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Regularity and h-polynomials of Binomial Edge Ideals

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Abstract
Let G be a finite simple graph on the vertex set $[n] = \{1, \dots, n\}$ and $K[\mathbf{x}, \mathbf{y}] = K[x_1, \dots, x_n, y_1, \dots, y_n]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_j = 1$. The binomial edge ideal of G is the binomial ideal $J_G \subset K[\mathbf{x}, \mathbf{y}]$ which is generated by those binomials $x_i y_j - x_j y_i$ for which $\{i, j\}$ is an edge of G. The Hilbert series $H_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$ of $K[\mathbf{x}, \mathbf{y}]/J_G$ is of the form $H_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)/(1-\lambda)^d$, where $d = \dim K[\mathbf{x}, \mathbf{y}]/J_G$ and where $h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ with each $h_i \in \mathbb{Z}$ and with $h_s \neq 0$ is the h-polynomial of $K[\mathbf{x}, \mathbf{y}]/J_G$. It is known that, when $K[\mathbf{x}, \mathbf{y}]/J_G$ is Cohen–Macaulay, one has $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$, where $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G)$ is the (Castelnuovo–Mumford) regularity of $K[\mathbf{x}, \mathbf{y}]/J_G$. In the present paper, given arbitrary integers r and s with $2 \leq r \leq s$, a finite simple graph G for which $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = r$ and $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = s$ will be constructed.

Keywords Binomial edge ideal · Castelnuovo–Mumford regularity · h-polynomial

Mathematics Subject Classification (2010) 05E40 · 13H10

1 Introduction

The binomial edge ideal of a finite simple graph was introduced in [2] and in [10] independently. (Recall that a finite graph G is *simple* if G possesses no loop and no multiple edge.) Let G be a finite simple graph on the vertex set $[n] = \{1, 2, \dots, n\}$ and $K[\mathbf{x}, \mathbf{y}] = K[x_1, \dots, x_n, y_1, \dots, y_n]$ the polynomial ring in $2n$ variables over a field K with each $\deg x_i = \deg y_j = 1$. The *binomial edge ideal* J_G of G is the binomial ideal of $K[\mathbf{x}, \mathbf{y}]$ which is generated by those binomials $x_i y_j - x_j y_i$ for which $\{i, j\}$ is an edge of G.

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Let, in general, $S = K[x_1, \ldots, x_n]$ denote the polynomial ring in n variables over a field K with each $\deg x_i = 1$ and $I \subset S$ a homogeneous ideal of S with $\dim S/I = d$. The Hilbert series $H_{S/I}(\lambda)$ of S/I is of the form $H_{S/I}(\lambda) = (h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s)/(1-\lambda)^d$, where each $h_i \in \mathbb{Z}$ [1, Proposition 4.4.1]. We say that $h_{S/I}(\lambda) = h_0+h_1\lambda+h_2\lambda^2+\cdots+h_s\lambda^s$ with $h_s \neq 0$ is the *h-polynomial* of S/I. Let $\mathrm{reg}(S/I)$ denote the (*Castelnuovo–Mumford*) *regularity* [1, p. 168] of S/I. It is known (e.g., [14, Corollary B.4.1]) that, when S/I is Cohen–Macaulay, one has $\mathrm{reg}(S/I) = \deg h_{S/I}(\lambda)$. Furthermore, in [3] and [4], for given integers r and s with $r, s \geq 1$, a monomial ideal I of $S = K[x_1, \ldots, x_n]$ with $n \gg 0$ for which $\mathrm{reg}(S/I) = r$ and $\deg h_{S/I}(\lambda) = s$ was constructed.

Let, as before, G be a finite simple graph on the vertex set $[n]$ with $d = \dim K[\mathbf{x}, \mathbf{y}]/J_G$ and $h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = h_0 + h_1\lambda + h_2\lambda^2 + \cdots + h_s\lambda^s$ the h-polynomial of $K[\mathbf{x}, \mathbf{y}]/J_G$.

Now, in the present paper, given arbitrary integers r and s with $2 \leq r \leq s$, a finite simple graph G on $[n]$ with $n \gg 0$ for which $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = r$ and $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = s$ will be constructed.

Theorem 1.1 *Given arbitrary integers r and s with $2 \leq r \leq s$, there exists a finite simple graph G on $[n]$ with $n \gg 0$ for which $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = r$ and $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = s$.*

2 Proof of Theorem 1.1

Our discussion starts in the computation of the regularity and the h-polynomial of the binomial edge ideal of a path graph.

