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Calderón's Problem for Some Classes of Conductivities in Circularly Symmetric Domains

Mai Thi Kim Dung[1] **· Dang Anh Tuan**[1]

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Abstract
In this note, we study Calderón's problem for certain classes of conductivities in domains with circular symmetry in two and three dimensions. Explicit formulas are obtained for the reconstruction of the conductivity from the Dirichlet-to-Neumann map. As a consequence, we show that the reconstruction is Lipschitz stable.

Keywords Inverse boundary problems · Dirichlet-to-Neumann map · Calderón problem · Lipschitz stability · Reconstruction

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1 Introduction

Consider a conductor in a domain $\Omega \subset \mathbb{R}^n$ with conductivity $\gamma(x)$. When a voltage potential $f \in H^{\frac{1}{2}}(\partial\Omega)$ is applied at the boundary $\partial\Omega$, the induced potential u in Ω is the unique weak solution in $H^1(\Omega)$ of

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet-to-Neumann map is given by $\Lambda_\gamma(f) = \gamma \partial_\nu u|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$. Here, ν denotes the exterior unit normal to $\partial\Omega$. The problem studied by Calderón in [8] is to determine the conductivity γ from Λ_γ.

✉ Dang Anh Tuan
datuan1105@gmail.com

Mai Thi Kim Dung
maithikimdung_t59@hus.edu.vn

[1] Department of Mathematics, University of Science, Vietnam National University, Hanoi, Vietnam

For $n \geq 3$ and $\gamma \in C^2$, that Λ_γ uniquely determines γ was proved by Sylvester and Uhlmann in [16]. Recently, based on the breakthrough work by Haberman and Tataru [13], Caro and Rogers [9] proved uniqueness for Lipschitz conductivities. There is also a related work by Haberman [12].

In two dimensions, and C^2 conductivities, the uniqueness was proved by Nachman [15]. Later, Astala and Päivärinta [4] proved uniqueness for bounded measurable conductivities.

After the uniqueness has been established, it is natural to study the stability of the reconstruction, i.e., we would like to estimate $\gamma_1 - \gamma_2$ in certain norm by

$$||\Lambda_1 - \Lambda_2||_\star = \sup_{\substack{f \in H^{\frac{1}{2}}(\partial\Omega) \\ f \neq 0}} \frac{||(\Lambda_1 - \Lambda_2) f||_{H^{-\frac{1}{2}}(\partial\Omega)}}{||f||_{H^{\frac{1}{2}}(\partial\Omega)}}.$$

In [1], Alessandrini proved that the following log-stability estimate holds:

$$||\gamma_1 - \gamma_2||_{L^\infty(\Omega)} \leq C\big(\log(1 + ||\Lambda_1 - \Lambda_2||_\star^{-1})\big)^{-\sigma},$$

where C, σ are positive constants and $\gamma_j \in H^{s+2}(\Omega), s > n/2$. Later, Mandache [14] showed that such estimate is optimal.

To improve the stability estimate, Alessandrini and Vessella [3] considered special classes of piecewise constant conductivities, for $n \geq 3$. The Lipschitz stability obtained therein has been generalized to other classes of conductivities in [2, 7] and [11].

The analog of the result of [3] was proven for the two-dimensional case in [5]. Subsequent generalizations of this result were obtained in [6] and [10].

In this paper, we prove Lipschitz stability estimate for two special cases of domains with circular symmetry. In the first case, we consider $\Omega = B(0, 1) \subset \mathbb{R}^2$ with conductivities of the form

$$\gamma_\alpha(x) = \begin{cases} \alpha_1 + \alpha_2(a - r) & \text{if } 0 \leq r < a, \\ \alpha_0 & \text{if } a \leq r < 1, \end{cases}$$

where $r = |x|$ and $\varepsilon_0 \leq \alpha_0, \alpha_1 \leq M$, $0 \leq \alpha_2 \leq N$. We denote this set of conductivities by $\mu(a, \varepsilon_0, M, N)$. In the second case, we consider $\Omega = B(0, 1) \times (0, +\infty) \subset \mathbb{R}^3$ with conductivities of the form

$$\gamma_\alpha(z) = \begin{cases} 1 + \alpha_1 & \text{if } h \leq z < \infty, \\ 1 + \alpha_2 & \text{if } 0 \leq z < h, \end{cases}$$

where $\alpha_j \in [0, M], j = 1, 2, M > 0, h > 0$. We denote this set of conductivities by $\mu(h, M)$. We give a formula for the Dirichlet-to-Neumann map in each case, together with a formula to recover the conductivity from the Dirichlet-to-Neumann map. As a consequence, we show that the map $\Lambda_\gamma \mapsto \gamma$ is Lipschitz. More precisely, our main results are as follows.

Theorem 1.1 *Let* $\Omega = B(0, 1)$ *and* $a \in (0, 1), \varepsilon_0, M > 0, N \geq 0$. *There exists a positive constant* $C = C(a, \varepsilon_0, M, N)$ *such that*

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star \geq C\left(\left|\alpha_0 - \beta_0\right| + \left|\alpha_1 - \beta_1\right| + \left|\alpha_2 - \beta_2\right|\right), \forall \gamma_\alpha, \gamma_\beta \in \mu(a, \varepsilon_0, M, N).$$

Theorem 1.2 *Let $\Omega = B(0,1) \times (0,\infty)$ and $h \in (0,\infty)$, $M > 0$. There exists a positive constant $C = C(h, M)$ such that*

$$\left\| \Lambda_\alpha - \Lambda_\beta \right\|_{H^{\frac{1}{2}}_{rad}(B) \to H^{-\frac{1}{2}}_{rad}(B)} \geq C \left(\left|\alpha_2 - \beta_2\right| + \left|\alpha_1 - \beta_1\right| \right), \forall \gamma_\alpha, \gamma_\beta \in \mu(h, M).$$

2 Proof of Theorem 1.1

Consider the Dirichlet problem in the unit disc $B = B(0,1)$ on the plane

$$\begin{cases} \nabla \cdot (\gamma_\alpha \nabla u) = 0 & \text{in } B, \\ u = f & \text{on } \partial B, \end{cases} \tag{2.1}$$

where the conductivity $\gamma_\alpha \in \mu(a, \varepsilon_0, M, N)$. In the polar coordinate, if $u(x) = \sum_{n\in\mathbb{Z}} u_n(r) e^{in\theta} \in H^1(B)$, then the equation in (2.1) is

$$\begin{cases} (\gamma_\alpha u_n')' + \dfrac{\gamma_\alpha}{r} u_n' - \dfrac{n^2 \gamma_\alpha}{r^2} u_n = 0, \forall n \in \mathbb{Z}, \\ \lim\limits_{r\to a^-} u_n(r) = \lim\limits_{r\to a^+} u_n(r), \\ \lim\limits_{r\to a^-} \left(\gamma u_n'\right)(r) = \lim\limits_{r\to a^+} \left(\gamma u_n'\right)(r). \end{cases}$$

