Acta Mathematica Vietnamica (2022) 47:293–304
https://doi.org/10.1007/s40306-021-00446-x

Odd-degree Rational Irreducible Characters

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Dedicated to Professor Nguyen Tu Cuong on the occasion of his 70th birthday

Received: 28 May 2021 / Revised: 4 June 2021 / Accepted: 17 June 2021 /
Published online: 28 July 2021

Abstract
We study finite groups whose rational-valued irreducible characters are all of odd degrees. We conjecture that in such groups, all rational elements must be 2-elements.

Keywords Finite groups · Rational irreducible characters · Rational conjugacy classes

Mathematics Subject Classification (2010) Primary 20C15 · Secondary 20D10

1 Introduction

Let G be a finite group and let $\mathbb{K}$ be a field with $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{C}$. We denote by $\mathrm{Irr}_{\mathbb{K}}(G)$ the set of complex irreducible characters $\chi \in \mathrm{Irr}(G)$ with values in $\mathbb{K}$, and refer to any character in $\mathrm{Irr}_{\mathbb{R}}(G)$ as a *real character*, respectively to any character in $\mathrm{Irr}_{\mathbb{Q}}(G)$ as a *rational character*.

The Itô-Michler theorem for the prime $p = 2$ states that the degrees of all complex irreducible characters of a finite group G are even if and only if G has a normal abelian Sylow 2-subgroup. A version of this theorem for real characters was obtained in [5] and another refinement to strongly real characters was proved in [15]. In both versions, it was shown that if all real (or strongly real) irreducible characters of a finite group G have odd degrees, then G has a normal Sylow 2-subgroup. Clearly, these groups have no non-trivial real elements of odd order. Unfortunately, when restricted to rational characters, a similar conclusion fails. The simple groups $\mathrm{L}_2(3^{2f+1})$, where $f \geq 1$ is an integer, have exactly two rational irreducible characters which are the trivial character and the Steinberg character of degree 3^f.

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However, Navarro and Sanus [16] managed to show that if all rational irreducible characters of G are linear, then G has a normal Sylow 2-subgroup. In this case, clearly, G has no non-trivial rational element of odd order. Note that all rational elements are real elements and since any power of rational elements remains rational, the two conditions G have no non-trivial rational element of odd order and all rational elements of G are 2-elements coincide. Navarro and Tiep show in [17] that if G has exactly two rational irreducible characters, then G has exactly two rational classes and since $|G|$ is even, all non-trivial rational elements of G are involutions. These results seem to suggest the following.

Conjecture *Let G be a finite group. If every rational irreducible character of G has odd degree, then all rational elements of G are 2-elements.*

In our first result, we will prove the following.

Theorem A *Let G be a finite group. Suppose that the simple group $\mathrm{L}_2(3^{2f+1})$ is not involved in G for any integer $f \geq 1$ and that all rational irreducible characters of G have odd degree. Then all rational elements of G are 2-elements.*

In order to prove Theorem A, we need the following solvability result.

Theorem B *Let G be a finite group. Suppose that no simple group $\mathrm{L}_2(3^{2f+1})$ with $f \geq 1$ is involved in G and that all rational irreducible characters of G have odd degree. Then G is solvable.*

The proof of Theorem B will depend on the following result on simple groups.

Theorem C *Let S be a normal non-abelian simple subgroup of a finite group G and $\mathbf{C}_G(S) = 1$. If S is not isomorphic to $\mathrm{L}_2(3^{2f+1})$ then there exists $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ of even degree not containing S in its kernel.*

Apparently, the non-existence of rational elements of certain orders might have stronger impact on the group structures (see, e.g., [4, Theorem C] and Theorem 3.1).

As mentioned earlier, [17, Theorem A] states that a finite group G has exactly two rational irreducible characters if and only if it has two rational classes. The non-trivial rational class of such groups consists of involutions and hence our conjecture clearly holds in this case. Extending this one step further, it turns out that there is no finite group having exactly three rational irreducible characters whose degrees are all odd.

Theorem D *Let G be a finite group. Suppose that G has exactly three rational irreducible characters. Then G has a rational irreducible character of even degree.*

In fact, it was conjectured by Navarro and Tiep that a finite group G has exactly three rational irreducible characters if and only if it has three rational conjugacy classes. One direction of this conjecture was proved by Rossi in [19]. For finite solvable groups G of even order, we can show that the number of odd degree rational irreducible characters of G is even (see Corollary 5.2). However, this is not true in general. The simple Janko group J_4 has exactly 13 rational, irreducible characters of odd degree.

Finally, we mention that the converse of our conjecture does not hold. As a counterexample, all rational elements of the dihedral group D_8 are of order 1 and 2 but D_8 has a rational irreducible character of degree 2. It would be an interesting problem to obtain a group-theoretical characterization of finite groups whose all rational irreducible characters have odd degree. Motivated by [16, Theorem A], where it is shown that if all rational irreducible characters of G are linear, then all rational irreducible characters of some Sylow 2-subgroup of G are linear, one might ask whether a similar conclusion holds under the hypothesis of our conjecture. Unfortunately, as pointed out to us by the reviewer, this is also not true. For example, one can take $G := \mathrm{SmallGroup}(160, 234) \cong C_2^4 \rtimes (C_5 \rtimes C_2)$. Then all rational irreducible characters of G have odd degree but a Sylow 2-subgroup of G has rational irreducible characters of degree 2.

Our notation is standard. We follow [11] for the character theory of finite group and [2] for the notation of non-abelian simple groups.

2 Preliminaries

We collect some useful results on rational characters and rational elements in this section.

