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Well-Posedness for Set Optimization Problems Involving Set Order Relations

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Dedicated to Professor Hoang Tuy

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Abstract
In this paper, we investigate set optimization problems with three types of set order relations. Various kinds of well-posedness for these problems and their relationship are concerned. Then, sufficient conditions for set optimization problems to be well-posed are established. Moreover, Kuratowski measure of noncompactness is applied to survey characterizations of well-posedness for set optimization problems. Furthermore, approximating solution maps and their stability are researched to propose the link between stability of the approximating problem and well-posedness of the set optimization problem.

Keywords Set order relation · Set optimization problem · Well-posedness · Stability · Measure of noncompactness

Mathematics Subject Classification (2010) 80M50 · 49K40 · 54C60

1 Introduction

Well-posedness plays an important role in both theory and numerical methods for optimization theory. This fact has motivated and inspired many researchers to study the well-posedness for problems related to optimization. In 1966, Tikhonov [29] introduced

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a definition of well-posedness for unconstrained optimization problems which is called Tikhonov well-posedness. This concept requires two conditions. The first condition is the existence and uniqueness of the solution, and the second one is the convergence of each minimizing sequence to the unique solution. Later on, several extensions of Tikhonov well-posedness have been introduced and investigated. The study of Tikhonov well-posedness and its extensions is interesting and very important in optimization theory. One of the most important extensions of Tikhonov well-posedness is relaxing the first condition. In general, practical problems, for example, vector optimization problems, have more than one solution, and hence, the extension in this way of Tikhonov well-posedness is meaningful and valuable. The main idea of this approach is based on the convergence of a subsequence of a minimizing sequence to a point in the solution set, and thus, it can be considered as an extension of Tikhonov well-posedness. Another generalization of Tikhonov well-posedness, called metrically well-setness, has also been studied in [3]. This concept requires minimizing sequences to metrically approach the solution set of the problem and is a relaxed form of the generalized well-posedness. For further reading and references, we refer to publications [7, 24].

As far as we know, there are two approaches to formulate optimality notions for set-valued optimization problems, namely the vector approach and the set approach. These criteria depend on the way that the notion of minimality is defined. In the first approach, optimal solutions are defined as the efficient points of the union of all images of the set-valued objective map [9]. In the second one, using set order relations defined on the power set of the objective space, we compare all images of the set-valued objective map [20]. Recently, optimization problems based on set approach, called set optimization problems [21], have attracted a great deal of attention of researchers because of their important roles and useful applications in the practical situations. An important socio-economic application of set order relations was presented by Neukel [28] in the project investigating relationship between noise disturbance and quality of life in the region surrounding the Frankfurt Airport in Germany. Another application of set order relations in the field of finance about measures of risk was found by Hamel and Heyde [11]. Many important and interesting results have been obtained in different topics in this area such as the existence conditions and optimality conditions [1, 13], nonlinear scalarization [14, 18], Lagrangian duality and saddle points [15], the Ekeland variational principle [10], and stability [12]. We would like to give a brief review of set order relations. The first introduction of set order relations was presented by Kuroiwa et al. [22] in 1997. Moreover, these relations were also independently studied by Young [30] and Nishnianidze [26]. Kuroiwa [20] showed six relations among sets and obtained duality theorems of set optimization problems. Relations $\leq^l$ and $\leq^u$ were studied in some publications [6, 23]. Many important and significant applications of set order relations were studied and discussed [4, 27].

In 2009, Zhang et al. [31] firstly introduced three kinds of well-posedness including one pointwise well-posedness and two global ones. The authors obtained some sufficient and necessary conditions for set optimization problems involving the relation $\leq^l$ to be well-posed. Moreover, criteria and characterizations of well-posedness for this problem were established by using the scalarization method. Well-posedness properties for such problems with a class of generalized convex set-valued maps were obtained by Crespi et al. [5]. Using assumptions on cone properness, Gutiérrez et al. [8] investigated pointwise well-posedness for set optimization problems involving the relation $\leq^l$. Recently, Dhingra and Lalitha [6] introduced a concept of well-setness and proved that it is an extension of generalized well-posedness which was considered in [31]. Furthermore, they gave sufficient conditions of well-setness for set optimization problems involving the relation $\leq^l$ and obtained characterizations of well-setness for them by the scalarization method. As mentioned in [11, 16, 17, 28], the relation $\leq^s$ plays an important role in real-life situations. To the best of our knowledge, there is no work devoted to well-posedness for set optimization problems involving the relation $\leq^s$. Hence, studying on well-posedness for problems involving these relations is significant.

Motivated and inspired by these works, in this paper, we aim to investigate various types of well-posedness for set optimization problems involving different kinds of set order relations. We introduce many kinds of well-posedness for such problems and study the relationship between them as well as their sufficient conditions. Moreover, Kuratowski measure of noncompactness is applied to survey characterization of well-posedness for set optimization problems. Finally, approximating solution maps and their stability properties are researched to propose the link between stability of the approximating problem and well-posedness of the set optimization problem.

This paper is organized as follows. In Section 2, we recall some necessary concepts and their properties used in what follows. Section 3 introduces various types of well-posedness for set optimization problems and analyzes their relationships. Moreover, in this section, sufficient conditions of these generalized well-posedness for set optimization problems are also studied. In the last section, characterization of well-posedness for set optimization problems is surveyed by using Kuratowski measure of noncompactness. Finally, approximating solution maps and their stability properties are studied to propose the connection between stability of the approximating problem and well-posedness of the set optimization problem.

2 Preliminaries

Let X be a metric space and Y be a Hausdorff topological vector space. Let K be a closed convex pointed cone in Y with $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} K$ denotes the interior of K. The space Y is endowed with an order relation induced by cone K in the following way:

$$x \leq_K y \Leftrightarrow y - x \in K,$$
$$x <_K y \Leftrightarrow y - x \in \operatorname{int} K.$$

The cone K induces various set orderings in Y. These orderings, given below, were introduced in [16, 20, 22]. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y. For $A, B \in \mathcal{P}(Y)$, lower set less relation $\leq^l$, upper set less relation $\leq^u$, and set less relation $\leq^s$, respectively, are defined by

$$A \leq^l B \text{ if and only if } B \subset A + K,$$
$$A \leq^u B \text{ if and only if } A \subset B - K,$$
$$A \leq^s B \text{ if and only if } A \subset B - K \text{ and } B \subset A + K.$$

Definition 1 [16] We say that the binary relation $\leq$ is

(a) Compatible with the addition if and only if $A \leq B$ and $D \leq E$ imply $A + D \leq B + E$ for all $A, B, D, E \in \mathcal{P}(Y)$.
(b) Compatible with the multiplication with a nonnegative real number if and only if $A \leq B$ implies $\lambda A \leq \lambda B$ for all scalars $\lambda \geq 0$ and all $A, B \in \mathcal{P}(Y)$.
(c) Compatible with the collinear structure of $\mathcal{P}(Y)$ if and only if it is compatible with both the addition and the multiplication with a nonnegative real number.