Solving these systems, we obtain

$$u_0(r) = c_0, \quad 0 \leq r < 1,$$

and for $n \neq 0$,

$$u_n(r) = \begin{cases} b_n r^{|n|} + c_n r^{-|n|} & \text{if } a \leq r < 1, \\ \sum\limits_{k\geq |n|} a_k r^k & \text{if } 0 < r < a, \end{cases}$$

where

$$\begin{aligned} a_{|n|+m} &= \frac{\alpha_2}{\alpha_1 + a\alpha_2} \frac{(2m-1)\,|n| + m(m-1)}{2m\,|n| + m^2} a_{|n|+m-1} \\ &= \left(\frac{\alpha_2}{\alpha_1 + a\alpha_2} \right)^m \prod_{j=1}^m \frac{(2j-1)\,|n| + j(j-1)}{2j\,|n| + j^2} a_{|n|}, m = 1, 2, \cdots. \end{aligned}$$

Note that $\alpha_2 \geq 0, \alpha_1 \geq \epsilon_0 > 0$, the power series $\sum_{k\geq|n|} a_k r^k$ is uniformly convergent on $[0, a]$. From that, we get

$$\frac{c_n}{b_n} = \frac{a^{2|n|} \left[|n|\, u_n(a^-) - a \frac{\alpha_1}{\alpha_0} u_n'(a^-) \right]}{|n|\, u_n(a^-) + a \frac{\alpha_1}{\alpha_0} u_n'(a^-)}.$$

The Dirichlet-to-Neumann map $\Lambda_\alpha : H^{\frac{1}{2}}(\partial B) \to H^{-\frac{1}{2}}(\partial B)$ is determined by

$$\Lambda_\alpha f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{\Lambda_\alpha f}(n) e^{in\theta},$$

where $f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{in\theta} \in H^{\frac{1}{2}}(\partial B)$ and

$$\widehat{\Lambda_\alpha f}(n) = \lim_{r\to 1^-} \gamma_\alpha(r)u_n'(r) = \alpha_0 |n| \widehat{f}(n)\frac{b_n - c_n}{b_n + c_n}$$
$$= \alpha_0 |n| \widehat{f}(n)\frac{1 - a^{2|n|} + (1 + a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)}{1 + a^{2|n|} + (1 - a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)}, \forall n \in \mathbb{Z},$$

$$B_n(b) = 1 + \frac{b}{2|n|+1} \times \frac{1 + \sum\limits_{m=2}^{\infty} mb^{m-1}h_{m,n}}{1 + \frac{|n|}{2|n|+1}b\left(1 + \sum\limits_{m=2}^{\infty} b^{m-1}h_{m,n}\right)},$$

$$h_{m,n} = \prod_{j=2}^{m} \frac{(2j-1)|n| + j(j-1)}{2j|n| + j^2}, \quad b = \frac{a\alpha_2}{\alpha_1 + a\alpha_2}.$$

Note that $0 \le b \le b_0 = \frac{aN}{\varepsilon_0 + aN} < 1$. To obtain some properties of $B_n(b)$, we need the following technical lemma.

Lemma 2.1 (i) $\sum_{m=2}^{\infty} mb^{m-1}\prod_{j=2}^{m}\frac{2j-1}{2j} = (1-b)^{-\frac{3}{2}} - 1$.
(ii) $\sum_{m=2}^{\infty} b^{m-1}\prod_{j=2}^{m}\frac{2j-1}{2j} = \frac{2}{1-b+\sqrt{1-b}} - 1$.
(iii) $\lim_{n\to\infty} h_{mn} = \prod_{j=2}^{m}\frac{2j-1}{2j}$.

We have the following proposition

Proposition 2.2 *B_n's satisfy the following properties.*
(i) $1 \le B_n(b) \le d_0$, *where* $d_0 = 1 + \frac{b_0}{(1-b_0)^{\frac{3}{2}}}$.
(ii) $\lim_{n\to\infty} B_n(b) = 1$.
(iii) $\lim_{n\to\infty}(2|n|+1)(B_n(b) - 1) = \frac{b}{1-b}$.
(iv) $\lim_{n\to\infty} \frac{\frac{\alpha_1}{\alpha_0}B_n(b)-1}{\frac{\alpha_1}{\alpha_0}B_{n+1}(b)-1} = 1, b \ne 0$.
(v) $\frac{1-b_0}{2|n|+1} \le B_n'(b) \le \frac{A}{2|n|+1}$, *where* $A = A(a, \varepsilon_0, N)$ *is a constant.*

Proof We rewrite $B_n(b)$ as follows

$$B_n(b) = 1 + \frac{b}{2|n|+1} \times \frac{1 + \sum\limits_{m=2}^{\infty} mb^{m-1}h_{m,n}}{1 + \frac{|n|}{2|n|+1}b\left(1 + \sum\limits_{m=2}^{\infty} b^{m-1}h_{m,n}\right)}. \tag{2.2}$$

(i) From (2.2), it is easy to see that $B_n(b) \ge 1$. We now show that $B_n(b) \le d_0$. Indeed, using (i) in Lemma 2.2, we have

$$B_n(b) = \frac{1 + \frac{|n|+1}{2|n|+1}b + \sum\limits_{m=2}^{\infty}\frac{|n|+m}{2|n|+1}b^m h_{m,n}}{1 + \frac{|n|}{2|n|+1}b + \sum\limits_{m=2}^{\infty}\frac{|n|}{2|n|+1}b^m h_{m,n}} \le 1 + b_0 + \sum_{m=2}^{\infty} mb_0^m \prod_{j=2}^{m}\frac{2j-1}{2j} = d_0.$$

(ii) From (2.2), it is not difficult to get $\lim_{n\to\infty} B_n(b) = 1$.