Lemma 2.1 *Let G be a finite group and let N be a normal subgroup of G. If χ is a real irreducible character of G of odd degree and θ is an irreducible constituent of χ_N, the restriction of χ to N, then θ is real of odd degree.*

Proof Assume that $\chi \in \mathrm{Irr}(G)$ is real with $\chi(1)$ odd. Let $\theta \in \mathrm{Irr}(N)$ be a constituent of χ_N. By Clifford's theorem, $\chi_N = e(\theta_1 + \theta_2 + \cdots + \theta_t)$, where all $\theta_i's$ are conjugate to $\theta = \theta_1$ and integers $e, t \geq 1$. Since $\chi(1) = et\theta(1)$ is odd, t is odd. As χ is real, we have $\overline{(\chi_N)} = (\overline{\chi})_N = \chi_N$, and so

$$e\left(\overline{\theta_1} + \overline{\theta_2} + \cdots + \overline{\theta_t}\right) = e\left(\theta_1 + \theta_2 + \cdots + \theta_t\right).$$

It follows that the G-orbit of all irreducible constituents of χ_N is closed under taking complex conjugate. Since t is odd, $\overline{\theta_j} = \theta_j$ for some j with $1 \leq j \leq t$. As θ_j is G-conjugate to θ, θ is real. □

Lemma 2.2 *Let G be a finite group and let $N \trianglelefteq G$. Let $\theta \in \mathrm{Irr}(N)$ be rational. Then there exists a rational character $\chi \in \mathrm{Irr}(G|\theta)$ if either $|G/N|$ is odd or $\theta(1)$ is odd and $o(\theta) = 1$.*

Proof These follow from [17, Corollaries 2.2 and 2.4]. □

Lemma 2.3 *Let S be a finite non-abelian simple group. Then all rational irreducible characters of S have odd degree if and only if $S \cong \mathrm{L}_2(3^{2f+1})$ for some integer $f \geq 1$. Moreover, the only non-trivial rational irreducible character of $\mathrm{L}_2(3^{2f+1})$ with $f \geq 1$ is the Steinberg character of degree 3^{2f+1}.*

Proof The first statement is [5, Theorem 2.7] and the second follows from [17, Lemma 9.4]. □

Let G be a finite group. Recall that an element $x \in G$ is rational (in G) if whenever $\langle y\rangle = \langle x\rangle$, then y is G-conjugate to x. Also $x \in G$ is real (in G) if $x^g = x^{-1}$ for some $g \in G$. Clearly, every rational element is real. Moreover, if $x \in G$ is an element of order 3, then x is real if and only if x is rational. We call a class x^G is rational if x is rational.

Lemma 2.4 *Let G be a finite group and let $N \trianglelefteq G$.*

(a) If $x \in G$ is rational, then xN is rational in G/N.
(b) If $g \in H \leq G$ and g is rational in H, then g is rational in G.
(c) Assume that $x \in G$ has prime order p. Then, x is rational in G if and only if there exists a p'-element $g \in G$ such that $x^g = x^t$, where $t \pmod{p}$ is any generator of $\mathbb{Z}_p^\times$.
(d) If $x \in G$ is rational, then every power of x is also rational; moreover, x_π is rational for every set of primes π.
(e) If $x \in G$, $\gcd(o(x), |N|) = 1$ and xN is rational in G/N, then x is rational in G.
(f) If G/N has a rational element of prime order p, then G also has a rational element of order p.

Proof These statements can be found in [17, Lemmas 5.1 and 5.2]. □

Lemma 2.5 *Let S be a finite non-abelian simple group. Then, either S contains a rational element of order 3, or $S = {}^2\mathrm{B}_2(2^{2f+1})$ and S contains a rational element of order 5, or $S \cong \mathrm{L}_2(3^{2f+1})$, where $f \geq 1$ is an integer.*

Proof This is [17, Theorem 11.1]. □

3 Finite Groups with No Even Rational Character Degrees

Let G be a finite group. If $U \trianglelefteq V$ are subgroups of G, then we call V/U a section of G. We say that a finite group T is involved in G if T is isomorphic to some section of G and G is said to be T-free if none of the section of G is isomorphic to T.

We first prove Theorem B, assuming Theorem C.

Proof of Theorem B Assume that every rational, irreducible character of G has odd degree and that $\mathrm{L}_2(3^{2f+1})$ is not involved in G for any integer $f \geq 1$. We prove by induction on $|G|$ that G is solvable. If $1 < N \trianglelefteq G$, then $\mathrm{Irr}_{\mathbb{Q}}(G/N) \subseteq \mathrm{Irr}_{\mathbb{Q}}(G)$ and $\mathrm{L}_2(3^{2f+1})$ is not involved in G/N, by induction, G/N is solvable. It follows that G has a unique minimal normal subgroup, say M. If M is solvable, then since G/M is solvable by the claim above, G is solvable. Thus, we assume that M is non-solvable.

