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# A Brief Survey on Pure Cohen–Macaulayness in a Fixed Codimension

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*Dedicated with gratitude to our colleague and friend Nguyen Tu Cuong on the occasion of his 70th birthday*

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## Abstract

A concept of Cohen–Macaulay in codimension $t$ is defined and characterized for arbitrary finitely generated modules and coherent sheaves by Miller, Novik, and Swartz in 2011. Soon after, Haghighi, Yassemi, and Zaare-Nahandi defined and studied $\mathrm{CM}_t$ simplicial complexes, which is the pure version of the abovementioned concept and naturally generalizes both Cohen–Macaulay and Buchsbaum properties. The purpose of this paper is to survey briefly recent results of $\mathrm{CM}_t$ simplicial complexes.

**Keywords** Cohen–Macaulay ring · Buchsbaum ring · Simplicial complex · Cohen–Macaulay simplicial complex · Buchsbaum simplicial complex · $\mathrm{CM}_t$ simplicial complex

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## 1 Introduction

We start the paper by explaining the motivation behind the notion of pure Cohen–Macaulayness in a fixed codimension. The study of Cohen–Macaulay rings, which historically goes back to about a century, plays a special role in commutative algebra. Indeed, early works on Cohen–Macaulay rings, named after the mathematicians Francis Sowerby Macaulay (1862–1937) and Irvin Sol Cohen (1917–1955), were inspired by polynomial rings (see [19] and [4]). Polynomial rings and formal power series rings over fields are examples of Cohen–Macaulay rings. Due to their special properties, the Cohen–Macaulay rings have found many applications in algebraic geometry. In [18, p. 887], Hochster wrote that "when a local ring is not Cohen–Macaulay, life is much harder." We refer the reader to the book by Bruns and Herzog [3] for a wealth of information on the theory of Cohen–Macaulay rings.

In 1973, Jürgen Stückrad and Wolfgang Vogel, by giving a negative answer to a question of David Buchsbaum (1929–2021), introduced the concept of Buchsbaum rings with the desire of having a better understanding of the connection between the different intersection multiplicities corresponding to the concept length and multiplicity of commutative algebra (see [32] and [33]). Fairly soon, it became apparent that Buchsbaum rings are the correct generalization of the Cohen–Macaulay rings and the class of Cohen–Macaulay rings is included in the larger class of Buchsbaum rings. In fact, every Cohen–Macaulay local ring is a Buchsbaum ring. For exploring the theory of Buchsbaum rings in full detail starting from elementary facts to more complex ones, we refer the readers to the book by Stückrad and Vogel [34], who are the founders of these rings.

A connection can be made between commutative algebra and combinatorics via the so-called Stanley–Reisner rings constructed from simplicial complexes. These rings are convenient and elegant tools for the study of the combinatorics of simplicial complexes. By definition, a simplicial complex is Cohen–Macaulay (resp. Buchsbaum) whenever its Stanley–Reisner ring is a Cohen–Macaulay (resp. Buchsbaum) ring. In 1974, Gerald Allen Reisner in his Ph.D. thesis [26] completely characterized Cohen–Macaulay simplicial complexes. This was then followed up by more precise homological results about Stanley–Reisner rings due to Melvin Hochster and then after a while Richard Stanley found a way to prove the Upper Bound Conjecture for simplicial spheres, which was open at the time, using the Stanley–Reisner ring construction and the Reisner's criterion of Cohen–Macaulayness. Stanley's idea of translating difficult conjectures in combinatorics into statements from commutative algebra and proving them by means of homological techniques was the origin of combinatorial commutative algebra, which is one of the fastest developing subfields within algebraic combinatorics.

In 2011, a concept of Cohen–Macaulay in codimension $t$ is defined and characterized for arbitrary finitely generated modules and coherent sheaves by Miller, Novik, and Swartz [21]. For the Stanley–Reisner ring of a simplicial complex $\Delta$, it is equivalent to nonsingularity of $\Delta$ in dimension $\dim \Delta - t$ and for a coherent sheaf on projective space, this condition is shown to be equivalent to the same condition on any single generic hyperplane section. Soon after, in 2012, the concept of $\mathrm{CM}_t$ simplicial complexes was introduced by Haghighi, Yassemi, and Zaare-Nahandi [11]. This latter concept is the pure version of the previous one studied by Miller, Novik, and Swartz, that is, simplicial complexes which are pure and Cohen–Macaulay in codimension $t$. In the hierarchy of families of simplicial complexes with respect to Cohen–Macaulay property, Buchsbaum simplicial complexes appear right after Cohen–Macaulay ones, and, indeed, $\mathrm{CM}_t$ simplicial complexes are naturally placed in the hierarchy. In fact, the $\mathrm{CM}_t$ property unifies and naturally generalizes both Cohen–Macaulay and Buchsbaum properties. The purpose of this paper is to survey briefly recent results of $\mathrm{CM}_t$ simplicial complexes.

## 2 Pure Cohen–Macaulayness in a Fixed Codimension

Let us start this section with some preliminaries. A *simplicial complex* $\Delta$ on the set of vertices $[n] := \{1, \dots, n\}$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of $\Delta$, the *size* of a face $F$ is defined to be $|F|$, and its *dimension* is defined to be $|F|-1$. (As usual, for a given finite set $X$, the number of elements of $X$ is denoted by $|X|$). The *dimension* of $\Delta$, which is denoted by $\dim \Delta$, is defined to be $d-1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of $\Delta$ is a maximal face of $\Delta$ with respect to inclusion. We say that $\Delta$ is *pure* if all facets of $\Delta$ have the same cardinality. The *link of* $\Delta$ *with respect to a face* $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$.

One of the connections between combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in $n$ variables over a field $\mathbb{K}$, and let $\Delta$ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i \in F} x_i$. The *Stanley–Reisner ideal of* $\Delta$ *over* $\mathbb{K}$ is the ideal $I_\Delta$ of $R$ which is generated by squarefree monomials $x_F$ with $F \notin \Delta$. Note that any squarefree monomial ideal is the Stanley–Reisner ideal of a suitable simplicial complex. The *Stanley–Reisner ring of* $\Delta$ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined as $\mathbb{K}[\Delta] = R/I_\Delta$. A simplicial complex $\Delta$ is called *Cohen–Macaulay over* $\mathbb{K}$ (resp. *Buchsbaum over* $\mathbb{K}$), if its Stanley–Reisner ring $\mathbb{K}[\Delta]$ is a Cohen–Macaulay ring (resp. a Buchsbaum ring). In the abovementioned notions and in other similar ones, one can simply omit "over $\mathbb{K}$" if there is no ambiguity.

