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An Algorithm for a Class of Bilevel Variational Inequalities with Split Variational Inequality and Fixed Point Problem Constraints

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Abstract

In this paper, we investigate the problem of solving strongly monotone variational inequality problems over the solution set of a split variational inequality and fixed point problem. Strong convergence of the iterative process is proved. In particular, the problem of finding a common solution to a variational inequality with pseudomonotone mapping and a fixed point problem involving demicontractive mapping is also studied. Besides, we get a strongly convergent algorithm for finding the minimum-norm solution to the split feasibility problem, which requires only two projections at each step. A simple numerical example is given to illustrate the proposed algorithm.

Keywords Split variational inequality and fixed point problem · Pseudomonotone mapping · Demicontractive mapping · Subgradient extragradient method · Strong convergence · Minimum-norm solution · Split feasibility problem

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1 Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. Let C be a nonempty closed convex subset of $\mathcal{H}_1$. Given mappings $G : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ and $S : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$, the split variational inequality and fixed point problem (in short, SVIFPP) is to find a solution x^* of the variational inequality problem in the space $\mathcal{H}_1$ so that the image Ax^*, under a given bounded linear operator A, is a fixed point of another mapping in the space $\mathcal{H}_2$. More specifically, the SVIFPP can be formulated as

$$\text{Find } x^* \in C : \langle G(x^*), x - x^* \rangle \geq 0, \forall x \in C \tag{1}$$

such that

$$S(Ax^*) = Ax^*.$$

When $G = 0$ and $S = P_Q$, the SVIFPP reduces to the split feasibility problem, shortly SFP,

$$\text{Find } x^* \in C \text{ such that } \ Ax^* \in Q.$$

The SFP was first introduced by Censor and Elfving [4] in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [1]. Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy [3, 5, 6], and many other practical problems.

If we consider only the problem (1) then (1) is a classical variational inequality problem. If $\mathcal{H}_1 = \mathcal{H}_2$ and A is the identity mapping in $\mathcal{H}_1$, then the SVIFPP becomes the problem of finding a common solution of a variational inequality problem and a fixed point problem, which can be written as follows

$$\text{Find } x^* \in \Omega := \text{Sol}(C, G) \cap \text{Fix}(S), \tag{2}$$

where the solution set of (1) is denoted by $\text{Sol}(C, G)$ and the set of fixed points of S is denoted by $\text{Fix}(S)$.

Problem (2) has been studied widely in recent years. The inspiration for studying this common solution problem is due to its possible applications to mathematical models whose constraints can be expressed as variational inequalities and/or fixed point problems. This happens, in particular, in the practical problems as network resource allocation, image recovery, signal processing (see, for instance, [9, 12]).

Very recently, Kraikaew and Saejung [11] combined the subgradient extragradient method and Halpern method to propose an algorithm which is called Halpern subgradient extragradient method to find a common element of the solution set of a variational inequality problem and the fixed point set of a quasi-nonexpansive mapping. Their algorithm is of the form

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda G(x^n)), \\ T_n = \{\omega \in \mathcal{H} : \langle x^n - \lambda G(x^n) - y^n, \omega - y^n \rangle \leq 0\}, \\ z^n = \alpha_n x^0 + (1 - \alpha_n) P_{T_n}(x^n - \lambda G(y^n)), \\ x^{n+1} = \beta_n x^n + (1 - \beta_n) S(z^n), \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$, $G : \mathcal{H} \longrightarrow \mathcal{H}$, and $S : \mathcal{H} \longrightarrow \mathcal{H}$ is a quasi-nonexpansive mapping. They proved that the sequence $\{x^n\}$ generated by (3) converges strongly to $P_{\text{Sol}(C,G)\cap\text{Fix}(S)}(x^0)$.

For finding a particular solution of (2), Mainge [12] considered the following variational inequality problem:

$$\text{Find } x^* \in \Omega \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \forall x \in \Omega, \tag{4}$$

where $F : \mathcal{H} \longrightarrow \mathcal{H}$ is η-strongly monotone and κ-Lipschitz continuous on $\mathcal{H}$, $G : \mathcal{H} \longrightarrow \mathcal{H}$ is monotone on C and L-Lipschitz continuous on $\mathcal{H}$ and $S : \mathcal{H} \longrightarrow \mathcal{H}$ is γ-demicontractive and demi-closed at zero. Also in [12], Mainge proposed the following hybrid extragradient-viscosity method

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = P_C(x^n - \lambda_n G(x^n)), \\ z^n = P_C(x^n - \lambda_n G(y^n)), \\ t^n = z^n - \alpha_n F(z^n), \\ x^{n+1} = (1-\omega)t^n + \omega S(t^n), \end{cases} \tag{5}$$

where $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, $\{\alpha_n\} \subset [0, 1)$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\omega \in (0, \frac{1-\gamma}{2}]$. The author proved that the sequence $\{x^n\}$ generated by (5) converges strongly to the unique solution x^* of (4).

In this paper, inspired by the abovementioned works, we suggest a method for solving the bilevel variational inequalities with split variational inequality and fixed point problem constraints. To be specified, we suppose that $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitz continuous on $\mathcal{H}_1$; G is pseudomonotone on C, L-Lipschitz continuous on $\mathcal{H}_1$; S is γ-demicontractive and demi-closed at zero. The problem to be considered in this paper then can be formulated as

$$\text{Find } x^* \in \Omega \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \forall x \in \Omega, \tag{6}$$

where $\Omega = \{x^* \in \mathrm{Sol}(C, G) : Ax^* \in \mathrm{Fix}(S)\}$. Here, A is a bounded linear operator between $\mathcal{H}_1$ and $\mathcal{H}_2$.

The remaining part of the paper is organized as follows. In Section 2, we recall some basic definitions and preliminary results that are needed. The third section is devoted to the description of our proposed algorithm and its strong convergence result. Finally, in Section 4, we illustrate the proposed method by considering a simple numerical experiment.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. We write $x^n \rightharpoonup x$ to indicate that the sequence $\{x^n\}$ converges weakly to x while $x^n \longrightarrow x$ to indicate that the sequence $\{x^n\}$ converges strongly to x.

A point $x \in \mathcal{H}$ is a fixed point of a mapping $S : \mathcal{H} \longrightarrow \mathcal{H}$ provided $S(x) = x$. Denote by $\mathrm{Fix}(S)$ the set of fixed points of S, i.e., $\mathrm{Fix}(S) = \{x \in \mathcal{H} : S(x) = x\}$. By P_C, we denote the projection onto C. Namely, for each $x \in \mathcal{H}$, $P_C(x)$ is the unique element in C such that

$$\|x - P_C(x)\| \leq \|x - y\|, \forall y \in C.$$

Some important properties of the projection operator P_C are gathered in the following lemma.