(iii) We have

$$(2|n|+1)(B_n(b)-1)=\frac{b\left(1+\sum\limits_{m=2}^{\infty} mb^{m-1}h_{m,n}\right)}{1+\frac{|n|}{2|n|+1}b\left(1+\sum\limits_{m=2}^{\infty} b^{m-1}h_{m,n}\right)}.$$

Hence, from Lemma 2.1, we obtain

$$\lim_{n\to\infty}(2|n|+1)(B_n(b)-1)=\frac{b(1-b)^{-\frac{3}{2}}}{(1-b)^{-\frac{1}{2}}}=\frac{b}{1-b}.$$

(iv) We consider two cases

Case 1 $\alpha_0 \neq \alpha_1$.
From (ii), we have

$$\lim_{n\to\infty}\frac{\frac{\alpha_1}{\alpha_0}B_n(b)-1}{\frac{\alpha_1}{\alpha_0}B_{n+1}(b)-1}=\frac{\frac{\alpha_1}{\alpha_0}-1}{\frac{\alpha_1}{\alpha_0}-1}=1.$$

Case 2 $\alpha_0=\alpha_1$.
We need to prove $\lim_{n\to\infty}\frac{B_n(b)-1}{B_{n+1}(b)-1}=1$. From (iii), we get

$$\lim_{n\to\infty}\frac{B_n(b)-1}{B_{n+1}(b)-1}=\lim_{n\to\infty}\frac{(2|n|+1)(B_n(b)-1)}{(2|n|+3)(B_{n+1}(b)-1)}=1.$$

(v) We denote by $M_n(b)$ and $N_n(b)$ the numerator and denominator of $B_n'(b)$, respectively. Direct computation gives

$$M_n(b)=\frac{1}{2|n|+1}+\sum_{m=2}^{\infty}\frac{|n|(m-1)^2}{(2|n|+1)^2}b^m h_{m,n}+\sum_{m=2}^{\infty}\frac{m^2}{2|n|+1}b^{m-1}h_{m,n}+I_n(b),$$

where

$$\begin{aligned}I_n(b)\ =&\left(\sum_{l=2}^{\infty}\frac{l^2b^{l-1}}{2|n|+1}h_{l,n}\right)\left(\sum_{k=2}^{\infty}\frac{|n|\,b^k}{2|n|+1}h_{k,n}\right)-\left(\sum_{l=2}^{\infty}\frac{l\,|n|\,b^{l-1}}{2|n|+1}h_{l,n}\right)\\&\times\left(\sum_{k=2}^{\infty}\frac{kb^k}{2|n|+1}h_{k,n}\right).\end{aligned}$$

The coefficient of b^m in $I_n(b)$ is

$$\begin{aligned}\sum_{k+l-1=m}\left(\frac{l^2|n|}{(2|n|+1)^2}-\frac{l\,|n|\,k}{(2|n|+1)^2}\right)h_{l,n}h_{k,n}\ &=\sum_{k+l-1=m}\frac{l\,|n|\,(l-k)}{(2|n|+1)^2}h_{l,n}h_{k,n}\\&=\frac{1}{2}\sum_{k+l-1=m}\frac{|n|\,(k-l)^2}{(2|n|+1)^2}h_{l,n}h_{k,n}.\end{aligned}$$

From this, we obtain

$$M_n(b)\geq\frac{1}{2|n|+1}. \tag{2.3}$$

Moreover, we have

$$\begin{aligned}\sum_{m=2}^{\infty}\frac{|n|\,(m-1)^2}{(2\,|n|+1)^2}b^m h_{m,n} &\le \frac{1}{2(2\,|n|+1)}\sum_{m=0}^{\infty}m^2 b^m\\ &= \frac{1}{2(2\,|n|+1)}\frac{b(b+1)}{(1-b)^3}\le\frac{1}{2(2\,|n|+1)}\frac{b_0(b_0+1)}{(1-b_0)^3}.\end{aligned}\tag{2.4}$$

Next, we have

$$\sum_{m=2}^{\infty}\frac{m^2}{2\,|n|+1}b^{m-1}h_{m,n}\le\frac{1}{2(2\,|n|+1)}\sum_{m=0}^{\infty}m^2 b^{m-1}\le\frac{1}{2(2\,|n|+1)}\frac{b_0+1}{(1-b_0)^3}.\tag{2.5}$$

We see that

$$\frac{1}{2}\sum_{k+l-1=m}\frac{|n|\,(k-l)^2}{(2\,|n|+1)^2}h_{l,n}h_{k,n}\le\frac{1}{4(2\,|n|+1)}(m+1)^3,\, m=3,4,\ldots$$

It follows that

$$\begin{aligned}I_n(b) &\le \frac{1}{4(2\,|n|+1)}\sum_{m=3}^{\infty}(m+1)^3 b^m\le\frac{1}{4(2\,|n|+1)}\sum_{m=0}^{\infty}(m+1)^3 b^m\\ &\le \frac{b_0}{4\,(2\,|n|+1)}\left(\frac{2b_0+1}{(1-b_0)^3}+\frac{3b_0(b_0+1)}{(1-b_0)^4}\right).\end{aligned}\tag{2.6}$$

From (2.3), (2.4), (2.5), and (2.6), we deduce that

$$\frac{1}{2\,|n|+1}\le M_n(b)\le\frac{A}{2\,|n|+1},\quad A \text{ is a constant depending on } a, \varepsilon_0, N.\tag{2.7}$$

On the other hand, we have

$$1\le N_n(b)=\left(1+\frac{|n|}{2\,|n|+1}b+\sum_{m=2}^{\infty}\frac{|n|}{2\,|n|+1}b^m h_{m,n}\right)^2\le\frac{1}{1-b}\le\frac{1}{1-b_0}.\tag{2.8}$$

From (2.7) and (2.8), we have

$$\frac{1-b_0}{2\,|n|+1}\le B_n'(b)\le\frac{A}{2\,|n|+1},$$

where A is a constant depending on a, ε_0, N. □

We now give an explicit formula to reconstruct the parameters a and α from the Dirichlet-to-Neumann map. We define

$$C_n=\frac{\Lambda_\alpha(e^{in\theta})}{|n|\,e^{in\theta}}=\alpha_0\frac{1-a^{2|n|}+(1+a^{2|n|})\frac{\alpha_1}{\alpha_0}B_n(b)}{1+a^{2|n|}+(1-a^{2|n|})\frac{\alpha_1}{\alpha_0}B_n(b)}.$$

If there is a strictly increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $C_{n_k}=\alpha_0$, it is easy to obtain $\alpha_0=\alpha_1, \alpha_2=0$; i.e., the conductor is homogeneous. Otherwise, we have the following proposition.

