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Extensions of Valuation Domains and Going-up

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Abstract

Suppose F is a field with valuation v and valuation domain O_v, E/F is a finite-dimensional field extension, and R is an O_v-subalgebra of E such that $F \cdot R = E$ and $R \cap F = O_v$. It is known that R satisfies LO, INC, GD and SGB over O_v; it is also known that under certain conditions R satisfies GU over O_v. In this paper, we present a necessary and sufficient condition for the existence of such R that does not satisfy GU over O_v. We also present an explicit example of such R that does not satisfy GU over O_v.

Keywords Valuation domain · Going-up · Quasi-valuation

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1 Introduction

Valuation theory has long been a key tool in commutative algebra, with applications in number theory and algebraic geometry. Several generalizations of the notion of valuation were made throughout the last few decades; the main purpose of these generalizations was to utilize them in the study of noncommutative algebra, especially in division rings. See [6] for a comprehensive survey.

In this paper we distinguish between a domain, which refers to an integral domain, and a ring, which may contain zero divisors and might not be commutative. Throughout this paper, for a valuation u on a field K we denote by O_u the valuation domain of K corresponding to u, by I_u the maximal ideal of O_u, and by Γ_u the value group of u. The symbol $\subset$ means proper inclusion and the symbol $\subseteq$ means inclusion or equality. Any unexplained terminology is as in [4].

Let S be a commutative ring and let R be an algebra over S. For subsets $I \subseteq R$ and $J \subseteq S$ we say that I is lying over J if $J = \{s \in S \mid s \cdot 1_R \in I\}$. By abuse of notation, we write $J = I \cap S$ (even when R is not faithful over S). We recall now from [5, Section 1.2]

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the definitions of the classical lifting conditions of ring extensions. We say that R satisfies LO (lying over) over S if for all $P \in \text{Spec}(S)$ there exists $Q \in \text{Spec}(R)$ lying over P. We say that R satisfies GD (going-down) over S if for any $P_1 \subset P_2$ in $\text{Spec}(S)$ and for every $Q_2 \in \text{Spec}(R)$ lying over P_2, there exists $Q_1 \subset Q_2$ in $\text{Spec}(R)$ lying over P_1. We say that R satisfies GU (going-up) over S if for any $P_1 \subset P_2$ in $\text{Spec}(S)$ and for every $Q_1 \in \text{Spec}(R)$ lying over P_1, there exists $Q_1 \subset Q_2$ in $\text{Spec}(R)$ lying over P_2. We say that R satisfies SGB (strong going-between) over S if for any $P_1 \subset P_2 \subset P_3$ in $\text{Spec}(S)$ and for every $Q_1 \subset Q_3$ in $\text{Spec}(R)$ such that Q_1 is lying over P_1 and Q_3 is lying over P_3, there exists Q_2, with $Q_1 \subset Q_2 \subset Q_3$ in $\text{Spec}(R)$, lying over P_2. We say that R satisfies INC (incomparability) over S if whenever $Q_1 \subset Q_2$ in $\text{Spec}(R)$, we have $Q_1 \cap S \subset Q_2 \cap S$.

Let K be a field and let L be an algebraic field extension of K. Let T be a valuation domain of K. It is well known (cf. [3, 13.2]) that there exists a valuation domain of L lying over T. Recall (cf. [3, Corollary 13.5]) that T is called indecomposed in L if there exists a unique valuation domain of L lying over T; otherwise, T is called decomposed in L. Moreover, by [3, Corollary 13.7]), whenever the separable degree of K over L is finite, the number of valuation domains of L that are lying over T is less than or equal to the separable degree of K over L.

Recall from [4] that a quasi-valuation on a ring A is a function $w : A \to M \cup \{\infty\}$, where M is a totally ordered abelian monoid, to which we adjoin an element ∞ greater than all elements of M, and w satisfies the following properties:

(B1) $w(0) = \infty$;
(B2) $w(xy) \geq w(x) + w(y)$ for all $x, y \in A$;
(B3) $w(x + y) \geq \min\{w(x), w(y)\}$ for all $x, y \in A$.