Lemma 1 ([8])
(i) *For given $x \in \mathcal{H}$ and $y \in C$, $y = P_C(x)$ if and only if*

$$\langle x - y, z - y \rangle \leq 0, \forall z \in C.$$

(ii) P_C *is nonexpansive, that is,*

$$\|P_C(x) - P_C(y)\| \leq \|x - y\|, \forall x, y \in \mathcal{H}.$$

(iii) *For all* $x \in \mathcal{H}$ *and* $y \in C$, *we have*

$$\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|P_C(x) - x\|^2.$$

Let us also recall some well-known definitions which will be used in this paper.

Definition 1 ([7, 10]) A mapping $F : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be
(i) η-strongly monotone on $\mathcal{H}$ if there exists $\eta > 0$ such that

$$\langle F(x) - F(y), x - y\rangle \geq \eta\|x - y\|^2, \forall x, y \in \mathcal{H};$$

(ii) κ-Lipschitz continuous on $\mathcal{H}$ if

$$\|F(x) - F(y)\| \leq \kappa\|x - y\|, \forall x, y \in \mathcal{H};$$

(iii) Monotone on C if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \forall x, y \in C;$$

(iv) Pseudomonotone on C if

$$\langle F(y), x - y\rangle \geq 0 \Longrightarrow \langle F(x), x - y\rangle \geq 0, \forall x, y \in C.$$

Definition 2 A mapping $S : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be
(i) γ-demicontractive if $\text{Fix}(S) \neq \emptyset$ and and there exists a constant $\gamma \in [0, 1)$ such that

$$\|S(x) - x^*\|^2 \leq \|x - x^*\|^2 + \gamma\|S(x) - x\|^2 \text{ for all } x \in \mathcal{H}, x^* \in \text{Fix}(S);$$

(ii) Quasi-nonexpansive if S is 0-demicontractive, that is, $\text{Fix}(S) \neq \emptyset$ and

$$\|S(x) - x^*\| \leq \|x - x^*\| \text{ for all } x \in \mathcal{H}, x^* \in \text{Fix}(S);$$

(iii) Nonexpansive if

$$\|S(x) - S(y)\| \leq \|x - y\| \text{ for all } x, y \in \mathcal{H};$$

(iv) Demi-closed at zero if, for every sequence $\{x^n\}$ contained in $\mathcal{H}$, the following implication holds

$$x^n \rightharpoonup x \text{ and } \|S(x^n) - x^n\| \longrightarrow 0 \implies x \in \text{Fix}(S).$$

We observe that the class of demicontractive mappings contains quasi-nonexpansive mappings as a special case. Besides, the set of quasi-nonexpansive mappings contains class of nonexpansive mappings with fixed points and the nonexpansive mappings are well known to be demi-closed at zero. The class of demicontractive mappings has been studied by some authors because of its interesting properties and applications (see, for example, [2] and the references therein).

Definition 3 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator. An operator $A^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with the property

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$$

for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ is called an adjoint operator.

The adjoint operator of a bounded linear operator A between Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ always exists and is uniquely determined. Furthermore, A^* is a bounded linear operator and $\|A^*\| = \|A\|$.

The next lemmas will be used for proving the convergence of the proposed algorithm described below.

Lemma 2 ([12, Remark 4.4]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that for any integer m, there exists an integer p such that $p \geq m$ and $a_p \leq a_{p+1}$. Let n_0 be an integer such that $a_{n_0} \leq a_{n_0+1}$ and define, for all integer $n \geq n_0$, by*

$$\tau(n) = \max\{n \in \mathbb{N} : n_0 \leq k \leq n, a_n \leq a_{n+1}\}.$$

Then, $\{\tau(n)\}_{n \geq n_0}$ is a nondecreasing sequence satisfying $\lim_{n \longrightarrow \infty} \tau(n) = \infty$ and the following inequalities hold true:

$$a_{\tau(n)} \leq a_{\tau(n)+1}, \ a_n \leq a_{\tau(n)+1}, \forall n \geq n_0.$$

Lemma 3 ([15, 16]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\{t_n\}$ be a sequence of real numbers with $\limsup_{n \longrightarrow \infty} t_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n, \forall n \geq 0.$$

Then, $\lim_{n \longrightarrow \infty} s_n = 0$.

3 The Algorithm and Convergence Analysis

In this section, we propose a strong convergence algorithm for solving the problem (6). We impose the following assumptions on the mappings F, G, and S associated with the problem (6).

(A_F): $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is η-strongly monotone and κ-Lipschitz continuous on $\mathcal{H}_1$.
(A_{G_1}): $G : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ is pseudomonotone on C, L-Lipschitz continuous on $\mathcal{H}_1$.
(A_{G_2}): $\limsup_{n \longrightarrow \infty} \langle G(x^n), y - y^n \rangle \leq \langle G(\overline{x}), y - \overline{y} \rangle$ for every sequence $\{x^n\}$, $\{y^n\}$ in $\mathcal{H}_1$ converging weakly to $\overline{x}$ and $\overline{y}$, respectively.
(A_S): $S : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ is γ-demicontractive and demi-closed at zero.

Remark 1 (i) In finite-dimensional spaces, assumption (A_{G_2}) is automatically followed from the Lipschitz continuity of G.
(ii) If G satisfies the assumptions (A_{G_1}) and (A_{G_2}), then the solution set $\text{Sol}(C, G)$ of the variational inequality problem $VIP(C, G)$ is closed and convex (see, e.g., [14]). Moreover, if S satisfies the assumption (A_S), then the set of fixed points $\text{Fix}(S)$ of S is closed and convex (see, e.g., [13]). Therefore, the solution set $\Omega = \{x^* \in \text{Sol}(C, G) : Ax^* \in \text{Fix}(S)\}$ of the SVIFPP is also closed and convex.

The algorithm can be expressed as follows.

Algorithm 1

Step 0 Choose $\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset (0, \frac{1-\gamma}{\|A\|^2+1})$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$ such that $\lim_{n\longrightarrow\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Step 1 Let $x^0 \in \mathcal{H}_1$. Set $n := 0$.

Step 2 Compute $u^n = A(x^n)$ and

$$y^n = x^n + \omega_n A^*(S(u^n) - u^n).$$

Step 3 Compute

$$z^n = P_C(y^n - \lambda_n G(y^n)), \quad t^n = P_{T_n}(y^n - \lambda_n G(z^n)),$$

where

$$T_n = \{\omega \in \mathcal{H}_1 : \langle y^n - \lambda_n G(y^n) - z^n, \omega - z^n\rangle \leq 0\}.$$

Step 4 Compute

$$x^{n+1} = t^n - \alpha_n F(t^n).$$

Step 5 Set $n := n + 1$, and go to Step 2.

We are now in position to prove our main strong convergence results.