Write $M = S_1 \times S_2 \times \cdots \times S_n$, where each S_i is non-abelian simple. Let $S = S_1$, $H = \mathbf{N}_G(S)$ and $C = \mathbf{C}_G(S)$. Since $\mathrm{L}_2(3^{2f+1})$ is not involved in G, $S \not\cong \mathrm{L}_2(3^{2f+1})$ for all integers $f \geq 1$. Let $\overline{H} = H/C$. Then, $\overline{S} \trianglelefteq \overline{H}$ and $\mathbf{C}_{\overline{H}}(\overline{S}) = 1$. By Theorem C, there exists a rational, irreducible character $\delta \in \mathrm{Irr}_{\mathbb{Q}}(\overline{H})$ of even degree such that $[\delta_{\overline{S}}, 1_{\overline{S}}] = 0$. Inflate δ to H, we still have that $[\delta_S, 1_S] = 0$. Hence, δ_S has an irreducible constituent $1 \neq \theta \in \mathrm{Irr}(S)$. Let $\phi = \theta \times 1_S \times \cdots \times 1_S \in \mathrm{Irr}(M)$. Let I be the inertia group of ϕ in G. Observe that δ lies over ϕ and $I \leq H$. Let ψ be the Clifford correspondence of δ over ϕ. Then, $\psi^H = \delta$ and $\psi^G \in \mathrm{Irr}(G)$. Thus, $\delta^G = \psi^G \in \mathrm{Irr}(G)$ and since δ is rational, $\psi^G \in \mathrm{Irr}_{\mathbb{Q}}(G)$. Furthermore, $\delta(1)$ is even and hence δ^G is a rational irreducible character of even degree of G, which is a contradiction. □

Using Theorem B, we can now prove Theorem A. Recall that for a prime p and a finite p-solvable group G, $\mathrm{B}_p(G)$ is a canonical subset of $\mathrm{Irr}(G)$ with values in $\mathbb{Q}_{|G|_p}$.

Proof of Theorem A We proceed by induction on $|G|$. By Theorem B, we know that G is solvable. If $1 < N \trianglelefteq G$, then $\mathrm{Irr}_{\mathbb{Q}}(G/N) \subseteq \mathrm{Irr}_{\mathbb{Q}}(G)$, by induction G/N has no rational element of odd order > 1.

Assume first that $\mathbf{O}_2(G) > 1$. Then, $G/\mathbf{O}_2(G)$ has no non-trivial rational element of odd order. Assume by contradiction that G has a non-trivial rational element x of odd order. By Lemma 2.4 (a), $x\mathbf{O}_2(G)$ is a non-trivial rational element of $G/\mathbf{O}_2(G)$ of odd order, which is a contradiction. Thus, we assume that $\mathbf{O}_2(G) = 1$. It follows that $\mathbf{O}_{2'}(G) > 1$ since G is solvable.

Let N be a minimal normal subgroup of G. Then, N is an elementary abelian p-group for some odd prime p. Note that G/N has no rational element of odd prime order. Again, assume that G has a rational element x of odd prime order. Since G/N has no rational element of odd order > 1, we deduce that $x \in N$ and thus $o(x) = p > 2$.

We now use the argument as in the proof of [17, Theorem 7.2] to produce an irreducible rational character of G which does not contain N in its kernel. By Lemma 2.4 (c), there exists a p'-element $g \in G$ such that $x^g = x^t$, where $t \pmod p$ is a generator for the multiplicative group $\mathbb{Z}_p^\times$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ be an element of order $p-1$ fixing p'-roots of unity and $\xi^\sigma = \xi^t$, where ξ is a primitive p-root of unity. By [17, Corollary 6.4], there exists $1_N \neq \psi \in \mathrm{B}_p(N)$ such that $\psi^\sigma = \psi^g$ and $\mathbb{Q}(\psi) \subseteq \mathbb{Q}_p$. By [17, Corollary 6.3], there exists a rational irreducible character χ of G lying over ψ.

From the hypothesis of the theorem, we know that $\chi(1)$ is odd. By Lemma 2.1, the constituent ψ of χ_N is real as χ is real. However, as $|N|$ is odd, the only real irreducible character of N is the trivial character 1_N which implies that $\psi = 1_N$, which is a contradiction. □

Observe that if G is a finite group and $x \in G$ is an element of order 3, then x is a real element of G if and only if x is a rational element of G as the only generators of the cyclic group $\langle x \rangle$ are x and $x^2 = x^{-1}$.

Theorem 3.1 *Let G be a finite group. Assume that* $\mathrm{L}_2(3^{2f+1})$ *is not involved in G for any integers $f \geq 1$. If G has no rational element of order* 3 *and* 5*, then G is solvable.*

Proof We proceed by induction on the order of G. From the definition, we observe that every subgroup and quotient of G are $\mathrm{L}_2(3^{2f+1})$-free for all integers $f \geq 1$. Let $\pi = \{3, 5\}$.

Assume first that G has a non-trivial proper normal subgroup N. By Lemma 2.4 (b), N has no rational element of order p with $p \in \pi$. If G/N has a rational element of order p for some $p \in \pi$, then by Lemma 2.4 (f), G has a rational element of order p, which is impossible. Thus, G/N has no rational element of order p for any $p \in \pi$. Since $|G/N| < |G|$ and $|N| < |G|$, by induction, both N and G/N are solvable and hence G is solvable. Therefore, we can assume that G is a finite simple group.

If G is abelian, then G is solvable. Thus, we can assume that G is a non-abelian simple group. Since G is not isomorphic to $\mathrm{L}_2(3^{2f+1})$ for any integer $f \geq 1$, by Lemma 2.5, G always contains a rational element of order 3 or 5. This contradiction proves that G is solvable. □

We recall the following result which describes the structure of finite groups having no rational element of order 3 (and which also implies Theorem 3.1).

Theorem 3.2 [4, Theorem C] *Let G be a finite non-solvable group. Assume that G has no rational element of order* 3. *Let $L = \mathbf{O}^{2'}(G)$ and $M = \mathbf{O}_{3'}(L)$. Then, $L/M = S_1 \times S_2 \times \cdots \times S_n$, where $n \geq 1$ is an integer and $S_i \cong \mathrm{L}_2(3^{2f_i+1})$, $f_i \geq 1$ for all $1 \leq i \leq n$.*

4 Proof of Theorem C

This section is devoted to proving Theorem C.

