injective hull of the residue field of $R_{\mathfrak{p}}$. We show that this is true when R is a Stanley–Reisner ring; moreover, in this case, we explicitly compute its closed complement, providing an algorithmic method for doing so.

Keywords Frobenius algebras · Stanley–Reisner rings

Mathematics Subject Classification (2010) 13A35 · 13F55

1 Introduction

The main goal of the present paper is to prove the following result:

✉ Alberto F. Boix
alberto.fernandez.boix@uva.es

Danny A.J. Gómez–Ramírez
daj.gomezramirez@gmail.com

Santiago Zarzuela
szarzuela@ub.edu

1 IMUVA–Mathematics Research Institute, Universidad de Valladolid, Paseo de Belen, s/n, 47011 Valladolid, Spain

2 Parque Tech at the Institución Universitaria Pascual Bravo, Medellín, Colombia

3 Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain

Theorem 1 *Let $S = \mathbb{K}[x_1, \ldots, x_n]/I$ be a Stanley-Reisner ring, where $\mathbb{K}$ is a field of prime characteristic p and I a squarefree monomial ideal. Then, the set*

$$\overline{W^S} = \{\mathfrak{p} \in \operatorname{Spec}(S) \,:\, \mathcal{F}^{E_{S_\mathfrak{p}}} \textit{ is finitely generated as a ring over its degree zero piece}\},$$

where $\mathcal{F}^{E_{S_\mathfrak{p}}}$ is the Frobenius algebra of the injective hull of the residue field of the local ring $S_\mathfrak{p}$, is an open set of $\operatorname{Spec}(S)$.

Moreover, we determine the defining ideal of its closed complement and provide an algorithmic method to compute it, both algebraic and combinatorial.

We explain now the context and the meaning of the above statement.

Let R be a commutative Noetherian ring of prime characteristic p, and let M be an R-module. For each integer $e \geq 0$, we denote by $\operatorname{End}_{p^e}(M)$ the set of p^e-linear maps of M, that is, $\operatorname{End}_{p^e}(M)$ is made up by abelian group endomorphisms $M \xrightarrow{\phi} M$ such that

$$\phi(rm) = r^{p^e}\phi(m) \text{ for all } (r, m) \in R \times M.$$

In this way, one can cook up the so-called Frobenius algebra of M

$$\mathcal{F}^M := \bigoplus_{e \geq 0} \operatorname{End}_{p^e}(M),$$

where multiplication is given by composition of maps. This algebra, introduced in [15] in the context of tight closure theory has attracted some attention in the last years.

One question raised in [15, p. 3156] is whether this algebra is finitely generated over its degree zero piece. We want to briefly summarize here some of the answers that have appeared in the last years.

(i) If $M = R$ and $R \xrightarrow{F} R$ denotes the Frobenius endomorphism of R, then $\mathcal{F}^R \cong R[\Theta; F]$, the algebra is finitely generated [15, Example 3.6]. The same conclusion holds when $(R, \mathfrak{m})$ is a local complete S_2-ring, $M = H_\mathfrak{m}^{\dim(R)}(R)$ is the top local cohomology module of R supported on $\mathfrak{m}$, and F denotes the natural Frobenius action on this module [15, Example 3.7].
(ii) If M is an R-module of finite length, then $\mathcal{F}^M$ is finitely generated [6, Theorem 2.11].
(iii) Let $R = \mathbb{K}[\![x_1, \ldots, x_n]\!]/I$, where $\mathbb{K}$ is a field of prime characteristic p, and I is a squarefree monomial ideal. If $M = E_R$ is the injective hull of the residue field of R, then $\mathcal{F}^M$ is either infinitely generated or $\mathcal{F}^M \cong R[u\theta; F]$ for some $u \in R$ [1, Theorem 3.5].
(iv) If R is a normal, excellent, $\mathbb{Q}$-Gorenstein local ring of prime characteristic p and order m, and again $M = E_R$ is the injective hull of the residue field of R, then $\mathcal{F}^M$ is finitely generated if and only if $\gcd(p, m) = 1$ (see [13, Proposition 4.1] and [6, Theorem 4.5]).

However, not too much is known about the following:

Question 1 Let R be a commutative Noetherian ring of prime characteristic p. Is it true that

$$\overline{W^R} := \{\mathfrak{p} \in \operatorname{Spec}(R) \,:\, \mathcal{F}^{E_\mathfrak{p}} \text{ is finitely generated as a ring over its degree zero piece}\},$$

where $E_\mathfrak{p}$ is the injective hull of the residue field of $R_\mathfrak{p}$, is an open set?

In [8], it is shown that this question has a positive answer in some particular cases; however, to the best of our knowledge, no systematic study of this problem seems to have been carried out so far.

The goal of this paper is to study Question 1 in some detail; more precisely, let $S = R/I$ be a commutative Noetherian ring which is a quotient of a regular Noetherian ring R (that is, locally regular) of prime characteristic p. Under these assumptions, we show that $\overline{W^S}$ is closed under generalization; moreover, when S is a Stanley–Reisner ring, we will show that $\overline{W^S}$ is really an open set in the Zariski topology and provide an algorithmic and explicit description of the defining ideal of its closed complement.

Our approach to the problem is by means of the study of a similar question over another non-commutative graded algebra, that we denote by $\mathcal{A}^S$ (see Definition 1). This algebra is much easier to handle and behaves well under flat morphisms. Moreover, thanks to a well-known result of R. Fedder [7], if R is a complete regular local ring, then the algebra $\mathcal{A}^S$ is isomorphic to $\mathcal{F}^E$, where E is the injective envelope of the residue field of S (see Sect. 2 for the details).

It turns out that, in the Stanley-Reisner case, the graded pieces of the algebra $\mathcal{A}^S$ are homogeneous for the natural multigrading of the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$. This allows to control their local vanishing and in the end the finite generation of $\mathcal{A}^{S_\mathfrak{p}}$, just by looking at the localization by homogeneous prime ideals of R, which are a finite set. Analyzing carefully this process, we are able to prove that the complement of the set $\overline{W^S}$ is the closed set defined by a squarefree monomial ideal (see Theorem 2).