Theorem 1 *Suppose that the assumptions* $(\mathrm{A_F})$, $(\mathrm{A_{G_1}})$, $(\mathrm{A_{G_2}})$, *and* $(\mathrm{A_S})$ *hold. Then, the sequence* $\{x^n\}$ *generated by Algorithm 1 converges strongly to the unique solution of the problem* (6), *provided* $\Omega = \{x^* \in \mathrm{Sol}(C, G) : Ax^* \in \mathrm{Fix}(S)\}$ *is nonempty.*

Proof Since $\Omega \neq \emptyset$, the problem (6) has a unique solution, denoted by x^*. In particular, $x^* \in \Omega$, i.e., $x^* \in \mathrm{Sol}(C, G) \subset C$, $Ax^* \in \mathrm{Fix}(S)$. The proof of the theorem is divided into several steps.

Step 1 For all $n \geq 0$, we have

$$\|t^n - x^*\|^2 \leq \|y^n - x^*\|^2 - (1 - \lambda_n L)\|y^n - z^n\|^2 - (1 - \lambda_n L)\|z^n - t^n\|^2. \quad (7)$$

By the definition of z^n and Lemma 1, it follows that

$$\langle y^n - \lambda_n G(y^n) - z^n, z - z^n\rangle \leq 0, \forall z \in C.$$

Combining this inequality and the definition of T_n, we get $C \subset T_n$. Since $x^* \in \mathrm{Sol}(C, G)$ and $z^n \in C$, we have, in particular, $\langle G(x^*), z^n - x^*\rangle \geq 0$. Using the pseudomonotonicity on C of G, we get

$$\langle G(z^n), z^n - x^*\rangle \geq 0. \quad (8)$$

From $t^n = P_{T_n}(y^n - \lambda_n G(z^n))$, we have $t^n \in T_n$. This together with the definition of T_n implies

$$\langle y^n - \lambda_n G(y^n) - z^n, t^n - z^n\rangle \leq 0. \quad (9)$$

Since $x^* \in C$ and $C \subset T_n$, we get $x^* \in T_n$. Thus, using Lemma 1, (8), and (9), we obtain

$$\begin{aligned}
\|t^n - x^*\|^2 &= \|P_{T_n}(y^n - \lambda_n G(z^n)) - x^*\|^2 \\
&\le \|y^n - \lambda_n G(z^n) - x^*\|^2 - \|y^n - \lambda_n G(z^n) - t^n\|^2 \\
&= \|y^n - x^*\|^2 - \|y^n - t^n\|^2 + 2\lambda_n\langle x^* - t^n, G(z^n)\rangle \\
&= \|y^n - x^*\|^2 - \|y^n - t^n\|^2 - 2\lambda_n\langle G(z^n), z^n - x^*\rangle + 2\lambda_n\langle z^n - t^n, G(z^n)\rangle \\
&\le \|y^n - x^*\|^2 - \|y^n - t^n\|^2 + 2\lambda_n\langle z^n - t^n, G(z^n)\rangle \\
&= \|y^n - x^*\|^2 + 2\lambda_n\langle z^n - t^n, G(z^n)\rangle \\
&\quad -\|y^n - z^n\|^2 - \|z^n - t^n\|^2 - 2\langle z^n - t^n, y^n - z^n\rangle \\
&= \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - \|z^n - t^n\|^2 + 2\langle z^n - t^n, \lambda_n G(z^n) - y^n + z^n\rangle \\
&= \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - \|z^n - t^n\|^2 + 2\langle y^n - \lambda_n G(y^n) - z^n, t^n - z^n\rangle \\
&\quad +2\lambda_n\langle G(y^n) - G(z^n), t^n - z^n\rangle \\
&\le \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - \|z^n - t^n\|^2 + 2\lambda_n\langle G(y^n) - G(z^n), t^n - z^n\rangle.
\end{aligned} \tag{10}$$

Using the Cauchy-Schwarz inequality and arithmetic and geometric means inequality and observing that G is L-Lipschitz continuous on $\mathcal{H}_1$, we obtain

$$\begin{aligned}
2\langle G(y^n) - G(z^n), t^n - z^n\rangle &\le 2\|G(y^n) - G(z^n)\|\|t^n - z^n\| \\
&\le 2L\|y^n - z^n\|\|t^n - z^n\| \le L(\|y^n - z^n\|^2 + \|t^n - z^n\|^2).
\end{aligned}$$

It follows from the above inequality and (10) that

$$\begin{aligned}
\|t^n - x^*\|^2 &\le \|y^n - x^*\|^2 - \|y^n - z^n\|^2 - \|z^n - t^n\|^2 + \lambda_n L(\|y^n - z^n\|^2 + \|t^n - z^n\|^2) \\
&= \|y^n - x^*\|^2 - (1 - \lambda_n L)\|y^n - z^n\|^2 - (1 - \lambda_n L)\|z^n - t^n\|^2.
\end{aligned}$$

Step 2 For all $n \ge 0$, we show that

$$\|y^n - x^*\|^2 \le \|x^n - x^*\|^2 - \omega_n(1 - \gamma - \omega_n\|A\|^2)\|S(u^n) - u^n\|^2. \tag{11}$$

Using the equality

$$\langle x, y\rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x - y\|^2), \forall x, y \in \mathcal{H}_2$$

and the γ-demicontractivity of S, we have

$$\begin{aligned}
&\langle A(x^n - x^*), S(u^n) - u^n\rangle \\
&= \langle A(x^n - x^*) + S(u^n) - u^n - (S(u^n) - u^n), S(u^n) - u^n\rangle \\
&= \langle S(u^n) - A(x^*), S(u^n) - u^n\rangle - \|S(u^n) - u^n\|^2 \\
&= \frac{1}{2}\left[\|S(u^n) - A(x^*)\|^2 + \|S(u^n) - u^n\|^2 - \|u^n - A(x^*)\|^2\right] - \|S(u^n) - u^n\|^2 \\
&= \frac{1}{2}\left[(\|S(u^n) - A(x^*)\|^2 - \|u^n - A(x^*)\|^2) - \|S(u^n) - u^n\|^2\right] \\
&\le \frac{1}{2}\left[\gamma\|S(u^n) - u^n\|^2 - \|S(u^n) - u^n\|^2\right].
\end{aligned}$$

This implies that

$$2\omega_n\langle A(x^n - x^*), S(u^n) - u^n\rangle \le -\omega_n(1 - \gamma)\|S(u^n) - u^n\|^2. \tag{12}$$

Using (12), we have

$$\begin{aligned}\|y^n - x^*\|^2 &= \|(x^n - x^*) + \omega_n A^*(S(u^n) - u^n)\|^2 \\ &= \|x^n - x^*\|^2 + \|\omega_n A^*(S(u^n) - u^n)\|^2 + 2\omega_n \langle x^n - x^*, A^*(S(u^n) - u^n)\rangle \\ &\le \|x^n - x^*\|^2 + \omega_n^2 \|A^*\|^2 \|S(u^n) - u^n\|^2 + 2\omega_n \langle A(x^n - x^*), S(u^n) - u^n\rangle \\ &\le \|x^n - x^*\|^2 + \omega_n^2 \|A\|^2 \|S(u^n) - u^n\|^2 - \omega_n(1-\gamma)\|S(u^n) - u^n\|^2 \\ &= \|x^n - x^*\|^2 - \omega_n(1 - \gamma - \omega_n\|A\|^2)\|S(u^n) - u^n\|^2.\end{aligned}$$