Let $\Delta$ be a $(d-1)$-dimensional simplicial complex and $t$ be an integer. In 2011, a concept of Cohen–Macaulayness in codimension $t$ for $\Delta$ is defined by Miller, Novik, and Swartz [21]. Indeed, by their definition, $\Delta$ is called Cohen–Macaulay of dimension $i$ along a face $F \in \Delta$ if $\mathrm{lk}_\Delta(F)$ is Cohen–Macaulay of dimension $i$ (see [21, Definition 6.1]). Also, $\Delta$ is called Cohen–Macaulay in codimension $t$ if $\Delta$ is either Cohen–Macaulay of dimension $t-1$ along every face $F$ with $|F| = d-t$, or Cohen–Macaulay whenever $t > d$ (see [21, Definition 6.3]). They remarked that if $\Delta$ is pure, then it would suffice to require that $\mathrm{lk}_\Delta(F)$ be Cohen–Macaulay for every face $F$ with $|F| = d-t$, but if $\Delta$ is not pure, then it is possible for $\mathrm{lk}_\Delta(F)$ to be Cohen–Macaulay without $\Delta$ being Cohen–Macaulay of dimension $d-1-|F|$ along $F$. Based on this observation, the pure version of the latter notion is defined by Haghighi, Yassemi, and Zaare-Nahandi as follows.

**Definition 2.1** ([11, Definition 2.1]) Let $\Delta$ be a $(d-1)$-dimensional simplicial complex and $0 \le t \le d-1$ be an integer. Then, $\Delta$ is called a $\mathrm{CM}_t$ *simplicial complex* provided $\Delta$ is pure and $\mathrm{lk}_\Delta(F)$ is Cohen–Macaulay for every $F \in \Delta$ with $|F| \ge t$.

We adopt the convention that $\mathrm{CM}_t$ means $\mathrm{CM}_0$ for any negative $t$. It is worthwhile to mention that for a $(d-1)$-dimensional simplicial complex $\Delta$, being $\mathrm{CM}_t$ implies Cohen–Macaulayness in codimension $d-t$ in the sense of Miller, Novik, and Swartz and the two concepts coincide if $\Delta$ is pure. Also, from the results by Reisner [27] and Schenzel [30] it follows that being $\mathrm{CM}_0$ is the same as being Cohen–Macaulay and the $\mathrm{CM}_1$ property is identical with the Buchsbaum property. Clearly for any $i \leq j$, being $\mathrm{CM}_i$ implies being $\mathrm{CM}_j$. These observations mean that the $\mathrm{CM}_t$ property naturally generalizes both Cohen–Macaulay and Buchsbaum properties. These two latter notions have been studied for many years by mathematicians. Therefore, it is far from expected to obtain a result whose special case gives us a result of new type in Cohen–Macaulay or Buchsbaum context. Nevertheless, we believe that generalizing familiar results for the $\mathrm{CM}_t$ case can be important. Indeed, it opens a variety of interesting questions which are already considered for Cohen–Macaulay and Buchsbaum ones.

We now include an example to illustrate the theory (see [11, Example 2.2]). Let $\Delta$ be the union of two $(d-1)$-simplices that intersect in a $(t-2)$-dimensional face $(1 \leq t \leq d-1)$. Then, $\Delta$ is a $\mathrm{CM}_t$ simplicial complex which is not $\mathrm{CM}_{t-1}$. Indeed, if $\Gamma$ is a finite union of $(d-1)$-simplices where any two of them intersect in a face of dimension at most $t-2$, then $\Gamma$ is a $\mathrm{CM}_t$ simplicial complex, and if at least two of the simplices have a $(t-2)$-dimensional face in common, then $\Gamma$ is not $\mathrm{CM}_{t-1}$. These include simplicial complexes corresponding to the transversal monomial ideals which happen to have linear resolutions (see [42]). Note that the condition $t \leq d-1$ is necessary because the union of two $(d-1)$-simplices which intersect in a $(d-2)$-dimensional face is Cohen–Macaulay.

We continue the paper by mentioning three results, each of which gives a characterization of $\mathrm{CM}_t$ simplicial complexes. It is known that the links of Cohen–Macaulay simplicial complexes are also Cohen–Macaulay (see [17]). A similar property holds true for $\mathrm{CM}_t$ simplicial complexes.

**Proposition 2.2** ([11, Lemma 2.3]) *Let $\Delta$ be a simplicial complex. Then, the following conditions are equivalent.*

(a) *$\Delta$ is $\mathrm{CM}_t$.*
(b) *$\Delta$ is pure and $\mathrm{lk}_\Delta(\{x\})$ is $\mathrm{CM}_{t-1}$ for every $\{x\} \in \Delta$.*

In analogy with the Reisner's characterization of Cohen–Macaulay simplicial complexes [27, Theorem 1], the following proposition provides equivalent conditions for $\mathrm{CM}_t$ simplicial complexes.

**Proposition 2.3** ([11, Theorem 2.6]) *Let $\Delta$ be a $(d-1)$-dimensional simplicial complex and $\mathbb{K}$ be a field. Then, the following conditions are equivalent.*

(a) *$\Delta$ is $\mathrm{CM}_t$ over $\mathbb{K}$.*
(b) *$\Delta$ is pure and $\widetilde{H}_i(\mathrm{lk}_\Delta(F); \mathbb{K}) = 0$ for all $F \in \Delta$ with $|F| \geq t$ and for all $i < d-1-|F|$.*

Before continuing the paper, let us write a few words about the key idea behind the proof of the above proposition. Indeed, one can prove it by the following Hochster's formula for local cohomology modules:

$$F\left(H^i_{\mathfrak{m}}(\mathbb{K}[\Delta]), t\right) = \sum_{F \in \Delta} \dim_{\mathbb{K}} \widetilde{H}_{i-|F|-1}\left(\mathrm{lk}_\Delta(F); \mathbb{K}\right) \left(\frac{t^{-1}}{1-t^{-1}}\right)^{|F|}.$$

Note that the formula expresses the Hilbert function of the local cohomology group $H^i_{\mathfrak{m}}(\mathbb{K}[\Delta])$ in terms of the reduced homology groups of subcomplexes of $\Delta$ ([31, Theorem 4.1]).

It is shown in [22, Corollary 3.4] that Cohen–Macaulayness is a topological property. Varbaro and Zaare-Nahandi [37, Theorem 2.4] have shown that the $\mathrm{CM}_t$ property is also topological. This is based on the following proposition together with the fact that the Krull dimension of $\mathrm{Ext}^i_R(\mathbb{K}[\Delta], R)$ is a topological invariant for all $i \in \mathbb{N}$ by Yanagawa [40].

**Proposition 2.4** ([21, Corollary 7.4]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on $[n]$. Then, $\Delta$ is $\mathrm{CM}_t$ if and only if $\Delta$ is pure and $\dim \mathrm{Ext}^i_R(\mathbb{K}[\Delta], R) \leq t$ for all $i > n-d$, where* dim *refers to the Krull dimension.*

Related with topological property, the next proposition gives a different characterization of $\mathrm{CM}_t$ property.