In [4] we studied quasi-valuations that extend a given valuation on a finite-dimensional field extension. We proved that the prime spectrum of the associated quasi-valuation domain and the prime spectrum of the valuation domain are intimately connected. Along these lines, let F be a field with a non-trivial valuation v and a corresponding valuation domain O_v, let E/F be a finite field extension, and let R be an O_v-subalgebra of E such that $FR = E$ and $R \cap F = O_v$; we call such R an O_v-nice subalgebra of E. By [4, Theorem 9.38, statements (2) and (3)], there exists a quasi-valuation w on E extending v on F, such that $R = O_w$ (we call $O_w = \{x \in E \mid w(x) \geq 0\}$ the quasi-valuation domain), and thus R satisfies LO, INC and GD over O_v. Note that if $FR \neq E$, then since FR is a finite-dimensional field extension of F, one can apply the results of [4] replacing E by FR. Moreover, by [4, Theorem 9.38, statement (6)], if there exists a quasi-valuation w' on E extending v with $R = O_{w'}$ such that $w'(E \setminus \{0\})$ is torsion over Γ_v, then R satisfies GU over O_v. In [5] we generalized some of the results that had been proved in [4], and proved that every algebra over a commutative valuation ring satisfies SGB over it (see [5, Theorem 3.8]). We also generalized [4, Theorem 9.38, statement (6)] and gave weaker conditions for a quasi-valuation ring to satisfy GU over a valuation domain. So, focusing on field extensions, one can say that if E/F is a finite field extension, and R is an O_v-nice subalgebra of E, then R satisfies LO, INC, GD, and SGB over O_v. In light of the above discussion, there exists a quasi-valuation w on E extending v on F such that $O_w = R$; one may suspect that such R also satisfies GU over O_v. We shall see that this is not the case.

In fact, in this paper we answer a question asked in [5, discussion after Corollary 4.26], by characterizing the existence of O_v-nice subalgebras of E that do not satisfy GU over O_v, and presenting an explicit example in which an O_v-nice subalgebra of E does not satisfy GU over O_v.

2 Going-up May Not Apply

Let F be a field with valuation v and corresponding valuation domain O_v; let E/F be a finite-dimensional field extension and let R be an O_v-nice subalgebra of E. In this section we characterize the existence of O_v-nice subalgebras of E that do not satisfy GU over O_v in terms of the decomposability in E of proper overrings of O_v (recall that an overring of O_v is a subring of F, the field of fractions of O_v, that contains O_v). Then, we present an example of an O_v-nice subalgebra R_0 of E that does not satisfy GU over O_v; equivalently, there exists a quasi-valuation w on E extending v on F with $FO_w = E$ and for which O_w does not satisfy GU over O_v. At the end of this section, we discuss the filter quasi-valuation induced by (R_0, v).

Theorem 2.1 *Let F be a field with valuation v and corresponding valuation domain O_v, and let E/F be a finite-dimensional field extension. There exists an O_v-nice subalgebra of E that does not satisfy GU over O_v iff there exists a proper overring of O_v that is decomposed in E.*

Proof Assume that there exists a proper overring B of O_v that is decomposed in E; then $B = (O_v)_P$, for some non-maximal prime ideal P of O_v. Let D_1 and D_2 be two valuation domains of E that are lying over B. It is clear that D_1 and D_2 are incomparable with respect to containment; this fact can be easily deduced by [3, Theorem 6.6] and [3, Proposition 13.1]. By [3, Theorem 13.2] there exists a valuation domain C_1 of E that is lying over O_v. By [3, Theorem 6.6], the set of all overrings of C_1 is totally ordered by inclusion; thus, C_1 cannot be contained in both D_1 and D_2. Without loss of generality, we assume that C_1 is not contained in D_2. Denote by I_1 the maximal ideal of C_1, and by I_2 the maximal ideal of D_2. Let $R = C_1 \cap D_2$; then, by [1, Ch. 6, § 7, no. 1, Proposition 2], R has two maximal ideals: $I_1 \cap R$ and $I_2 \cap R$. However, since D_2 is lying over $B = (O_v)_P$, its maximal ideal is lying over P. So, R has a maximal ideal that does not lie over I_v, the maximal ideal of O_v. In particular, R does not satisfy GU over O_v.

In the other direction, assume that there exists an O_v-nice subalgebra R of E that does not satisfy GU over O_v. So, there exist $P_1 \subset P_2$ in Spec(O_v) and $Q_1 \in$ Spec(R) lying over P_1, such that there exists no $Q_1 \subset Q_2$ in Spec(R) lying over P_2. By [4, Theorem 9.38, statement (3)], R satisfies LO over O_v; thus, there exists ${Q_2}'$ in Spec(R) lying over P_2. Again, by [4, Theorem 9.38, statement (3)], R satisfies GD over O_v; thus, there exists ${Q_1}' \subset {Q_2}'$ in Spec(R) lying over P_1. It is clear that $Q_1 \neq {Q_1}'$. By [3, Corollary 9.7] there exists a valuation domain O_u of E, containing R, and having a maximal ideal I_u such that $Q_1 = I_u \cap R$; likewise, there exists a valuation domain O_w of E, containing R, and having a maximal ideal I_w such that ${Q_1}' = I_w \cap R$. Since both Q_1 and ${Q_1}'$ are lying over P_1, we deduce that I_u and I_w are lying over P_1; thus, O_u and O_w are two different valuation domains of E that are lying over $(O_v)_{P_1}$; i.e., $(O_v)_{P_1}$ is a proper overring of O_v that is decomposed in E. □