Step 3 The sequences $\{x^n\}$, $\{y^n\}$, $\{z^n\}$, $\{t^n\}$, and $\{F(t^n)\}$ are bounded. Since $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, $\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset (0, \frac{1-\gamma}{\|A\|^2+1})$, by (7) and (11), we have

$$\|t^n - x^*\| \le \|y^n - x^*\| \le \|x^n - x^*\|. \tag{13}$$

Combining $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, the nonexpansiveness of P_C, the L-Lipschitz continuity on $\mathcal{H}_1$ of G, and (13), we obtain

$$\begin{aligned}\|z^n - x^*\| &= \|P_C(y^n - \lambda_n G(y^n)) - P_C(x^*)\| \\ &\le \|y^n - x^* - \lambda_n G(y^n)\| \\ &= \|y^n - x^* - \lambda_n(G(y^n) - G(x^*)) - \lambda_n G(x^*)\| \\ &\le \|y^n - x^*\| + \lambda_n\|G(y^n) - G(x^*)\| + \lambda_n\|G(x^*)\| \\ &\le \|y^n - x^*\| + \lambda_n L\|y^n - x^*\| + \lambda_n\|G(x^*)\| \\ &\le (1 + bL)\|y^n - x^*\| + b\|G(x^*)\| \\ &\le (1 + bL)\|x^n - x^*\| + b\|G(x^*)\|.\end{aligned} \tag{14}$$

Since F is κ-Lipschitz continuous and η-strongly monotone on $\mathcal{H}_1$, we have

$$\|F(t^n)\| \le \|F(t^n) - F(x^*)\| + \|F(x^*)\| \le \kappa\|t^n - x^*\| + \|F(x^*)\|, \tag{15}$$

and

$$\begin{aligned}&\|t^n - x^* - \mu(F(t^n) - F(x^*))\|^2 \\ &= \|t^n - x^*\|^2 - 2\mu\langle t^n - x^*, F(t^n) - F(x^*)\rangle + \mu^2\|F(t^n) - F(x^*)\|^2 \\ &\le \|t^n - x^*\|^2 - 2\mu\eta\|t^n - x^*\|^2 + \mu^2\kappa^2\|t^n - x^*\|^2 \\ &= [1 - \mu(2\eta - \mu\kappa^2)]\|t^n - x^*\|^2.\end{aligned} \tag{16}$$

Since $\lim_{n\longrightarrow\infty} \alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $\alpha_n < \mu$ for all $n \ge n_0$. So, from (16), we get, for all $n \ge n_0$

$$\begin{aligned}&\|t^n - \alpha_n F(t^n) - (x^* - \alpha_n F(x^*))\| \\ &= \left\| \left(1 - \frac{\alpha_n}{\mu}\right)(t^n - x^*) + \frac{\alpha_n}{\mu}[t^n - x^* - \mu(F(t^n) - F(x^*))] \right\| \\ &\le \left(1 - \frac{\alpha_n}{\mu}\right)\|t^n - x^*\| + \frac{\alpha_n}{\mu}\|t^n - x^* - \mu(F(t^n) - F(x^*))\| \\ &\le \left(1 - \frac{\alpha_n}{\mu}\right)\|t^n - x^*\| + \frac{\alpha_n}{\mu}\sqrt{1 - \mu(2\eta - \mu\kappa^2)}\|t^n - x^*\| \\ &= \left[1 - \frac{\alpha_n}{\mu}\left(1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}\right)\right]\|t^n - x^*\| \\ &= \left(1 - \frac{\alpha_n\tau}{\mu}\right)\|t^n - x^*\|,\end{aligned} \tag{17}$$

where

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1].$$

Using (17) and (13), we obtain, for all $n \geq n_0$

$$\begin{aligned}
\|x^{n+1} - x^*\| &= \|t^n - \alpha_n F(t^n) - x^*\| \\
&= \|t^n - \alpha_n F(t^n) - (x^* - \alpha_n F(x^*)) - \alpha_n F(x^*)\| \\
&\leq \|t^n - \alpha_n F(t^n) - (x^* - \alpha_n F(x^*))\| + \alpha_n \|F(x^*)\| \\
&\leq \left(1 - \frac{\alpha_n \tau}{\mu}\right) \|t^n - x^*\| + \alpha_n \|F(x^*)\| \qquad (18) \\
&\leq \left(1 - \frac{\alpha_n \tau}{\mu}\right) \|x^n - x^*\| + \alpha_n \|F(x^*)\| \\
&= \left(1 - \frac{\alpha_n \tau}{\mu}\right) \|x^n - x^*\| + \frac{\alpha_n \tau}{\mu} \frac{\mu \|F(x^*)\|}{\tau}. \qquad (19)
\end{aligned}$$

We obtain from (19) that

$$\|x^{n+1} - x^*\| \leq \max\left\{\|x^n - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\right\}, \forall n \geq n_0.$$

So, by induction, we obtain, for every $n \geq n_0$, that

$$\|x^n - x^*\| \leq \max\left\{\|x^{n_0} - x^*\|, \frac{\mu\|F(x^*)\|}{\tau}\right\}.$$

Hence, the sequence $\{x^n\}$ is bounded and so are the sequences $\{y^n\}$, $\{z^n\}$, $\{t^n\}$, and $\{F(t^n)\}$ thanks to (13), (14), and (15).

Step 4 We prove that $\{x^n\}$ converges strongly to x^*. Using the inequality

$$\|x - y\|^2 \leq \|x\|^2 - 2\langle y, x - y\rangle, \forall x, y \in \mathcal{H}_1,$$

(17) and (13), we obtain, for all $n \geq n_0$

$$\begin{aligned}
\|x^{n+1} - x^*\|^2 &= \|t^n - \alpha_n F(t^n) - x^*\|^2 \\
&= \|t^n - \alpha_n F(t^n) - (x^* - \alpha_n F(x^*)) - \alpha_n F(x^*)\|^2 \\
&\leq \|t^n - \alpha_n F(t^n) - (x^* - \alpha_n F(x^*))\|^2 - 2\alpha_n \langle F(x^*), x^{n+1} - x^*\rangle \\
&\leq \left[\left(1 - \frac{\alpha_n \tau}{\mu}\right) \|t^n - x^*\|\right]^2 - 2\alpha_n \langle F(x^*), x^{n+1} - x^*\rangle \\
&\leq \left(1 - \frac{\alpha_n \tau}{\mu}\right) \|t^n - x^*\|^2 - 2\alpha_n \langle F(x^*), x^{n+1} - x^*\rangle \qquad (20) \\
&\leq \left(1 - \frac{\alpha_n \tau}{\mu}\right) \|x^n - x^*\|^2 + 2\alpha_n \langle F(x^*), x^* - x^{n+1}\rangle. \qquad (21)
\end{aligned}$$

Let us consider two cases.