Proposition 1 [16]

(i) *The order relations* $\leq^l$, $\leq^u$, *and* $\leq^s$ *are pre-order (i.e., the relations are reflexive and transitive).*
(ii) *The order relations* $\leq^l$, $\leq^u$, *and* $\leq^s$ *are compatible with the collinear structure of* $\mathcal{P}(Y)$.
(iii) *In general, the order relations* $\leq^l$, $\leq^u$, *and* $\leq^s$ *are not antisymmetric; more precisely, for arbitrary sets* $A, B \in \mathcal{P}(Y)$, *we have*

$$(A \leq^l B \quad and \quad B \leq^l A) \Leftrightarrow A + K = B + K,$$

$$(A \leq^u B \quad and \quad B \leq^u A) \Leftrightarrow A - K = B - K,$$

$$(A \leq^s B \quad and \quad B \leq^s A) \Leftrightarrow (A + K = B + K \quad and \quad A - K = B - K).$$

For $\alpha \in \{u, l, s\}$, we say that

$$A \sim^\alpha B \text{ if and only if } A \leq^\alpha B \text{ and } B \leq^\alpha A.$$

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values on X. For each $\alpha \in \{u, l, s\}$, we consider the following set optimization problem:

$$(\mathrm{P}_\alpha) \qquad \alpha - \mathrm{Min}\, F(x)$$
$$\text{subject to} \quad x \in M,$$

where M is a nonempty closed subset of X. A point $\bar{x} \in M$ is called an $\alpha-$minimal solution of (P_α) if for any $x \in M$ such that $F(x) \leq^\alpha F(\bar{x})$, then $F(\bar{x}) \leq^\alpha F(x)$. The set of all $\alpha-$minimal solutions of (P_α) is denoted by $\alpha - \mathrm{Min}\, F$.

Remark 1 It can be seen that if $\bar{x} \in \alpha - \mathrm{Min}\, F$ and $F(\bar{x}) \sim^\alpha F(\hat{x})$ for some $\hat{x} \in M$, then $\hat{x} \in \alpha - \mathrm{Min}\, F$.

We recall the following definitions of semicontinuity for a set-valued map and their properties used in the sequel.

Definition 2 [2, pp. 38, 39] A set-valued map $F : X \rightrightarrows Y$ is said to be

(a) Upper semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \subset U$ there is a neighborhood N of x_0 such that $F(x) \subset U$ for every $x \in N$.
(b) Lower semicontinuous at $x_0 \in X$ if and only if for any open subset U of Y with $F(x_0) \cap U \neq \emptyset$ there is a neighborhood N of x_0 such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.
(c) Lower (upper) semicontinuous on a subset S of X if and only if it is lower (upper) semicontinuous at every $x \in S$

Lemma 1 [2] *Let* $F : X \rightrightarrows Y$ *be a set-valued map.*

(i) F *is lower semicontinuous at* $x_0 \in X$ *if and only if for any net* $\{x_\gamma\} \subset X$ *converging to* x_0 *and for any* $y \in F(x_0)$, *there exist* $y_\gamma \in F(x_\gamma)$ *such that* $\{y_\gamma\}$ *converges to* y.
(ii) *If* $F(x_0)$ *is compact, then* F *is upper semicontinuous at* $x_0 \in X$ *if and only if for any net* $\{x_\gamma\}$ *converging to* x_0 *and for any* $y_\gamma \in F(x_\gamma)$, *there exist* $y_0 \in F(x_0)$ *and a subnet of* $\{y_\gamma\}$ *converging to* y_0.

Next, we recall the concepts of Hausdorff distance and Hausdorff convergence of sequence of sets. If S is a nonempty subset of X and $x \in X$, then the distance d between x and S is defined as

$$d(x, S) := \inf_{u \in S} d(x, u).$$

If S_1 and S_2 are two nonempty subsets of X, then Hausdorff distance between S_1 and S_2, denoted by $H(S_1, S_2)$, is defined as

$$H(S_1, S_2) := \max\{H^*(S_1, S_2), H^*(S_2, S_1)\},$$

where $H^*(S_1, S_2) := \sup_{x \in S_1} d(x, S_2)$.

Definition 3 [19, p. 359] Let $\{A_n\}$ be a sequence of subsets of X. We say that $\{A_n\}$ converges to A in the sense of Hausdorff, denoted by $A_n \to A$, if and only if $H(A_n, A) \to 0$ as $n \to \infty$.

Now, we recall the concept of the Kuratowski measure of noncompactness and its properties used in the sequel.

Definition 4 [25, Definition 2.1] Let M be a nonempty subset of a metric space X. The Kuratowski measure of noncompactness μ of the set M is defined by

$$\mu(M) := \inf\left\{\varepsilon > 0 \mid \exists n \in \mathbb{N}, \exists M_i, \operatorname{diam} M_i < \varepsilon, i = 1, \ldots, n, \text{ s.t. } M \subset \bigcup_{i=1}^{n} M_i\right\},$$

where diam M_i is the diameter of M_i.

Lemma 2 [25, Proposition 2.3] *The following assertions are true:*

(i) $\mu(M) = 0$ *if* M *is compact.*
(ii) $\mu(M) \le \mu(N)$ *whenever* $M \subset N$.
(iii) *If* $\{M_n\}$ *is a sequence of closed subsets in* X *satisfying* $M_{n+1} \subset M_n$ *for every* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \mu(M_n) = 0$, *then* $K := \bigcap_{n\in\mathbb{N}} M_n$ *is a nonempty compact set and* $\lim_{n\to\infty} H(M_n, K) = 0$.

It is easy to check the following property. We omit the proof.

Lemma 3 *Let* Y *be a Hausdorff topological vector space and* A, B *be subsets of* Y. *If* A *is compact and* B *is closed, then* $A + B$ *is closed.*

3 Various Kinds of Well-Posedness for Set Optimization Problems

Motivated by the study in [31], we introduce concepts of generalized minimizing sequence and employ them to study several types of well-posedness for (P_α). Let e be a fixed element of int K.

Definition 5 A sequence $\{x_n\} \subset M$ is called a generalized minimizing sequence of (P_α) if and only if there exist sequences $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $\{z_n\} \subset \alpha - \operatorname{Min} F$ satisfying $F(x_n) \le^{\alpha} F(z_n) + \varepsilon_n e$ for all n.

Definition 6 Problem (P_α) is said to be generalized e-well-posed (shortly, generalized well-posed) if and only if for every generalized minimizing sequence $\{x_n\}$ of (P_α) there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in \alpha - \mathrm{Min}\, F$ such that $\{x_{n_k}\}$ converges to $\bar{x}$.

Remark 2 When $\alpha = l$, Definitions 5 and 6 reduce to [31, Definition 2.2] and corresponding concepts in [6].