Example 2.1 Let P_n be the path on the vertex set $[n]$ with $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}$ its edges. Since $K[\mathbf{x}, \mathbf{y}]/J_{P_n}$ is a complete intersection, it follows that the Hilbert series of $K[\mathbf{x}, \mathbf{y}]/J_{P_n}$ is $H_{K[\mathbf{x},\mathbf{y}]/J_{P_n}}(\lambda) = (1+\lambda)^{n-1}/(1-\lambda)^{n+1}$ and that $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_{P_n}) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_{P_n}}(\lambda) = n - 1$.

Let G be a finite simple graph on the vertex set $[n]$ and $E(G)$ its edge set. The *suspension* of G is the finite simple graph $\widehat{G}$ on the vertex set $[n+1]$ whose edge set is $E(\widehat{G}) = E(G) \cup \{\{i, n+1\} : i \in [n]\}$. Given a positive integer $m \geq 2$, the *m-th suspension* of G is the finite simple graph $\widehat{G}^m$ on $[n+m]$ which is defined inductively by $\widehat{G}^m = \widehat{\widehat{G}^{m-1}}$, where $\widehat{G}^1 = \widehat{G}$.

Lemma 2.2 *Let G be a finite connected simple graph on $[n]$ which is not complete. Suppose that $\dim K[\mathbf{x}, \mathbf{y}]/J_G = n + 1$ and $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) \geq 2$. Then*

$$\mathrm{reg}\left(K[\mathbf{x}, \mathbf{y}, x_{n+1}, y_{n+1}]/J_{\widehat{G}}\right) = \mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G),$$
$$\deg h_{K[\mathbf{x},\mathbf{y},x_{n+1},y_{n+1}]/J_{\widehat{G}}}(\lambda) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) + 1.$$

In particular, if $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$, then

$$\mathrm{reg}\left(K[\mathbf{x}, \mathbf{y}, x_{n+1}, y_{n+1}]/J_{\widehat{G}}\right) < \deg h_{K[\mathbf{x},\mathbf{y}]/J_{\widehat{G}}}(\lambda).$$

Proof The suspension $\widehat{G}$ is the join product [12, p. 3] of G and $\{n+1\}$, and $\widehat{G}$ is not complete. Hence, by virtue of [11, Theorem 2.1] and [12, Theorem 2.1 (a)], one has

$$\mathrm{reg}\left(K[\mathbf{x}, \mathbf{y}, x_{n+1}, y_{n+1}]/J_{\widehat{G}}\right) = \max\{\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G), 2\} = \mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G).$$

Furthermore, [8, Theorem 4.6] says that

$$\begin{aligned} H_{K[\mathbf{x},\mathbf{y},x_{n+1},y_{n+1}]/J_{\widehat{G}}}(\lambda) &= H_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) + \frac{2\lambda + (n-1)\lambda^2}{(1-\lambda)^{n+2}} \\ &= \frac{h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)}{(1-\lambda)^{n+1}} + \frac{2\lambda + (n-1)\lambda^2}{(1-\lambda)^{n+2}} \\ &= \frac{h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)\cdot(1-\lambda) + 2\lambda + (n-1)\lambda^2}{(1-\lambda)^{n+2}}. \end{aligned}$$

Thus, $\deg h_{K[\mathbf{x},\mathbf{y},x_{n+1},y_{n+1}]/J_{\widehat{G}}}(\lambda) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) + 1$, as desired. □

We are now in the position to give a proof of Theorem 1.1.

Proof of Theorem 1.1 Each of the following three cases is discussed.

Case 1 Let $2 \le r = s$. Let $G = P_{r+1}$. As was shown in Example 2.1, one has

$$\mathrm{reg}(K[\mathbf{x},\mathbf{y}]/J_G) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = r.$$

Case 2 Let $r = 2$ and $3 \le s$. Let $G = K_{s-1,s-1}$ denote the complete bipartite graph on the vertex set $[2s-2]$. By using [13, Theorem 1.1 (c) together with Theorem 5.4 (a)], one has $\mathrm{reg}(K[\mathbf{x},\mathbf{y}]/J_G) = 2$ and

$$\begin{aligned} H_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) &= \frac{1+(2s-3)\lambda}{(1-\lambda)^{2s-1}} + \frac{2}{(1-\lambda)^{2s-2}} - \frac{2\{1+(s-2)\lambda\}}{(1-\lambda)^s} \\ &= \frac{1+(2s-3)\lambda + 2(1-\lambda) - 2\{1+(s-2)\lambda\}(1-\lambda)^{s-1}}{(1-\lambda)^{2s-1}}. \end{aligned}$$

Hence, $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = s$, as required.