Case 1 There exists $n_* \in \mathbb{N}$ such that $\{\|x^n - x^*\|\}$ is decreasing for $n \geq n_*$. In that case, the limit of $\{\|x^n - x^*\|\}$ exists. So, it follows from (20) and (13), for all $n \geq n_0$, that

$$\begin{aligned}
&\frac{\alpha_n \tau}{\mu} \|t^n - x^*\|^2 + 2\alpha_n \langle F(x^*), x^{n+1} - x^*\rangle + (\|x^{n+1} - x^*\|^2 - \|x^n - x^*\|^2) \\
&\leq \|t^n - x^*\|^2 - \|x^n - x^*\|^2 \leq \|t^n - x^*\|^2 - \|y^n - x^*\|^2 \leq 0.
\end{aligned}$$

Since the limit of $\{\|x^n - x^*\|\}$ exists, $\lim_{n\longrightarrow\infty} \alpha_n = 0$, and $\{x^n\}$ and $\{t^n\}$ are two bounded sequences, it follows from the above inequality that

$$\lim_{n\longrightarrow\infty}(\|t^n - x^*\|^2 - \|x^n - x^*\|^2) = 0, \quad \lim_{n\longrightarrow\infty}(\|t^n - x^*\|^2 - \|y^n - x^*\|^2) = 0. \tag{22}$$

From (22), we get

$$\lim_{n\longrightarrow\infty}(\|x^n - x^*\|^2 - \|y^n - x^*\|^2) = 0. \tag{23}$$

From (11) and $\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset (0, \frac{1-\gamma}{\|A\|^2+1})$, we get

$$\underline{\omega}(1 - \gamma - \overline{\omega}\|A\|^2)\|S(u^n) - u^n\|^2 \leq \|x^n - x^*\|^2 - \|y^n - x^*\|^2. \tag{24}$$

Then, from (23) and (24), we obtain

$$\lim_{n\longrightarrow\infty}\|S(u^n) - u^n\| = 0. \tag{25}$$

Note that, for all n,

$$\|y^n - x^n\| = \|\omega_n A^*(S(u^n) - u^n)\| \leq \omega_n\|A^*\|\|S(u^n) - u^n\| \leq \overline{\omega}\|A\|\|S(u^n) - u^n\|.$$

It follows from the above inequality and (25) that

$$\lim_{n\longrightarrow\infty}\|y^n - x^n\| = 0. \tag{26}$$

From (7) and $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, we have

$$(1 - bL)\|y^n - z^n\|^2 + (1 - bL)\|z^n - t^n\|^2 \leq \|y^n - x^*\|^2 - \|t^n - x^*\|^2.$$

It follows from the above inequality and (22) that

$$\lim_{n\longrightarrow\infty}\|y^n - z^n\| = 0, \quad \lim_{n\longrightarrow\infty}\|z^n - t^n\| = 0. \tag{27}$$

From the triangle inequality, we get

$$\begin{aligned} \|y^n - t^n\| &\leq \|y^n - z^n\| + \|z^n - t^n\|, \\ \|x^n - t^n\| &\leq \|x^n - y^n\| + \|y^n - z^n\| + \|z^n - t^n\|, \end{aligned}$$

from which, by (26) and (27), it follows that

$$\lim_{n\longrightarrow\infty}\|y^n - t^n\| = 0, \quad \lim_{n\longrightarrow\infty}\|x^n - t^n\| = 0. \tag{28}$$

Now, we prove that

$$\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n\rangle \leq 0. \tag{29}$$

Take a subsequence $\{t^{n_k}\}$ of $\{t^n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n\rangle = \lim_{k\longrightarrow\infty}\langle F(x^*), x^* - t^{n_k}\rangle.$$

Since $\{t^{n_k}\}$ is bounded, we may assume that t^{n_k} converges weakly to some $\overline{t} \in \mathcal{H}_1$. Therefore,

$$\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n\rangle = \lim_{k\longrightarrow\infty}\langle F(x^*), x^* - t^{n_k}\rangle = \langle F(x^*), x^* - \overline{t}\rangle. \tag{30}$$

From $t^{n_k} \rightharpoonup \overline{t}$ and (27) and (28), we imply $z^{n_k} \rightharpoonup \overline{t}$, $y^{n_k} \rightharpoonup \overline{t}$, and $x^{n_k} \rightharpoonup \overline{t}$.

We now prove $\overline{t} \in \mathrm{Sol}(C, G)$. Indeed, let $x \in C$. It follows from the definition of z^{n_k} and Lemma 1 that

$$\langle y^{n_k} - \lambda_{n_k}G(y^{n_k}) - z^{n_k}, x - z^{n_k}\rangle \leq 0, \forall k \in \mathbb{N}.$$

Since $\lambda_{n_k} > 0$ for every $k \in \mathbb{N}$, it follows from the above inequality that

$$\langle G(y^{n_k}), x - z^{n_k}\rangle \geq \frac{\langle y^{n_k} - z^{n_k}, x - z^{n_k}\rangle}{\lambda_{n_k}}. \tag{31}$$

Since $\lim_{k\longrightarrow\infty} \|y^{n_k} - z^{n_k}\| = 0$, $\{\lambda_{n_k}\} \subset [a, b]$, and $\{z^{n_k}\}$ is bounded, we get

$$\lim_{k\longrightarrow\infty} \frac{\langle y^{n_k} - z^{n_k}, x - z^{n_k}\rangle}{\lambda_{n_k}} = 0.$$

So, using (31), condition (A_{G_2}), and the weak convergence of two sequences $\{y^{n_k}\}$, $\{z^{n_k}\}$ to $\bar{t}$, we have

$$0 \leq \limsup_{k\longrightarrow\infty}\langle G(y^{n_k}), x - z^{n_k}\rangle \leq \langle G(\bar{t}), x - \bar{t}\rangle,$$

i.e., $\bar{t} \in \mathrm{Sol}(C, G)$.

Next, we prove that $A(\bar{t}) \in \mathrm{Fix}(S)$. From $x^{n_k} \rightharpoonup \bar{t}$, we get $u^{n_k} = A(x^{n_k}) \rightharpoonup A(\bar{t})$. This together with (25) and the demiclosedness of S imply $A(\bar{t}) \in \mathrm{Fix}(S)$. In view of $\bar{t} \in \mathrm{Sol}(C, G)$, it implies that $\bar{t} \in \Omega$. Since x^* is the solution of problem (6), we have $\langle F(x^*), \bar{t} - x^*\rangle \geq 0$. Which together with (30) implies $\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n\rangle \leq 0$.