In order to find explicitly the above defining ideal, first, we need to pass from the local case to the Stanley–Reisner case. This we do by relating through a faithfully flat morphism the localization of S at a homogeneous prime ideal to an adequate monomial localization (see Definition 4), which provides the needed frame to apply the criteria from [1] and [2] to determine whether the corresponding Frobenius algebra of the injective hull of the residue field is finitely generated or not in the Stanley–Reisner case. Then, we introduce a very simple algebraic algorithm that computes explicitly our defining ideal (see Theorem 4). From the combinatorial point of view, this process corresponds to consider all the possible links of the simplicial complex determined by S and to look for the existence of the so-named free faces inside them. The details are developed in Sect. 4.

Finally, in Sect. 5, we provide several examples computed by means of the implementation of our algorithm in Macaulay2.

2 The Non-finitely Generated Locus: Definition and General Properties

Definition 1 Let $\mathbb{K}$ be a field of prime characteristic $p > 0$, let R be a commutative Noetherian ring containing $\mathbb{K}$, $S = R/I$, where $I \subset R$ is an ideal, and set

$$\mathcal{A}^S := \bigoplus_{e \geq 0} \left(\frac{(I^{[p^e]} :_R I)}{I^{[p^e]}} \right) = \bigoplus_{e \geq 0} \mathcal{A}_e.$$

This is a graded abelian group that we can endow with a non-commutative ring structure by setting its multiplication as

$$a \cdot b = ab^{p^e}, \; a \in \mathcal{A}_e, \; b \in \mathcal{A}_{e'}.$$

Finally, set

$$\mathcal{A}^{S_{\mathfrak{p}}} := \bigoplus_{e\geq 0}\left(\frac{(I_{\mathfrak{p}}^{[p^e]} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}})}{I_{\mathfrak{p}}^{[p^e]}}\right),$$

where again this is regarded as a non-commutative ring with multiplication defined as before. Then, we refer to

$$W := \{\mathfrak{p} \in \operatorname{Spec}(R) : \ \mathfrak{p} \supseteq I, \ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is a not finitely generated ring over } S_{\mathfrak{p}}\},$$

as the **non-finitely generated locus** of the algebra $\mathcal{A}^S$. Moreover, we denote by $\overline{W}$ the complement of W inside $V(I)$.

The reader might ask why to care about the non-commutative algebra defined above; actually, this ring is very close to the Frobenius algebra attached to the injective hull of the residue field of a complete local ring, as the below discussion explains.

Discussion 1 Let $(R, \mathfrak{m})$ be a commutative Noetherian regular local ring of prime characteristic p, let $I \subset R$ be an ideal, and set $S := R/I$. Moreover, let $T := \widehat{S}$ be the completion of S with respect to the $\mathfrak{m}$-adic topology. Now, consider

$$\mathcal{G}^S = \bigoplus_{e\geq 0}\left(\frac{(I^{[p^e]} :_R I)}{I^{[p^e]}}\right)F^e = \bigoplus_{e\geq 0}\mathcal{G}_e F^e,$$

where F^e denotes the e-th iteration of the Frobenius map on the injective hull of the residue field of R. It is clear that $\mathcal{G}^S$ is an $\mathbb{N}_0$-graded ring, not necessarily commutative with $\mathcal{G}_0 = S$ which is degree-wise finitely generated and a left S-skew algebra; now, we also consider

$$\mathcal{G}^T := \bigoplus_{e\geq 0}\mathcal{B}_e F_T^e, \text{ where } \mathcal{B}_e := T\otimes_S \mathcal{G}_e,$$

and F_T^e denotes the e-th iteration of the Frobenius map on T. However, notice that, by Fedder's Lemma [7, p. 465], $\mathcal{G}^T$ is isomorphic to $\mathcal{F}^{E_S}$, the Frobenius algebra attached to the injective hull of the residue field of S, which in turn is also isomorphic to $\mathcal{F}^{E_T}$ because of [15, Proposition 3.3]. On the other hand, we also have a graded ring isomorphism

$$\mathcal{A}^S \cong \mathcal{G}^S.$$

These facts allow us to regard $\mathcal{A}^S$ as a subring of $\mathcal{F}^{E_T}$.

Before going on, we need to review the following notions borrowed from [6].

Discussion 2 Let $\mathcal{A} = \bigoplus_{e\geq 0}\mathcal{A}_e$ be an $\mathbb{N}_0$-graded ring, not necessarily commutative.

(i) Let $G_e := G_e(\mathcal{A})$ be the subring of $\mathcal{A}$ generated by the homogeneous elements of degree less than or equal than e; we agree that $G_{-1} = \mathcal{A}_0$. Notice also that $G_0 = \mathcal{A}_0$.
(ii) Let $k_e := k_e(\mathcal{A})$ be the minimal number of homogeneous generators of G_e as a subring of $\mathcal{A}$ over $\mathcal{A}_0$; we agree that $k_{-1} = 0$. One says that $\mathcal{A}$ is **degree-wise finitely generated** if $k_e < \infty$ for any e.
(iii) Again, assume that $\mathcal{A}$ is a degree-wise finitely generated ring and set, for each integer $e \geq 0$, $c_e(\mathcal{A}) := k_e - k_{e-1}$; as pointed out in [6, Remark 2.6], $\mathcal{A}$ is finitely generated as a ring over $\mathcal{A}_0$ if and only if the sequence $\{c_e(\mathcal{A})\}_{e\geq 0}$ is eventually zero.

(iv) Let R be a commutative ring, and let $\mathcal{A}$ be a degree-wise finitely generated ring such that $R = \mathcal{A}_0$; moreover, assume that $\mathcal{A}$ is a left **R-skew algebra** (i.e., $aR \subseteq Ra$ for all homogeneous elements $a \in \mathcal{A}$). Then, by [6, Corollary 2.10], $c_e(\mathcal{A})$ equals the minimum number of generators of $\mathcal{A}_e/(G_{e-1})_e$ as a left R-module for any e.

As observed along Discussion 2, we know that when $\mathcal{A}$ is a non-commutative, degree-wise finitely generated ring, $\mathcal{A}$ is finitely generated as a ring over its degree zero piece if and only if the sequence $\{c_e(\mathcal{A})\}_{e\geq 0}$ is eventually zero. This fact motivates us to introduce the following:

Definition 2 Given $\mathcal{A}$ a non-commutative, degree-wise finitely generated ring, and given an integer $k \geq 1$, we say that $\mathcal{A}$ is ***k*-generated** provided $c_e(\mathcal{A}) = 0$ for all $e > k$; when $k = 1$, we say that $\mathcal{A}$ is **principally generated**.