4.1 Alternating Groups and Sporadic Groups

Suppose $S = \mathsf{A}_n$, where $n \geq 7$. Then, $S \leq G \leq \mathrm{Aut}(S) \cong \mathsf{S}_n$. Consider the irreducible characters $\alpha = \chi^{(n-2,2)}$ and $\beta = \chi^{(n-2,1^2)}$ of S_n, labeled by partitions $(n-2, 2)$ and $(n-2, 1^2)$, and of degree $n(n-3)/2$ and $n(n-3)/2+1$. Since both partitions are non-associate, α and β restrict to rational, irreducible characters of G, and one of them has even degree, yielding the desired character χ.

If $S = \mathsf{A}_5$, A_6, ${}^2F_4(2)'$, or one of the 26 sporadic simple groups, then one can check directly using [2] that G always contains a rational irreducible character of even degree.

4.2 Lie-type Groups in Characteristic 2

Suppose S is a Lie-type group in characteristic 2. By [8], the Steinberg character St of S extends to a rational irreducible character of even degree of G.

From now on, we will assume that S is a simple group of Lie type, defined over a field $\mathbb{F}_q$ of odd characteristic p, $q = p^f$. Our proof is largely based on [18, Lemma 5.1], which we reformulate for the case of rational characters.

Lemma 4.1 *Suppose that $G \lhd A$. Let ρ be a rational character of A, not necessarily irreducible. Assume that $\rho|_G$ contains an A-invariant rational irreducible constituent α such that* $[\alpha, \rho|_G] = 1$. *Then, there is a rational* $\chi \in \mathrm{Irr}(A)$ *such that* $\chi|_G = \alpha$.

4.3 The Case $S = \mathrm{L}_n(q)$, $n \geq 2$

In the subsequent treatment of $\mathrm{L}_n(q)$, it is convenient to adopt the labeling of irreducible $\mathbb{C}\,\mathrm{GL}_n(q)$-modules as given in [12], which uses Harish-Chandra induction $\circ$. Each such a module is labeled as $S(s_1, \lambda_1) \circ \ldots \circ S(s_m, \lambda_m)$, where $s_i \in \overline{\mathbb{F}}_q^\times$ has degree d_i (over $\mathbb{F}_q$), λ_i is a partition of k_i, and $\sum_{i=1}^m k_i d_i = n$ (cf. [12, 13]).

Lemma 4.2 [18, Lemma 5.4] *Let* $B = \mathrm{GL}_2(q)$ *and* $\alpha \in \mathrm{Irr}(B)$ *be of form* $S(s, (1)) \circ S(s^{-1}, (1))$ *for some s of order* 4 *and degree* 1, *or* $S(t, (1))$ *for some t of degree* 2 *and order* 4. *Then, α is rational and invariant under any field automorphism of B.*

We view S as $L/Z(L)$, where $L = \mathrm{SL}_n(q) \lhd H = \mathrm{GL}_n(q)$. Consider the natural module $\langle e_1, \ldots e_n\rangle_{\mathbb{F}_q}$ for H, the subgroup

$$T = \mathrm{Stab}_H(\langle e_1\rangle_{\mathbb{F}_q}, \langle e_2, \ldots, e_n\rangle_{\mathbb{F}_q}) \cong \mathrm{GL}_1(q) \times \mathrm{GL}_{n-1}(q),$$

and the induced character $\rho_n = (1_T)^H$. Then, ρ_n is a rational character of degree $q^{n-1}(q^n - 1)/(q-1)$. Since $T > \mathbf{Z}(H)$, ρ_n can be viewed as a character of $\bar{H} = \mathrm{PGL}_n(q)$. Recall that $A = \mathrm{Aut}(\mathrm{L}_n(q))$ is a semidirect product $\bar{H} \rtimes F$, where F is generated by a field automorphism σ, and also the transpose-inverse τ if $n > 2$. We can define σ and τ such that they stabilize T. It follows that ρ_n extends to the rational A-character $(1_{\bar{T} \rtimes F})^A$, where $\bar{T} = \mathbf{Z}(G)T/\mathbf{Z}(G)$.

If $\lambda \vdash n$, then let χ^λ denote the unipotent character of G labeled by λ (cf. [1]). As shown in the proof of [18, Proposition 5.5], if $n \geq 2$ then ρ_n has the following decomposition into distinct irreducible constituents:

$$\begin{aligned}\rho_n \;=\; & \chi^{(n)} + 2\chi^{(n-1,1)} + \chi^{(n-2,1^2)} + S(-1,(1^2)) \circ S(1,(n-2)) \\ & + \sum_{a \in \mathbb{F}_q^\times,\ a \neq \pm 1} S(a,(1)) \circ S(a^{-1},(1)) \circ S(1,(n-2)) \\ & + \sum_{b \in \mathbb{F}_{q^2}^\times,\ b^{q+1}=1,\ b\neq \pm 1} S(b,(1)) \circ S(1,(n-2)),\end{aligned} \tag{1}$$

where in $\sum_a$, there is one summands for the pair $\{a, a^{-1}\}$, and similarly for $\sum_b$; also, the summand $\chi^{(n-2,1^2)}$ occurs only for $n \geq 3$.