Considering the problem (P_α), we define a set-valued map $L_F^\alpha : M \times \mathbb{R}^+ \rightrightarrows M$ as, for all $(x, \varepsilon) \in M \times \mathbb{R}^+$,

$$L_F^\alpha(x, \varepsilon) := \{\hat{x} \in M \mid F(\hat{x}) \leq^\alpha F(x) + \varepsilon e\}.$$

We refer to the set $L_F^\alpha(x, \varepsilon)$ as level set at x with level ε and L_F^α as level set-valued map. It is clear that $\{x_n\}$ is a generalized minimizing sequence of (P_α), if there exist $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $z_n \in \alpha - \mathrm{Min}\, F$ such that $x_n \in L_F^\alpha(z_n, \varepsilon_n)$.

The following proposition plays an important role in our analysis.

Proposition 2 *Let L_F^α be a level set-valued map. Then the following statements hold:*

(i) $x \in L_F^\alpha(x, \varepsilon)$ *for all* $x \in M$.
(ii) *If* $\varepsilon_1 < \varepsilon_2$, *then* $L_F^\alpha(x, \varepsilon_1) \subset L_F^\alpha(x, \varepsilon_2)$.
(iii) $\bigcup_{z \in \alpha - \mathrm{Min}\, F} L_F^\alpha(z, 0) = \alpha - \mathrm{Min}\, F$.

Proof We only prove the assertions (i)–(iii) for the case $\alpha = s$; the proofs of these assertions for the cases $\alpha = l$ and $\alpha = u$ are similar.

(i) For any $\varepsilon > 0$ and $x \in M$, because $\varepsilon e \in \mathrm{int}\, K \subset K$, we have $F(x) + \varepsilon e \subset F(x) + K$, i.e., $F(x) \leq^l F(x) + \varepsilon e$.

On the other hand, by the convexity of K, one gets $-K \subset \varepsilon e - K$. Therefore,

$$F(x) \subset F(x) - K \subset F(x) + \varepsilon e - K,$$

i.e., $F(x) \leq^u F(x) + \varepsilon e$. So, $x \in L_F^s(x, \varepsilon)$.
(ii) For $\varepsilon_1 < \varepsilon_2$ and $\bar{x} \in L_F^s(x, \varepsilon_1)$, we have $F(\bar{x}) \leq^s F(x) + \varepsilon_1 e$, i.e.,

$$F(x) + \varepsilon_1 e \subset F(\bar{x}) + K \quad \text{and} \quad F(\bar{x}) \subset F(x) + \varepsilon_1 e - K.$$

Obviously,

$$F(x) + \varepsilon_2 e = F(x) + \varepsilon_1 e + (\varepsilon_2 - \varepsilon_1)e,$$

and

$$F(x) + \varepsilon_2 e - K = F(x) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of cone K with Proposition 1 (ii), we get

$$F(x) + \varepsilon_2 e \subset F(\bar{x}) + K, \quad F(\bar{x}) \subset F(x) + \varepsilon_2 e - K,$$

and hence $F(\bar{x}) \leq^s F(x) + \varepsilon_2 e$. So, $L_F^s(x, \varepsilon_1) \subset L_F^s(x, \varepsilon_2)$.
(iii) Let $\bar{x} \in s - \mathrm{Min}\, F$, we always get $F(\bar{x}) \leq^s F(\bar{x})$ because $\leq^s$ is reflexive. By (i), we have $\bar{x} \in L_F^s(\bar{x}, 0)$. So, $\bar{x} \in \bigcup_{z \in s - \mathrm{Min}\, F} L_F^s(z, 0)$.

Conversely, let $\bar{x} \in \bigcup_{z \in s - \mathrm{Min}\, F} L_F^s(z, 0)$ and suppose that there exists $x \in M$ satisfying $F(x) \leq^s F(\bar{x})$, we need to prove that $F(\bar{x}) \leq^s F(x)$. Since $\bar{x} \in \bigcup_{z \in s - \mathrm{Min}\, F} L_F^s(z, 0)$, there exists $z \in s - \mathrm{Min}\, F$ such that $\bar{x} \in L_F^s(z, 0)$. Equivalently, $F(\bar{x}) \leq^s F(z)$. Since $F(x) \leq^s F(\bar{x})$, by the transitivity property of $\leq^s$, we get $F(x) \leq^s F(z)$. This implies $F(z) \leq^s F(x)$ as $z \in s - \mathrm{Min}\, F$. Using the transitivity property, we conclude that $\bar{x} \in s - \mathrm{Min}\, F$. □

Remark 3 When $\alpha = l$, Proposition 2 reduces to [6, Proposition 3.1 (without proof)], and Proposition 2 is new for cases where $\alpha = u$ and $\alpha = s$.

Inspired by [6], we next introduce notions of metrically α-well-posedness for (P$_\alpha$) by using the Hausdorff distance.

Definition 7 Problem (P$_\alpha$) is said to be metrically α-well-posed if and only if $\alpha -$ Min F is nonempty, and for every generalized minimizing sequence $\{x_n\}$ of (P$_\alpha$),

$$H^*(L_F^\alpha(x_n, \varepsilon_n), \alpha - \text{Min}\, F) \to 0,$$

where $\{\varepsilon_n\} \subset \mathbb{R}^+$ is the sequence corresponding to $\{x_n\}$.

Next, we propose a new kind of well-posedness for (P$_\alpha$) which is a relaxed form of metrically α-well-posedness and useful to improve some known results.

Definition 8 Problem (P$_\alpha$) is said to be weak metrically α-well-posed if and only if $\alpha -$ Min F is nonempty, and for every generalized minimizing sequence $\{x_n\}$ of (P$_\alpha$),

$$d(x_n, \alpha - \text{Min}\, F) \to 0.$$

By Proposition 2 (i), it is clear that if the problem (P$_\alpha$) is metrically α-well-posed, then it is weak metrically α-well-posed.

The existence conditions of the solutions for (P$_\alpha$) have been studied intensively (see, e.g., [1, 13]). In this paper, we focus on necessary and sufficient conditions of well-posedness for (P$_\alpha$). Therefore, we here assume that $\alpha -$ Min F is nonempty.

Firstly, we study a necessary condition of the generalized well-posedness for (P$_\alpha$).

Theorem 1 *If* (P$_\alpha$) *is generalized well-posed, then* $\alpha -$ Min F *is compact.*

Proof For $\{x_n\} \subset \alpha - \text{Min}\, F$ and $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0, for each n, we have $F(x_n) \preceq^\alpha F(x_n) + \varepsilon_n e$ as $e \in \text{int}\, K$. So, $\{x_n\}$ is a generalized minimizing sequence of (P$_\alpha$). By the generalized well-posedness of (P$_\alpha$), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in \alpha - \text{Min}\, F$ such that $\{x_{n_k}\}$ converges to $\bar{x}$. Hence, $\alpha - \text{Min}\, F$ is compact. □

Combining Theorem 1 with [6, Theorem 3.1], we get relationships between generalized well-posedness and (weak) metrically α-well-posedness for (P$_\alpha$).