**Proposition 2.5** ([11, Theorem 2.8]) *Let $\Delta$ be a pure $(d-1)$-dimensional simplicial complex and $\mathbb{K}$ be a field. Then, the following conditions are equivalent.*

(a) *$\Delta$ is $\mathrm{CM}_t$ over $\mathbb{K}$.*
(b) *$H_i(|\Delta|, |\Delta| \setminus p; \mathbb{K}) = 0$ for all $p \in |\Delta| \setminus |\Delta_{t-2}|$ and for all $i < d-1$, where $\Delta_{t-2}$ is the $(t-2)$-skeleton of $\Delta$ and $|\Delta_{t-2}|$ is induced from a fixed geometric realization of $\Delta$.*

We now gather together some useful results concerning the join of $\mathrm{CM}_t$ simplicial complexes. First, let us recall what the join is. For two simplicial complexes $\Delta$ and $\Delta'$ with disjoint vertex sets, the *simplicial join* $\Delta * \Delta'$ of $\Delta$ and $\Delta'$ is defined to be the simplicial complex whose faces are in the form of $F \cup F'$, where $F \in \Delta$ and $F' \in \Delta'$. The algebraic and combinatorial properties of the simplicial join $\Delta * \Delta'$ through the properties of $\Delta$ and $\Delta'$ have been studied by a number of authors (see, for example, [1, 2, 8, 25]). For instance, Fröberg [8] has shown that the join is closed with respect to the Cohen–Macaulay and Gorenstein properties (see also [29]). This means that the simplicial join of two simplicial complexes is Cohen–Macaulay (resp. Gorenstein) if and only if both of them are Cohen–Macaulay (resp. Gorenstein). But the story is different for Buchsbaumness: if $\Gamma$ is the triangulation of a cylinder and $\Gamma'$ is a single vertex simplicial complex, then both $\Gamma$ and $\Gamma'$ are Buchsbaum (the first one is Buchsbaum by [28, Corollary 2.9] and the second one is Buchsbaum since, by [34, Example II.2.14 (ii)], it is Cohen–Macaulay), whereas $\Gamma * \Gamma'$ is not. Indeed, in [28, Theorem 2.6], it is shown that the simplicial join of two simplicial complexes is Buchsbaum if and only if both of them are Cohen–Macaulay.

Based on the abovementioned observations, it is natural to ask what would happen to $\Delta$ and $\Delta'$ when $\Delta * \Delta'$ is $\mathrm{CM}_t$. Below, we present two relevant results in this regard. In the following proposition, we use the convention that $\mathrm{CM}_s$ is just $\mathrm{CM}_0$ for any negative $s$.

**Proposition 2.6** ([11, Proposition 2.10]) *Let $\Delta$ be a $(d-1)$-dimensional and $\Delta'$ be a $(d'-1)$-dimensional simplicial complexes. Then, $\Delta * \Delta'$ is $\mathrm{CM}_t$ if and only if $\Delta$ is $\mathrm{CM}_{t-d'}$ and $\Delta'$ is $\mathrm{CM}_{t-d}$.*

It is worth mentioning that the Künneth tensor formula, together with the third preceding proposition, gives us a proof for the above one. We recall that Künneth tensor formula (see, for example, [28, Lemma 2.1]) states that for all $j$, the isomorphism

$$\mathrm{Ext}^j_{R''}\left(\mathbb{K}[\Delta * \Delta'], R''\right) \cong \bigoplus_{p+q=j} \mathrm{Ext}^p_R\left(\mathbb{K}[\Delta], R\right) \otimes_{\mathbb{K}} \mathrm{Ext}^q_{R'}\left(\mathbb{K}[\Delta'], R'\right)$$

holds true. Here, $R$ and $R'$ are polynomial rings over a field $\mathbb{K}$ corresponding to the vertex sets of $\Delta$ and $\Delta'$, respectively, and $R'' = R \otimes_{\mathbb{K}} R'$.

In fact, in the above proposition, if $\Delta$ is a $(d-1)$-dimensional $\mathrm{CM}_r$ simplicial complex and $\Delta'$ is a $(d'-1)$-dimensional $\mathrm{CM}_{r'}$ simplicial complex, then $\Delta * \Delta'$ is a $\mathrm{CM}_t$ simplicial complex with $t = \max\{d + r', d' + r\}$. However, if one of the simplicial complexes is Cohen–Macaulay, this result could be strengthened.

**Proposition 2.7** ([12, Theorem 3.1]) *Let $\Delta$ be a $(d-1)$-dimensional and $\Delta'$ be a $(d'-1)$-dimensional simplicial complexes. Then, the following conditions hold true.*

(a) *If $\Delta$ is Cohen–Macaulay and $\Delta'$ is $\mathrm{CM}_{r'}$ for some $r' \geq 1$, then $\Delta * \Delta'$ is $\mathrm{CM}_{d+r'}$. Moreover, if $\Delta'$ is not $\mathrm{CM}_{r'-1}$, then $\Delta * \Delta'$ is not $\mathrm{CM}_{d+r'-1}$. In particular, a cone on $\Delta'$ is $\mathrm{CM}_{r'+1}$.*
(b) *If $\Delta$ is $\mathrm{CM}_r$ and $\Delta'$ is $\mathrm{CM}_{r'}$ for some $r, r' \geq 1$, then $\Delta * \Delta'$ is $\mathrm{CM}_t$ with $t = \max\{d + r', d' + r\}$. Conversely, if $\Delta * \Delta'$ is $\mathrm{CM}_t$, then $\Delta$ is $\mathrm{CM}_{t-d'}$ and $\Delta'$ is $\mathrm{CM}_{t-d}$.*

We are now going to deal with Alexander duality. We recall that for a given simplicial complex $\Delta$ on $[n]$, the *Alexander dual* $\Delta^\vee$ of $\Delta$ is defined by $\Delta^\vee = \{[n] \setminus F \mid F \notin \Delta\}$. Note that $\Delta^\vee$ is a simplicial complex on $[n]$.

The following result characterizes the vanishing of Betti numbers of the Stanley–Reisner ideal of a simplicial complex $\Delta$ for which $\Delta^\vee$ is $\mathrm{CM}_t$. This is a generalization of the Eagon–Reiner's theorem [6] as well as a generalization of a result of Yanagawa [39]. One can prove this result by using Hochster's formula on Betti numbers of $I_\Delta$ [15, Corollary 8.1.4], namely,

$$\beta_{i,i+j}(I_\Delta) = \sum_{\substack{F \in \Delta^\vee \\ |F| = n-(i+j)}} \dim_{\mathbb{K}} \widetilde{H}_{i-1}\,(\mathrm{lk}_{\Delta^\vee}(F); \mathbb{K})\,.$$

**Proposition 2.8** ([9, Theorem 3.1]) *Let $\Delta$ be a simplicial complex on $[n]$, $\Delta^\vee$ its Alexander dual and let $I_\Delta \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be the Stanley–Reisner ideal of $\Delta$ over a field $\mathbb{K}$. Then, the following conditions are equivalent.*

(a) *$\Delta^\vee$ is a $(d-1)$-dimensional $\mathrm{CM}_t$ simplicial complex.*
(b) *$\beta_{0,j}(I_\Delta) = 0$ for all $j \neq n - d$ and $\beta_{i,i+j}(I_\Delta) = 0$ for all $i, j$ with $i + j \leq n - t$ and $j > n - d$.*

The previous proposition for $t = 1$ leads to the following well-known result (see [9, Corollary 3.2]): Let $\Delta$ be a simplicial complex on $[n]$ and suppose that $\dim \Delta^\vee = d - 1$. Then, $\Delta^\vee$ is Buchsbaum if and only if $\beta_{0,j}(I_\Delta) = 0$ for all $j \neq n - d$ and $\beta_{i,j}(I_\Delta) = 0$ for all $i, j$ with $i + j \leq n - 1$ and $j > n - d$.