Remark 2.2 Note that, by [5, Corollary 4.24], an O_v-nice subalgebra R of E that does not satisfy GU over O_v, is not finitely generated as an algebra over O_v. Moreover, by [4, Theorem 9.38, statement (6)], for any quasi-valuation w on E extending v on F with $O_w = R$, one has $w(E \setminus \{0\})$ is not torsion over Γ_v.

We present now an explicit example demonstrating the situation discussed in the previous theorem.

Example 2.3 Let C be a field with $\mathrm{Char}(C) \neq 2$ and let $F = C(x, y)$ denote the field of rational functions with two indeterminates x and y. Let Γ_v denote the group $\mathbb{Z} \times \mathbb{Z}$ with the left to right lexicographic order and let v denote the rank 2 valuation on F defined by

$$v(0) = \infty, \quad v\left(\sum_{0 \leq i, j \leq k} \alpha_{ij} y^i x^j\right) = \min\{(i, j) \mid \alpha_{ij} \neq 0\}$$

for every nonzero $\sum_{0 \leq i, j \leq k} \alpha_{ij} y^i x^j \in C[x, y]$, and $v(\frac{f}{g}) = v(f) - v(g)$ for every $f, g \in C[x, y],\ g \neq 0$. Let $\{0\} \neq P$ denote the non-maximal prime ideal of O_v, namely $P = yO_v$; and denote $(O_v)_P$, the localization of O_v at P, by $O_{\widetilde{v}}$. Note that $O_{\widetilde{v}}$ is a valuation domain of F of rank 1. Let $E = F[\sqrt{1-y}]$ and let u be a valuation on E extending v; it is well known that there exists such u (see [2, Corollary 14.1.2]). By the fundamental inequality of valuation theory (see [2, Theorem 17.1.5]), there exist either one or two extensions of v to E. Note that

$$\sqrt{1-y} \in O_u, \quad (1 + \sqrt{1-y}) + (1 - \sqrt{1-y}) = 2 \notin I_v$$

and

$$u((1 + \sqrt{1-y})(1 - \sqrt{1-y})) = u(y) = (1, 0).$$

Hence, exactly one of the elements $1 + \sqrt{1-y}$ or $1 - \sqrt{1-y}$ has u-value $(1, 0)$. Now, since the map $a + b\sqrt{1-y} \to a - b\sqrt{1-y}$, $a, b \in F$, is an automorphism of E, there exist two extensions of v to E. We denote them by u_1 and u_2; where $u_1(1 + \sqrt{1-y}) = (1, 0)$, $u_1(1 - \sqrt{1-y}) = (0, 0)$ and $u_2(1 - \sqrt{1-y}) = (1, 0)$, $u_2(1 + \sqrt{1-y}) = (0, 0)$. Since $(\mathbb{Z} \times \mathbb{Z})/(\{0\} \times \mathbb{Z}) \cong \mathbb{Z}$, we may view $\mathbb{Z}$ as the value group of $\widetilde{v}$. Using the same argument as above, we deduce that $\widetilde{v}$ has two extensions to E. We denote them by $\widetilde{u_1}$ and $\widetilde{u_2}$; where $\widetilde{u_1}(1 + \sqrt{1-y}) = 1$, $\widetilde{u_1}(1 - \sqrt{1-y}) = 0$ and $\widetilde{u_2}(1 - \sqrt{1-y}) = 1$, $\widetilde{u_2}(1 + \sqrt{1-y}) = 0$. It is easy to see that $O_{u_1} \nsubseteq O_{\widetilde{u_2}}$. Let $R_0 = O_{u_1} \cap O_{\widetilde{u_2}}$ and note that R_0 is an O_v-nice subalgebra of E. Using the same reasoning as in the previous theorem, we get that R_0 has two maximal ideals: $I_{u_1} \cap R_0$ and $I_{\widetilde{u_2}} \cap R_0$. However, since $O_{\widetilde{u_2}}$ is lying over $O_{\widetilde{v}}$, its maximal ideal, $I_{\widetilde{u_2}}$, is lying over P. So, R_0 has a maximal ideal that does not lie over I_v.