Corollary 1 *If* (P$_\alpha$) *is generalized well-posed, then* (P$_\alpha$) *is metrically* α*-well-posed.*

Remark 4 Corollary 1 improves [6, Theorem 3.1] by removing the closedness of $\alpha -$ Min F.

The below example illustrates that the converse of Corollary 1 is not true.

Example 1 Let $X = Y = \mathbb{R}$, $M = [-1, 1]$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} (0, 1) & \text{if } x \in (0, 1), \\ (0, 2) & \text{otherwise.} \end{cases}$$

Clearly, $\alpha - \text{Min}\, F = (0, 1)$ and (P$_\alpha$) is metrically α-well-posed. However, (P$_\alpha$) is not generalized well-posed by Theorem 1.

Note that Example 1 also shows that [6, Theorem 3.2] is not true. The following result is a correction version of this theorem.

Theorem 2 *If* $\alpha - \text{Min}\, F$ *is compact and* (P_α) *is weak metrically* α*-well-posed, then* (P_α) *is generalized well-posed.*

Proof Let $\{x_n\}$ be a generalized minimizing sequence of (P_α), we have $d(x_n, \alpha-\text{Min}\, F) \to 0$ as (P_α) is weak metrically α-well-posed. By the compactness of $\alpha - \text{Min}\, F$, there exists a sequence $\{\bar{x}_n\} \subset \alpha - \text{Min}\, F$ such that

$$d(x_n, \bar{x}_n) = d(x_n, \alpha - \text{Min}\, F) \to 0.$$

Then, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to some $\bar{x} \in \alpha - \text{Min}\, F$ as $\alpha - \text{Min}\, F$ is compact. Due to

$$d(x_{n_k}, \bar{x}) \leq d(x_{n_k}, \bar{x}_{n_k}) + d(\bar{x}_{n_k}, \bar{x}),$$

$\{x_{n_k}\}$ converges to $\bar{x}$. We conclude that (P_α) is generalized well-posed. □

Next, we now give sufficient conditions for (P_α) to be generalized well-posed.

Theorem 3 *Suppose that the following conditions hold:*

(i) M *and* $\alpha - \text{Min}\, F$ *are compact.*
(ii) F *is continuous and compact-valued on* M.

Then, (P_α) *is generalized well-posed.*

Proof We only demonstrate the proof of the statement for the case $\alpha = u$ since the technique to prove the statement for the cases $\alpha = l$ and $\alpha = s$ is similar. Suppose that (P_u) is not generalized well-posed, it follows from Theorem 2 that (P_u) is not weak metrically u-well-posed. Then, there exists a generalized minimizing sequence $\{x_n\}$ of (P_u) such that

$$d(x_n, u - \text{Min}\, F) \not\to 0. \tag{1}$$

Since $\{x_n\}$ is a generalized minimizing sequence, there exists $\{z_n\} \subset u - \text{Min}\, F$ such that

$$F(x_n) \preceq^u F(z_n) + \varepsilon_n e. \tag{2}$$

Since M is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $\bar{x} \in M$. It follows from (2) that

$$F(x_{n_k}) \subset F(z_{n_k}) + \varepsilon_{n_k} e - K. \tag{3}$$

By the compactness of $u - \text{Min}\, F$, we can assume that $\{z_{n_k}\}$ converges to some $\bar{z} \in u - \text{Min}\, F$. Let $\bar{v} \in F(\bar{x})$ be arbitrary, there exists $\{v_{n_k}\}$ with $v_{n_k} \in F(x_{n_k})$ converging to $\bar{v}$ because of the lower semicontinuity of F at $\bar{x}$. By (3), we get $v_{n_k} \in F(z_{n_k}) + \varepsilon_{n_k} e - K$, and hence there exist $u_{n_k} \in F(z_{n_k})$ such that

$$v_{n_k} \in u_{n_k} + \varepsilon_{n_k} e - K. \tag{4}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_{n_k}\}$, denoted by the same indexes, such that $\{u_{n_k}\}$ converges to $\bar{u}$. Taking limit as $n \to \infty$ in (4), we get $\bar{v} \in \bar{u} - K$. Thus, $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, we conclude that $F(\bar{x}) \subset F(\bar{z}) - K$, i.e., $F(\bar{x}) \preceq^u F(\bar{z})$. Since $\bar{z} \in u - \text{Min}\, F$, we have $F(\bar{x}) \sim^u F(\bar{z})$, and hence $\bar{x} \in u - \text{Min}\, F$ which contradicts (1). So, (P_α) is generalized well-posed. □

Theorem 3 gives sufficient conditions of the generalized well-posedness for (P_α) in the case the constraint set is compact. The following result devotes to the noncompactness case of this set.

Theorem 4 *Suppose that the following conditions hold:*

(i) *X is locally compact and $\alpha - \text{Min}\, F$ is compact.*
(ii) *F is compact-valued on M.*
(iii) *There exists $\delta > 0$ such that the level set $L_F^\alpha(x, \varepsilon)$ is connected for every $x \in \alpha - \text{Min}\, F$ and for every $\varepsilon \in (0, \delta)$. Then,*

(a) (P_u) *is generalized well-posed if F is upper semicontinuous on $u - \text{Min}\, F$ and F is lower semicontinuous on M.*
(b) (P_l) *is generalized well-posed if F is lower semicontinuous on $l - \text{Min}\, F$ and F is upper semicontinuous on M.*
(c) (P_s) *is generalized well-posed if F is continuous on M.*

Proof By the similarity, we here only demonstrate the proof for the statement (a). Suppose that (P_u) is not generalized well-posed. By the assumption (i) and Theorem 2, (P_u) is also not metrically well-posed. Then, there exists a generalized minimizing sequence $\{x_n\}$ of (P_u) such that

$$H^*(L_F^u(x_n, \varepsilon_n), u - \text{Min}\, F) \not\to 0,$$

where $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 is the sequence corresponding to $\{x_n\}$. Because $\{x_n\}$ is a generalized minimizing sequence, for each $n \in \mathbb{N}$ there exists $z_n \in u - \text{Min}\, F$ such that $F(x_n) \leq^u F(z_n) + \varepsilon_n e$. Since $H^*(L_F^u(x_n, \varepsilon_n), u - \text{Min}\, F) \not\to 0$, we can assume that there is $\beta > 0$ satisfying $H^*(L_F^u(x_n, \varepsilon_n), u - \text{Min}\, F) \geq \beta$ for all n (take a subsequence if necessary). By (i), there exists an open neighborhood U of $u - \text{Min}\, F$ such that its closure, $\overline{U}$, is compact and $L_F^u(x_n, \varepsilon_n) \not\subset \overline{U}$. Hence, for each $n \in \mathbb{N}$, there exists $\hat{x}_n \in L_F^u(x_n, \varepsilon_n)$ such that