Finally, we close this section by pointing out the connection between the Serre's condition $(S_r)$ and the $\mathrm{CM}_t$ property. Assume that a $(d-1)$-dimensional simplicial complex satisfies the Serre's condition $(S_r)$. Then, $\Delta$ is $\mathrm{CM}_{d-r}$. Indeed, for any face $F \in \Delta$ with $|F| = s \geq d - r$ on $\{i_1, \ldots, i_s\}$, we have $\dim(\mathbb{K}[\mathrm{lk}_\Delta(F)]) = \dim(\mathbb{K}[\Delta]_P) \leq r$, where $P = (x_j \mid j \notin \{i_1, \ldots, i_s\})$. Therefore, by the definition of the Serre's condition $(S_r)$, $\mathbb{K}[\Delta]_P$ is Cohen–Macaulay. But Cohen–Macaulayness does not change under an extension of the base field. Therefore, $\mathbb{K}[\mathrm{lk}_\Delta(F)]$ is Cohen–Macaulay if and only if $\mathbb{K}[\Delta]_P$ is so. Hence, $\mathrm{lk}_\Delta(F)$ is Cohen–Macaulay, i.e., $\Delta$ is $\mathrm{CM}_{d-r}$. Note that the converse is false. Indeed, it is enough to think about a disconnected Buchsbaum simplicial complex $\Delta$ which is $\mathrm{CM}_1$ and does not even satisfy the Serre's condition $(S_2)$ (see [9, Remark 3.3] and [37, Remark 2.1]).

Yanagawa and Terai [39] have generalized the Eagon–Reiner's theorem by showing that $\Delta^\vee$ satisfies the Serre's condition $(S_r)$ if and only if the minimal free resolution of $\mathbb{K}[\Delta]$ is linear in the first $r$ steps. Since the Serre's condition $(S_r)$ implies $\mathrm{CM}_{d-r}$, the above proposition is also a generalization of Yanagawa's result (see [9, Corollary 3.7]). According to Yanagawa's result, if $\Delta^\vee$ is of dimension $d-1$, for any integer $0 \le t \le d-1$, $\Delta^\vee$ is $(S_{d-t})$ if and only if the minimal free resolution of $\mathbb{K}[\Delta]$ is linear in the first $d-t$ steps. Therefore, $\beta_{i,j}(I_\Delta) = 0$ for all $i, j$ with $i < d-t$ and $j > n-d$. Thus, by the above proposition, $\Delta^\vee$ is $\mathrm{CM}_t$. This is another proof of the fact that the Serre's condition $(S_r)$ implies $\mathrm{CM}_t$ property. This proof is concerning the comparison of the shape of the Betti diagram of $I_\Delta$ when $\Delta^\vee$ is $(S_r)$ with the case when $\Delta^\vee$ is $\mathrm{CM}_t$ (see also [9, Figures 2 and 3]).

According to [7] a simplicial complex $\Delta$ is said to satisfy $N_{d,p}$ if $I_\Delta$ is generated in degree $\le d$ and the first $p$ steps of the minimal free resolution of $(I_\Delta)_{\ge d}$ are linear, in the sense that in the first $p$ steps of the resolution, the boundary maps are represented by matrices of linear forms. Here, $(I_\Delta)_{\ge d}$ is the ideal generated by elements of $I_\Delta$ of degree $\ge d$. The following proposition is an important result about the relationship between $\mathrm{CM}_t$ property and the condition $N_{d,p}$.

**Proposition 2.9** ([37, Theorem 3.3]) *Let $\Delta$ be a $(d-1)$-dimensional $\mathrm{CM}_t$ simplicial complex on n vertices. Then, $\Delta^\vee$ satisfies the $N_{n-d,2d-n-t+2}$ condition.*

As we mentioned, by a result of Yanagawa [39, Corollary 3.7], for $r \ge 2$ and a simplicial complex $\Delta$ of codimension $c$, $\mathbb{K}[\Delta]$ satisfies the Serre's condition $(S_r)$ if and only if $I_{\Delta^\vee}$ satisfies the $N_{c,r}$ condition. Therefore, one can find an interesting consequence of the previous proposition which connects $\mathrm{CM}_t$ property to the Serre's condition $(S_r)$ from another aspect.

**Proposition 2.10** ([37, Corollary 3.5]) *Let $\Delta$ be a simplicial complex of dimension $d-1$ on n vertices. Assume that $\Delta$ is $\mathrm{CM}_t$ for some $t \ge 0$. Then, $\Delta$ satisfies the Serre's condition $(S_{2d-n-t+2})$. In particular, if $\Delta$ is Buchsbaum, then* depth $\mathbb{K}[\Delta] \ge 2d-n+1$.

## 3 Graphs and the $\mathrm{CM}_t$ Property

In this section, we deal with simplicial complexes that come from graph theory. Let us recall a few things concerning graphs. Let $G$ be a finite undirected graph without loops or multiple edges and let $V(G) = [n]$ be its vertex set. An *independent set* in $G$ is a set $I$ of vertices such that for any two vertices in $I$, there is no edge connecting them. The *independence simplicial complex* of $G$, denoted by $\Delta_G$, is the simplicial complex on the set $[n]$ whose faces are all the independent sets of $G$. A graph $G$ is called Cohen–Macaulay (resp. $\mathrm{CM}_t$, etc.) if $\Delta_G$ is Cohen–Macaulay (resp. $\mathrm{CM}_t$, etc.). In this context, the Stanley–Reisner ideal of $\Delta_G$ is called the *edge ideal* of $G$ because of its structure related to the edges of $G$. Indeed, one can show that $I_{\Delta_G}$ is the ideal generated by $x_i x_j$'s in $\mathbb{K}[x_1, \ldots, x_n]$, where $ij$ is an edge of $G$.

The following proposition gives a basic tool for checking the $\mathrm{CM}_t$ property of graphs. We recall that for a graph $G$ and a vertex $v \in V(G)$, the set of neighbors of $v$ is denoted by $N_G(v)$. We also set $N_G[v] = \{v\} \cup N_G(v)$.

**Proposition 3.1** ([12, Lemma 2.2]) *Let G be a graph and $t \ge 1$ be an integer. Then, the following conditions are equivalent.*

(a) *G is* $\mathrm{CM}_t$.
(b) *G is unmixed and* $G \setminus N_G[v]$ *is* $\mathrm{CM}_{t-1}$ *for every vertex* $v \in G$.

Let $G$ and $G'$ be two graphs and denote by $G \sqcup G'$ their disjoint union. By using the fact that the equality $\Delta_{G \sqcup G'} = \Delta_G * \Delta_{G'}$ holds true, one can obtain the following proposition.