Now, we are in a position to introduce the following:

Definition 3 Under the assumptions and notations of Definition 1, we can define

$$\overline{W}_k := \{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is } k\text{-generated}\}.$$

The reader will easily note that the $\overline{W}_k$'s provide a stratification of the set

$$\overline{W} = \{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is finitely generated over } S_{\mathfrak{p}}\};$$

in other words,

$$\overline{W} = \bigcup_{k\geq 1} \overline{W}_k.$$

Our next goal is to compute the sequence $\{c_e(\mathcal{A})\}_{e\geq 0}$ for the ring $\mathcal{A}^S$ and to characterize when it is eventually zero.

Proposition 1 *Let R be a commutative Noetherian ring of prime characteristic p, let $I \subset R$ be an ideal, and set $S := R/I$. Moreover, for each $e \geq 1$, write $K_e := (I^{[p^e]} :_R I)$, and set*

$$L_e := \sum_{\substack{1\leq a_1,\dots,\, a_s\leq e-1 \\ a_1+\cdots+a_s=e}} K_{a_1} K_{a_2}^{[p^{a_1}]} K_{a_3}^{[p^{a_1+a_2}]} \cdots K_{a_s}^{[p^{a_1+\cdots+a_{s-1}}]}.$$

Then, the following assertions hold.

(i) *For any $e \geq 1$, one has that $c_e(\mathcal{A}^S)$ equals the minimum number of generators as a left S-module of*

$$\frac{(I^{[p^e]} :_R I)}{L_e}.$$

(ii) *The sequence $\{c_e(\mathcal{A}^S)\}_{e\geq 0}$ is eventually zero if and only if, for all $e \gg 0$, one has that*

$$(I^{[p^e]} :_R I) = L_e.$$

(iii) *Given an integer $k \geq 1$, the ring $\mathcal{A}^S$ is k-generated if and only if, for all $e > k$, one has that*

$$(I^{[p^e]} :_R I) = L_e.$$

Proof Since parts (ii) and (iii) follow immediately from part (i), we only plan to prove part (i); indeed, as viewed in Discussion 2, $c_e(\mathcal{A}^S)$ equals the minimum number of generators as a left S-module of $\mathcal{A}_e/(G_{e-1}\cap\mathcal{A}_e)$, where G_{e-1} is as defined in Discussion 2. However, thanks to [12, Proposition 2.1], we know that

$$G_{e-1}\cap\mathcal{A}_e = L_e.$$

In this way, the result follows directly from this fact. □

Remark 1 Notice that [12, Proposition 2.1] was only proved for a formal power series ring in three indeterminates; however, the reader can easily verify that its proof also holds in our setting.

Now, our aim is to show how the algebra $\mathcal{A}^S$ behaves under certain base change; this behavior will play a key role in several places of this paper later.

Proposition 2 *Let $R\xrightarrow{\phi}R'$ be a faithfully flat ring homomorphism between commutative Noetherian regular rings of prime characteristic p, let $I\subset R$ be an ideal, and set $I':=\phi(I)R'$ and $S':=R'/I'$. Now, set*

$$\mathcal{A}^S:=\bigoplus_{e\geq 0}\left(\frac{(I^{[p^e]}:_R I)}{I^{[p^e]}}\right),\quad \mathcal{A}^{S'}=\bigoplus_{e\geq 0}\left(\frac{(I'^{[p^e]}:_{R'} I')}{I'^{[p^e]}}\right).$$

Finally, we assume, for any integer $e\geq 0$, that

$$\phi((I^{[p^e]}:_R I))R'=(I'^{[p^e]}:_{R'} I').$$

Then, $\mathcal{A}^S$ is k-generated for some integer $k\geq 1$ if and only if $\mathcal{A}^{S'}$ is k-generated; in particular, this implies that $\mathcal{A}$ is finitely generated as a ring over S, if and only if $\mathcal{A}^{S'}$ is finitely generated as a ring over S'.

Proof We know that $\mathcal{A}^S$ is k-generated as a ring over S if and only if $c_e(\mathcal{A}^S)=0$ for all $e>k$, which is equivalent to say, thanks to Proposition 1, that for all $e>k$, one has $(I^{[p^e]}:_R I)=L_e$, where

$$L_e=\sum_{\substack{1\leq a_1,\ldots,a_s\leq e-1\\ a_1+\cdots+a_s=e}}(I^{[p^{a_1}]}:_R I)\cdot(I^{[p^{a_2}]}:_R I)^{[p^{a_1}]}\cdots(I^{[p^{a_s}]}:_R I)^{[p^{a_1+\cdots+a_{s-1}}]}.$$

In this way, if for all $e>k$, one has that $(I^{[p^e]}:_R I)=L_e$, then this is equivalent to say, thanks to our assumptions, that for all $e>k$ $(I'^{[p^e]}:_{R'} I')=L'_e$, where

$$L'_e=\sum_{\substack{1\leq a_1,\ldots,a_s\leq e-1\\ a_1+\cdots+a_s=e}}(I'^{[p^{a_1}]}:_{R'} I')\cdot(I'^{[p^{a_2}]}:_{R'} I')^{[p^{a_1}]}\cdots(I'^{[p^{a_s}]}:_{R'} I')^{[p^{a_1+\cdots+a_{s-1}}]}.$$

The proof is therefore complete. □

By applying Proposition 2 to the completion map, we immediately obtain the following:

Corollary 1 *Let $(R,\mathfrak{m})$ be a commutative Noetherian regular local ring of prime characteristic p, let $I\subset R$ be an ideal, and set $S:=R/I$. Given an integer $k\geq 1$, one has that $\mathcal{A}^S$ is k-generated if and only if $\mathcal{F}^{E_S}$ is k-generated; in particular, $\mathcal{A}^S$ is finitely generated as a ring over its degree zero piece if and only if so is $\mathcal{F}^{E_S}$.*

Our next goal is to show that the set W introduced in Definition 1 is closed under specialization. This is exactly the content of the next:

Proposition 3 *Let $k \geq 1$ be an integer, and let*

$$\overline{W}_k = \{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \; \mathcal{A}^{S_\mathfrak{p}} \text{ is } k\text{-generated}\}.$$

Then, $\overline{W}_k$ is closed under generalization. Equivalently, given prime ideals $\mathfrak{q} \subset \mathfrak{p}$ with $\mathfrak{p} \in \overline{W}_k$, one has that $\mathfrak{q} \in \overline{W}_k$.