First consider the case $n = 2$ and $2 \nmid f$, whence $\mathrm{Aut}(S) = K \rtimes C_2$ with $K = S \rtimes C_f$. By assumption, $p \neq 3$. Hence, by Lemma 2.3, S admits a rational irreducible character θ of even degree, and θ extends to a rational character of H by Lemma 2.2. In particular, we are done if $G \leq K$. Suppose $G \not\leq K$. Note that $\tilde{K} := \mathrm{SL}_2(q) \rtimes C_f \leq \mathrm{Sp}_{2nf}(p)$, and $\mathrm{Sp}_{2nf}(p)$ admits irreducible Weil characters ξ_1, ξ_2 of degree $(q^n+1)/2$ and η_1, η_2 of degree $(q^n-1)/2$, all remaining irreducible upon restriction to $\tilde{H}$ and $\mathrm{SL}_2(q)$. Moreover, $\xi_1 + \xi_2$ and $\eta_1 + \eta_2$ are rational-valued. Furthermore, the outer diagonal automorphisms of S, and so any element of $G \smallsetminus K$, fuse $(\xi_1)|_S$ with $(\xi_2)|_S$, and $(\eta_1)|_S$ with $(\eta_2)|_S$. Choosing the one, say η_1, that is trivial at $\mathbf{Z}(\tilde{K})$, we obtain an irreducible character ψ of $G \cap K$. Setting $\chi := \psi^G$, we see that $\chi = 0$ on $G \smallsetminus K$, and $\chi(g) = (\eta_1 + \eta_2)(g)$ for $g \in G \cap K$. Thus, χ is an even-degree rational irreducible character of G.

Assume now that $n = 2$ and $2|f$. Then, $8|(q-1)$, and we can find $a \in \mathbb{F}_q^\times$ of order 4. By Lemma 4.2, $\alpha := S(a,(1)) \circ S(a^{-1},(1))$ is rational and σ-invariant, of even degree $q+1$. It is easy to check α, viewed as PGL_2-character, is irreducible upon restriction to S. By Lemma 4.1, α extends to a rational character φ of A, and now we can take $\chi := \varphi|_G$.

Next we consider the case $n \geq 3$. If $q \equiv 1 \pmod 4$, choose $a \in \mathbb{F}_q^\times$ of order 4 and take $\alpha := S(a,(1)) \circ S(a^{-1},(1)) \circ S(1,(n-2))$ of even degree

$$(q+1)\frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

If $q \equiv 3 \pmod 4$, choose $b \in \mathbb{F}_{q^2}^\times$ of order 4 and take $\alpha := S(b,(1)) \circ S(1,(n-2))$ of even degree

$$(q-1)\frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

Arguing as in the proof of [18, Proposition 5.5], using [12, (7.33)] and [13, Lemmas 3.2, 4.1], we see that α is A-invariant, rational-valued, and irreducible upon restriction to S. By Lemma 4.1, α extends to a rational character φ of A, and now we can take $\chi := \varphi|_G$.

4.4 The Case $S = \mathrm{U}_3(q)$, $q \geq 3$

We will view $S = L/\mathbf{Z}(L)$ with $L = \mathrm{SU}_3(q)$, and use the notation for irreducible characters of L as given in [9]. Also, recall that $\mathrm{Aut}(S) = H \rtimes \langle\sigma\rangle$, with $H = \mathrm{PGU}_3(q)$, and σ is an outer automorphism of order $2f$ induced by the field automorphism $x \mapsto x^p$. Note that S has a unique (unipotent) character of even degree $q(q-1)$, so rational. Hence, we are done by Lemma 2.2 if $2 \nmid |G/S|$. It remains to consider the case $|G/S|$ is even.

First assume that $q \equiv 3 \pmod 4$; in particular, f is odd. Then, L has a pair of Weil characters $\chi^{(u)}_{q^2-q+1}$ of degree $q^2 - q + 1$, with $u = (q + 1)/4$ and $3(q + 1)/4$, which are dual to each other, and fused by σ^{2f}, and trivial at $\mathbf{Z}(L)$. In fact, it is straightforward to check that the semisimple characters of $GU_3(q)$ labeled by the elements $\mathrm{diag}(b, b, -1)$ with $b \in \mathbb{F}^\times_{q^2}$ of order 4 are trivial at $\mathbf{Z}(GU_3(q))$, so can be viewed as H-characters θ and $\overline{\theta}$, restrict to the previous two Weil characters of S, dual to each other, with $\mathbb{Q}(\sqrt{-1})$ as field of values, and fused by σ. It follows that $\psi := \theta^{H \rtimes \langle \sigma^f \rangle}$ is a rational irreducible character of $H \rtimes \langle \sigma^f \rangle$, which is of odd index f in $\mathrm{Aut}(S)$. By Lemma 2.2, ψ extends to a rational character φ of $\mathrm{Aut}(S)$. The construction of φ shows that

$$\varphi|_S = \chi^{((q+1)/4)}_{q^2-q+1} + \chi^{(3(q+1)/4)}_{q^2-q+1};$$

hence, the inertia subgroup of $\chi^{((q+1)/4)}_{q^2-q+1}$ in $\mathrm{Aut}(S)$ is precisely $H \rtimes \langle \sigma^f \rangle$. As $|G/S|$ is even, $\chi^{((q+1)/4)}_{q^2-q+1}$ is not G-invariant. Hence, we can take $\chi := \varphi|_G$.