Proposition 2.3 *The following formulas hold:*
(i) $\alpha_0=\lim_{n\to\infty}C_n$.

(ii) $a^{-2} = \lim_{n\to\infty} \frac{C_n-\alpha_0}{C_{n+1}-\alpha_0}$.
(iii) $\alpha_1 = \alpha_0 D$, *where*

$$D = \frac{\lim\limits_{n\to\infty} \frac{C_n-\alpha_0}{2a^{2|n|}\alpha_0} + 1}{1 - \lim\limits_{n\to\infty} \frac{C_n-\alpha_0}{2a^{2|n|}\alpha_0}}.$$

(iv) $a\alpha_2 = \alpha_1 E$, *where*

$$E = \lim_{n\to\infty} (2\,|n| + 1)\left[\frac{\alpha_0\left(2\alpha_0 a^{2|n|} + (C_n-\alpha_0)(1+a^{2|n|})\right)}{\alpha_1\left(2\alpha_0 a^{2|n|} - (C_n-\alpha_0)(1-a^{2|n|})\right)} - 1\right].$$

Proof (i) From (ii) in Proposition 2.2

$$\lim_{n\to\infty} C_n = \alpha_0.$$

(ii) Next, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{C_n-\alpha_0}{C_{n+1}-\alpha_0} \\
&= \lim_{n\to\infty} \frac{2a^{2|n|}\left(\frac{\alpha_1}{\alpha_0}B_n(b)-1\right)}{2a^{2|n|+2}\left(\frac{\alpha_1}{\alpha_0}B_{n+1}(b)-1\right)} \frac{1+a^{2|n|+2}+(1-a^{2|n|+2})\frac{\alpha_1}{\alpha_0}B_{n+1}(b)}{1+a^{2|n|}+(1-a^{2|n|})\frac{\alpha_1}{\alpha_0}B_n(b)}.
\end{aligned}$$

Using (ii) and (iv) in Proposition 2.2, we obtain

$$\lim_{n\to\infty} \frac{C_n-\alpha_0}{C_{n+1}-\alpha_0} = \frac{1}{a^2}.$$

(iii) Using (ii) in Proposition 2.2, we have

$$\lim_{n\to\infty} \frac{C_n-\alpha_0}{2a^{2|n|}\alpha_0} = \lim_{n\to\infty} \frac{\frac{\alpha_1}{\alpha_0}B_n(b)-1}{1+a^{2|n|}+(1-a^{2|n|})\frac{\alpha_1}{\alpha_0}B_n(b)} = \frac{\frac{\alpha_1}{\alpha_0}-1}{\frac{\alpha_1}{\alpha_0}+1}.$$

This leads to

$$\alpha_1 = \frac{\alpha_0\left(\lim\limits_{n\to\infty} \frac{C_n-\alpha_0}{2a^{2|n|}\alpha_0}+1\right)}{1-\lim\limits_{n\to\infty} \frac{C_n-\alpha_0}{2a^{2|n|}\alpha_0}}.$$

(iv) We now calculate α_2. From

$$C_n - \alpha_0 = \alpha_0 \frac{2a^{2|n|}\left(\frac{\alpha_1}{\alpha_0}B_n(b)-1\right)}{1+a^2|n|+(1-a^2|n|)\frac{\alpha_1}{\alpha_0}B_n(b)}$$

we calculate

$$B_n(b) = \frac{\alpha_0\left(2\alpha_0 a^{2|n|} + (C_n-\alpha_0)(1+a^{2|n|})\right)}{\alpha_1\left(2\alpha_0 a^{2|n|} - (C_n-\alpha_0)(1-a^{2|n|})\right)}.$$

From that and (iii) in Proposition 2.2, we get

$$\begin{aligned}
E &= \lim_{n\to\infty} (2\,|n| + 1)\left[\frac{\alpha_0\left(2\alpha_0 a^{2|n|} + (C_n-\alpha_0)(1+a^{2|n|})\right)}{\alpha_1\left(2\alpha_0 a^{2|n|} - (C_n-\alpha_0)(1-a^{2|n|})\right)} - 1\right] \\
&= \lim_{n\to\infty} (2\,|n| + 1)(B_n(b) - 1) = \frac{b}{1-b}.
\end{aligned}$$

From $b = \frac{a\alpha_2}{\alpha_1 + a\alpha_2}$, we obtain $a\alpha_2 = \alpha_1 E$. □

We now prove Theorem 1.1.

Proof of Theorem 1.1 For $\gamma_\alpha, \gamma_\beta \in \mu(a, \varepsilon_0, M, N)$, $f \in H^{\frac{1}{2}}(\partial B)$, we have

$$\left\| \left(\Lambda_\alpha - \Lambda_\beta\right) f \right\|^2_{H^{-\frac{1}{2}}(\partial B)} = \sum_{n\in\mathbb{Z}} \frac{n^2}{(1+n^2)^{\frac{1}{2}}} (A_n - B_n)^2 \left|\widehat{f}(n)\right|^2,$$

where $b = a\alpha_2/(\alpha_1 + a\alpha_2)$, $c = a\beta_2/(\beta_1 + a\beta_2)$, and

$$A_n = \alpha_0 \frac{1 - a^{2|n|} + (1 + a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)}{1 + a^{2|n|} + (1 - a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)},\ B_n = \beta_0 \frac{1 - a^{2|n|} + (1 + a^{2|n|})\frac{\beta_1}{\beta_0} B_n(c)}{1 + a^{2|n|} + (1 - a^{2|n|})\frac{\beta_1}{\beta_0} B_n(c)}.$$

By direct computation, we obtain

$$\begin{aligned} &A_n - B_n \\ &= \frac{(\alpha_0 + \beta_0)\left(\frac{\alpha_1}{\alpha_0} B_n(b) - \frac{\beta_1}{\beta_0} B_n(c)\right) 2a^{2|n|}}{\left(1 + a^{2|n|} + (1 - a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)\right)\left(1 + a^{2|n|} + (1 - a^{2|n|})\frac{\beta_1}{\beta_0} B_n(c)\right)} \\ &\quad + \frac{(\alpha_0 - \beta_0)\left[(1 - a^{4|n|})\left(1 + \frac{\alpha_1}{\alpha_0}\frac{\beta_1}{\beta_0} B_n(b)B_n(c)\right) + (1 + a^{4|n|})\left(\frac{\alpha_1}{\alpha_0} B_n(b) + \frac{\beta_1}{\beta_0} B_n(c)\right)\right]}{\left(1 + a^{2|n|} + (1 - a^{2|n|})\frac{\alpha_1}{\alpha_0} B_n(b)\right)\left(1 + a^{2|n|} + (1 - a^{2|n|})\frac{\beta_1}{\beta_0} B_n(c)\right)}. \end{aligned}$$