From the boundedness of $\{F(t^n)\}$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$, and (29), we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - x^{n+1}\rangle &= \limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n + \alpha_n F(t^n)\rangle\\ &= \limsup_{n\longrightarrow\infty}\left[\langle F(x^*), x^* - t^n\rangle + \alpha_n\langle F(x^*), F(t^n)\rangle\right]\\ &= \limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^n\rangle\\ &\leq 0.\end{aligned} \tag{32}$$

From (21), we get

$$\|x^{n+1} - x^*\|^2 \leq \left(1 - \frac{\alpha_n\tau}{\mu}\right)\|x^n - x^*\|^2 + \frac{\alpha_n\tau}{\mu}t_n, \quad \forall n \geq n_0, \tag{33}$$

where

$$t_n = \frac{2\mu\langle F(x^*), x^* - x^{n+1}\rangle}{\tau}.$$

Using (32), we get $\limsup_{n\longrightarrow\infty} t_n \leq 0$. From $0 < \alpha_n < \mu\ \forall n \geq n_0$ and $0 < \tau \leq 1$, we get $\{\frac{\alpha_n\tau}{\mu}\}_{n\geq n_0} \subset (0, 1)$. So, from (33), $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\limsup_{n\longrightarrow\infty} t_n \leq 0$, and Lemma 3, we have $\lim_{n\longrightarrow\infty}\|x^n - x^*\|^2 = 0$, i.e., $x^n \longrightarrow x^*$ as $n \longrightarrow \infty$.

Case 2 Suppose that for any integer m, there exists an integer n such that $n \geq m$ and $\|x^n - x^*\| \leq \|x^{n+1} - x^*\|$. According to Lemma 2, there exists a nondecreasing sequence $\{\tau(n)\}_{n\geq n_2}$ of $\mathbb{N}$ such that $\lim_{n\longrightarrow\infty}\tau(n) = \infty$ and the following inequalities hold:

$$\|x^{\tau(n)} - x^*\| \leq \|x^{\tau(n)+1} - x^*\|, \quad \|x^n - x^*\| \leq \|x^{\tau(n)+1} - x^*\|, \forall n \geq n_2. \tag{34}$$

Choose $n_3 \geq n_2$ such that $\tau(n) \geq n_0$ for all $n \geq n_3$. From (18), (34), and (13), we get, for all $n \geq n_3$

$$\begin{aligned}\frac{\alpha_{\tau(n)}\tau}{\mu}\|t^{\tau(n)} - x^*\| - \alpha_{\tau(n)}\|F(x^*)\| &\leq \|t^{\tau(n)} - x^*\| - \|x^{\tau(n)+1} - x^*\|\\ &\leq \|t^{\tau(n)} - x^*\| - \|x^{\tau(n)} - x^*\|\\ &\leq \|t^{\tau(n)} - x^*\| - \|y^{\tau(n)} - x^*\| \leq 0.\end{aligned}$$

Thus, from the boundedness of $\{t^n\}$ and $\lim_{n\longrightarrow\infty}\alpha_n = 0$, we have

$$\lim_{n\longrightarrow\infty}(\|t^{\tau(n)} - x^*\| - \|x^{\tau(n)} - x^*\|) = 0, \quad \lim_{n\longrightarrow\infty}(\|t^{\tau(n)} - x^*\| - \|y^{\tau(n)} - x^*\|) = 0. \quad (35)$$

From (35) and the boundedness of $\{x^n\}$, $\{y^n\}$, and $\{t^n\}$, we obtain

$$\lim_{n\longrightarrow\infty}(\|t^{\tau(n)} - x^*\|^2 - \|x^{\tau(n)} - x^*\|^2) = 0, \quad \lim_{n\longrightarrow\infty}(\|t^{\tau(n)} - x^*\|^2 - \|y^{\tau(n)} - x^*\|^2) = 0.$$

Arguing similarly as in the first case, we can conclude that

$$\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^{\tau(n)}\rangle \leq 0.$$

Then, the boundedness of $\{F(t^n)\}$ and $\lim_{n\longrightarrow\infty}\alpha_n = 0$ yield

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - x^{\tau(n)+1}\rangle &= \limsup_{n\longrightarrow\infty}\left\langle F(x^*), x^* - t^{\tau(n)} + \alpha_{\tau(n)}F(t^{\tau(n)})\right\rangle \\ &= \limsup_{n\longrightarrow\infty}\left[\langle F(x^*), x^* - t^{\tau(n)}\rangle + \alpha_{\tau(n)}\langle F(x^*), F(t^{\tau(n)})\rangle\right] \\ &= \limsup_{n\longrightarrow\infty}\langle F(x^*), x^* - t^{\tau(n)}\rangle \\ &\leq 0. \end{aligned} \quad (36)$$

From (21) and (34), we have, for all $n \geq n_3$

$$\begin{aligned}\|x^{\tau(n)+1} - x^*\|^2 &\leq \left(1 - \frac{\alpha_{\tau(n)}\tau}{\mu}\right)\|x^{\tau(n)} - x^*\|^2 + 2\alpha_{\tau(n)}\langle F(x^*), x^* - x^{\tau(n)+1}\rangle \\ &\leq \left(1 - \frac{\alpha_{\tau(n)}\tau}{\mu}\right)\|x^{\tau(n)+1} - x^*\|^2 + 2\alpha_{\tau(n)}\langle F(x^*), x^* - x^{\tau(n)+1}\rangle.\end{aligned}$$

In particular, since $\alpha_{\tau(n)} > 0$

$$\|x^{\tau(n)+1} - x^*\|^2 \leq \frac{2\mu}{\tau}\langle F(x^*), x^* - x^{\tau(n)+1}\rangle, \forall n \geq n_3.$$

From (34) and the above inequality, we get

$$\|x^n - x^*\|^2 \leq \frac{2\mu}{\tau}\langle F(x^*), x^* - x^{\tau(n)+1}\rangle, \forall n \geq n_3. \quad (37)$$

Taking the limit in (37) as $n \longrightarrow \infty$, and using (36), we obtain that

$$\limsup_{n\longrightarrow\infty}\|x^n - x^*\|^2 \leq 0,$$

which implies $x^n \longrightarrow x^*$. This completes the proof of the theorem. □

Applying Theorem 1 and Algorithm 1 when $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ and A is the identity mapping in $\mathcal{H}$, we obtain the following result for problem (4).