Assume now that $q \equiv 1 \pmod 4$. Then, L has a rational character $\chi^{((q^2-1)/4)}_{q^3+1}$ of degree $q^3 + 1$, which is trivial at $\mathbf{Z}(L)$. This character extends to the rational character $\alpha := \chi^{(0,(q^2-1)/4)}_{q^3+1}$ of $M := GU_3(q)$, in the notation of [7]. Direct calculations using the character table of $GU_2(q)$, also given in [7], show that $[\alpha|_N, 1_N] = 1$, where $N := GU_1(q) \times GU_2(q)$ embedded naturally in M. It follows that α is a multiplicity-one constituent of $(1_N)^M$. Note that σ can be defined to fix both N and M, so $(1_N)^M$ extends to the permutation character $(1_{N \rtimes \langle \sigma \rangle})^{M \rtimes \langle \sigma \rangle}$, and moreover α is σ-invariant. By Lemma 4.1, α extends to a rational irreducible character ψ of $M \rtimes \langle \sigma \rangle$. Note that α is trivial at $\mathbf{Z}(M)$, so, after modding out by $\mathbf{Z}(M)$, ψ yields a rational irreducible character φ of $(M \rtimes \langle \sigma \rangle)/\mathbf{Z}(M) \cong \mathrm{Aut}(S)$, which is irreducible over S. Now we can take $\chi := \varphi|_G$.

4.5 Other Classical Groups

For the remaining simple classical groups $S \ncong P\Omega^+_8(q)$, we can follow the proof of [5, Theorem 2.1], which produces two irreducible constituents α, β of a rank 3 permutation character of S (see, e.g., [20]), one of even degree and another of odd degree, which extend to rational-valued irreducible characters of $\mathrm{Aut}(S)$. One of this extensions, say φ, has even degree, so we can take $\chi := \varphi|_G$.

Suppose $S = P\Omega^+_8(q)$. We consider the parabolic subgroups S_i with $i = 1, 2$ in the notation of [3, Lemma (6.4)], and let $\pi_i := (1_{S_i})^S$. By [3, Lemma (6.4)], π_1 and π_2 are both multiplicity-free, $[\pi_1, \pi_1] = 3$, $[\pi_2, \pi_2] = 6$, and π_2 contains π_1. It is well-known (see, e.g., [20, Table 1]) that the rank 3 permutation character $\pi_1 = 1_G + \alpha + \beta$, with $\alpha(1) = q(q^2+1)^2$ and $\beta(1) = q^2(q^4 + q^2 + 1)$. It follows that α is a multiplicity-one constituent of π_2, and it is well-known that α is $\mathrm{Aut}(S)$-invariant and rational (as the unique unipotent character of this degree). Since S_2 corresponds to the branching node of the Dynkin diagram D_4 of S, π_2 extends to a permutation character of $\mathrm{Aut}(S)$. Hence, α extends to a rational character of $G \le \mathrm{Aut}(S)$ by Lemma 4.1.

4.6 Exceptional Groups

Suppose that $S = {}^2G_2(q)$ with $q = 3^f \geq 27$. Here $|\mathrm{Aut}(S)/S| = f$ is odd. By Lemma 2.3, S has an even-degree rational character θ, which then extends to a rational character of G by Lemma 2.2.

Suppose that $S = G_2(q)$ with $q \geq 3$, or ${}^3D_4(q)$. Let B denote the Borel subgroup of G. By [3, Proposition (7.22) (iv)], $(1_B)^G$ is the sum of 5 distinct irreducible characters, among which one constituent, say α, is the unique character $\phi_{2,1}$ (in the notation of [1, Section 13.9]) of S of even degree $(1/6)q(q+1)^2(q^2+q+1)$, respectively $(1/2)q^3(q^3+1)^2$, hence rational and Aut(S)-invariant. As B can be chosen to be invariant under (suitable) outer automorphisms of S, α extends to a rational character of $G \leq \mathrm{Aut}(S)$ by Lemma 4.1.

Suppose that $S = E_6(q)$ or ${}^2E_6(q)$. The proof of [5, Theorem 2.1] shows that S has a rank 5 permutation character, which is the sum of 5 distinct irreducible characters, among which three are of even degree, and all extendible to rational characters of $\mathrm{Aut}(S)$.

Suppose that $S = F_4(q)$, respectively $E_8(q)$. As shown in [3, Proposition (4.2)], S has a parabolic subgroup S_i, with $i = 1$, respectively 8, such that the permutation character $(1_{S_i})^S$ is the sum of 5 distinct irreducible characters, among which one constituent, say α, is the so-called reflection character, of even degree

$$\frac{1}{2}q(q^3+1)^2(q^4+1), \text{ respectively } q(q^{10}+1)\frac{q^{24}-1}{q^6-1}.$$

Now, α is $\mathrm{Aut}(S)$-invariant (as it is the unique unipotent character of this degree), and S_i can be chosen to be invariant under (suitable) outer automorphisms of S. Hence, α extends to a rational character of $G \leq \mathrm{Aut}(S)$ by Lemma 4.1.

Finally, suppose that $S = E_7(q)$. We consider the parabolic subgroups S_1 and S_7, in the notation of [3, Propositions (4.3) and (5.2)]. By these propositions of [3], $[\pi_7, \pi_7] = 4$ and $[\pi_1, \pi_7] = 3$, for $\pi_i := (1_{S_i})^S$, and

$$\pi_1 = 1_S + \phi_{7,1} + \chi_1 + \chi_2 + \chi_3,$$

a sum of 5 distinct irreducible characters, with $\phi_{7,1}$ (in the notation of [1, Section 13.9]) being the reflection character, but now of odd degree. Next,

$$\pi_7(1) = (q^5+1)(q^9+1)(q^{14}-1)/(q-1),$$

and π_7 is the sum of 4 unipotent characters of the principal series. Checking the degrees of the latter as given in [1, Section 13.9], we see that $\pi_7 = 1_S + \phi_{7,1} + \phi_{27,2} + \phi_{21,3}$. Again using the unipotent degrees listed in [1, Section 13.9] and $[\pi_1, \pi_7] = 3$, we obtain

$$\pi_1 = 1_S + \phi_{7,1} + \phi_{27,2} + \phi_{35,4} + \phi_{56,3},$$

with $\alpha := \phi_{56,3}$ rational, $\mathrm{Aut}(S)$-invariant, and of even degree. Since S_1 can be chosen to be invariant under (suitable) outer automorphisms of S, α extends to a rational character of $G \leq \mathrm{Aut}(S)$ by Lemma 4.1, completing the proof of Theorem C.