We denote by K_n and H_n the numerator and denominator of $A_n - B_n$, respectively. We have $H_n \le (2 + \frac{M}{\varepsilon_0} d_0)^2$ and

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star = \sup_{\substack{f \in H^{\frac{1}{2}}(\partial B) \\ f \neq 0}} \frac{\left\|\left(\Lambda_\alpha - \Lambda_\beta\right) f\right\|_{H^{-\frac{1}{2}}(\partial B)}}{\|f\|_{H^{\frac{1}{2}}(\partial B)}} \ge \sup_{n\neq 0} \frac{|K_n|}{|2H_n|} \ge \sup_{n\neq 0} \frac{|K_n|}{2\left(2 + \frac{M}{\varepsilon_0} d_0\right)^2},$$

$$|K_n| \ \ge\ \frac{2\varepsilon_0}{M} \left|\alpha_0 - \beta_0\right| - \frac{8M^2 d_0}{\varepsilon_0} a^{2|n|}.$$

When $\alpha_0 \neq \beta_0$, for n big enough, we obtain

$$\frac{8M^2 d_0}{\varepsilon_0} a^{2|n|} \le \frac{\varepsilon_0}{M} \left|\alpha_0 - \beta_0\right|.$$

Hence,

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star \ge \frac{\varepsilon_0}{2M}\left(2 + \frac{M}{\varepsilon_0} d_0\right)^{-2} \left|\alpha_0 - \beta_0\right|. \tag{2.9}$$

For $\alpha_0 = \beta_0$, we also have (2.9).

Next, we have

$$|K_n| \ge 4\varepsilon_0 a^{2|n|} \left|\frac{\alpha_1}{\alpha_0} B_n(b) - \frac{\beta_1}{\beta_0} B_n(c)\right| - \left|\alpha_0 - \beta_0\right| \left|1 + \frac{M^2}{\varepsilon_0^2} d_0 + 4\frac{M}{\varepsilon_0}\right|.$$

From (2.9), we have

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star \ge C_1 4\varepsilon_0 a^{2|n|} \left|\frac{\alpha_1}{\alpha_0} B_n(b) - \frac{\beta_1}{\beta_0} B_n(c)\right|, \tag{2.10}$$

where $C_1 = C_1(a, \varepsilon_0, M)$ is a constant. We now consider

$$\begin{aligned}
&\frac{\alpha_1}{\alpha_0}B_n(b) - \frac{\beta_1}{\beta_0}B_n(c)\\
&= \frac{\alpha_1}{\alpha_0}\left(B_n(b) - B_n(c)\right) + \left(\frac{\alpha_1}{\alpha_0} - \frac{\beta_1}{\beta_0}\right)B_n(c)\\
&= \frac{\alpha_1}{\alpha_0}(b-c)B_n'(\xi) + \frac{\beta_0(\alpha_1 - \beta_1) + \beta_1(\beta_0 - \alpha_0)}{\alpha_0\beta_0}B_n(c) \quad (\text{for } \xi \in (b, c))\\
&= -\frac{\beta_1 B_n(c)}{\alpha_0\beta_0}(\alpha_0 - \beta_0) + \left[\frac{B_n(c)}{\alpha_0} - \frac{\alpha_1\beta_2 a B_n'(\xi)}{\alpha_0(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)}\right](\alpha_1 - \beta_1)\\
&\quad + \frac{\alpha_1\beta_1 a B_n'(\xi)}{\alpha_0(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)}(\alpha_2 - \beta_2).
\end{aligned}$$

So from (2.9) and (2.10), we have

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star \geq C_2 a^{2|n|} D_n, \tag{2.11}$$

where $C_2 = C_2(a, \varepsilon_0, M)$ and

$$\begin{aligned}
D_n = \frac{1}{\alpha_0}&\left|\left[B_n(c) - \frac{\alpha_1\beta_2 a B_n'(\xi)}{(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)}\right](\alpha_1 - \beta_1)\right.\\
&\left. + \frac{\alpha_1\beta_1 a B_n'(\xi)}{(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)}(\alpha_2 - \beta_2)\right|.
\end{aligned}$$

Using (i) and (v) in Proposition 2.2, we get

$$\begin{aligned}
&\frac{1}{M}\left(1 - \frac{A}{2|n| + 1}\right) \leq \frac{B_n(c)}{\alpha_0} - \frac{\alpha_1\beta_2 a B_n'(\xi)}{\alpha_0(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)} \leq \frac{d_0}{\varepsilon_0},\\
&0 \leq \frac{\varepsilon_0^2 a(1 - b_0)}{M(\varepsilon_0 + Na)^2(2|n| + 1)} \leq \frac{\alpha_1\beta_1 a B_n'(\xi)}{\alpha_0(\alpha_1 + \alpha_2 a)(\beta_1 + \beta_2 a)} \leq \frac{A}{\varepsilon_0(2|n| + 1)}.
\end{aligned}$$

There exists an $n_0 = n_0(a, \varepsilon_0, N)$ such that for every $n \geq n_0$ then

$$0 \leq \frac{1}{2M} \leq \frac{1}{M}\left(1 - \frac{A}{2|n| + 1}\right).$$

We now show that

$$\left\|\Lambda_\alpha - \Lambda_\beta\right\|_\star \geq C(a, \varepsilon_0, M, N)\left(\left|\alpha_1 - \beta_1\right| + \left|\alpha_2 - \beta_2\right|\right). \tag{2.12}$$

We consider three cases.

Case 1 $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) \geq 0.$
We have

$$\begin{aligned}
D_{n_0} &\geq \frac{1}{2M}\left|\alpha_1 - \beta_1\right| + \frac{\varepsilon_0^2 a(1 - b_0)}{M(\varepsilon_0 + Na)^2(2|n_0| + 1)}\left|\alpha_2 - \beta_2\right|\\
&\geq \min\left\{\frac{1}{2M}, \frac{\varepsilon_0^2 a(1 - b_0)}{M(\varepsilon_0 + Na)^2(2|n_0| + 1)}\right\}\left(\left|\alpha_1 - \beta_1\right| + \left|\alpha_2 - \beta_2\right|\right).
\end{aligned} \tag{2.13}$$

From (2.11) and (2.13), we obtain (2.12).

