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Singularities and Perfectoid Geometry

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Abstract
These notes give a quick overview of recent work by the speaker and of some subsequent works introducing perfectoid geometry into homological commutative algebra and singularity theory. The emphasis is on big Cohen-Macaulay algebras and applications. The progresses take place primarily in mixed characteristic, but sometimes provide a bridge between characteristic p and characteristic 0.

Keywords Big Cohen-Macaulay algebra · Perfectoid algebra · Singularities

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1 Singularities: Resolution Versus Homological Study

1.1 Introduction

In his second letter to Leibniz (1677), Newton answered his correspondent's query about the foundation of his methods in this way: "the foundation is evident enough, in fact; but because I cannot proceed with the explanation of it now, I have preferred to conceal it thus: 6accdae13eff7i3l9n4o4qrr4s8t12ux".

This anagram was decoded as: *Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa*[1], which is usually grossly translated as:

It is useful to solve differential equations.

History has largely done justice to this watchword. But some three centuries later, another watchword emerged:

It is (also) useful not to solve differential equations... but to study their structure.

Proponents of this approach were Grothendieck who, reinterpreting the classical resolvent as descent datum, gave birth to the crystalline viewpoint on differential modules;

[1] Given an equation involving any number of fluent quantities to find the fluxions, and vice versa.

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and the Kyoto school (Sato, Kashiwara) who, treating solutions and cosolutions on equal footing, gave birth to the homological study of differential modules known as algebraic analysis.

This pattern "resolution versus homological study" also occurs in the algebraic theory of singularities.

Viewed as nuisances, one tries to get rid of them, which leads to the theory of resolution of singularities. Viewed as jewels of commutative algebra, gardening and classifying them leads to the homological study of singularities.

1.2 Cohen-Macaulay Versus Non-Cohen-Macaulay Singularities

This is a natural dichotomy: Cohen-Macaulay (CM) singularities form a rather broad class with lots of interesting examples from various origin, and they share many nice properties: they satisfy Serre duality, they allow concrete calculations of syzygies, etc.

Let us remind their definition, and their homological characterization (reference: [6]).

We adopt the viewpoint and language of commutative algebra, where spaces are replaced by the commutative rings of their functions, locality being reflected by the uniqueness of the maximal ideal. Let S be a (Noetherian) local ring with maximal ideal $\mathfrak{m}$. A sequence $\underline{x} = (x_1, \ldots, x_n)$ of elements of $\mathfrak{m}$ is *secant* if $S/\underline{x}S$ has dimension $\dim S - n$.

Let M be an S-module. A sequence $\underline{x}$ is *M-regular* if

$$M/(x_1, \ldots, x_{i-1})M \overset{\cdot x_i}{\to} M/(x_1, \ldots, x_{i-1})M$$

is injective ($i = 1, \ldots, n$), and $M \neq \underline{x}M$.

M is a *Cohen-Macaulay module* if any secant sequence is M-regular.

S is a *Cohen-Macaulay ring* if any secant sequence is S-regular. This is in particular the case if S is *regular*, i.e., if $\mathfrak{m}$ is generated by some secant sequence. Moreover, *if S is regular, an S-module M is Cohen-Macaulay if and only if it is faithfully flat* [16, 2.1.d].

One has the following characterizations:

[Auslander-Serre] S is regular $\Leftrightarrow$ every finite S-module has a finite free resolution $\Leftrightarrow$ $S/\mathfrak{m}$ has a finite free resolution.

[Peskine-Szpiro-Roberts] S is Cohen-Macaulay $\Leftrightarrow$ some nonzero S-module of *finite length* has a finite free resolution.

1.3 Big Cohen-Macaulay Algebras

What to do in front of a non-Cohen-Macaulay ring S?

A first attitude (initiated by Faltings) is to try to get rid of the problem, finding a *Cohen-Macaulay resolution*. Such a weak resolution of singularities is now known to exist in great generality:

Theorem 1.3.1 (Kawasaki [18], Cesnavicius [8]) *Let S be a quasi-excellent noetherian ring. There exists a projective morphism $Y \to X = \mathrm{Spec} S$ with Y CM, which is an isomorphism over the CM locus of X.*

A second attitude, prompted by Hochster, is to look for *(big) CM algebras*: namely, an S-algebra T (not necessarily of finite type) which is a CM S-module.

In the first approach (CM resolution), any secant sequence on Y (i.e., in the local rings of Y) is regular, but a secant sequence on X needs not remain secant on Y.

In the second approach, any secant sequence on X becomes regular on $Y = \mathrm{Spec} T$, but a secant sequence on Y needs not be regular.

Big CM algebras turn out to be very important tools in the homological study of singularities (and surprisingly, even in the study of a CM singularity, see Sections 2.2 and 3.2 below).