We will need the following result for the proof of Theorem D.

Lemma 4.3 *Let G be a finite group and let $S \trianglelefteq G$ with $\mathbf{C}_G(S) = 1$. Assume that $S \cong \mathrm{L}_2(3^{2f+1})$ for some integer $f \geq 1$. If $|\mathrm{Irr}_{\mathbb{Q}}(G)| \geq 3$, then G has a rational irreducible character of even degree.*

Proof Assume that $|\mathrm{Irr}_{\mathbb{Q}}(G)| \geq 3$. Suppose by contradiction that all rational irreducible characters of G have odd degree. We have that $S \trianglelefteq G \leq \mathrm{Aut}(S)$. Note that $\mathrm{Out}(S)$ is cyclic of order $2(2f+1)$. Assume that $|G/S|$ is even. Then, G has a normal subgroup G_1 such that $G_1 \cong \mathrm{PGL}_2(3^{2f+1})$. In this case, G_1 has a rational, irreducible character α of degree $q-1$ (lying over the characters labeled by η_1, η_2 as in [6, Theorem 38.1]). As $|G/G_1|$ is odd, there exists $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ lying over α and has even degree.

Assume that G/S is odd. Let $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ be such that χ is non-trivial and is not the extension of the Steinberg character St_S of S to G. Clearly $[\chi, 1_S] = 0$ as G/S is of odd order. Let $1_S \neq \theta \in \mathrm{Irr}(S)$ be an irreducible constituent of χ_S. By Lemma 2.1, $\theta \in \mathrm{Irr}_{\mathbb{R}}(S)$. Since $\theta \notin \{1_S, \mathsf{St}_S\}$, by [6, Theorem 38.1] $\theta(1) \in \{3^{2f+1} \pm 1\}$ and so $\theta(1)$ is even and hence $\chi(1)$ is even. This completes the proof. □

5 The Number of Rational Irreducible Characters of Odd Degrees

We first record the following result which should be well-known.

Lemma 5.1 *Let G be a finite group of even order. Then, the number of irreducible real characters of G of odd degree must be even.*

Proof Let G be a finite group of even order. Clearly, if χ is not real, then $\chi \neq \overline{\chi}$. Since $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$ is even, we deduce easily that the number of real characters in $\mathrm{Irr}(G)$ of odd degree must be even. □

The next result will be needed in the proof of Theorem D.

Corollary 5.2 *Let G be a finite solvable group of even order. Then, the number of rational irreducible characters of G of odd degree must be even.*

Proof By the main result in [10], every real irreducible character of odd degree of G is rational. Hence, the number of rational irreducible characters of G of odd degree coincides with the number of irreducible real characters of G of odd degree. Now, the corollary follows from the previous lemma. □

Proof of Theorem D Let G be a counterexample to the theorem with minimal order. Then, $|\mathrm{Irr}_{\mathbb{Q}}(G)| = 3$ and all rational irreducible characters of G have odd degree. Clearly $|G|$ is even and so by Corollary 5.2, G is non-solvable. It follows from Lemma 2.1 that $\mathbf{O}_{2'}(G)$ lies in the kernel of all rational irreducible characters of odd degrees of G. Thus, $|\mathrm{Irr}_{\mathbb{Q}}(G/\mathbf{O}_{2'}(G))| = 3$. By the minimality of $|G|$, we can assume that $\mathbf{O}_{2'}(G) = 1$.

Let N be a minimal normal subgroup of G. Since $\mathrm{Irr}_{\mathbb{Q}}(G/N) \subseteq \mathrm{Irr}_{\mathbb{Q}}(G)$ and $|G/N| < |G|$, by the minimality of G, $|\mathrm{Irr}_{\mathbb{Q}}(G/N)| \leq 2$. We consider the following cases.

(a) N is non-solvable. Write $N = S_1 \times S_2 \times \cdots \times S_n$, where each S_i is conjugate in G to $S = S_1$, where S is a non-abelian simple group. By Theorem B and its proof, we deduce that $S \cong \mathrm{L}_2(3^{2f+1})$ for some integer $f \geq 1$. Let θ be the Steinberg character of S. We know that θ is rational and $o(\theta) = 1$ since N is perfect.

Assume that $n \geq 3$. Let $\psi_1 = \theta \times 1 \times \cdots \times 1$, $\psi_2 = \theta \times \theta \times \cdots \times 1$, and $\psi_3 = \theta \times \theta \times \cdots \times \theta$. Then, $\psi_i \in \mathrm{Irr}(N)$ are all rational irreducible characters of odd degree and $o(\psi_i) = 1$ for all i. By Lemma 2.2, for each i, there exists $\chi_i \in \mathrm{Irr}_{\mathbb{Q}}(G)$ lying above ψ_i. Since ψ_i, $1 \leq i \leq 3$, lie in different G-orbits of irreducible characters of N, all $\chi_i's$ are pairwise distinct and thus $|\mathrm{Irr}_{\mathbb{Q}}(G)| \geq 4$, a contradiction.

Assume that $n = 2$. Then, $\psi = \theta \times 1 \in \mathrm{Irr}_{\mathbb{R}}(N)$ is rational, irreducible character of odd degree > 1. Let $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ be lying over ψ. Observe that the inertia group of ψ lies inside $\mathbf{N}_G(S_1)$ and since $|G : \mathbf{N}_G(S_1)| = 2$, from Clifford's theory, $\chi(1)$ is even, which is a contradiction.