Therefore, W_k is closed under specialization, and therefore, by [19, Tag 0EES], *W_k can be expressed as a directed union of closed subsets of $V(I)$.*

In particular, the set

$$\overline{W} = \{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \; \mathcal{A}^{S_\mathfrak{p}} \text{ is a finitely generated ring over } S_\mathfrak{p}\}$$

is closed under generalization.

Proof Let $\mathfrak{q} \subset \mathfrak{p}$ be prime ideals with $\mathfrak{p} \in \overline{W}$, and we assume that $\mathcal{A}^S$ is not k-generated; otherwise, the statement is obvious (indeed, in that case, $W_k = \emptyset$). Let $R \xrightarrow{l_\mathfrak{p}} R_\mathfrak{p} \xrightarrow{l_{\mathfrak{p},\mathfrak{q}}} R_\mathfrak{q}$ be the natural localization maps, notice that $l_\mathfrak{q} = l_{\mathfrak{p},\mathfrak{q}} \circ l_\mathfrak{p}$. Since $\mathfrak{p} \in \overline{W}$, one has that $c_e(\mathcal{A}^{S_\mathfrak{p}}) = 0$ for all $e > k$, which means, due to Proposition 1, that for all $e > k$,

$$(I^{[p^e]} :_R I) R_\mathfrak{p} = L_e R_\mathfrak{p}.$$

Applying to this equality the map $l_{\mathfrak{p},\mathfrak{q}}$ and using the equality $l_\mathfrak{q} = l_{\mathfrak{p},\mathfrak{q}} \circ l_\mathfrak{p}$, one finally obtains that for all $e > k$,

$$(I^{[p^e]} :_R I) R_\mathfrak{q} = L_e R_\mathfrak{q}.$$

In this way, this is equivalent to say, using once again Proposition 1, that $c_e(\mathcal{A}^{S_\mathfrak{q}}) = 0$ for all $e > k$, and therefore, $\mathfrak{q} \in \overline{W}_k$, as claimed. □

As an immediate consequence of Proposition 3, one gets the below:

Corollary 2 *Let R be a commutative Noetherian regular ring of prime characteristic p, let $I \subset R$ be an ideal, and set $S := R/I$. Then, for any integer $k \geq 1$,*

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \; \mathcal{F}^{E_{S_\mathfrak{p}}} \text{ is } k\text{-generated}\}$$

is closed under generalization; in particular,

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \, : \, \mathfrak{p} \supseteq I, \; \mathcal{F}^{E_{S_\mathfrak{p}}} \text{ is a finitely generated ring over } \widehat{S_\mathfrak{p}}\}$$

is closed under generalization.

3 The Case of the Stanley–Reisner Ring

In what follows in this section, let $\mathbb{K}$ be a field, and let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over $\mathbb{K}$. We regard R as an $\mathbb{N}^n$-graded ring, where $\deg(x_i)$ is the i-th canonical basis vector in $\mathbb{N}^n$. With this grading, the graded R-submodules of R (aka the graded ideals of R) are exactly the monomial ideals, and the graded prime ideals of R are the ideals generated by a subset of the variables; in this way, we denote by $^* \operatorname{Spec}(R)$ the set of graded prime ideals of R.

In this setting, given $\mathfrak{p}$ a (not necessarily graded) prime ideal of R containing a monomial ideal $J \subset R$, it is known that $\mathfrak{p} \supseteq \mathfrak{p}^* \supseteq J$, where $\mathfrak{p}^*$ is the graded prime ideal of R generated by the homogeneous elements of $\mathfrak{p}$.

The first technical result we need to establish is that, in order to calculate the non-finitely generated locus of a Stanley–Reisner ring, it is enough to restrict to face ideals; this holds because of the following:

Lemma 1 *Let $I \subset R$ be a squarefree monomial ideal, set $S := R/I$, and assume that there is a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathcal{A}^{S_{\mathfrak{p}^*}}$ is k-generated for some integer $k \geq 1$. Then, $\mathcal{A}^{S_{\mathfrak{p}}}$ is also k-generated.*

In particular, if there is a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathcal{A}^{S_{\mathfrak{p}^*}}$ is finitely generated as a ring over $S_{\mathfrak{p}^*}$, then we have that $\mathcal{A}^{S_{\mathfrak{p}}}$ is also finitely generated as a ring over $S_{\mathfrak{p}}$.

Proof Assume that $\mathcal{A}^{S_{\mathfrak{p}^*}}$ is k-generated as a ring over $S_{\mathfrak{p}^*}$; this means that $c_e(\mathcal{A}^{S_{\mathfrak{p}^*}}) = 0$ for all $e > k$. Hence, for all $e > k$, one has that

$$\left(\frac{(I^{[p^e]} :_R I)}{L_e}\right)_{\mathfrak{p}^*} = 0.$$

This means, setting $M_e := \frac{(I^{[p^e]}:_R I)}{L_e}$, that $\mathfrak{p}^* \notin \operatorname{Supp}(M_e)$, and this implies, by [5, 13.1.6 (i)], that $\mathfrak{p} \notin \operatorname{Supp}(M_e)$ for all $e > k$. But, this is equivalent to say that $c_e(\mathcal{A}^{S_{\mathfrak{p}}}) = 0$ for all $e > k$, which implies that $\mathcal{A}^{S_{\mathfrak{p}}}$ is k-generated; the proof is therefore complete. □

Now, we are in a position to prove one of the main results of this paper, namely, the following:

Theorem 2 *Let $\mathbb{K}$ be a field of prime characteristic p, let $I \subseteq \mathbb{K}[x_1, \ldots, x_n] = R$ be a squarefree monomial ideal, and set $S := R/I$. Then, the set*

$$W = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I,\ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is not finitely generated as a ring over } S_{\mathfrak{p}}\}$$

is closed in the Zariski topology.

Proof If $W = \emptyset$, then we are done, so we can assume $W \neq \emptyset$; now, let $\mathfrak{p} \in W$. Thanks to Lemma 1, we have that $\mathfrak{p}^* \in W$. This shows that the minimal members of the set W are face ideals, which turn out to be a finite set. Thus, let $\mathfrak{p}_1^*, \ldots, \mathfrak{p}_t^*$ be the face ideals that belong to W, and set

$$J := \bigcap_{a=1}^{t} \mathfrak{p}_a^*.$$

Our above argument shows that $W \subseteq V(J)$; conversely, let $\mathfrak{q} \supset J$, in particular $\mathfrak{q} \supset \mathfrak{p}_a$ for some $a = 1, \ldots, t$. In this way, since $\mathfrak{p}_a \in W$ and W is closed under specialization by Proposition 3, we have that $\mathfrak{q} \in W$.