2 Big Cohen-Macaulay Algebras in Mixed Characteristic via Perfectoid Theory

2.1 Existence and Weak Functoriality

The main result is the following:

Theorem 2.1.1 [2, 4] *(Big) Cohen-Macaulay algebras exist, and are weakly functorial.*

More precisely: 1) *for any complete local ring S, there is a CM S-algebra T,*

2) *For any chain of local homomorphisms $S_1 \to \cdots \to S_n$ of complete local domains, there is a compatible chain $T_1 \to \cdots \to T_n$ of CM algebras for $S_1, \ldots, S_n$ respectively.*

It turns out that 1) is equivalent to the following more geometric statement:

1') *For any regular ring R and any finite extension S, there is an S-algebra T which is faithfully flat over R.*

Both 1) and 2) were conjectured by Hochster and proved by him in equal characteristic (partly with Huneke).

2.2 Three Consequences of Theorem 2.1.1 in Commutative Algebra

The direct summand conjecture [Hochster '69]: any finite extension S of a regular ring R splits (as R-module).

Another direct summand conjecture: any ring S which is a direct summand (as S-module) of a regular ring R is Cohen-Macaulay.

The syzygy conjecture [Evans-Griffiths '81]: any n-th syzygy module of a finite module M of projective dimension $> n$ has rank $\geq n$.

We refer to [2–4, 14] and to the earlier papers [6, Chapter 9][10, 15, 16] for this circle of ideas.

2.3 The Role of Perfectoids in the Proof

In mixed characteristic, Theorem 2.1.1 is proved using "deep ramification" techniques; more precisely, using the theory of perfectoid spaces introduced by Scholze [22].

Let us review the basic notions and constructions used in the proof.

2.3.1 Perfectoid Valuation Rings

This is a complete, non discrete valuation ring K^o of mixed characteristic $(0, p)$, such that for some (equivalently: any) $\varpi \in K^o$ with $p \in \varpi^p K^o$, *the Frobenius map* $F : K^o/\varpi \overset{x \mapsto x^p}{\to} K^o/\varpi^p$ *is an isomorphism*[2].

[2]It turns out that this property is equivalent to deep ramification: $\Omega_{\bar{K}^o/K^o} = 0$ (Gabber-Ramero).

Example 2.3.1 Let $W(k)$ be the Witt ring of a perfect residue field k. Then $K^o = W(k)\langle p^{1/p^\infty}\rangle$ is a perfectoid valuation ring (one can take $\varpi = p^{1/p}$).

2.3.2 Perfectoid K^o-algebras

These are p-adically complete, p-torsionfree K^o-algebras A such that *the Frobenius map* $F : A/\varpi \overset{x\mapsto x^p}{\to} A/\varpi^p$ *is an isomorphism.*

Using non-archimedean geometry, one can attach to them some spaces, and by glueing, construct the geometry of perfectoid spaces. This geometry will not appear explicitly in the sequel: it is hidden in the proof of the next two theorems. Nor will appear the fundamental tilting equivalence relating (perfectoid) geometry over K^o and over the valuation ring $K^{\flat o} := \lim_F K^o/\varpi K^o$ of characteristic p (also hidden in the proofs).

Example 2.3.2 Let K^o be as in the previous example. The p-adic completion A of $\cup_i W(k)[p^{1/p^i}][[x_1^{1/p^i}, \dots, x_n^{1/p^i}]]$ is a perfectoid K^o-algebra.

One can construct further perfectoid algebras by adjoining p^∞-roots of elements of A in a suitable sense:

Theorem 2.3.3 [2] *(improved in* [12, Section 16.9]*) The completed p-root closure*[3] *of* $A[g^{1/p^\infty}]$ *is perfectoid and faithfully flat over* A*.*

2.3.3 Almost Algebra

(Faltings, Gabber-Ramero [11]): given a commutative ring $\mathfrak{V}$ and an idempotent ideal $\mathfrak{I}$, almost algebra systematically "neglects" $\mathfrak{V}$-modules which are killed by $\mathfrak{I}$ (almost-zero modules). A morphism is an almost-isomorphism if its kernel and cokernel are almost-zero, etc. Whereas this is a special case of Gabriel localization, many notions of almost algebra go well beyond and are not of categorical nature: almost finite, almost etale, etc.

The usual set-up in the perfectoid theory is $(\mathfrak{V}, \mathfrak{I}) = (K^o, p^{\frac{1}{p^\infty}} K^o)$. However, for our constructions, we need to work with the following non-valuative set-up:

$$(\mathfrak{V}, \mathfrak{I}) = (K^o[t^{1/p^\infty}], (pt)^{\frac{1}{p^\infty}} K^o[t^{1/p^\infty}]).$$

One obtains the notion of almost perfectoid algebra A (containing a compatible sequence of p-power roots of some $g \in A$) on considering A as a $K^o[t^{1/p^\infty}]$-algebra ($t^{1/p^i} \mapsto g^{1/p^i}$) and relaxing the condition on Frobenius (only required to be an almost-isomorphism). Similarly, one obtains the notion of almost CM algebra A (containing a compatible sequence of p-power roots of some $g \in A$) on interpreting the injectivity and inequality in the definition in the almost sense.