Case 3 Let $3 \le r < s$. Let $G = \widehat{P_{r+1}}^{\,s-r}$ be the $(s-r)$-th suspension of the path P_{r+1}. Applying Lemma 2.2 repeatedly shows $\mathrm{reg}(S/J_G) = r$ and

$$\begin{aligned} &H_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) \\ &= \frac{(1+\lambda)^r(1-\lambda)^{s-r} + 2\lambda\left\{\sum_{i=0}^{s-r-1}(1-\lambda)^i\right\} + \lambda^2\sum_{i=0}^{s-r-1}(s-1-i)(1-\lambda)^i}{(1-\lambda)^{s+2}} \\ &= \frac{(1+\lambda)^r(1-\lambda)^{s-r} + 2\lambda\cdot\frac{1-(1-\lambda)^{s-r}}{\lambda} + \lambda^2\cdot\frac{-1+(1-\lambda)^{s-r}+\lambda\{s-r(1-\lambda)^{s-r}\}}{\lambda^2}}{(1-\lambda)^{s+2}} \\ &= \frac{(1+\lambda)^r(1-\lambda)^{s-r} + 2\left\{1-(1-\lambda)^{s-r}\right\} - 1 + (1-\lambda)^{s-r} + \lambda\{s-r(1-\lambda)^{s-r}\}}{(1-\lambda)^{s+2}} \\ &= \frac{(1+\lambda)^r(1-\lambda)^{s-r} + 1 - (1-\lambda)^{s-r} + \lambda\{s-r(1-\lambda)^{s-r}\}}{(1-\lambda)^{s+2}} \\ &= \frac{1+s\lambda+(1-\lambda)^{s-r}\{(1+\lambda)^r - 1 - r\lambda\}}{(1-\lambda)^{s+2}}. \end{aligned}$$

Hence, $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = s$, as desired. □

3 Examples

Proposition 3.1 *The cycle C_n of length $n \geq 3$ satisfies*

$$\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_{C_n}) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_{C_n}}(\lambda).$$

Proof Since the length of the longest induced path of C_n is $n - 2$, it follows from [9, Theorem 1.1] and [7, Theorem 3.2] that $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_{C_n}) = n - 2$. Furthermore, [15, Theorem 10 (b)] says that

$$\deg h_{K[\mathbf{x},\mathbf{y}]/J_{C_n}} = \begin{cases} 1 & (n = 3), \\ n - 1 & (n > 3). \end{cases}$$

Hence, the desired inequality follows. □

Let $k \geq 1$ be an integer and $p_1, p_2, \ldots, p_k$ a sequence of positive integers with $p_1 \geq p_2 \geq \cdots \geq p_k \geq 1$ and $p_1 + p_2 + \cdots + p_k = n$. Let $V_1, V_2, \ldots, V_k$ denote a partition of $[n]$ with each $|V_i| = p_i$. In other words, $[n] = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ and $V_i \cap V_j = \emptyset$ if $i \neq j$. Suppose that

$$V_i = \left\{ \sum_{j=1}^{i-1} p_j + 1, \sum_{j=1}^{i-1} p_j + 2, \ldots, \sum_{j=1}^{i-1} p_j + p_i - 1, \sum_{j=1}^{i} p_j \right\}$$

for each $1 \leq i \leq k$. The *complete multipartite graph* $K_{p_1,\ldots,p_k}$ is the finite simple graph on the vertex set $[n]$ with the edge set

$$E\left(K_{p_1,\ldots,p_k}\right) = \{\{k, \ell\} : k \in V_i, \ \ell \in V_j, \ 1 \leq i < j \leq k\}.$$

Proposition 3.2 *The complete multipartite graph $G = K_{p_1,\ldots,p_k}$ satisfies*

$$\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda).$$

Proof We claim $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$ by induction on k. If $k = 1$, then $G = K_{p_1}$ is the complete graph and $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = 1$.

Let $k > 1$. If $p_k = 1$, then $G = \widehat{G'}$, where $G' = K_{p_1,\ldots,p_{k-1}}$. Lemma 2.2 as well as the induction hypothesis then guarantees that $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$. Hence, one can assume that $p_k > 1$. In particular, G is not complete. It then follows from [12, Theorem 2.1 (a)] that $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) = 2$. Furthermore, [8, Corollary 4.14] says that

$$\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = \begin{cases} n - p_k + 1 & (2p_1 < n + 1), \\ 2p_1 - p_k & (2p_1 \geq n + 1). \end{cases}$$

Since $k > 1$ and $p_k > 1$, one has $\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) \geq n - p_k + 1 \geq p_1 + 1 \geq 3$. Thus, the desired inequality follows. □

Let $t \geq 3$ be an integer and $K_{1,t}$ the complete bipartite graph on $\{1, v_1, \ldots, v_t\}$ with the edge set $E(K_{1,t}) = \{\{1, v_i\} : 1 \leq i \leq t\}$. Let $p_1, p_2, \ldots, p_t$ be a sequence of positive integers and $P^{(i)}$ the path of length p_i on the vertex set $\{w_{i,1}, w_{i,2}, \ldots, w_{i,p_i+1}\}$ for each