**Proposition 3.2** ([12, Theorem 3.2]) *Let $G$ and $G'$ be two $(d-1)$-dimensional and $(d'-1)$-dimensional graphs, respectively, on disjoint sets of vertices. Then, the following conditions hold true.*

(a) *The graph $G \sqcup G'$ is Cohen–Macaulay if and only if both $G$ and $G'$ are Cohen–Macaulay.*
(b) *If $G$ is Cohen–Macaulay and $G'$ is* $\mathrm{CM}_{r'}$ *for some $r' \geq 1$, then $G \sqcup G'$ is* $\mathrm{CM}_{d+r'}$*. If $G'$ is not* $\mathrm{CM}_{r'-1}$*, then $G \sqcup G'$ is not* $\mathrm{CM}_{d+r'-1}$.
(c) *If $G$ is* $\mathrm{CM}_r$ *and $G'$ is* $\mathrm{CM}_{r'}$ *for some $r, r' \geq 1$, then $G \sqcup G'$ is* $\mathrm{CM}_t$*, where $t = \max\{d + r', d' + r\}$. Conversely, if $G \sqcup G'$ is* $\mathrm{CM}_t$*, then $G$ is* $\mathrm{CM}_{t-d'}$ *and $G'$ is* $\mathrm{CM}_{t-d}$.

There are many results about relation of bipartite graphs to Cohen–Macaulayness, Buchsbaumness, etc. For instance, unmixed bipartite graphs have been already characterized by Villarreal (see [38, Theorem 1.1]). There is a relevant theorem to this point in [12], due to Haghighi, Yassemi, and Zaare-Nahandi, which states that every $(d-1)$-dimensional unmixed bipartite graph $G$ having $K_{n,n}$ with $n \geq 2$ as a maximal complete bipartite subgraph of minimum dimension, is $\mathrm{CM}_{d-n+1}$, but it is not $\mathrm{CM}_{d-n}$ (see [12, Theorem 4.1]). There seems to be an ambiguity in the notation used there which leads to an error. For example, suppose that $G$ is a graph with the edge set $E(G) = \{12, 34, 56, 14, 16, 36\}$. Note that $G$ is Cohen–Macaulay bipartite, and so it is $\mathrm{CM}_0$. As $K_{2,2}$ takes the edge set $E(K_{2,2}) = \{14, 16, 34, 36\}$, we have $d = 3$ and $n = 2$, and thus the latter-mentioned result says that $G$ is not $\mathrm{CM}_1$. But this is not true since $G$ is $\mathrm{CM}_0$. Fortunately, they have provided a new definition and have given a new statement for the theorem. We state the definition and the theorem below, but before getting to this, let us first recall the notion of a pure ordering of a graph. Indeed, when we say a graph $G$ is bipartite on a partition of vertices $V_1 = \{x_1, \ldots, x_d\}$ and $V_2 = \{y_1, \ldots, y_d\}$ with a *pure order*, it means that the vertices lie in $V_1$ and $V_2$ are ordered in such a way that for all $1 \leq i \leq d$, $x_i y_i$ is an edge of $G$, and for every distinct $1 \leq i, j, k \leq d$, if $x_i y_j$ and $x_j y_k$ are edges of $G$, then $x_i y_k$ is so.

**Definition 3.3** [13, Definition 0.3] Let $G$ be a $(d-1)$-dimensional non-Cohen–Macaulay unmixed bipartite graph on a partition of vertices $V_1 = \{x_1, \ldots, x_d\}$ and $V_2 = \{y_1, \ldots, y_d\}$ with a pure order. Let $\{i_1, \ldots, i_n\} \subset \{1, \ldots, d\}$ with $n \geq 2$. A complete bipartite subgraph of $G$ on $(\{x_{i_1}, \ldots, x_{i_n}\}, \{y_{i_1}, \ldots, y_{i_n}\})$ is called a *multi-cross* of $G$ on the given subpartition. It is denoted by $M_{i_1,\ldots,i_n}$ or simply by $M_{n,n}$ if no confusion occurs.

Note that an unmixed bipartite graph is Cohen–Macaulay if it does not have any multi-cross (see, for example, [14, Theorem 3.4]).

**Proposition 3.4** ([13, Theorem 0.4]) *Let $G$ be a $(d-1)$-dimensional non-Cohen–Macaulay unmixed bipartite graph on a partition of vertices $V_1 = \{x_1, \ldots, x_d\}$ and $V_2 = \{y_1, \ldots, y_d\}$ with a pure order. Let $M_{n,n}$ with $n \geq 2$ be a maximal multi-cross of $G$ of minimum dimension. Then, $G$ is* $\mathrm{CM}_{d-n+1}$*, whereas it is not* $\mathrm{CM}_{d-n}$.

The example that was given before the above definition does not have a multi-cross and thus the preceding proposition cannot be applied for it. Let us give another example in which this proposition can be applied. To this end, let $G$ be an unmixed bipartite graph with the vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$ and the edge set $E(G) = \{12, 34, 56, 25, 45, 36\}$. Then, $G$ contains the complete bipartite subgraph $M$ with $E(M) = \{34, 56, 45, 36\}$ as a maximal multi-cross and $G$ is $\mathrm{CM}_2$, whereas it is not $\mathrm{CM}_1$ by the above proposition.

It is worth mentioning that in the above proposition, if $n = d$ is the case, then result of Cook and Nagel regarding Buchsbaumness of an unmixed bipartite graph would be recovered (see [5, Theorem 4.10] and [10, Theorem 1.3]).

There are also at least two different characterization of Cohen–Macaulay bipartite graphs, one given by Herzog and Hibi in [14, Theorem 3.4] and the other given by Cook and Nagel in [5, Proposition 4.8]. It has been shown that a simplicial complex is Buchsbaum if and only if it is pure and the link of every vertex is Cohen–Macaulay (see [30]). This means that a graph $G$ is Buchsbaum if and only if $G$ is unmixed and for every vertex $v \in G$, $G \setminus N[v]$ is Cohen–Macaulay. Moreover, there is a sharper result for bipartite graphs by Cook and Nagel. In fact, complete bipartite graphs are well-known to be Buchsbaum (see [42, Proposition 2.3]) and indeed, the converse is also true: Let $G$ be a bipartite graph. Then, $G$ is Buchsbaum if and only if $G$ is either Cohen–Macaulay or the complete bipartite graph $K_{n,n}$ for some $n \geq 2$ (see [5, Theorem 4.10] and [10, Theorem 1.3]).

The following proposition generalizes the results of Cook and Nagel in light of the result of Herzog and Hibi. Note that in its statement, by a Macaulay order, we mean the order which appeared in the characterization given by Herzog and Hibi.