2.3.4 Perfectoid Abhyankar Lemma

The classical Abhyankar lemma asserts that under appropriate assumptions (tameness...), a ramified extension can be made etale by adjoining roots of the discriminant, rather than inverting it.

[3]The p-root closure of a p-adic ring R: elements r of $R[1/p]$ such that $r^{p^j} \in R$ for some $j > 0$.

There is an analog for finite ramified extensions of perfectoid algebras:

Theorem 2.3.4 [1] *Let A be a perfectoid $K^o[t^{1/p^\infty}]$-algebra, such that the image g of t in A is a nonzero divisor.*

Let B' be a finite etale $A[1/pg]$-algebra, and let B be the integral closure of A in B'.

Then B is $(pt)^{\frac{1}{p^\infty}}$-almost perfectoid, and for any $n > 0$, B/p^n is $(pt)^{\frac{1}{p^\infty}}$-almost faithfully flat and almost finite etale over A/p^n.

2.3.5 Application of These Tools to the Construction of CM Algebras

Let us come back to our complete local domain S of characteristic $(0, p)$. For simplicity, we assume that its residue field k is perfect. We want to construct a (big) CM S-algebra.

By Cohen's theorem, we may view S as a finite extension of some regular ring of the form $R = W(k)[[\underline{x}]]$.

Then an S-algebra is a CM S-algebra if and only if it is faithfully flat over R.

Let $g \in R$ be such that $S[1/pg]$ finite etale over $R[1/pg]$.

Let us take $K^o = W(k)\langle p^{1/p^\infty}\rangle$, and $A_0 = \hat{\cup}_i W(k)[p^{1/p^i}][[x_1^{1/p^i}, \dots, x_n^{1/p^i}]]$ be as in the above examples (A_0 is a perfectoid K^o-algebra).

Let A be completed p-root closure of $A_0[g^{1/p^\infty}]$: this is a perfectoid and faithfully flat R-algebra by Theorem 2.3.3.

Let $B' := A[1/pg] \otimes_R S$, which is finite etale extension of $A[1/pg]$.

It then follows from Theorem 2.3.4 that the integral closure B of A in B' is an *almost* perfectoid *almost* CM S-algebra.

How to get rid of "almost"?

There are two ways: 1) Hochster's modifications (which predate our work) [16].

2) Gabber's trick: replacing B by $\Sigma^{-1}(B^{\mathbb{N}}/B^{(\mathbb{N})})$, where Σ is the multiplicative system $(pg)^{\epsilon_i}$, $\epsilon_i \to 0 \in \mathbb{N}[1/p]$, one gets a genuine perfectoid CM S-algebra [12, Section 17.5].

Weak functoriality uses similar techniques, but is more difficult. Here is a strong version:

Theorem 2.3.5 [4] *Any finite sequence $R_0 \overset{f_1}{\to} R_1 \overset{f_2}{\to} \cdots \overset{f_n}{\to} R_n$ of local homomorphisms of complete Noetherian local domains, with R_0 of mixed characteristic, fits into a commutative diagram*

$$\begin{array}{ccccccc}
R_0 & \xrightarrow{f_1} & R_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_n} & R_n \\
\downarrow & & \downarrow & & & & \downarrow \\
R_0^+ & \xrightarrow{f_1^+} & R_1^+ & \xrightarrow{f_2^+} & \cdots & \xrightarrow{f_n^+} & R_n^+ \\
\downarrow & & \downarrow & & & & \downarrow \\
C_0 & \longrightarrow & C_1 & \longrightarrow & \cdots & \longrightarrow & C_n
\end{array} \tag{2.1}$$

where

R_i^+ is the absolute integral closure of R_i,

C_i is a perfectoid CM R_i-algebra if R_i is of mixed characteristic (resp. a perfect CM R_i-algebra if R_i is of positive characteristic). Moreover, the f_i^+ can be given in advance.

3 Applications to Singularities

3.1 Symbolic Powers

Let S be a Noetherian ring, $\mathfrak{p}$ a prime ideal, and $V(\mathfrak{p})$ the corresponding subvariety of Spec$S$. Symbolic powers are defined by

$$\mathfrak{p}^{(n)} := (\mathfrak{p}^n S_{\mathfrak{p}}) \cap \mathfrak{p}.$$

If S is an algebra of finite type over a field of characteristic 0, $\mathfrak{p}^{(n)} =$ is the ideal of functions which vanish at $V(\mathfrak{p})$ at order at least n (Zariski).