Summing up, we have finally checked that $W = V(J)$, hence a Zariski closed set.

Since we know that, in the Stanley–Reisner case, the finite generation of the Frobenius algebra is equivalent to its principal generation (equivalently, to its 1-generation using the terminology that we employ in this paper), Theorem 2 implies the following:

Theorem 3 *Let $\mathbb{K}$ be a field of prime characteristic p, let $I \subseteq \mathbb{K}[x_1, \ldots, x_n] = R$ be a squarefree monomial ideal, and set $S := R/I$. Then, we have*

$$\overline{W} = \overline{W}_1 = \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{p} \supseteq I,\ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is } 1\text{-generated}\}.$$

Remark 2 Theorem 2 is equivalent to say that

$$\overline{W} = \{\mathfrak{p} \in \operatorname{Spec}(R) \,:\, \mathfrak{p} \supseteq I,\ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is finitely generated as a ring over } S_{\mathfrak{p}}\}$$

is open in the Zariski topology. The reader will easily note that our proof does not involve the use of the so-called (the terminology is borrowed from [14]) topological Nagata criterion [16, Theorem 24.2].

Remark 3 Similarly, as we have already done in Corollary 2, both Theorems 2 and 3 can also be equally formulated for the corresponding Frobenius algebra.

4 An Algorithmic Description of the Non-finitely Generated Locus

Our goal now is to give, in the case of a Stanley–Reisner ring, an explicit and algorithmic description of the non-finitely generated locus of its corresponding Frobenius algebra. We continue with the same notations as in Sect. 3.

Hereafter, in this section, $[n]$ will denote the subset $\{1, \dots, n\}$ of $n \geq 1$ elements, let $\Delta \subset [n]$ be a simplicial complex, and let $I = I_\Delta \subseteq R$ be the squarefree monomial ideal attached to Δ via the Stanley–Reisner correspondence [18, 1.6 and 1.7]; moreover, given $F \subset \Delta$ a face, we denote by $\mathbf{p}_F$ the prime ideal of R generated by the variables whose indices are not in F. In other words,

$$\mathfrak{p}_F := (x_i \,:\, i \notin F).$$

On the other hand, given a monomial $m = x_1^{a_1} \cdots x_n^{a_n}$, we denote by $\operatorname{supp}(m)$ its support, that is,

$$\operatorname{supp}(m) = \{i \in [n] \,:\, a_i \neq 0\}.$$

Finally, set

$$\mathbf{x}_F := \prod_{i \in F} x_i.$$

Lemma 2 *If F is a face of Δ, then $I \subseteq (I :_R \mathbf{x}_F) \subseteq \mathfrak{p}_F$.*

Proof Since the inclusion $I \subseteq (I :_R \mathbf{x}_F)$ holds by the definition of colon ideals, it is enough to check that $(I :_R \mathbf{x}_F) \subseteq \mathfrak{p}_F$.

Indeed, let $m_1, \dots, m_t$ be a system of squarefree monomial generators of I, and it is known [10, 1.2.2] that $(I :_R \mathbf{x}_F) = (g_1, \dots, g_t)$, where

$$g_j = \frac{m_j}{\gcd(m_j, \mathbf{x}_F)},\ 1 \leq j \leq t.$$

Therefore, it is enough to check that, given $1 \leq j \leq t$, g_j is divisible by a variable x_i with $i \notin F$.

So, fix $1 \leq j \leq t$, and set $S_j := \operatorname{supp}(m_j)$; notice that, if $S_j \subset F$, then m_j would divide $\mathbf{x}_F$, and therefore, $\mathbf{x}_F \in I$, a contradiction because F is a face of Δ. Hence, there is $i \in S_j$ such that $i \notin F$; thus, x_i divides, not only m_j, but also g_j, as claimed. □

The reader might ask why one needs to consider the above colon ideal. Next statement gives a combinatorial reason; recall that, given a simplicial complex Δ and a face F of it, the link of F inside Δ is defined as $\operatorname{link}(F) := \{G \subset \Delta :\ F \cap G = \emptyset,\ F \cup G \subset \Delta\}$.

Lemma 3 *Let $F \subset \Delta$ be a face. Then, $(I :_R \mathbf{x}_F) = I_{\operatorname{link}(F)}$.*

Proof First of all, let $m_1, \dots, m_t$ be a system of squarefree monomial generators of I; since [10, 1.2.2] $(I :_R \mathbf{x}_F) = (g_1, \dots, g_t)$, where

$$g_j = \frac{m_j}{\gcd(m_j, \mathbf{x}_F)}, \ 1 \leq j \leq t, \tag{1}$$

one has that $(I :_R \mathbf{x}_F)$ is also a squarefree monomial ideal, so $(I :_R \mathbf{x}_F) = I_{\Delta'}$ for some simplicial complex Δ'. Moreover, as $I \subseteq (I :_R \mathbf{x}_F)$, one has, again as consequence of the Stanley–Reisner correspondence, that $\Delta' \subset \Delta$. Finally, given $G \subset [n]$ such that $\mathbf{x}_G$ is a squarefree minimal monomial generator of $(I :_R \mathbf{x}_F)$, one has that $G \notin \Delta'$ if and only if $\mathbf{x}_G \in (I :_R \mathbf{x}_F)$, which is equivalent to say that $\mathbf{x}_F \cdot \mathbf{x}_G \in I$, which is equivalent to say (notice that $F \cap G = \emptyset$ because of (1)) that $\mathbf{x}_{F \cup G} \in I$, which is equivalent to say that $F \cup G \notin \Delta$. □

Now, before establishing the main result of this section, we want to review for the convenience of the reader the notion of monomial localization as introduced in [11, p. 293].