**Proposition 3.5** ([12, Theorem 4.4] and [13, Remark 0.6]) *Let $G$ be a Cohen–Macaulay bipartite graph with a Macaulay order on the vertex set $V(G) = V \cup W$, where $V = \{x_1, \ldots, x_d\}$ and $W = \{y_1, \ldots, y_d\}$. Let $n_1, \ldots, n_d$ be any positive integers with $n_i \geq 2$ for at least one $i$. Suppose that $G' = G(n_1, \ldots, n_d)$ is the graph obtained by replacing each edge $x_i y_i$ with the multi-cross $M_{n_i,n_i}$ for all $i = 1, \ldots, d$. Let*

$$n_{i_0} = \min\{n_i > 1 \mid i = 1, \ldots, d\}$$

*and set $n = \sum_{i=1}^{d} n_i$. Then, $G'$ is exclusively a $\mathrm{CM}_{n-n_{i_0}+1}$ graph. Furthermore, any bipartite $\mathrm{CM}_t$ graph is obtained by such a replacement of complete bipartite graphs in a unique bipartite Cohen–Macaulay graph.*

At this point it is useful to remark the following facts. Let $H$ be a bipartite Cohen–Macaulay graph and let $G = H'$ be a bipartite $\mathrm{CM}_t$ graph obtained from $H$ by the replacing process in the statement of the above proposition. Assume that $G$ is not $\mathrm{CM}_{t-1}$ and $t \geq 2$. One can show the following observations. Using these observations, it may be easily distinguished all bipartite $\mathrm{CM}_t$ graphs for $t = 2, 3, 4$ (see [12, Examples 4.6, 4.7, and 4.8]).
(1) First of all, $1 \leq \dim H \leq t-1$. Because if $\dim H \geq t$ and we replace just one edge with $K_{n,n}$ where $n \geq 2$, then $G$ is strictly $\mathrm{CM}_r$ with $r \geq t+1$. On the other hand, if $\dim H = 0$, then $G$ is $\mathrm{CM}_1$.
(2) If $\dim H = t-1$, then only one edge can be replaced with $K_{n,n}$ where $n \geq 2$. Because if we replace at least two edges with $K_{n,n}$'s, $n \geq 2$, then $G$ will be strictly $\mathrm{CM}_r$ where $r \geq t+1$.
(3) If $\dim H = t-1$, for replacing just one edge with $K_{n,n}$, $n \geq 2$ can be arbitrary and hence $G$ will be of dimension $n+t-2$.
(4) If $\dim H \leq t-2$, the number of replacements should be at least 2. Again, because if we replace one edge with $K_{n,n}$, $n \geq 2$, then $G$ would be $\mathrm{CM}_r$ for $r \leq t-1$.

(5) When $\dim H \leq t-2$, the maximum number of replacements of edges with $K_{n,n}$, $n \geq 2$, is at most $t - \dim H$ which may occur replacing $K_{2,2}$'s.
(6) For $\dim H \leq t-2$, the maximum size of $K_{n,n}$ to be used for replacements is also $n = t - \dim H$ which may occur when we have two replacements.

A simplicial complex $\Delta$ is called *bi-Cohen–Macaulay* (resp. *bi*-$\mathrm{CM}_t$, etc.) if both $\Delta$ and $\Delta^\vee$ are Cohen–Macaulay (resp. $\mathrm{CM}_t$, etc.). As usual, a graph $G$ is called *bi-Cohen–Macaulay* (resp. *bi*-$\mathrm{CM}_t$, etc.) if $\Delta_G$ is bi-Cohen–Macaulay (resp. bi-$\mathrm{CM}_t$, etc.). We are now going to give two characterizations of bi-$\mathrm{CM}_t$ bipartite graphs and bi-$\mathrm{CM}_t$ chordal graphs generalizing results of Herzog and Rahimi on bi-Cohen–Macaulay bipartite graphs and bi-Cohen–Macaulay chordal graphs (see [16]). The first one is as follows.

**Proposition 3.6** ([9, Theorem 4.3]) *Let $G$ be a bipartite graph and let $t$ be a nonnegative integer. Then, the following conditions hold true.*

(a) *If $|V(G)| \leq t+3$, then $G$ is bi-$\mathrm{CM}_t$ if and only if it is $\mathrm{CM}_t$.*
(b) *If $|V(G)| \geq t+4$, then $G$ is bi-$\mathrm{CM}_t$ if and only if $G$ is $\mathrm{CM}_t$ and the edge ideal $I(G)$ of $G$ has a linear resolution.*

The second characterization is also as follows.

**Proposition 3.7** ([9, Theorem 4.5]) *Let $G$ be a bi-Cohen–Macaulay bipartite graph with a Macaulay order on the vertex set $V(G) = V \cup W$, where $V = \{x_1, \ldots, x_d\}$ and $W = \{y_1, \ldots, y_d\}$. Let $n_1, \ldots, n_d$ be any positive integers with $n_i \geq 2$ for at least one $i$. Suppose that $G' = G(n_1, \ldots, n_d)$ is the graph obtained by replacing each edge $x_i y_i$ with the complete bipartite graph $K_{n_i,n_i}$ for all $i = 1, \ldots, d$. Let*

$$n_{i_0} = \min\{n_i > 1 \mid i = 1, \ldots, d\}$$

*and set $n = \sum_{i=1}^{d} n_i$. Then, $G'$ is exclusively a $\mathrm{CM}_{n-n_{i_0}+1}$ graph and the edge ideal $I(G')$ of $G'$ has a linear resolution. In particular, $G'$ is bi-$\mathrm{CM}_t$ with $t = n - n_{i_0} + 1$. Furthermore, any bi-$\mathrm{CM}_t$ bipartite graph is obtained by such a replacement of complete bipartite graphs in a unique bi-$\mathrm{CM}_t$ bipartite graph.*

The next one is also a characterization for bi-$\mathrm{CM}_t$ chordal graphs.

**Proposition 3.8** ([9, Theorem 4.8]) *Let $G$ be a chordal graph and let $t$ be a nonnegative integer. Then, the following conditions hold true.*

(a) *If $|V(G)| \leq t+3$, then $G$ is bi-$\mathrm{CM}_t$ if and only if it is unmixed.*
(b) *If $|V(G)| \geq t+4$, then $G$ is bi-$\mathrm{CM}_t$ if and only if one of the following equivalent conditions hold:* (1) *$G$ is bi-Cohen–Macaulay.* (2) *If $\{F_1, \ldots, F_m\}$ is the set of all facets of the clique complex of $G$ which contain at least a free vertex, then either $m = 1$ or $m > 1$ with $V(G) = V(F_1) \cup \cdots \cup V(F_m)$, which is a disjoint union and each $F_i$ has exactly one free vertex $j_i$ and the restriction of $G$ to $[n] \setminus \{j_1, \ldots, j_m\}$ is a clique.*

The following result reflects the relation between the minimum length of chordless cycles of graphs and the $\mathrm{CM}_t$ property.

**Proposition 3.9** ([37, Corollary 3.14]) *Let $G$ be a simple graph on $[n] = \{1, \ldots, n\}$ with no isolated vertices. Let $\Delta = \Delta(G)$ be the clique complex of $G$. Let $r \geq 3$ be an integer. Then, $\Delta^\vee$ is $\mathrm{CM}_{n-r}$ if and only if every cycle of $G$ of length at most $r$ has a chord.*

Fröberg [8] has proved that $I_\Delta = I(\overline{G})$ admits a linear resolution if and only if $G$ is chordal. Using this result together with the above proposition and [37, Theorem 3.11], one may conclude the following proposition.