$$\hat{x}_n \notin \overline{U}. \tag{5}$$

Since $\{x_n\}$ is a generalized minimizing sequence and $\hat{x}_n \in L_F^u(x_n, \varepsilon_n)$, i.e., $F(\hat{x}_n) \leq^u F(x_n) + \varepsilon_n e$, we conclude that $F(\hat{x}_n) \leq^u F(z_n) + 2\varepsilon_n e$. Hence, $\hat{x}_n \in L_F^u(z_n, 2\varepsilon_n)$. Combining this with (5), we get

$$L_F^u(z_n, 2\varepsilon_n) \cap (\overline{U})^c \neq \emptyset, \tag{6}$$

where $(\overline{U})^c$ denotes the complement of $\overline{U}$ in X. Also, we obtain

$$z_n \in L_F^u(z_n, 2\varepsilon_n) \cap \text{int}\, \overline{U}. \tag{7}$$

We next claim that $L_F^u(z_n, 2\varepsilon_n) \cap \partial(\overline{U}) \neq \emptyset$ for every $n \in \mathbb{N}$ satisfying $2\varepsilon_n < \delta$. Suppose on the contrary that there exists $\bar{m}_n \in \mathbb{N}$ such that $L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \subset \text{int}\, \overline{U} \cup \text{int}(\overline{U})^c$. This leads to

$$L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) = (L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \cap \text{int}\, \overline{U}) \cup (L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \cap (\text{int}(\overline{U})^c)). \tag{8}$$

We note that $L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \cap \text{int}\, \overline{U}$ and $L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \cap (\text{int}(\overline{U})^c)$ are separated since $\overline{\text{int}\overline{U}} \cap \text{int}(\overline{U})^c = \emptyset$ and $\text{int}\, \overline{U} \cap \overline{\text{int}(\overline{U})^c} = \emptyset$. Employing (6)–(8) and $L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n}) \cap \text{int}\, \overline{U} \cap (\overline{U})^c = \emptyset$, we arrive at a contradiction of the fact that $L_F^u(z_{\bar{m}_n}, 2\varepsilon_{\bar{m}_n})$ is a connected set. Therefore, there exists a sequence $\{w_n\}$ such that

$$w_n \in L_F^u(z_n, 2\varepsilon_n) \cap \text{bd}(\overline{U}), \tag{9}$$

where $\mathrm{bd}(A)$ denotes the boundary of a given set A. By the compactness of $\overline{U}$, there exists a subsequence of $\{w_n\}$ which is still denoted by $\{w_n\}$ converging to some $\bar{w} \in \overline{U}$. Since $w_n \in L_F^u(z_n, 2\varepsilon_n)$, $F(w_n) \subset F(z_n) + 2\varepsilon_n e - K$. Due to the compactness of $u - \mathrm{Min}\, F$, there is a subsequence of $\{z_n\}$ which is still denoted by $\{z_n\}$ converging to some $\bar{z} \in u - \mathrm{Min}\, F$. Now, we show that $F(\bar{w}) \subset F(\bar{z}) - K$. Let $\bar{v} \in F(\bar{w})$ be arbitrary, by the lower semicontinuity of F at $\bar{w}$, there exists a sequence $\{v_n\}$ converging to $\bar{v}$ where $v_n \in F(w_n)$ for all n. We get $v_n \in F(z_n) + 2\varepsilon_n e - K$. Thus, there exists $u_n \in F(z_n)$ such that

$$v_n \in u_n + 2\varepsilon_n e - K. \tag{10}$$

Since F is upper semicontinuous and compact-valued at $\bar{z}$, there exist $\bar{u} \in F(\bar{z})$ and a subsequence of $\{u_n\}$, denoted by the same indexes, converging to $\bar{u}$. Taking limit as $n \to \infty$ in (10), we get $\bar{v} \in \bar{u} - K$. Therefore, $\bar{v} \in F(\bar{z}) - K$. By the arbitrariness of $\bar{v}$, we have $F(\bar{w}) \subset F(\bar{z}) - K$, i.e., $F(\bar{w}) \leq^u F(\bar{z})$. Since $\bar{z} \in u - \mathrm{Min}\, F$, we have $F(\bar{w}) \sim^u F(\bar{z})$, and hence $\bar{w} \in u - \mathrm{Min}\, F$ which contradicts (9). □

The following examples show that Theorems 3 and 4 are not comparable.

Example 2 Let $X = Y = \mathbb{R}$, $M = [0, 1]$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by $F(x) = [-x^2 + x, -2x^2 + 2x]$. Clearly, all conditions of Theorem 3 are satisfied but the condition (iv) of Theorem 4 does not hold. Indeed, let $\delta = \frac{1}{4}$, direct calculations give us $\alpha - \mathrm{Min}\, F = \{0, 1\}$ and the level set $L_F^\alpha(x, \varepsilon) = [0, \frac{1-\sqrt{1-4\varepsilon}}{2}] \cup [\frac{1+\sqrt{1-4\varepsilon}}{2}, 1]$ is not connected for every $x \in \alpha - \mathrm{Min}\, F$ and every $\varepsilon \in (0, \delta)$.

Theorem 4 does not require the compactness of constraint set M, and hence, when M is not compact, Theorem 3 does not work while Theorem 4 can apply. Furthermore, the below example show that even in the case M is compact, they are also not comparable.

Example 3 Let $X = Y = \mathbb{R}$, $M = [-2, 2]$, $K = \mathbb{R}^+$, $e = 1$ and $F, G : X \rightrightarrows Y$ be defined by

$$F(x) = \begin{cases} [x^2, 2x^2] & \text{if } -1 \leq x \leq 1, \\ [0, x^2 + 4] & \text{otherwise.} \end{cases}$$

$$G(x) = \begin{cases} [x^2 - 1, x^2 + 1] & \text{if } x = 0, \\ \left[\dfrac{x^2}{4}, \dfrac{x^2}{2}\right] & \text{otherwise.} \end{cases}$$

Then, $u - \mathrm{Min}\, F = l - \mathrm{Min}\, G = \{0\}$, and hence, they are compact. For $\delta = 1$ and for each $\varepsilon \in (0, \delta)$, we have $L_F^u(0, \varepsilon) = [-\sqrt{\frac{\varepsilon}{2}}, \sqrt{\frac{\varepsilon}{2}}]$, $L_F^l(0, \varepsilon) = [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$, and thus, they are connected. Therefore, all conditions of Theorem 4 are satisfied. Employing this theorem, we conclude that the problems (P_u) and (P_l) with respect to F and G, respectively, are generalized well-posed. However, Theorem 3 does not work as F and G are not continuous on M.

The next example illustrates that the assumption (i) of Theorem 4 cannot be dropped.