Definition 4 Let $J \subseteq R$ be a (not necessarily squarefree) monomial ideal. We define the **monomial localization of** J **at the prime ideal** $\mathfrak{p}_F$ as the monomial ideal $J(\mathfrak{p}_F) \subset \mathbb{K}[x_i : i \notin F]$ obtained from J by setting $x_j = 1$ for all variables $j \in F$. In other words, $J(\mathfrak{p}_F)$ is the extension of J with respect to the $\mathbb{K}$-algebra map

$$\varphi_{\mathfrak{p}_F} : R \to \mathbb{K}[x_i : i \notin F],$$
$$x_j \longmapsto \begin{cases} x_j & \text{if } j \notin F, \\ 1 & \text{if } j \in F. \end{cases}$$

We also need to establish the below technical statement that will play a key role along the proof of the main result of this section. In the following result, given J a monomial ideal with minimal monomial generating set $\{m_1, \dots, m_r\}$, we denote by LCM_J the following monomial ideal:

$$\mathrm{LCM}_J := (\mathrm{LCM}(m_i, m_j) : 1 \leq i < j \leq r).$$

Finally, given $I = (f_1, \dots, f_s)$ an ideal inside a commutative Noetherian ring, we set $I^{[2]} := (f_1^2, \dots, f_s^2)$.

Lemma 4 *Let $I \subset R$ be a squarefree monomial ideal, and let $\mathfrak{p} = \mathfrak{p}_F \in {}^*\operatorname{Spec}(R)$ be a face ideal. Then, the following statements are equivalent.*

(i) *$\mathcal{A}^{S_\mathfrak{p}}$ is a finitely generated ring over $S_\mathfrak{p}$.*
(ii) *$\mathcal{A}^{\widetilde{R}}$ is a finitely generated ring over $\widetilde{R}$, where $\widetilde{R} := \mathbb{K}[x_i : i \notin F]_{(x_i :\, i \notin F)}$.*
(iii) *$\mathcal{A}^{R'}$ is a finitely generated ring over R', where $R' := \mathbb{K}[x_i : i \notin F]$.*
(iv) *We have $(I'^{[2]} :_{R'} I') = I'^{[2]} + (\mathrm{LCM}_{I'})$, where $I' := I(\mathfrak{p}_F)$.*
(v) *We have $(K^{[2]} :_R K) = K^{[2]} + (\mathrm{LCM}_K)$, where $K := (I :_R \mathbf{x}_F)$.*

Proof We consider the following maps, where starting from the left, the first one is just $\varphi := \varphi_{\mathfrak{p}_F}$ as defined in Definition 4, the second one is the obvious inclusion, and the last one is localization at $\mathfrak{p}$.

$$R \xrightarrow{\varphi} R' \hookrightarrow R \longrightarrow R_\mathfrak{p}.$$

Since the inclusion $R' \hookrightarrow R$ is given by adjoining variables and $R \longrightarrow R_\mathfrak{p}$ is a localization, the composition $R' \longrightarrow R_\mathfrak{p}$ is flat; moreover, the ideal $(x_i : i \notin F) \subset R'$ maps under this composition to $\mathfrak{p}R_{\mathfrak{p}}$. Combining these two facts, one gets that the induced inclusion

$$\widetilde{R} = \mathbb{K}[x_i : i \notin F]_{(x_i : i \notin F)} \longrightarrow R_{\mathfrak{p}}$$

is faithfully flat; this shows, combined with Proposition 2, that parts (i) and (ii) are equivalent.

Moreover, since $(x_i : i \notin F) \subset R'$ is the unique homogeneous maximal ideal of the graded ring R', one has that the localization $R' \longrightarrow \widetilde{R}$, restricted to graded R'-modules, is also faithfully flat; hence, parts (ii) and (iii) are also equivalent again using Proposition 2.

Now, notice that $R' \otimes_R (R/I) \cong R'/I' := S'$ is also a Stanley–Reisner ring; therefore, by [1, Theorem 3.5] and [2, Remark 2 and Lemma 3], the algebra $\mathcal{A}^{S'}$ is finitely generated as ring over S' if and only if

$$(I'^{[2]} : I') = I'^{[2]} + (\mathrm{LCM}_{I'}),$$

which proves the equivalence between parts (iii) and (iv).

Finally, set $K := (I :_R \mathbf{x}_F)$. Along Lemma 3, we saw that K admits a minimal monomial generating set G such that, if a monomial m belongs to G, then the support of m is disjoint from F. On the other hand, notice, by the definition of φ, that $\varphi(K)R' = I'$; actually, both K and I' admit G as minimal generating set. This implies that

$$(I'^{[2]} :_{R'} I') = I'^{[2]} + (\mathrm{LCM}_{I'}),$$

if and only if

$$(K^{[2]} :_R K) = K^{[2]} + (\mathrm{LCM}_K),$$

just what we finally wanted to show. □

Finally, we are in a position to establish the second main result of this paper, which is the following:

Theorem 4 *Let $\mathbb{K}$ be a field of prime characteristic p, let $R = \mathbb{K}[x_1, \ldots, x_n]$, $S = R/I$, and I is a squarefree monomial ideal. Then, the set*

$$W = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I,\ \mathcal{A}^{S_{\mathfrak{p}}} \text{ is not finitely generated as a ring over } S_{\mathfrak{p}}\}$$
$$= \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I,\ \mathcal{F}^{E_{S_{\mathfrak{p}}}} \text{ is not finitely generated as a ring over } \widehat{S_{\mathfrak{p}}}\}$$

is closed in the Zariski topology; more precisely, it is equal to $V(J)$, where

$$J = \bigcap_{F \in \mathrm{IGL}(\Delta)} \mathfrak{p}_F,$$

$\mathfrak{p}_F$ is the prime ideal generated by the variables whose indices are not in F, and

$$\mathrm{IGL}(\Delta) := \{F \subset \Delta : (K^{[2]} :_R K) \neq K^{[2]} + (\mathrm{LCM}_K),\ K := (I :_R \mathbf{x}_F)\}.$$

In particular, $W \cap {}^*\mathrm{Spec}(R) = \{\mathfrak{p}_F : F \in \mathrm{IGL}(\Delta)\}$.

Proof Let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \supset I$ and such that $\mathcal{A}^{S_{\mathfrak{p}}}$ is a not finitely generated ring over $S_{\mathfrak{p}}$. Thanks to Lemma 1, we know that $\mathfrak{p} \supset \mathfrak{p}^*$, and $\mathcal{A}^{S_{\mathfrak{p}^*}}$ is also a not finitely generated ring over $S_{\mathfrak{p}^*}$. Now, write $\mathfrak{p}^* = \mathfrak{p}_F$ for some $F \subset [n]$. Since $\mathfrak{p}_F \supset I$, by [18, Theorem 1.7], there is a minimal prime ideal $\mathfrak{p}_G$ of I such that $\mathfrak{p}_F \supset \mathfrak{p}_G$ for some $G \subset \Delta$. This implies that $F \subset G$, and therefore, since $G \subset \Delta$ and Δ is a simplicial complex, $F \subset \Delta$. This shows that $\mathfrak{p} \supset J$, and therefore, $W \subset V(J)$.