$1 \le i \le t$. Then the *t-starlike graph* $T_{p_1,p_2,\dots,p_t}$ is defined as the finite simple graph obtained by identifying v_i with $w_{i,1}$ for each $1 \le i \le t$. Thus, the vertex set of $T_{p_1,p_2,\dots,p_t}$ is

$$\{1\} \cup \bigcup_{i=1}^{t} \{w_{i,1}, w_{i,2}, \dots, w_{i,p_i+1}\}$$

and its edge set is

$$E(T_{p_1,p_2,\dots,p_t}) = \bigcup_{i=1}^{t} \left\{\{w_{i,j}, w_{i,j+1}\} \mid 0 \le j \le p_i\right\},$$

where $w_{i,0} = 1$ for each $1 \le i \le t$.

Proposition 3.3 *The t-starlike graph $G = T_{p_1,p_2,\dots,p_t}$ satisfies*

$$\mathrm{reg}(K[\mathbf{x},\mathbf{y}]/J_G) < \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda).$$

Proof It follows from [5, Corollary 3.4 (2)] that $\mathrm{reg}(K[\mathbf{x},\mathbf{y}]/J_G) = 2 + \sum_{i=1}^{t} p_i$. Furthermore, [13, Theorem 5.4 (a)] guarantees that

$$\begin{aligned} H_{K[\mathbf{x},\mathbf{y}]/J_{K_{1,t}}}(\lambda) &= \frac{1}{(1-\lambda)^{2t}} - \frac{1+(t-1)\lambda}{(1-\lambda)^{t+1}} + \frac{1+t\lambda}{(1-\lambda)^{t+2}} \\ &= \frac{1-\{1+(t-1)\lambda\}(1-\lambda)^{t-1} + (1+t\lambda)(1-\lambda)^{t-2}}{(1-\lambda)^{2t}} \\ &= \frac{1+(1-\lambda)^{t-2}\left\{2\lambda+(t-1)\lambda^2\right\}}{(1-\lambda)^{2t}}. \end{aligned}$$

Hence, by virtue of [8, Corollary 3.3], one has

$$h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = \left[1+(1-\lambda)^{t-2}\left\{2\lambda+(t-1)\lambda^2\right\}\right] \cdot (1-\lambda)^{\sum_{i=1}^{t} p_i}.$$

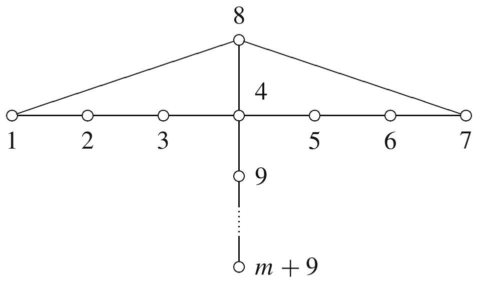
Thus

$$\deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda) = t + \sum_{i=1}^{t} p_i > 2 + \sum_{i=1}^{t} p_i = \mathrm{reg}(K[\mathbf{x},\mathbf{y}]/J_G),$$

as required. □

Example 3.4 Let $m \ge 0$ be an integer and G_m the finite simple graph on the vertex set $[m+9]$ drawn below

$G_m =$



Then $K[\mathbf{x}, \mathbf{y}]/J_{G_m}$ is not unmixed. In fact, for each subset $S \subset [m+9]$, we define

$$P_S = \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(S)}} \right),$$

where $G_1, \dots, G_{c(S)}$ are connected components of $G_{[m+9] \setminus S}$ and where $\tilde{G}_1, \dots, \tilde{G}_{c(S)}$ is the complete graph on the vertex set $V(G_1), \dots, V\left(G_{c(S)}\right)$, respectively. It then follows from [2, Lemma 3.1 and Corollary 3.9] that $P_\emptyset$ and $P_{\{3,8\}}$ are minimal primes of J_{G_m} and that height $P_\emptyset = m + 8 <$ height $P_{\{3,8\}} = m + 9$. Thus, $K[\mathbf{x}, \mathbf{y}]/J_{G_m}$ is not unmixed. In particular, $K[\mathbf{x}, \mathbf{y}]/J_{G_m}$ is not Cohen-Macaulay. However, one has $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_{G_m}) = \deg h_{K[\mathbf{x},\mathbf{y}]/J_{G_m}}(\lambda) = m + 6$.

A lot of computational experience encourages the authors to propose the conjecture that, for an arbitrary finite simple graph G, one has $\mathrm{reg}(K[\mathbf{x}, \mathbf{y}]/J_G) \leq \deg h_{K[\mathbf{x},\mathbf{y}]/J_G}(\lambda)$. However, the conjecture turns out to be false. A counterexample is constructed in [6].

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