Case 2 $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) < 0$ and

$$D_{n_0} = \left[\frac{B_{n_0}(c)}{\alpha_0} - \frac{\alpha_1\beta_2 a B'_{n_0}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\right]\left|\alpha_1-\beta_1\right| - \frac{\alpha_1\beta_1 a B'_{n_0}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\left|\alpha_2-\beta_2\right|.$$

From that, we have

$$\frac{d_0}{\varepsilon_0}\left|\alpha_1-\beta_1\right| - \frac{\varepsilon_0^2 a(1-b_0)}{M(\varepsilon_0+Na)^2\,(2\left|n_0\right|+1)}\left|\alpha_2-\beta_2\right| \geq D_{n_0} \geq 0.$$

Then there exists an $n_1 = n_1(a, \varepsilon_0, M, N) > n_0$ such that

$$\frac{\left|\alpha_1-\beta_1\right|}{4M} \geq \frac{d_0 AM(\varepsilon_0+Na)^2}{\varepsilon_0^2 a(1-b_0)}\frac{(2\left|n_0\right|+1)}{(2\left|n_1\right|+1)}\left|\alpha_1-\beta_1\right| \geq \frac{A}{\varepsilon_0\,(2\left|n_1\right|+1)}\left|\alpha_2-\beta_2\right|.$$

We get

$$D_{n_1} \geq \frac{\left|\alpha_1-\beta_1\right|}{2M} - \frac{A}{\varepsilon_0\,(2\left|n_1\right|+1)}\left|\alpha_2-\beta_2\right| \geq \frac{\left|\alpha_1-\beta_1\right|}{4M} > \frac{A}{\varepsilon_0\,(2\left|n_1\right|+1)}\left|\alpha_2-\beta_2\right|. \tag{2.14}$$

From (2.11) and (2.14), we have (2.12)

Case 3 $(\alpha_1 - \beta_1)(\alpha_2 - \beta_2) < 0$ and

$$D_{n_0} = \frac{\alpha_1\beta_1 a B'_{n_0}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\left|\alpha_2-\beta_2\right| - \left[\frac{B_{n_0}(c)}{\alpha_0} - \frac{\alpha_1\beta_2 a B'_{n_0}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\right]\left|\alpha_1-\beta_1\right|.$$

There exists an $n_2 = n_2(a, \varepsilon_0, M, N) > n_0$ such that

$$\frac{\varepsilon_0^2 a(1-b_0)}{2M(\varepsilon_0+Na)^2\,(2\left|n_0\right|+1)}\left|\alpha_2-\beta_2\right| \geq \frac{2d_0 MA}{\varepsilon_0^2(2\left|n_2\right|+1)}\left|\alpha_2-\beta_2\right|. \tag{2.15}$$

If

$$D_{n_2} = \left[\frac{B_{n_2}(c)}{\alpha_0} - \frac{\alpha_1\beta_2 a B'_{n_2}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\right]\left|\alpha_1-\beta_1\right| - \frac{\alpha_1\beta_1 B'_{n_2}(\xi)}{\alpha_0(\alpha_1+\alpha_2 a)(\beta_1+\beta_2 a)}\left|\alpha_2-\beta_2\right|,$$

we return to Case 2. Otherwise,

$$\frac{A}{\varepsilon_0\,(2\left|n_2\right|+1)}\left|\alpha_2-\beta_2\right| - \frac{1}{2M}\left|\alpha_1-\beta_1\right| \geq D_{n_2} \geq 0. \tag{2.16}$$

From (2.15) and (2.16), we obtain

$$\frac{\varepsilon_0^2 a(1-b_0)}{2M(\varepsilon_0+Na)^2\,(2\left|n_0\right|+1)}\left|\alpha_2-\beta_2\right| \geq \frac{d_0}{\varepsilon_0}\left|\alpha_1-\beta_1\right|.$$

Moreover, we have

$$\begin{aligned} D_{n_0} &\geq \frac{\varepsilon_0^2 a(1-b_0)}{M(\varepsilon_0+Na)^2\,(2\left|n_0\right|+1)}\left|\alpha_2-\beta_2\right| - \frac{d_0}{\varepsilon_0}\left|\alpha_1-\beta_1\right| \\ &\geq \frac{\varepsilon_0^2 a(1-b_0)}{2M(\varepsilon_0+Na)^2\,(2\left|n_0\right|+1)}\left|\alpha_2-\beta_2\right| \geq \frac{d_0}{\varepsilon_0}\left|\alpha_1-\beta_1\right|. \end{aligned} \tag{2.17}$$

From (2.11) and (2.17), we have (2.12). From (2.9) and (2.12), the conclusion follows. □

3 Proof of Theorem 1.2

Consider the Dirichlet problem

$$\begin{cases} \nabla \cdot (\gamma_\alpha \nabla u) = 0 & \text{in } B \times (0, +\infty), \\ u = 0 & \text{on } \partial B \times (0, +\infty), \\ u = f & \text{on } B \times \{0\}, \end{cases} \tag{3.1}$$

where the conductivity $\gamma_\alpha \in \mu(h, M)$.

Definition 3.1 (i) We denote

$$L^2_{rad}(B) = \left\{ u \in L^2(B), u(x, y) = f\left(\sqrt{x^2 + y^2}\right) \right\}.$$

$$L^2_{rad}(B \times (0, +\infty)) = \left\{ u \in L^2(B \times (0, +\infty)), u(x, y, z) = f\left(\sqrt{x^2 + y^2}, z\right) \right\}.$$

(ii) Let

$$H^{\frac{1}{2}}_{rad}(B) = \left\{ f \in L^2_{rad}(B), \sum_{n=1}^{\infty} (1 + |\lambda_n|^2)^{\frac{1}{2}} \left|\widehat{f}(n)\right|^2 J_1^2(\lambda_n) < \infty \right\},$$

where

$$\widehat{f}(n) = \frac{2\int_0^1 f(r) J_0(\lambda_n r) r dr}{(J_1(\lambda_n))^2},$$

$J_0(\lambda_n r)$ is Bessel function of order zero, λ_n is positive zero of function J_0,

$$\lambda_1 < \lambda_2 < \ldots \lambda_n \ldots, \lambda_n \sim \left(n - \frac{1}{4}\right)\pi, \text{ when } n \to \infty.$$