In view of Remark 2.2 and the previous example, let w_f denote the filter quasi-valuation induced by (R_0, v) (cf. [4, Section 9] for the construction of the filter quasi-valuation); then there exists $m \in w_f(E \setminus \{0\})$ such that for all $n \in \mathbb{N}$, $nm \notin \mathbb{Z} \times \mathbb{Z}$. Recall that a cut $m = (m^L, m^R)$ of Γ_v is a partition of Γ_v into two subsets m^L and m^R, such that, for every $\alpha \in m^L$ and $\beta \in m^R$, $\alpha < \beta$. Also recall that $w_f(E \setminus \{0\})$ is contained in $\mathcal{M}(\Gamma_v)$, where $\mathcal{M}(\Gamma_v)$ is the set of all cuts of Γ_v; $\mathcal{M}(\Gamma_v)$ is called the cut monoid of Γ_v. Let $m_0 \in \mathcal{M}(\Gamma_v)$ be the cut defined by ${m_0}^L = \{\alpha \in \mathbb{Z} \times \mathbb{Z} \mid \alpha \leq (0, z) \text{ for some } z \in \mathbb{Z}\}$; clearly, $m_0 + m_0 = m_0$ and thus m_0 is not torsion over Γ_v. Denote $1 + \sqrt{1-y}$ by r_0; we show now that $w_f(r_0) = m_0$. Note that for all $t \in \mathbb{Z}$, $u_1(r_0 x^t) = (1, t) > (0, 0)$, and $\widetilde{u_2}(r_0 x^t) = 0$. Thus, for all $t \in \mathbb{Z}$, $r_0 x^t \in R_0$. Moreover, $\widetilde{u_2}(r_0 y^{-1}) = -1$; thus, $r_0 y^{-1} \notin R_0$. More generally, $r_0 a^{-1} \in R_0$ for every $a \in O_v \setminus P$ and $r_0 a^{-1} \notin R_0$ for every $a \in P$; in particular, the support of r_0, $S_{r_0} = \{a \in O_v \mid r_0 \in aR_0\}$, satisfies: for all $a \in S_{r_0}$ there exists $b \in S_{r_0}$ with $v(b) > v(a)$. Therefore, by the definition of the filter quasi-valuation, $w_f(r_0) = m_0$. In addition, $(\Gamma_v, \emptyset)$ and $(\emptyset, \Gamma_v)$ are not in $w_f(E \setminus \{0\})$; indeed, by the definition of the filter quasi-valuation, $(\emptyset, \Gamma_v) \notin w_f(E \setminus \{0\})$, and it is easy to check that in our case $(\Gamma_v, \emptyset) \notin w_f(E \setminus \{0\})$. Indeed, otherwise there exists $r \in R_0$ such that for all

$a \in O_v$ we have $ra^{-1} \in R_0 \subset O_{u_1}$, but then r would have an infinite value by u_1; we note that one can also use [4, Theorem 8.14] to deduce that $(\Gamma_v, \emptyset) \notin w_f(E \setminus \{0\})$. Moreover, by [4, Definition 1.5 and Lemma 1.6], for every $t \in \mathbb{Z}$, $w_f(r_0 y^t) = (t, 0) + m_0$; where, of course,

$$((t,0) + m_0)^L = \{\alpha \in \mathbb{Z} \times \mathbb{Z} \mid \alpha \leq (t, z) \text{ for some } z \in \mathbb{Z}\}.$$

Thus, $w_f(E \setminus \{0\}) = \mathcal{M}(\Gamma_v) \setminus \{(\Gamma_v, \emptyset), (\emptyset, \Gamma_v)\}$.

Finally, we show that Example 2.3 easily provides us with an example, in a similar setting as in Example 2.3, of a noncommutative algebra lying over a valuation domain and not satisfying GU.

Example 2.4 Let the notation be as in Example 2.3 and let $A = M_n(E)$. Then $M_n(R_0)$ is an O_v-subalgebra of A such that $FM_n(R_0) = A$ and $M_n(R_0) \cap F = O_v$; $M_n(R_0)$ has maximal ideals $M_n(I_{u_1} \cap R_0)$ and $M_n(I_{\widetilde{u_2}} \cap R_0)$, while obviously $M_n(I_{\widetilde{u_2}} \cap R_0)$ is not lying over I_v.

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