Example 4 Let $X = l^\infty$ be the space of all bounded sequences of real numbers with the sup norm, $\|\boldsymbol{x}\|_\infty = \sup |x_n|$ for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n, \ldots)$. Let $Y = \mathbb{R}$, $M = X$, $K = \mathbb{R}^+$, $e = 1$ and $F : X \rightrightarrows Y$ be defined by

$$F(\boldsymbol{x}) = [|\|\boldsymbol{x}\| - 1|, |\|\boldsymbol{x}\| - 1| + 1].$$

Then, X is a metric space with the metric $d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. Since X is an infinite dimensional space, we conclude that (X, d_∞) is not locally compact, and hence, the assumption (i) of Theorem 4 is not satisfied. It is obvious that $\alpha - \operatorname{Min} F = \{\boldsymbol{x} \in X \mid \|\boldsymbol{x}\| = 1\}$, and F is continuous and compact-valued on M. Moreover, for $\delta = 1$, the level set $L_F^\alpha(\boldsymbol{x}, \varepsilon) = \{\boldsymbol{x} \in X \mid 1 - \varepsilon \le \|\boldsymbol{x}\| \le 1 + \varepsilon\}$ is connected for every $\boldsymbol{x} \in \alpha - \operatorname{Min} F$ and for every $\varepsilon \in (0, \delta)$. Let $\{\boldsymbol{x}^i\} \subset M$, $\boldsymbol{x}^i$ here is the sequence which is zero everywhere except for a 1 at the ith position. Then, $\{\boldsymbol{x}^i\}$ is a minimizing sequence, but it has no convergent subsequence. So, the problem (P_α) is not generalized well-posedness and the assumption (i) is crucial.

4 Links Between Well-Posedness and Stability

In this section, we study some characterizations and criteria of well-posedness for (P_α). The generalized well-posedness can be characterized by the behavior of $S_{\alpha-\operatorname{Min} F}(\varepsilon)$ as $\varepsilon \to 0$, which is given by

$$S_{\alpha-\operatorname{Min} F}(\varepsilon) := \{x \in M \mid \exists z \in \alpha - \operatorname{Min} F,\ F(x) \le^\alpha F(z) + \varepsilon e\}.$$

The set-valued map $S_{\alpha-\operatorname{Min} F} : \mathbb{R}^+ \rightrightarrows M$ is considered as approximating solution map of (P_α), and it has a closed relationship with the level set, more precisely, $S_{\alpha-\operatorname{Min} F}(\varepsilon) = \bigcap_{x\in\alpha-\operatorname{Min} F} L_F^\alpha(x, \varepsilon)$.

The following proposition gives some properties of the map $S_{\alpha-\operatorname{Min} F}$ which are useful in the sequel.

Proposition 3 *The following statements hold:*

(i) $S_{\alpha-\operatorname{Min} F}(0) = \alpha - \operatorname{Min} F$.
(ii) *If* $\varepsilon_1 \le \varepsilon_2$, *then* $S_{\alpha-\operatorname{Min} F}(\varepsilon_1) \subset S_{\alpha-\operatorname{Min} F}(\varepsilon_2)$.
(iii) $\bigcap_{\varepsilon>0} S_{\alpha-\operatorname{Min} F}(\varepsilon) = \alpha - \operatorname{Min} F$ *if* F *is compact-valued on* M.

Proof We only prove the assertions (i)–(iii) for the case $\alpha = s$; proofs of these assertions for cases $\alpha = l$ and $\alpha = u$ are given by similar arguments.

(i) Obviously, $\alpha - \operatorname{Min} F \subset S_{\alpha-\operatorname{Min} F}(0)$. Conversely, let $x \in S_{s-\operatorname{Min} F}(0)$, there exists $z \in s - \operatorname{Min} F$ such that $F(x) \le^s F(z)$. Taking $y \in M$ satisfying $F(y) \le^s F(x)$, we show that $F(x) \le^s F(y)$. By the transitivity property of $\le^s$, we have $F(y) \le^s F(z)$, and hence, $F(z) \le^s F(y)$ as $z \in s - \operatorname{Min} F$. Again by the transitivity property, one gets $F(x) \le^s F(y)$. So, $x \in s - \operatorname{Min} F$.
(ii) Assume $\varepsilon_1 \le \varepsilon_2$. Let $x \in S_{s-\operatorname{Min} F}(\varepsilon_1)$, there exists $z \in s - \operatorname{Min} F$ such that $F(x) \le^s F(z) + \varepsilon_1 e$, i.e.,

$$F(z) + \varepsilon_1 e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon_1 e - K.$$

We have

$$F(z) + \varepsilon_2 e = F(z) + \varepsilon_1 e + (\varepsilon_2 - \varepsilon_1)e,$$

and

$$F(z) + \varepsilon_2 e - K = F(z) + \varepsilon_1 e - K + (\varepsilon_2 - \varepsilon_1)e.$$

Combining the convexity of K with Proposition 1, we obtain that

$$F(z) + \varepsilon_2 e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon_2 e - K,$$

i.e., $F(x) \leq^s F(z) + \varepsilon_2 e$. Consequently, $x \in S_{s-\mathrm{Min}\,F}(\varepsilon_2)$ as $z \in s - \mathrm{Min}\,F$. Hence, $S_{s-\mathrm{Min}\,F}(\varepsilon_1) \subset S_{s-\mathrm{Min}\,F}(\varepsilon_2)$.
(iii) Let $x \in s - \mathrm{Min}\,F$ and $\varepsilon > 0$ be arbitrary. We have $x \in S_{s-\mathrm{Min}\,F}(\varepsilon)$ for any $\varepsilon > 0$, and so $x \in \bigcap_{\varepsilon>0} S_{s-\mathrm{Min}\,F}(\varepsilon)$. Conversely, let $x \in \bigcap_{\varepsilon>0} S_{s-\mathrm{Min}\,F}(\varepsilon)$, we get $x \in S_{s-\mathrm{Min}\,F}(\varepsilon)$ for any $\varepsilon > 0$. By the definition of $S_{s-\mathrm{Min}\,F}(\varepsilon)$, there exists $z \in s - \mathrm{Min}\,F$ such that $F(x) \leq^s F(z) + \varepsilon e$, i.e.,

$$F(z) + \varepsilon e \subset F(x) + K, \quad \text{and} \quad F(x) \subset F(z) + \varepsilon e - K. \tag{11}$$

Since F is compact-valued, $F(x) - K$ and $F(z) - K$ are closed by Lemma 3. From (11), let $\varepsilon \to 0$, we have

$$F(z) \subset F(x) + K \quad \text{and} \quad F(x) \subset F(z) - K.$$

Equivalently, $F(x) \leq^s F(z)$. This together with $z \in s - \mathrm{Min}\,F$ implies that $x \in s - \mathrm{Min}\,F$. Indeed, suppose that $F(y) \leq^s F(x)$ for some $y \in M$. Then, $F(y) \leq^s F(z)$ because $F(x) \leq^s F(z)$. Moreover, as $z \in \alpha - \mathrm{Min}\,F$, we get $F(z) \leq^s F(y)$. It yields that $F(x) \leq^s F(y)$, and hence, $x \in s - \mathrm{Min}\,F$. So, $\bigcap_{\varepsilon>0} S_{s-\mathrm{Min}\,F}(\varepsilon) \subset s - \mathrm{Min}\,F$. □

Using the Kuratowski measure of noncompactness of approximate solution sets, we now establish a metric characterization of the generalized well-posedness for (P_α).