$J_1(\lambda_n)$ is Bessel function of order 1 and

$$J_1(\lambda_n) = \sum_{m=0}^{\infty} \frac{(-1)^m \lambda_n^{2m+1}}{2^{2m+1}(m+1)!m!}, \quad J_1(\lambda_n) = -J_0'(\lambda_n)$$

with $J_1(\lambda_n) \sim \sqrt{\frac{2}{\pi \lambda_n}} \cos(\lambda_n - \frac{3\pi}{4}) + O(\frac{1}{\lambda_n^{3/2}})$ when $n \to \infty$. The norm of $f \in H^{\frac{1}{2}}_{rad}(B)$ is given by

$$\|f\|_{H^{\frac{1}{2}}_{rad}(B)} = \left(\sum_{n=1}^{\infty} (1 + |\lambda_n|^2)^{\frac{1}{2}} \left|\widehat{f}(n)\right|^2 J_1^2(\lambda_n) \right)^{\frac{1}{2}}.$$

(iii) The dual space of $H^{\frac{1}{2}}_{rad}(B)$ is defined by

$$H^{-\frac{1}{2}}_{rad}(B) = \left(H^{\frac{1}{2}}_{rad}(B) \right)^* = \left\{ f : H^{\frac{1}{2}}_{rad}(B) \to \mathbb{C} \text{ bounded linear functional} \right\}$$

with the norm

$$\|f\|_{H^{-\frac{1}{2}}_{rad}(B)} = \left(\sum_{n=1}^{\infty} (1 + |\lambda_n|^2)^{-\frac{1}{2}} \left|\widehat{f}(n)\right|^2 J_1^2(\lambda_n) \right)^{\frac{1}{2}}.$$

(iv) We denote

$$H^1_{rad}(B \times (0, +\infty)) = \left\{ u \in L^2_{rad}(B \times (0, +\infty)) : |\nabla u| \in L^2_{rad}(B \times (0, \infty)) \right\}.$$

In the cylindrical coordinates, if $u(r,z)=\sum_{n=1}^{\infty} u_n(z)J_0(\lambda_n r)$ we have

$$\|u\|_{H^1_{rad}(B\times(0,+\infty))}=\pi\sum_{n=1}^{\infty} J_1^2(\lambda_n)\int_0^{\infty}[(1+\lambda_n^2)|u_n(z)|^2+\left|u_n'(z)\right|^2]dz.$$

For $f\in H^{\frac{1}{2}}_{rad}(B)$, the Dirichlet problem (3.1) in cylindrical coordinates is

$$\begin{cases}\gamma_\alpha u_{rr}+\dfrac{\gamma_\alpha}{r}u_r+\partial_z(\gamma_\alpha u_z)=0, & B\times(0,\infty),\\ u(1,z)=0, & 0<z<\infty,\\ u(r,0)=f, & 0\le r<1,\end{cases}$$

and have a unique solution $u\in H^1_{rad}(B\times(0,\infty))$.

We expand $u=\sum_{n=1}^{\infty} u_n(z)J_0(\lambda_n r)$. By direct computation, we have

$$u_n(z)=\begin{cases}a_n e^{-\lambda_n z} & \text{if } h\le z<\infty,\\ b_n e^{-\lambda_n z}+c_n e^{\lambda_n z} & \text{if } 0\le z<h.\end{cases}$$

At $z=h$, we have

$$\begin{cases}\lim\limits_{z\to h^+} u_n(z)=\lim\limits_{z\to h^-} u_n(z),\\ \lim\limits_{z\to h^+}\left(\gamma_\alpha u_n'\right)(z)=\lim\limits_{z\to h^-}\left(\gamma_\alpha u_n'\right)(z).\end{cases}$$

It follows that

$$\frac{c_n}{b_n}=\frac{\alpha_2-\alpha_1}{(2+\alpha_1+\alpha_2)e^{2\lambda_n h}}.$$

The Dirichlet-to-Neumann map $\Lambda_\alpha: H^{\frac{1}{2}}_{rad}(B)\to H^{-\frac{1}{2}}_{rad}(B)$ is determined by

$$\Lambda_\alpha f(r)=-\sum_{n=1}^{\infty}(1+\alpha_2)\frac{(\alpha_2-\alpha_1)e^{-2\lambda_n h}-(2+\alpha_1+\alpha_2)}{(\alpha_2-\alpha_1)e^{-2\lambda_n h}+2+\alpha_1+\alpha_2}\lambda_n\hat{f}(n)J_0(\lambda_n r).$$

We now give an explicit formula to reconstruct the parameters h, α from the Dirichlet-to-Nemann map. Define

$$A_n=-\frac{\Lambda_\alpha(J_0(\lambda_n r))}{\lambda_n J_0(\lambda_n r)}=(1+\alpha_2)\frac{2+\alpha_1+\alpha_2-(\alpha_2-\alpha_1)e^{-2\lambda_n h}}{2+\alpha_1+\alpha_2+(\alpha_2-\alpha_1)e^{-2\lambda_n h}}. \tag{3.2}$$

If $A_1=1+\alpha_2$ then $\alpha_1=\alpha_2$; i.e., the conductor is homogeneous. Otherwise, $A_n\neq 1+\alpha_2, \forall n\in\mathbb{N}$ and we have the following proposition.

Proposition 3.2 *We reconstruct h,α_j as follows:*
(i) $\alpha_2=\lim_{n\to\infty}A_n-1$.
(ii) $h=\frac{1}{2\pi}\ln(\lim_{n\to\infty}\frac{A_n-1-\alpha_2}{A_{n+1}-1-\alpha_2})$.
(iii) $\alpha_1=\frac{2A+(A+2)\alpha_2}{2-A}$, *where*

$$A=\lim_{n\to\infty}\frac{(A_n-1-\alpha_2)\,e^{2\lambda_n h}}{1+\alpha_2}.$$

Proof (i) It is easy to show that $\alpha_2=\lim_{n\to\infty}A_n-1$.