**Proposition 3.10** ([37]) *Under the assumptions of the previous proposition, assume that $G$ is $r$-chordal, that is, it has no chordless cycles of length greater than $r$. Then, $\Delta^\vee$ is $\mathrm{CM}_{n-r}$ if and only if $I_\Delta = I(\overline{G})$ has a linear resolution.*

It is easy to see that if $G$ is either a bipartite graph or a chordal graph, then $\overline{G}$ can only have chordless four-cycles (see, for example, [9, Lemmas 4.2 and 4.7]). Combining this fact with the previous proposition, one can obtain the following result.

**Proposition 3.11** ([37]) *Let $G$ be a graph on $n$ vertices which is either bipartite or chordal. If the Alexander dual of $\Delta(\overline{G}) = \Delta_G$ is $\mathrm{CM}_{n-4}$, then $I(G)$ has a linear resolution.*

We close this section by mentioning that a variety of interesting questions and topics may arise from $\mathrm{CM}_t$ property of simplicial complexes and graphs which are already considered for Cohen–Macaulay and Buchsbaum ones. As a further example, Pournaki, Seyed Fakhari, and Yassemi [23] have studied the $h$-vector of $\mathrm{CM}_t$ simplicial complexes extending a result of Terai [35].

## 4 General Monomial Ideals and the $\mathrm{CM}_t$ Property

Let us start this section by introducing the $\mathrm{CM}_t$ property for an unmixed monomial ideal $I$ of a polynomial ring $R$. Note that if $R/I$ is $d$-dimensional, then always there exists a $(d-1)$-dimensional simplicial complex $\Delta$ such that $\sqrt{I} = I_\Delta$. Based on this observation, we give the following definition.

**Definition 4.1** ([24, Definition 3.1]) Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $I$ be an unmixed monomial ideal of $R$ such that $R/I$ is $d$-dimensional and set $\Delta$ as a $(d-1)$-dimensional simplicial complex such that $\sqrt{I} = I_\Delta$. If $0 \le t \le d-1$ is an integer, then $R/I$ is called a $\mathrm{CM}_t$ ring provided the localized ring $(R/I)_{x_F}$ is Cohen–Macaulay for every face $F \in \Delta$ with $|F| \ge t$. Moreover, the monomial ideal $I$ is called $\mathrm{CM}_t$ if the ring $R/I$ is $\mathrm{CM}_t$.

Let $I$ be a *squarefree* unmixed monomial ideal of the polynomial ring $R$ such that $R/I$ is $d$-dimensional and set $\Delta$ as a $(d-1)$-dimensional simplicial complex such that $\sqrt{I} = I_\Delta$. In this case, $R/I$ is $\mathrm{CM}_t$ in the sense of this section means that $\Delta$ is $\mathrm{CM}_t$. Also, for the ring $R/I$, where $I$ is not necessarily squarefree, the $\mathrm{CM}_0$ property is the same as Cohen–Macaulayness of $R/I$, whereas the $\mathrm{CM}_1$ property is identical with the generalized Cohen–Macaulay property. Note that the generalized Cohen–Macaulay property is weaker than the Buchsbaum property for a general monomial ideal, while these two notions are equivalent for a squarefree monomial ideal.

The following proposition shows that the radical preserves the $\mathrm{CM}_t$ property.

**Proposition 4.2** ([24, Proposition 3.2]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $I$ be an unmixed monomial ideal of $R$ such that $R/I$ is $d$-dimensional and $0 \le t \le d-1$ be an integer. If $R/I$ is* $\mathrm{CM}_t$*, then $R/\sqrt{I}$ is also* $\mathrm{CM}_t$.

We now pose the following question.

**Question 4.3** Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex. Let $0 \le t \le t' \le d-1$ be integers. Suppose that $\Delta$ is $\mathrm{CM}_t$ but not $\mathrm{CM}_{t-1}$. Does there exist an unmixed monomial ideal $I$ of $R$ with $\sqrt{I} = I_\Delta$ such that $I$ is $\mathrm{CM}_{t'}$ but not $\mathrm{CM}_{t'-1}$?

In the following definition, we introduce the notion of Cohen–Macaulayness in a fixed codimension for finitely generated modules over Noetherian rings. We recall that for a Noetherian ring $R$ and an $R$-module $M$, $\mathrm{supp}_R(M)$ denotes the support of $M$ over $R$.

**Definition 4.4** [21, Definition 6.8] Let $R$ be a Noetherian ring and $M$ be a $d$-dimensional finitely generated $R$-module. If $t$ is an integer and $t \le d$, then $M$ is called *Cohen–Macaulay in codimension t* provided for every $\mathfrak{p} \in \mathrm{supp}_R(M)$ such that $\dim R/\mathfrak{p} = d-t$, the localized module $M_\mathfrak{p}$ is Cohen–Macaulay of dimension $t$.

The following proposition provides a necessary and sufficient condition for the $\mathrm{CM}_t$ property based on the Krull dimension of Ext-modules (cf. [21, Corollary 7.3]). We recall that for a Noetherian ring $R$ and an $R$-module $M$, $\dim_R M$ denotes the Krull dimension of $M$ over $R$.

**Proposition 4.5** ([24, Proposition 3.4]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $I$ be an unmixed monomial ideal of $R$ and $0 \le t \le d-1$ be an integer. Then, $R/I$ is* $\mathrm{CM}_t$ *if and only if* $\dim_R \mathrm{Ext}^i_R(R/I, R) < t$ *for every $i > n-d$.*

The following proposition indicates when tensor products are Cohen–Macaulay in a fixed codimension.

**Proposition 4.6** ([24, Proposition 3.5]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $R' = \mathbb{K}[y_1, \ldots, y_{n'}]$ be the polynomial rings over a field $\mathbb{K}$. Let $M$ (resp. $M'$) be a $d$-dimensional (resp. $d'$-dimensional) finitely generated $R$-module (resp. $R'$-module). Then, $M \otimes_\mathbb{K} M'$ is Cohen–Macaulay in codimension $d + d' - t$ as an $(R \otimes_\mathbb{K} R')$-module if and only if $M$ and $M'$ are both Cohen–Macaulay in codimension $d + d' - t$.*

The following statement is a corollary to the above proposition and generalizes [11, Proposition 2.10].

**Proposition 4.7** ([24, Corollary 3.6]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and $R' = \mathbb{K}[y_1, \ldots, y_{n'}]$ be the polynomial rings over a field $\mathbb{K}$. Let $I$ (resp. $I'$) be a monomial ideal of $R$ (resp. $R'$) with* $\dim R/I = d$ *(resp.* $\dim R'/I' = d'$*). Then, $(R/I) \otimes_\mathbb{K} (R'/I')$ is* $\mathrm{CM}_t$ *if and only if $R/I$ is* $\mathrm{CM}_{t-d'}$ *and $R'/I'$ is* $\mathrm{CM}_{t-d}$.

In the following two definitions, $\mathbf{e}_i$ denotes the $i$th basis vector of $\mathbb{Z}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Z}^n$. We also denote by $a_i$ the $i$th coordinate of a vector $\mathbf{a} \in \mathbb{Z}^n$.