One has $\mathfrak{p}^{(n)} \supset \mathfrak{p}^n$, and $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if $\mathfrak{p}$ is generated by a regular sequence (or more generally if $\mathrm{gr}_{\mathfrak{p}} S$ is a domain).

The comparison of $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^m$, and its extension to non-prime ideals, is a classical problem (the containment problem), which has applications in complex analysis, interpolation theory (fat points) and transcendental number theory (Waldschmidt constants).

Theorem 3.1.1 [19] *Let S be an excellent regular ring of dimension d.*
Then for any prime $\mathfrak{p}$ and any n, $\mathfrak{p}^{(dn)} \subset \mathfrak{p}^n$.

This was proved by Ein-Lazarsfeld-Smith in characteristic 0, using subadditivity of the "multiplier ideal", and by Hochster in characteristic p. In mixed characteristic, Ma and Schwede use a new notion of multiplier ideal in which the complex $R\Gamma(Y, \mathcal{O}_Y)$ attached to a resolution of $V(\mathfrak{p})$ is replaced by a perfectoid Cohen-Macaulay algebra for $S/\mathfrak{p}$.

Let us explain this last point in general terms: let now S be a local domain, essentially of finite type over $\mathbb{C}$, and let $\pi : Y \to \mathrm{Spec} S$ be a (log-)resolution of singularities. By Grauert-Riemenschneider, $R^i\Gamma(Y, \omega_Y) = 0$ for $i > 0$, whence, by local duality: $\mathbb{H}^j_{\mathfrak{m}}(R\Gamma(Y, \mathcal{O}_Y)) = 0$ for $j < \dim S$. Thus $R\Gamma(Y, \mathcal{O}_Y) \in D^b(S)$ appears as a "derived avatar" of a CM algebra. In mixed characteristic or in characteristic p, one can replace the missing $R\Gamma(Y, \mathcal{O}_Y)$ by a suitable a (big) Cohen-Macaulay S-algebra.

3.2 Rational Singularities

Remind that S (as before) "is" a *rational singularity* if and only if $R\Gamma(Y, \mathcal{O}_Y) \cong S$. By Grauert-Riemenschneider and local duality, any rational singularity is CM.

Question How to check that a singularity is rational without computing a resolution?

A criterion by reduction mod. p, after spreading out, has been known for some time (Hara, Smith, Mehta-Srinivas): S rational singularity $\Leftrightarrow$ (S mod. p) F-rational singularity for all $p >> 0$.

For a Cohen-Macaulay singularity in characteristic p, F-rationality has several equivalent definitions (e.g., the top local cohomology is a simple Frobenius module), and can be checked algorithmically (Macaulay2). But the above criterion requires all $p >> 0$.[4]

Using perfectoid Cohen-Macaulay S-algebras, Ma and Schwede have shown that it suffices to check F-rationality for some p (such that spreading out and reduction mod. p make sense), which leads to an effective criterion of rationality:

[4] Or at least, in the Smith version, for some p big enough, but with an ineffective lower bound.

Theorem 3.2.1 [20, Section 8] *S rational singularity* $\Leftrightarrow$ (*S mod. p*) *F-rational singularity for some p*.

This may be viewed as an application of p-adic techniques to effective complex algebraic geometry. However, perfectoid CM algebras do not appear in the statement: they are hidden in the proof that the algorithm works, in the passage from characteristic p to characteristic 0 through mixed characteristic.

There are similar results for specific classes of rational singularities (log-terminal singularities).

Rational singularities were introduced by Artin in dimension 2. In the context of their work on rational singularities in mixed characteristic, Ma and Schwede, together with Carvajal-Rosas, Polstra and Tucker, came back to the case of dimension 2 and extended Artin's theory of (Gorenstein complete) *rational singularities S of dimension 2* from characteristic p to *mixed characteristic* $(0, p > 5)$ (with separably closed residue field). The classification is the same:

They admit a split finite regular extension $S \subset R$. They are "Du Val singularities", i.e., of the form $S = R'/fR'$, where R' is a 3-dimensional complete regular ring and f can be written in one of the forms

$$\begin{array}{ll} x^2 + y^2 + z^{n+1} & (A_n) \\ x^2 + y^2 z + z^{n-1} & (D_n) \\ x^2 + y^3 + z^4 & (E_6) \\ x^2 + y^3 + yz^3 & (E_7) \\ x^2 + y^3 + z^5 & (E_8), \end{array}$$

the label referring to the graph of a minimal resolution of the double point. One difference with the equicharacteristic case is that the label does not determine S, due to the fact that the parameter p is special, being fixed by automorphisms of S: choosing $p = x$, y or z may lead to different singularities with the same label.

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