(ii) We have

$$\frac{A_n - 1 - \alpha_2}{A_{n+1} - 1 - \alpha_2} = \frac{e^{-2\lambda_n h}}{e^{-2\lambda_{n+1} h}} \frac{2 + \alpha_1 + \alpha_2 + (\alpha_2 - \alpha_1)e^{-2\lambda_{n+1} h}}{2 + \alpha_1 + \alpha_2 + (\alpha_2 - \alpha_1)e^{-2\lambda_n h}}.$$

Note that $\lambda_n \sim \left(n - \frac{1}{4}\right)\pi$, when $n \to \infty$. We obtain

$$\lim_{n\to\infty} \frac{A_n - 1 - \alpha_2}{A_{n+1} - 1 - \alpha_2} = e^{2\pi h}.$$

Hence,

$$h = \frac{1}{2\pi} \ln\left(\lim_{n\to\infty} \frac{A_n - 1 - \alpha_2}{A_{n+1} - 1 - \alpha_2}\right).$$

(iii) Since

$$A = \lim_{n\to\infty} \frac{(A_n - 1 - \alpha_2)\, e^{2\lambda_n h}}{1 + \alpha_2} = \frac{2(\alpha_1 - \alpha_2)}{2 + \alpha_1 + \alpha_2},$$

so $\alpha_1 = (2A + (A + 2)\alpha_2)/(2 - A)$. □

Remark 3.3 We can reconstruct h, α_1 from α_2, A_1, A_2 as follows

$$h = \frac{1}{2(\lambda_1 - \lambda_2)} \ln\left(\frac{(A_1 + 1 + \alpha_2)(A_2 - 1 - \alpha_2)}{(A_1 - 1 - \alpha_1)(A_2 + 1 + \alpha_2)}\right)$$

$$\alpha_1 = \frac{A_1(2 + \alpha_2(1 + e^{-2\lambda_1 h})) - (1 + \alpha_2)(2 + \alpha_2(1 - e^{-2\lambda_2 h}))}{(1 + \alpha_2)(1 + e^{-2\lambda_1 h}) - A_1(1 - e^{-2\lambda_1 h})}.$$

We now prove Theorem 1.2.

Proof of Theorem 1.2 Firstly, for each $\gamma_\alpha, \gamma_\beta \in \mu(h, M)$, $f \in H^{\frac{1}{2}}_{rad}(B)$, we have

$$\left\| \left(\Lambda_\alpha - \Lambda_\beta\right) f \right\|^2_{H^{-\frac{1}{2}}_{rad}(B)} = \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1 + \lambda_n^2)^{\frac{1}{2}}} (A_n - B_n)^2 |\hat{f}(n)|^2 (J_1(\lambda_n))^2,$$

where

$$A_n = -(1 + \alpha_2) \frac{(\alpha_2 - \alpha_1)e^{-2\lambda_n h} - (2 + \alpha_1 + \alpha_2)}{(\alpha_2 - \alpha_1)e^{-2\lambda_n h} + 2 + \alpha_1 + \alpha_2},$$

$$B_n = -(1 + \beta_2) \frac{(\beta_2 - \beta_1)e^{-2\lambda_n h} - (2 + \beta_1 + \beta_2)}{(\beta_2 - \beta_1)e^{-2\lambda_n h} + 2 + \beta_1 + \beta_2}.$$

By direct computation we obtain

$$A_n - B_n = \frac{(A - Be^{-2\lambda_n h} - Ce^{-4\lambda_n h})(\alpha_2 - \beta_2) + De^{-2\lambda_n h}(\alpha_1 - \beta_1)}{(2 + \alpha_1 + \alpha_2 + (\alpha_2 - \alpha_1)e^{-2\lambda_n h})(2 + \beta_1 + \beta_2 + (\beta_2 - \beta_1)e^{-2\lambda_n h})},$$

where

$$A = (2 + \alpha_1 + \alpha_2)(2 + \beta_1 + \beta_2) \in [4, 4(M + 1)^2],$$
$$B = (2 + \alpha_2 + \beta_2)(2 + \beta_1) \in [4, 4(M + 1)^2],$$
$$C = (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) \in [-M^2, M^2],$$
$$D = (2 + \alpha_2 + \beta_2)(2 + \beta_2) \in [4, 4(M + 1)^2].$$

We denote by K_n and H_n the numerator and denominator of $(A_n - B_n)$, respectively. We have $H_n \leq (2+3M)^2$ and

$$\left\| \Lambda_\alpha - \Lambda_\beta \right\|_{H^{\frac{1}{2}}_{rad}(B) \to H^{-\frac{1}{2}}_{rad}(B)} = \sup_{\substack{f \in H^{\frac{1}{2}}_{rad}(B) \\ f \neq 0}} \frac{\left\| (\Lambda_\alpha - \Lambda_\beta) f \right\|_{H^{-\frac{1}{2}}_{rad}(B)}}{\|f\|_{H^{\frac{1}{2}}_{rad}(B)}}$$
$$\geq \sup_{n \neq 0} \frac{|K_n|}{|2H_n|} \geq \sup_{n \neq 0} \frac{|K_n|}{2\,(2+3M)^2}. \tag{3.3}$$

Hence,

$$\sup_n |K_n| \geq (A + Be^{-2\lambda_n h} + |C| e^{-4\lambda_n h}) \left| \alpha_2 - \beta_2 \right| - De^{-2\lambda_n h} \left| \alpha_1 - \beta_1 \right|.$$

For $\alpha_2 \neq \beta_2$, we choose n big enough so that

$$\sup_n |K_n| \geq 2 \left| \alpha_2 - \beta_2 \right|. \tag{3.4}$$

From (3.4), (3.3) becomes

$$\left\| \Lambda_\alpha - \Lambda_\beta \right\|_{H^{\frac{1}{2}}_{rad}(B) \to H^{-\frac{1}{2}}_{rad}(B)} \geq \frac{1}{(2+3M)^2} \left| \alpha_2 - \beta_2 \right|. \tag{3.5}$$

For $\alpha_2 = \beta_2$, we also have (3.5). It is easy to get

$$|K_1| \geq 4e^{-2\lambda_1 h} |\alpha_1 - \beta_1| - 4(M+1)^2 (1 + e^{-2\lambda_1 h})^2 |\alpha_2 - \beta_2|.$$

Therefore, from (3.3) and (3.5), we have

$$\left\| \Lambda_\alpha - \Lambda_\beta \right\|_{H^{\frac{1}{2}}_{rad}(B) \to H^{-\frac{1}{2}}_{rad}(B)} \geq \frac{e^{-2\lambda_1 h}}{2(2+3M)^2 (M+1)^2 (1 + e^{-2\lambda_1 h})^2} \left| \alpha_1 - \beta_1 \right|. \tag{3.6}$$

From (3.5) and (3.6), we are done. □

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