Theorem 5 (i) *If the problem* (P_α) *is generalized well-posed, then* $\mu(S_{\alpha-\mathrm{Min}\,F}(\varepsilon)) \to 0$ *as* $\varepsilon \to 0$.
(ii) *If the following conditions hold:*

(a) *F is compact-valued on M.*
(b) *$S_{\alpha-\mathrm{Min}\,F}(\varepsilon)$ is closed for all $\varepsilon > 0$.*
(c) *$\mu(S_{\alpha-\mathrm{Min}\,F}(\varepsilon)) \to 0$ as $\varepsilon \to 0$.*

Then, the problem (P_α) *is generalized well-posed.*

Proof (i) Suppose that (P_α) is generalized well-posed. Notice that $\alpha - \mathrm{Min}\,F$ is compact by Theorem 1. Hence, for any $\varepsilon > 0$, there are M_i $(i = 1, \ldots, n)$ with $\mathrm{diam}\, M_i \leq \varepsilon$ and $\alpha - \mathrm{Min}\,F \subset \bigcup_{i=1}^n M_i$. For each $i \in \{1, \ldots, n\}$, we denote

$$N_i := \{x \in X \mid d(x, M_i) \leq H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F)\}.$$

Firstly, we show that $S_{\alpha-\mathrm{Min}\,F}(\varepsilon) \subset \bigcup_{i=1}^n N_i$. Letting $x \in S_{\alpha-\mathrm{Min}\,F}(\varepsilon)$, we have

$$d(x, \alpha - \mathrm{Min}\,F) \leq H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F).$$

Since $\alpha - \mathrm{Min}\,F \subset \bigcup_{i=1}^n M_i$, we conclude that

$$d(x, \cup_{i=1}^n M_i) \leq d(x, \alpha - \mathrm{Min}\,F) \leq H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F).$$

So, there is $k_0 \in \{1, \ldots, n\}$ such that

$$d(x, M_{k_0}) \leq H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F),$$

i.e., $x \in N_{k_0}$. Hence, $S_{\alpha-\mathrm{Min}\,F}(\varepsilon) \subset \bigcup_{i=1}^n N_i$. Notice further that

$$\begin{aligned} \mathrm{diam}\, N_i &= \mathrm{diam}\, M_i + 2H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F) \\ &\leq \varepsilon + 2H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F). \end{aligned}$$

Therefore,

$$\mu(S_{\alpha-\mathrm{Min}\,F}(\varepsilon)) \leq \mu(\alpha - \mathrm{Min}\,F) + 2H(S_{\alpha-\mathrm{Min}\,F}(\varepsilon), \alpha - \mathrm{Min}\,F).$$

Since $\alpha - \text{Min}\, F$ is compact, $\mu(\alpha - \text{Min}\, F) = 0$. Hence,

$$\mu(S_{\alpha-\text{Min}\, F}(\varepsilon)) \leq 2H(S_{\alpha-\text{Min}\, F}(\varepsilon), \alpha - \text{Min}\, F).$$

Finally, we show that $H(S_{\alpha-\text{Min}\, F}(\varepsilon), \alpha - \text{Min}\, F) \to 0$ as $\varepsilon \to 0$. Because $\alpha - \text{Min}\, F \subset S_{\alpha-\text{Min}\, F}(\varepsilon)$, $H^*(\alpha - \text{Min}\, F, S_{\alpha-\text{Min}\, F}(\varepsilon)) = 0$, and hence, we only need to prove that $H^*(S_{\alpha-\text{Min}\, F}(\varepsilon), \alpha - \text{Min}\, F) \to 0$ as $\varepsilon \to 0$. Assume, by contradiction, that there exist a real number $r > 0$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging 0 such that for each n, there exists $x_n \in S_{\alpha-\text{Min}\, F}(\varepsilon_n)$ satisfying

$$d(x_n, \alpha - \text{Min}\, F) \geq r. \tag{12}$$

Because $x_n \in S_{\alpha-\text{Min}\, F}(\varepsilon_n)$, for each n, there exists $z_n \in \alpha - \text{Min}\, F$ such that $F(x_n) \leq^{\alpha} F(z_n) + \varepsilon_n e$. This means that $\{x_n\}$ is a generalized minimizing sequence of (P_α). So, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $x \in \alpha - \text{Min}\, F$. Therefore, for n_k sufficiently large, we have $d(x_{n_k}, x) < r$ which contradicts (12). Hence, $\mu(S_{\alpha-\text{Min}\, F}(\varepsilon)) \to 0$ as $\varepsilon \to 0$.

(ii) Let $\{x_n\}$ be a generalized minimizing sequence of (P_α), then there exist $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $z_n \in \alpha - \text{Min}\, F$ satisfying $F(x_n) \leq^{\alpha} F(z_n) + \varepsilon_n e$. Thus, $x_n \in S_{\alpha-\text{Min}\, F}(\varepsilon_n)$. Using Proposition 3, we have that $\alpha - \text{Min}\, F$ is compact and $H(S_{\alpha-\text{Min}\, F}(\varepsilon_n), \alpha - \text{Min}\, F) \to 0$. We get $d(x_n, \alpha - \text{Min}\, F) \to 0$ as $d(x_n, \alpha - \text{Min}\, F) \leq H(S_{\alpha-\text{Min}\, F}(\varepsilon_n), \alpha - \text{Min}\, F)$. Therefore, there exists a sequence $\{\bar{x}_n\} \subset \alpha - \text{Min}\, F$ such that $d(x_n, \bar{x}_n) \to 0$ as $n \to \infty$. By the compactness of $\alpha - \text{Min}\, F$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to some $\bar{x} \in \alpha - \text{Min}\, F$. Then, the corresponding subsequence $\{x_{n_k}\}$ converges to $\bar{x}$. So, (P_α) is generalized well-posed. □

Remark 5 For the necessary conditions of generalized well-posedness, our result in Theorem 5 (i) improves [31, Proposition 4.1]. More precisely, the generalized well-posedness of (P_α) is obtained without using compactness of solution set, closedness of approximating solution set, and compact values of F imposed in [31, Proposition 4.1].

In Theorem 5 (ii), we use assumption about the closedness of approximating solution sets. In the next result, we give sufficient conditions for this assumption.

Proposition 4 *Assume that the following conditions are satisfied:*

(i) *F is continuous and compact-valued on M.*
(ii) *$\alpha - \text{Min}\, F$ is compact.*

Then, $S_{\alpha-\text{Min}\, F}(\varepsilon)$ is closed for each $\varepsilon \geq 0$.