Corollary 1 *Let $F : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone and Lipschitz continuous on $\mathcal{H}$; $S : \mathcal{H} \longrightarrow \mathcal{H}$ be γ-demicontractive, demi-closed at zero; and $G : \mathcal{H} \longrightarrow \mathcal{H}$ be pseudomonotone on C, L-Lipschitz continuous on $\mathcal{H}$,* $\limsup_{n\longrightarrow\infty}\langle G(x^n), y - y^n\rangle \leq \langle G(\overline{x}), y - \overline{y}\rangle$

for every sequence $\{x^n\}$, $\{y^n\}$ *in* $\mathcal{H}$ *converging weakly to* $\overline{x}$ *and* $\overline{y}$, *respectively. Suppose that* $\mathrm{Sol}(C, G) \cap \mathrm{Fix}(S) \neq \emptyset$. *Let* $\{x^n\}$ *be the sequence generated by*

$$\begin{cases} x^0 \in \mathcal{H}, \\ y^n = (1-\omega_n)x^n + \omega_n S(x^n), \\ z^n = P_C(y^n - \lambda_n G(y^n)), \\ T_n = \{\omega \in \mathcal{H} : \langle y^n - \lambda_n G(y^n) - z^n, \omega - z^n \rangle \leq 0\}, \\ t^n = P_{T_n}(y^n - \lambda_n G(z^n)), \\ x^{n+1} = t^n - \alpha_n F(t^n), \end{cases}$$

where the sequences $\{\omega_n\}$, $\{\lambda_n\}$, *and* $\{\alpha_n\}$ *satisfy the following conditions:*
(i) $\{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset (0, \frac{1-\gamma}{2})$;
(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$;
(iii) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, $\{x^n\}$ *converges strongly to a solution* $x^* \in \mathrm{Sol}(C, G) \cap \mathrm{Fix}(S)$, *where* x^* *is the unique solution of the following variational inequality problem*

$$\text{Find } x^* \in \mathrm{Sol}(C, G) \cap \mathrm{Fix}(S) \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0, \forall x \in \mathrm{Sol}(C, G) \cap \mathrm{Fix}(S).$$

When $G = 0$, $S = P_Q$, and $F = I_{\mathcal{H}_1}$, we have the following corollary from Theorem 1 and Algorithm 1.

Corollary 2 *Let C and Q be two nonempty closed convex subsets of two real Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$, *respectively. Suppose that positive sequences* $\{\alpha_n\}$, $\{\omega_n\}$ *satisfy the following conditions:*

$$\begin{cases} \{\alpha_n\} \subset (0, 1), \lim\limits_{n\longrightarrow\infty} \alpha_n = 0, \sum\limits_{n=0}^{\infty} \alpha_n = \infty, \\ \{\omega_n\} \subset [\underline{\omega}, \overline{\omega}] \subset \left(0, \dfrac{1}{\|A\|^2 + 1}\right). \end{cases}$$

Let $\{x^n\}$ *be the sequence defined by* $x^0 \in \mathcal{H}_1$ *and*

$$x^{n+1} = (1-\alpha_n)P_C(x^n + \omega_n A^*(P_Q(Ax^n) - Ax^n)), \forall n \geq 0.$$

Then, the sequence $\{x^n\}$ *converges strongly to the minimum-norm solution of the split feasibility problem, provided that the solution set* $\Gamma = \{x^* \in C : Ax^* \in Q\}$ *of the split feasibility problem is nonempty.*

4 Numerical Results

Example 1 Let $\mathcal{H}_1 = \mathbb{R}^4$ with the norm $\|x\| = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $\mathcal{H}_2 = \mathbb{R}^2$ with the norm $\|y\| = (y_1^2 + y_2^2)^{\frac{1}{2}}$ for $y = (y_1, y_2)^T \in \mathbb{R}^2$. Consider the mapping $F : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $F(x) = x$ for all $x \in \mathbb{R}^4$. It is easy to see that F is strongly monotone with $\eta = 1$ and Lipschitz continuous with $\kappa = 1$ on $\mathbb{R}^4$. In this case, the problem (6) becomes the problem of finding the minimum-norm solution of the SVIFPP.

Let $A(x) = (x_1 + x_3 + x_4, x_2 + x_3 - x_4)^T$ for all $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ then A is a bounded linear operator from $\mathbb{R}^4$ into $\mathbb{R}^2$ with $\|A\| = \sqrt{3}$. For $y = (y_1, y_2)^T \in \mathbb{R}^2$, let $B(y) = (y_1, y_2, y_1 + y_2, y_1 - y_2)^T$, then B is a bounded linear operator from $\mathbb{R}^2$ into $\mathbb{R}^4$

with $\|B\| = \sqrt{3}$. Moreover, for any $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ and $y = (y_1, y_2)^T \in \mathbb{R}^2$, $\langle A(x), y\rangle = \langle x, B(y)\rangle$, so $B = A^*$ is an adjoint operator of A.

Let

$$C = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 \geq 9\}$$

and define a mapping $G : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ by $G(x) = (\sin\|x\| + 2)a^0$ for all $x \in \mathbb{R}^4$, where $a^0 = (12, -4, 4, -4)^T \in \mathbb{R}^4$. It is easy to verify that G is pseudomonotone on $\mathbb{R}^4$.

Furthermore, for all $x, y \in \mathbb{R}^4$, we have

$$\begin{aligned}\|G(x) - G(y)\| &= \|a^0\||\sin\|x\| - \sin\|y\|| = 8\sqrt{3}|\sin\|x\| - \sin\|y\||\\ &\leq 8\sqrt{3}|\|x\| - \|y\|| \leq 8\sqrt{3}\|x - y\|.\end{aligned}$$

So G is $8\sqrt{3}$-Lipschitz continuous on $\mathbb{R}^4$.

It is easy to see that the solution set $\mathrm{Sol}(C, G)$ of $VIP(C, G)$ is given by

$$\mathrm{Sol}(C, G) = \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 = 9\}.$$

Assume that $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by, for all $y = (y_1, y_2)^T \in \mathbb{R}^2$

$$S(y) = \begin{cases}(y_1, y_2)^T & \text{if } y_1 \leq 0,\\ (-2y_1, y_2)^T & \text{if } y_1 > 0.\end{cases}$$

Clearly, $\mathrm{Fix}(S) = (-\infty, 0] \times \mathbb{R}$. To see that S is $\frac{1}{3}$-demicontractive, observe that if $y^* = (y_1^*, y_2^*)^T \in \mathrm{Fix}(S)$, then

$$\|S(y) - y^*\|^2 \leq \|y - y^*\|^2 + \frac{1}{3}\|S(y) - y\|^2 \text{ for all } y = (y_1, y_2)^T \in \mathbb{R}^2. \tag{38}$$

If $y_1 \leq 0$ then $S(y) = y$. Thus, (38) holds.