Now, let $\mathfrak{p} \in V(J)$, so $\mathfrak{p} \supset \mathfrak{p}_F$ for some $F \in \mathrm{IGL}(\Delta)$. By construction, $\mathfrak{p}_F \in W$; moreover, since W is closed under specialization by Proposition 3, one has that $\mathfrak{p} \in W$, just what we finally wanted to prove. □

Theorem 4 leads to a very naive algorithm for computing the infinitely generated locus of the Frobenius algebra attached to a Stanley–Reisner ring; in this method, our input is a simplicial complex Δ as before, and our output will be the ideal J described along the statement of Theorem 4. This method has been implemented in Macaulay2 [9].

(i) **Step 1**: Initialize K as I, $\mathfrak{p}$ as the ideal $(x_1, \ldots, x_n)$, and L as the empty list.
(ii) **Step 2**: If $(K^{[2]} : K) \neq K^{[2]} + \langle \mathrm{LCM}_K \rangle$, then add $\mathfrak{p}$ to the list L. Otherwise, return the empty list.
(iii) **Step 3**: For each non-empty face $F \subset \Delta$, assign to K the value $(I : \mathbf{x}_F)$. If $(K^{[2]} : K) \neq K^{[2]} + \langle \mathrm{LCM}_K \rangle$, then add $\mathfrak{p} = \mathfrak{p}_F$ to the list L, where $\mathfrak{p}$ denotes the ideal generated by the variables that do not belong to F.
(iv) **Step 4**: Output the intersection of the elements of L.

Discussion 3 Thanks to [2, Theorem 4], we know that the Frobenius algebra attached to the injective hull of a complete Stanley–Reisner ring is finitely generated if and only if Δ has no free faces. In this way, we can rewrite the above algorithm in terms of the simplicial complex Δ as follows:

(i) If Δ has no free faces, then stop and output the empty list.
(ii) For each non-empty face $F \subset \Delta$, let $\mathrm{link}(F)$ be its corresponding link. If $\mathrm{link}(F)$ has at least one free face, then add $\mathfrak{p} = \mathfrak{p}_F$ to the list L.

We conclude this section by exhibiting a family of ideals with a quite simple infinitely generated locus; before so, we want to review the following notion [4, Definition 2.5].

Definition 5 Let $\mathbb{K}$ be any field, let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$, let $I \subseteq R$ be a squarefree monomial ideal, and let $\mathrm{supp}(I)$ be the support of I, that is,

$$\mathrm{supp}(I) := \{i \in [n] : x_i \text{ divides at least one minimal monomial generator of } I\}.$$

We say that I is a **nearly complete intersection** (hereafter, NCI for short) if it is generated in degree at least two and is not a complete intersection, and for each $i \in \mathrm{supp}(I)$, the monomial localization $I(\mathfrak{p}_{([n]\setminus \mathrm{supp}(I))\cup\{i\}})$ is a complete intersection.

The interested reader in this family of ideals may also like to consult [17] where a classification of these ideals in the degree two case is given. Our reason for considering this class of ideals is given in the following:

Proposition 4 *Let $\mathbb{K}$ be a field of prime characteristic p, let $R = \mathbb{K}[x_1, \ldots, x_n]$, $S = R/I$, and I is a NCI ideal. Then, the set*

$$W = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I,\ \mathcal{A}^{S_\mathfrak{p}} \text{ is not finitely generated as a ring over } S_\mathfrak{p}\}$$

is either empty or is given by the closed set $V(J)$, where $J := (x_i : i \in \mathrm{supp}(I))$.

Proof If $W = \emptyset$, then we are done, so we assume hereafter $W \neq \emptyset$. First of all, notice that

$$J = (x_i : i \in \mathrm{supp}(I)) = \mathfrak{p}_{[n]\setminus \mathrm{supp}(I)}$$

is a prime ideal. Secondly, since $W \neq \emptyset$, in particular, one has that $J \in W$. Now, we assume, to reach a contradiction, that there is a face ideal $\mathfrak{p} \in W$ such that $\mathfrak{p} \subsetneq J$. This implies that there is $i \in \mathrm{supp}(I)$ such that

$$\mathfrak{p} \subseteq (x_j : j \in \mathrm{supp}(I)\setminus\{i\}) = \mathfrak{p}_{([n]\setminus \mathrm{supp}(I))\cup\{i\}}.$$

However, since W is closed under specialization and $\mathfrak{p} \in W$, we have that $\mathfrak{p}_{([n]\setminus \mathrm{supp}(I))\cup\{i\}} \in W$, which is equivalent to say, thanks to Lemma 4, that

$$(I'^{[2]} : I') \neq I'^{[2]} + (\mathrm{LCM}_{I'}),$$

where $I' = I(\mathfrak{p}_{([n]\setminus \mathrm{supp}(I))\cup\{i\}})$. But this is a contradiction because, since I is NCI by assumption, I' is a complete intersection, and therefore, in this case, we know that

$$(I'^{[2]} : I') = I'^{[2]} + (\mathrm{LCM}_{I'}).$$

Summing up, we have checked that if $W \neq \emptyset$, then

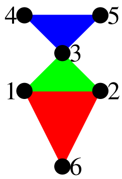
$$W = V(\mathfrak{p}_{[n]\setminus \mathrm{supp}(I)}) = V((x_i \ : \ i \in \mathrm{supp}(I))),$$

as claimed. □

5 Examples

The goal of this section is to show several examples of how to use our method in Macaulay2; the first one is directly borrowed from [3, Example 5].

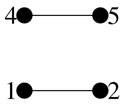
Example 1 Let Δ be the simplicial complex given by facets $\{1, 2, 3\}$, $\{1, 2, 6\}$ and $\{3, 4, 5\}$. In the following calculation, the set of vertices $\{1, 2, 3, 4, 5, 6\}$ is identified with the set of variables $\{x, y, z, w, a, b\}$.



We use our method to determine the non-finitely generated locus of the corresponding Stanley–Reisner ideal.