Assume that $n = 1$. Then, $N = S \cong \mathrm{L}_2(3^{2f+1})$. By a result in [8], θ extends to $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$. If $|\mathrm{Irr}_{\mathbb{Q}}(G/S)| = 2$, then by Gallagher's theorem, G would have at least 4 distinct rational irreducible characters. Hence, G/S is of odd order. Let $C = \mathbf{C}_G(S)$. Then, $|C|$ is odd and so $C \subseteq \mathbf{O}_{2'}(G) = 1$. Thus, $S \trianglelefteq G$ and $\mathbf{C}_G(S) = 1$. By Lemma 4.3, G has a rational irreducible character of even degree, which is a contradiction.

(b) Assume N is solvable. Since $\mathbf{O}_{2'}(G) = 1$, N is an elementary abelian subgroup of order 2^n for some integer $n \geq 1$. Moreover G/N is non-solvable and $|\mathrm{Irr}_{\mathbb{Q}}(G/N)| = 2$. By [17, Theorem 10.2], there exist normal subgroups $N \leq K \leq L$ such that $K/N = \mathbf{O}_{2'}(G/N)$, $L/N = \mathbf{O}^{2'}(G/N)$ is perfect and $L/K \cong S = \mathrm{L}_2(3^{2f+1})$ for some integer $f \geq 1$. Write $\mathrm{Irr}_{\mathbb{Q}}(G/N) = \{1, \chi\}$, where χ is an extension of the Steinberg character of L/K which is rational of degree 3^{2f+1}. Let ψ be the remaining rational irreducible character of G. Note that $[\psi_N, 1_N] = 1$. Since $\mathbf{O}_2(G/N) = \mathbf{O}_2(L/N) = 1$, $N = \mathbf{O}_2(G)$ is a unique minimal normal subgroup of G.

Since $N \cap L' \trianglelefteq G$ and N is a unique minimal normal subgroup of G, $N \leq L'$ which implies that $L = L'$ is perfect. Since G/L is of odd order, by Lemma 2.2, each rational irreducible character of L lies under some rational irreducible character of G and thus every rational irreducible character of L has odd degree.

Assume first that G has a component V, that is, $V = V'$ is perfect and $V/\mathbf{Z}(V)$ is a non-abelian simple group and V is subnormal in G. Then, the layer $\mathbf{E}(G)$ of G, which is the normal subgroup of G generated by all components of G, is non-trivial. We know that $\mathbf{E}(G)/\mathbf{Z}(\mathbf{E}(G))$ is a direct product of non-abelian simple groups. It follows that $\mathbf{E}(G)/\mathbf{Z}(\mathbf{E}(G)) \cong \mathrm{L}_2(3^{2f+1})$. If $\mathbf{Z}(\mathbf{E}(G)) > 1$, then $\mathbf{E}(G) \cong \mathrm{SL}_2(3^{2f+1})$. In this case, $\mathbf{E}(G)$ has a rational irreducible character μ of degree $q-1$ (labelled by θ_j with $j = (q+1)/4$ as in [6, Theorem 38.1]) and $o(\mu) = 1$ (since $\mathbf{E}(G)$ is perfect). By Lemma 2.2, G has a rational irreducible character lying over μ and so this character has even degree, a contradiction. Thus, $\mathbf{Z}(\mathbf{E}(G)) = 1$ and so $\mathbf{E}(G)$ is a minimal normal subgroup of G, contradicting the uniqueness of N. Hence, $\mathbf{F}^*(G) = \mathbf{F}(G)\mathbf{E}(G) = \mathbf{O}_2(G) = N$, where $\mathbf{F}^*(G)$ is the generalized Fitting subgroup of G. By Bender's theorem, we have $\mathbf{C}_G(N) = N$.

Assume that $K > N$. Note that N is a normal Sylow 2-subgroup of K. It follows that $\mathbf{O}^2(K) = K$ since K is not a 2-group and N is the unique minimal normal subgroup of G. Let $1 \neq \lambda \in \mathrm{Irr}_{\mathbb{Q}}(K)$. Then, $\lambda(1)$ is odd and $o(\lambda) = 1$. By Lemma 2.2, $\psi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ lies over λ. Thus, all non-trivial rational irreducible characters of K are G-conjugate. Let $V = \mathrm{Irr}(N) \cong N$. Then, G/N acts transitively on $V - \{1_N\}$ (note that $\mathbf{C}_G(N) = \mathbf{C}_G(V) = N$). Thus, the semidirect product $V \rtimes G/N$ is a doubly transitive permutation group. However, this cannot occur by Hering's theorem. (See [14, Appendix 1]).

Assume that $K = N$. As $\mathbf{C}_G(N) = N$, $\mathbf{C}_L(N) = N$. So $V = \mathrm{Irr}(N)$ is a non-trivial GF(2)-module for S. By [17, Theorem 10.1], there exists $1 \neq \lambda \in V$ such that T/N is of odd order, where T is the inertia group of λ in L. Let $\nu \in \mathrm{Irr}(T)$ be the canonical extension of λ. Then, ν^L is a rational irreducible character of L of even degree as $|L : T|$ is even, which is a contradiction. □

Acknowledgements The authors are grateful to the referee for careful reading of and helpful comments on the paper.

Funding The first author gratefully acknowledges the support of the NSF (grant DMS-1840702), the Joshua Barlaz Chair in Mathematics, and the Charles Simonyi Endowment at the Institute for Advanced Study (Princeton).

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