**Definition 4.8** ([20, Definition 2.1]) Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$, $M$ be a finitely generated $\mathbb{N}^n$-graded $R$-module and let $\mathbf{a} \in \mathbb{N}^n$. Then, $M$ is called *positively* $\mathbf{a}$*-determined* if the multiplication map $\cdot x_i : M_{\mathbf{b}} \longrightarrow M_{\mathbf{b}+\mathbf{e}_i}$ is bijective for every $\mathbf{b} \in \mathbb{N}^n$ and for every $i \in [n]$ with $b_i \geq a_i$.

**Definition 4.9** ([39, Definition 2.1]) Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$ and $M$ be a finitely generated $\mathbb{N}^n$-graded $R$-module. Then, $M$ is called *squarefree* if it is positively $\mathbf{1}$-determined.

It is worthwhile to mention that in the case of monomial ideals, the above definition agrees with the usual one given in the second section.

For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, set $|\mathbf{a}| = \sum_{i=1}^n a_i$ and suppose that

$$\widetilde{R} = \mathbb{K}[x_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq a_i]$$

is the polynomial ring over a field $\mathbb{K}$ with $|\mathbf{a}|$ variables. Let $I$ be a monomial ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$ with the set of minimal monomial generators $\{u_1, \ldots, u_m\}$, where for every $1 \leq i \leq m$,

$$u_i = \prod_{j=1}^{n} x_j^{a_{ij}}.$$

Then, for every $1 \leq j \leq n$, we define $a(I)_j = \max\{a_{ij} \mid 1 \leq i \leq m\}$, we set $\mathbf{a}(I) = (a(I)_1, \ldots, a(I)_n)$, and we denote the polarization of $I$ by

$$I^{\mathrm{pol}} \subset R^{\mathrm{pol}} := \mathbb{K}[x_{i,j} \mid 1 \leq i \leq n,\ 1 \leq j \leq a(I)_i].$$

Yanagawa [41] has constructed the polarization functor $\mathrm{pol}_{\mathbf{a}}$ from the category of positively $\mathbf{a}$-determined modules to the category of squarefree modules. It is well-known that (see [41, Section 4]) if $I$ is a monomial ideal of $R$, then

$$\mathrm{pol}_{\mathbf{a}(I)}(R/I) \cong R^{\mathrm{pol}}/I^{\mathrm{pol}}.$$

**Proposition 4.10** ([24, Theorem 3.9]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$, $\mathbf{a} \in \mathbb{N}^n$ and let $M$ be a positively $\mathbf{a}$-determined $R$-module with $\dim_R M = d$. Then, $M$ is Cohen–Macaulay in codimension $d-t$ if and only if $\mathrm{pol}_{\mathbf{a}} M$ is Cohen–Macaulay in codimension $d-t$.*

The following statement is a corollary to the above proposition.

**Proposition 4.11** ([24, Corollary 3.10]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$ and let $I$ be a monomial ideal of $R$. Then, $R/I$ is $\mathrm{CM}_t$ if and only if $R^{\mathrm{pol}}/I^{\mathrm{pol}}$ is $\mathrm{CM}_{t+|\mathbf{a}(I)|-n}$.*

Let us give an example illustrating the abovementioned points. Suppose that $R = \mathbb{K}[x_1, x_2, x_3, x_4]$ is the polynomial ring over a field $\mathbb{K}$ and

$$I = \left(x_1^2 x_2, x_2 x_3, x_1 x_4\right)$$

is an ideal of $R$. It is easy to see that $I$ is not unmixed, $\dim R/I = 2$ and depth $R/I = 1$. We have $\dim_R \mathrm{Ext}_R^3(R/I, R) = 1$, and $R/I$ is Cohen–Macaulay in codimension one. In fact, the localized ring $(R/I)_{x_2}$ is not Cohen–Macaulay, whereas for every $F \subset [4]$ with $|F| \geq 2$, $(R/I)_{x_F}$ is Cohen–Macaulay. However, $\dim_{R^{\mathrm{pol}}} \mathrm{Ext}^3_{R^{\mathrm{pol}}}(R^{\mathrm{pol}}/I^{\mathrm{pol}}, \widetilde{R}) = 2$ and

$(R^{\mathrm{pol}}/I^{\mathrm{pol}})_{x_F}$ is Cohen–Macaulay for every $F$ with $|F| \geq 2$, while both $I$ and $I^{\mathrm{pol}}$ are not unmixed.

The following theorem generalizes [36, Theorem 4.3]. We recall that for a monomial ideal $I$ of a polynomial ring $R$ and for a positive integer $r$, the ring $R/I$ satisfies the *Serre's condition* $(S_r)$ if $\mathrm{depth}(R/I)_{\mathfrak{p}} \geq \min\{\dim(R/I)_{\mathfrak{p}}, r\}$ holds true for every $\mathfrak{p} \in \mathrm{Spec}(R/I)$.

**Proposition 4.12** ([24, Theorem 3.11]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$. If $R/I_\Delta$ satisfies the Serre's condition $(S_2)$, then the following conditions are equivalent.*

(a) $R/I_\Delta^\ell$ *is Cohen–Macaulay for every* $\ell \geq 1$.
(b) $R/I_\Delta^\ell$ *is* $\mathrm{CM}_t$ *for every* $\ell \geq 1$ *and for every* $0 \leq t \leq d-2$.
(c) $R/I_\Delta^\ell$ *is* $\mathrm{CM}_t$ *for some* $\ell \geq 3$ *and for some* $0 \leq t \leq d-2$.
(d) $\mathbb{K}[\Delta]$ *is a complete intersection.*

We now state a similar result to the above proposition which corresponds to the symbolic powers and generalizes [36, Theorem 3.6]. Let us first recall the notion of a matroid. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex on $[n]$. Then, $\Delta$ is called a *matroid* if the induced simplicial complex $\Delta_W = \{F \in \Delta \mid F \subseteq W\}$ is pure for every $W \subseteq [n]$.

**Proposition 4.13** ([24, Theorem 3.13]) *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$. If $R/I_\Delta$ satisfies the Serre's condition $(S_2)$, then the following conditions are equivalent.*

(a) $R/I_\Delta^{(\ell)}$ *is Cohen–Macaulay for every* $\ell \geq 1$.
(b) $R/I_\Delta^{(\ell)}$ *is* $\mathrm{CM}_t$ *for every* $\ell \geq 1$ *and for every* $0 \leq t \leq d-2$.
(c) $R/I_\Delta^{(\ell)}$ *is* $\mathrm{CM}_t$ *for some* $\ell \geq 3$ *and for some* $0 \leq t \leq d-2$.
(d) $\Delta$ *is a matroid.*

We finally close this paper with the following two questions and by inviting interested people to work on this topic.

**Question 4.14** [24, Question 3.14] Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. Let $\Delta$ be a $(d-1)$-dimensional simplicial complex with $d \geq 2$ and let $1 \leq t \leq d-2$ be an integer.

(a) Characterize the $\mathrm{CM}_t$ property for $R/I_\Delta^2$ and $R/I_\Delta^{(2)}$.
(b) If $R/I_\Delta$ satisfies the Serre's condition $(S_2)$, then is it true that the Cohen–Macaulay and $\mathrm{CM}_t$ properties for $R/I_\Delta^2$ and $R/I_\Delta^{(2)}$ are equivalent?

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