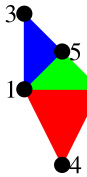
```
loadPackage "SimplicialComplexes";

load "non_finitely_generated_locus_algorithm.m2";

R=QQ[x,y,z,w,a,b];

A= simplicialComplex {x*y*z,x*y*b,z*w*a};

I=monomialIdeal(A);

I

monomialIdeal (x*w, y*w, x*a, y*a, z*b, w*b, a*b)

nonfglocus(I)

monomialIdeal (x, y, w, a, b)
```

Notice that link(3) is a subsimplicial complex of Δ with facets $\{1, 2\}$ and $\{4, 5\}$, and its corresponding squarefree monomial ideal has an infinitely generated Frobenius algebra [3, Example 1]. Here, we have depicted this link:



The below example is exactly [8, Example 2.10].

Example 2 Let Δ be the simplicial complex given by facets $\{1, 2, 5\}$, $\{1, 3, 5\}$ and $\{1, 2, 4\}$.



We use our method to determine the non-finitely generated locus of the corresponding Stanley–Reisner ideal.

```
R=QQ[x_1,x_2,x_3,x_4,x_5];

A= simplicialComplex {x_1*x_2*x_5,x_1*x_3*x_5,x_1*x_2*x_4};

I=monomialIdeal(A);

I

monomialIdeal (x x , x x , x x )
                2 3    3 4    4 5

nonfglocus(I)

monomialIdeal (x , x , x , x )
                2   3   4   5
```

We can also explore how far are from Theorem 4 some of the components calculated in [8] using our method in the following way. The first example is the one studied in [8, Proposition 2.6].

Example 3 Let Δ be the simplicial complex given by facets $\{2\}$ and $\{1, 3\}$. As it is explained there, the upper bound given by [8, Theorem 2.4] yields the ideal (x_2). Our algorithm shows, in this particular example, that the whole non-finitely generated locus is given by a smaller ideal.

```
R=QQ[x_1,x_2,x_3];
```

```
I=monomialIdeal(x_1*x_2,x_2*x_3);

{closedcomponent1(I),nonfglocus(I)}

{monomialIdeal x , monomialIdeal (x , x , x )}
                2                  1   2   3
```

The next example is the one studied in [8, Example 2.9]. This example shows, in particular, that the upper bound given by [8, Theorem 2.7] is, in general, not always equal to the defining ideal of the non-finitely generated locus.

Example 4 Let Δ be the simplicial complex given by facets $\{1, 2, 4\}$, $\{1, 3\}$ and $\{2, 3\}$. We proceed as before.

```
R=QQ[x_1,x_2,x_3,x_4];

I=monomialIdeal(x_1*x_2*x_3,x_3*x_4);

{closedcomponent1(I),nonfglocus(I)}

{monomialIdeal x , monomialIdeal (x x , x , x )}
                3                  1 2   3   4
```

Acknowledgements The authors would like to especially thank the referee for his/her useful comments and remarks concerning the contents of this paper.

D. A. J. Gómez Ramírez thanks the Institución Universitaria Pascual Bravo, Visión Real Cognitiva S.A.S, and Juan Carlos Díaz and Liliana Lopera for all their kindness and support.

Funding Alberto F. Boix and S. Zarzuela are partially supported by the Spanish Ministerio de Economía y Competitividad grant PID2019-104844GB-I00.

References

1. Álvarez Montaner, J., Boix, A.F., Zarzuela, S.: Frobenius and Cartier algebras of Stanley-Reisner rings. J. Algebra **358**, 162–177 (2012)
2. Álvarez Montaner, J., Yanagawa, K.: Addendum to Frobenius and Cartier algebras of Stanley–Reisner rings [J. Algebra **358**, 162–177 (2012)]. J. Algebra **414**, 300–304 (2014)
3. Boix, A.F., Zarzuela, S.: Frobenius and Cartier algebras of Stanley-Reisner rings (II). Acta Math. Vietnam. **44**(3), 571–586 (2019)
4. Boocher, A., Seiner, J.: Lower bounds for Betti numbers of monomial ideals. J. Algebra **508**, 445–460 (2018)
5. Brodmann, M.P., Sharp, R.Y.: Local Cohomology: An Algebraic Introduction with Geometric Applications. Cambridge Studies in Advanced Mathematics, vol. 136, second edn. Cambridge University Press, Cambridge (2013)
6. Enescu, F., Yao, Y.: The Frobenius complexity of a local ring of prime characteristic. J. Algebra **459**, 133–156 (2016)
7. Fedder, R.: F-purity and rational singularity. Trans. Amer. Math. Soc. **278**(2), 461–480 (1983)
8. Gallego, E., Vélez, J.D., Molina-Aristizábal, S.D., Hernández-Rodas, J.P., Gómez-Ramírez, D.A.J.: On the topological structure of the (non)finitely-generated locus of Frobenius algebras emerging from Stanley-Reisner rings. Comm. Algebra **51**(1), 414–423 (2023)
9. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry (2013). Available at http://www.math.uiuc.edu/Macaulay2/
10. Herzog, J., Hibi, T.: Monomial Ideals. Graduate Texts in Mathematics, vol. 260. Springer-Verlag, London Ltd., London (2011)

11. Herzog, J., Rauf, A., Vladoiu, M.: The stable set of associated prime ideals of a polymatroidal ideal. J. Algebr. Comb. **37**(2), 289–312 (2013)
12. Katzman, M.: A non-finitely generated algebra of Frobenius maps. Proc. Amer. Math. Soc. **138**(7), 2381–2383 (2010)
13. Katzman, M., Schwede, K., Singh, A.K., Zhang, W.: Rings of Frobenius operators. Math. Proc. Cambridge Philos. Soc. **157**(1), 151–167 (2014)
14. Kimura, K.: Openness of various loci over Noetherian rings. J. Algebra **633**, 403–424 (2023)
15. Lyubeznik, G., Smith, K.E.: On the commutation of the test ideal with localization and completion. Trans. Amer. Math. Soc. **353**(8), 3149–3180 (2001). electronic
16. Matsumura, H.: Commutative Ring Theory. Cambridge Studies in Advanced Mathematics, vol. 8, second edn. Cambridge University Press, Cambridge (1989). Translated from the Japanese by M. Reid
17. Miller, C., Stone, B.: Classifying nearly complete intersection ideals generated in degree two (2023). Available at https://arxiv.org/pdf/2101.07901v1.pdf
18. Miller, E., Sturmfels, B.: Combinatorial Commutative Algebra. Graduate Texts in Mathematics, vol. 227. Springer-Verlag, New York (2005)
19. The Stacks project authors: the Stacks project (2021). https://stacks.math.columbia.edu