Proof By the similarity, we only prove the assertion for the case $\alpha = u$. For each $\varepsilon \geq 0$, let $\{x_n\} \subset S_{u-\text{Min}\, F}(\varepsilon)$ be a sequence converging to some $x \in M$, we need to prove that $x \in S_{u-\text{Min}\, F}(\varepsilon)$. Since $\{x_n\} \subset S_{u-\text{Min}\, F}(\varepsilon)$, there exist $z_n \in u - \text{Min}\, F$ such that

$$F(x_n) \leq^{u} F(z_n) + \varepsilon e. \tag{13}$$

By the compactness of $u - \text{Min}\, F$, there exist a subsequence of $\{z_n\}$ which is still denoted by $\{z_n\}$ and $z \in u - \text{Min}\, F$ such that $\{z_n\}$ converges to z. Next, we show that $F(x) \leq^{u} F(z) + \varepsilon e$. Indeed, from (13), we get

$$F(x_n) \subset F(z_n) + \varepsilon e - K. \tag{14}$$

Let $y \in F(x)$, by the lower semicontinuity of F, there exists $\{y_n\}$ converging to y where $y_n \in F(x_n)$. Combining this with (14), for each n, there exists $w_n \in F(z_n)$ such that

$$y_n \in w_n + \varepsilon e - K. \tag{15}$$

Since F is upper semicontinuous and compact-valued at z, there exist $w \in F(z)$ and a subsequence of $\{w_n\}$ which is still denoted by $\{w_n\}$ such that $\{w_n\}$ converges to w. From (15), there exist $k_n \in K$ satisfying $y_n = w_n + \varepsilon e - k_n$, i.e., $k_n = w_n + \varepsilon e - y_n$. Then, k_n converge to $w + \varepsilon e - y \in K$ as K is closed. Therefore, there exists $k \in K$ such that $y = w + \varepsilon e - k \in w + \varepsilon e - K \subset F(z) + \varepsilon e - K$ as $w \in F(z)$. We arrive at the fact that $F(x) \subset F(z) + \varepsilon e - K$. It means that $F(x) \leq^u F(z) + \varepsilon e$. So, $x \in S_{u-\mathrm{Min}\,F}(\varepsilon)$ as $z \in u - \mathrm{Min}\,F$. □

The next result gives sufficient and necessary conditions for generalized well-posedness of (P$_\alpha$) via upper semicontinuity of approximating solution map of (P$_\alpha$). When $\alpha = l$, this result coincides with [31, Proposition 4.3 (i)].

Theorem 6 *$S_{\alpha-\mathrm{Min}\,F}$ is upper semicontinuous and compact-valued at* 0 *if and only if the problem* (P$_\alpha$) *is generalized well-posed.*

Proof Suppose that (P$_\alpha$) is generalized well-posed. By Theorem 1, $\alpha - \mathrm{Min}\,F$ is compact. By contradiction, suppose that $S_{\alpha-\mathrm{Min}\,F}$ is not upper semicontinuous at 0. Then, there exists a neighborhood N of $S_{\alpha-\mathrm{Min}\,F}(0)$ such that for any neighborhood U of 0, $S_{\alpha-\mathrm{Min}\,F}(U) \not\subset N$. It means that there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 such that for each n, $S_{\alpha-\mathrm{Min}\,F}(\varepsilon_n) \not\subset N$. Then, there exist $x_n \in S_{\alpha-\mathrm{Min}\,F}(\varepsilon_n) \setminus N$ for all n, and hence, there exist $z_n \in \alpha - \mathrm{Min}\,F$ such that $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. This implies that $\{x_n\}$ is a generalized minimizing sequence of (P$_\alpha$). Because (P$_\alpha$) is generalized well-posed, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in \alpha - \mathrm{Min}\,F$ such that $\{x_{n_k}\}$ converges to z which is impossible since $x_{n_k} \notin N$ for all n_k.

Conversely, let $\{x_n\} \subset M$ be a generalized minimizing sequence of (P$_\alpha$), there are $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 and $z_n \in \alpha - \mathrm{Min}\,F$ satisfying $F(x_n) \leq^\alpha F(z_n) + \varepsilon_n e$. This means that $x_n \in S_{\alpha-\mathrm{Min}\,F}(\varepsilon_n)$. Since $S_{\alpha-\mathrm{Min}\,F}(\cdot)$ is upper semicontinuous at 0 and N is a neighborhood of $S_{\alpha-\mathrm{Min}\,F}(0)$, $x_n \in N$ for n sufficiently large. Equivalently, for every neighborhood W of 0, there exists $n_0 \in \mathbb{N}$ such that $x_n \in \alpha - \mathrm{Min}\,F + W$ for any $n \geq n_0$. By the compactness of $\alpha - \mathrm{Min}\,F$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in \alpha - \mathrm{Min}\,F$ such that $\{x_{n_k}\}$ converges to $\bar{x}$. So, (P$_\alpha$) is generalized well-posed. □

To end up this section, we give the sufficient conditions for the upper semicontinuity of approximating solution map of (P$_\alpha$) used in the previous result.

Proposition 5 *If M is compact and the conditions of Proposition 4 are satisfied, then $S_{\alpha-\mathrm{Min}\,F}$ is upper semicontinuous at* 0.

Proof We only give the proof of the assertion for the case $\alpha = u$. Suppose to the contrary that $S_{u-\mathrm{Min}\,F}$ is not upper semicontinuous at 0. Then, there exist an open neighborhood W_0 of $S_{u-\mathrm{Min}\,F}(0)$ and a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ converging to 0 such that $S_{u-\mathrm{Min}\,F}(\varepsilon_n) \not\subset W_0$ for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, there exists

$$x_n \in S_{u-\mathrm{Min}\,F}(\varepsilon_n) \setminus W_0. \tag{16}$$

Since M is compact, we can assume that $\{x_n\}$ converges to an element $x_0 \in M$. Moreover, by (16), for each $n \in \mathbb{N}$, there exists $z_n \in u - \text{Min}\, F$ such that $F(x_n) \leq^u F(z_n) + \varepsilon_n e$. Hence,

$$F(x_n) \subset F(z_n) + \varepsilon_n e - K. \tag{17}$$

By the compactness of $u - \text{Min}\, F$, we can assume that $\{z_n\}$ converges to some $z_0 \in u - \text{Min}\, F$. Next, we prove that $F(x_0) \subset F(z_0) - K$. Indeed, for any $u_0 \in F(x_0)$, by the lower semicontinuity of F on M and Lemma 3, there exists a sequence $\{u_n\}$, $u_n \in F(x_n)$, such that $\{u_n\}$ converges to u_0. It follows from (17) that, for each $n \in \mathbb{N}$, there exists $v_n \in F(z_n)$ such that

$$u_n - v_n - \varepsilon_n e \in -K. \tag{18}$$

Since F is upper semicontinuous and compact-valued at z_0, we can assume that $\{v_n\}$ converges to $v_0 \in F(z_0)$. By (18) and the closedness of K, $u_0 - v_0 \in -K$. Therefore, $u_0 \in F(z_0) - K$. Since $u_0 \in F(x_0)$ is arbitrary, $F(x_0) \subset F(z_0) - K$, and hence $x_0 \in S_{u-\text{Min}\, F}(0) \subset W_0$ which is impossible as x_n is not in W_0 for all n. So, $S_{u-\text{Min}\, F}$ is upper semicontinuous at 0. □

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