For $y_1 > 0$, we have

$$\begin{aligned}\|y - y^*\|^2 + \frac{1}{3}\|S(y) - y\|^2 &= \|(y_1 - y_1^*, y_2 - y_2^*)^T\|^2 + \frac{1}{3}\|(-2y_1, y_2)^T - (y_1, y_2)^T\|^2\\ &= (y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 + \frac{1}{3}\|(-2y_1 - y_1, 0)^T\|^2\\ &= (y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 + 3y_1^2\\ &= 4y_1^2 - 2y_1y_1^* + (y_1^*)^2 + (y_2 - y_2^*)^2\\ &\geq 4y_1^2 + 4y_1y_1^* + (y_1^*)^2 + (y_2 - y_2^*)^2\\ &= (-2y_1 - y_1^*)^2 + (y_2 - y_2^*)^2\\ &= \|(-2y_1, y_2)^T - (y_1^*, y_2^*)^T\|^2 = \|S(y) - y^*\|^2.\end{aligned}$$

Now we prove that S is demi-closed at zero. Suppose that $\{z^n = (x^n, y^n)^T\} \subset \mathbb{R}^2$, $z^n \longrightarrow z = (x, y)^T$, and $\lim_{n\longrightarrow\infty}\|S(z^n) - z^n\| = 0$. Thus, $\lim_{n\longrightarrow\infty} x^n = x$ and $\lim_{n\longrightarrow\infty} y^n = y$. It is clear that $S(z^n) = (g(x^n), y^n)^T$, where

$$g(y_1) = \begin{cases}y_1 & \text{if } y_1 \leq 0,\\ -2y_1 & \text{if } y_1 > 0.\end{cases}$$

Therefore, $\lim_{n\longrightarrow\infty}|g(x^n) - x^n| = 0$. If $x > 0$, since $\lim_{n\longrightarrow\infty} x^n = x$ then there exists $n_0 \geq 0$ such that $x^n > 0$ for all $n \geq n_0$. Thus for all $n \geq n_0$, $|g(x^n) - x^n| = |-2x^n - x^n| = 3x^n$. So $\lim_{n\longrightarrow\infty}|g(x^n) - x^n| = 3\lim_{n\longrightarrow\infty} x^n = 3x$. Combine with $\lim_{n\longrightarrow\infty}|g(x^n) - x^n| = 0$, we get $x = 0$, which contradicts to $x > 0$. Thus, $x \leq 0$, so $z = (x, y)^T \in (-\infty, 0] \times \mathbb{R} = \mathrm{Fix}(S)$. So S is demi-closed at zero.

Table 1 Algorithm 1 for Example 1, with different starting points, $\omega_n = \frac{n+1}{8n+10}$, $\lambda_n = \frac{n+1}{16n+18}$, $\alpha_n = \frac{1}{n+2}$, and tolerance $\varepsilon = 10^{-7}$

Starting point	Iter (n)	CPU time(s)	x^n
$(0.6, -0.2, 0.1, -0.4)^T$	3235	1.2480	$(0.49987, -0.24989, 0.00007, -0.49973)^T$
$(-1, 3, 2, 1)^T$	5525	2.1372	$(0.49990, -0.24954, 0.00028, -0.50006)^T$
$(-3, -5, -6, -4)^T$	7093	2.6832	$(0.49991, -0.25055, -0.00020, -0.49961)^T$
$(12, 30, -25, 21)^T$	20,694	7.1916	$(0.50095, -0.24879, -0.00133, -0.49959)^T$
$(-94, 70, 142, -356)^T$	57,075	18.5485	$(0.49838, -0.24788, 0.00429, -0.50266)^T$

The solution set Ω of the SVIFPP is given by

$$\begin{aligned}\Omega &= \{(x_1, x_2, x_3, x_4)^T \in \mathrm{Sol}(C, G) : A(x_1, x_2, x_3, x_4) \in \mathrm{Fix}(S)\}\\ &= \{(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 : 12x_1 - 4x_2 + 4x_3 - 4x_4 = 9, x_1 + x_3 + x_4 \le 0\}.\end{aligned}$$

Suppose $x = (x_1, x_2, x_3, x_4)^T \in \Omega$ then

$$\begin{aligned}\|x\| &= \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\\ &= \sqrt{\frac{(2x_1-1)^2}{4} + \frac{(4x_2+1)^2}{16} + x_3^2 + \frac{(2x_4+1)^2}{4} + \frac{1}{8}(12x_1 - 4x_2 + 4x_3 - 4x_4) - \frac{1}{2}(x_1 + x_3 + x_4) - \frac{9}{16}}\\ &= \sqrt{\frac{(2x_1-1)^2}{4} + \frac{(4x_2+1)^2}{16} + x_3^2 + \frac{(2x_4+1)^2}{4} + \frac{9}{8} - \frac{1}{2}(x_1 + x_3 + x_4) - \frac{9}{16}}\\ &= \sqrt{\frac{(2x_1-1)^2}{4} + \frac{(4x_2+1)^2}{16} + x_3^2 + \frac{(2x_4+1)^2}{4} - \frac{1}{2}(x_1 + x_3 + x_4) + \frac{9}{16}} \ge \frac{3}{4}.\end{aligned}$$

The above equality holds if and only if $x_1 = \frac{1}{2}$, $x_2 = -\frac{1}{4}$, $x_3 = 0$, and $x_4 = -\frac{1}{2}$. Therefore, the minimum-norm solution x^* of the SVIFPP is $x^* = (\frac{1}{2}, -\frac{1}{4}, 0, -\frac{1}{2})^T$. We choose $\omega_n = \frac{n+1}{8n+10}$, $\lambda_n = \frac{n+1}{16n+18}$, and $\alpha_n = \frac{1}{n+2}$. An elementary computation shows that $\{\omega_n\} \subset [\frac{1}{10}, \frac{1}{8}] \subset (0, \frac{1}{6}) = (0, \frac{1-\gamma}{\|A\|^2+1})$, $\{\lambda_n\} \subset [\frac{1}{18}, \frac{1}{16}] \subset (0, \frac{1}{8\sqrt{3}}) = (0, \frac{1}{L})$, $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\longrightarrow\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Tables 1 and 2 present the numerical results of Algorithm 1 with different starting points and different tolerances.

From the preliminary numerical results reported in the tables, we observe that the running time of Algorithm 1 depends very much on the initial point and the tolerance.

Table 2 Algorithm 1 for Example 1, with different tolerances, $\omega_n = \frac{n+1}{8n+10}$, $\lambda_n = \frac{n+1}{16n+18}$, $\alpha_n = \frac{1}{n+2}$, and starting point $x^0 = (-2, 3, 5, -4)^T$

Tolerance	Iter (n)	CPU time(s)	x^n
$\varepsilon = 10^{-6}$	2719	3.4164	$(0.49915, -0.24874, 0.00204, -0.50095)^T$
$\varepsilon = 10^{-7}$	8598	5.8032	$(0.49973, -0.24960, 0.00065, -0.50030)^T$
$\varepsilon = 10^{-8}$	27,189	9.8749	$(0.49991, -0.24987, 0.00020, -0.50009)^T$
$\varepsilon = 10^{-9}$	85,980	28.0178	$(0.49997, -0.24996, 0.00006, -0.50003)^T$
$\varepsilon = 10^{-10}$	271,891	122.4452	$(0.49999, -0.24999, 0.00002, -0.50001)^T$

We perform the iterative schemes in MATLAB R2018a running on a laptop with Intel(R) Core(TM) i5-3230M CPU @ 2.60GHz, 4 GB RAM.

5 Conclusion

In this paper, we presented a method for solving strongly monotone variational inequality problems with split variational inequality and fixed point problem constraints. As a consequence, we have obtained an algorithm for finding a common solution to a variational inequality with pseudomonotone mapping and a fixed point problem involving demicontractive mapping. When applied to the split feasibility problem, our method is reduced to a strongly convergent algorithm, which requires only two projections at each iteration step.

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