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Congruences for Partition Quadruples with t-Cores

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Abstract

Let $C_t(n)$ denote the number of partition quadruples of n with t-cores for $t = 3, 5, 7, 25$. We establish some Ramanujan type congruences modulo 5, 7, 8 for $C_t(n)$. For example, $n \geq 0$, we have

$$\begin{aligned} C_5(5n+4) &\equiv 0 \pmod{5}, \\ C_7(7n+6) &\equiv 0 \pmod{7}, \\ C_3(16n+14) &\equiv 0 \pmod{8}. \end{aligned}$$

Keywords Congruences · Partition quadruples · t-core partition

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1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The Ferrers-Young diagram of the partition λ of n is obtained by arranging n nodes in k left aligned rows so that the ith row has λ_i nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. Let λ'_j denote the number of nodes in column j. The hook number $H(i, j)$ of the (i, j) node is defined as the number of nodes directly below and to the right of the node including the node itself, i.e., $H(i, j) = \lambda_i + \lambda'_j - j - i + 1$. A t-core is a partition with no hook number that are divisible by t.

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For example, the Ferrers-Young diagram of the partition $\lambda = (5, 3, 2)$ of 10 is

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{matrix}$$

The nodes (1, 1), (1, 2), (1, 3), (1, 4), (1,5), (2, 1), (2, 2), (2, 3), (3, 1), and (3, 2) have hook numbers 7, 6, 4, 2, 1, 4, 3, 1, 2, and 1, respectively. Therefore, λ is a t-core partition for $t = 5$ and for all $t \geq 8$.

Let $a_t(n)$ be the number of partitions of n that are t-cores. Then, its generating function is given by [4, Eq. (2.1)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t; q^t)_\infty^t}{(q; q)_\infty}.$$

Ramanujan's three famous congruences of $p(n)$ are as follows:

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

In [5, 6], Hischhorn and Sellers have studied the 4-core partition (i.e., $a_4(n)$) and established some infinite families of arithmetic relations for $a_4(n)$. Baruah and Nath [1] have proved some more infinite families of arithmetic identities for $a_4(n)$.

A bipartition of n is a pair of partitions (λ_1, λ_2) such that the sum of all parts of λ_1 and λ_2 equals n. A bipartition with t-core of n is a bipartition (λ_1, λ_2) of n such that λ_1 and λ_2 are both t-cores. Let $A_t(n)$ denote the number of bipartitions with t-cores of n. The generating function for $A_t(n)$ is given by

$$\sum_{n=0}^{\infty} A_t(n)q^n = \frac{(q^t; q^t)_\infty^{2t}}{(q; q)_\infty^2}.$$

Recently, Lin [8] has established some congruence and infinite families for $A_3(n)$. In [2], Baruah and Nath have found three infinite families of $A_3(n)$.

A partition $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a positive integer n is a k-tuple of partitions such that the sum of all the parts equals to n. A partition k-tuple of n with t-cores is a partition k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n where each λ_i is t-core for $i = 1, 2, 3, \ldots, k$.

In 2015, Wang [10] has found several infinite families of arithmetic identities and congruences for partition triples with t-cores.

Motivated by the above works, we define $C_t(n)$ to be the number of partition quadruples of n with t-cores. The generating function is given by

$$\sum_{n=0}^{\infty} C_t(n)q^n = \frac{(q^t; q^t)_\infty^{4t}}{(q; q)_\infty^4}. \tag{1.1}$$

In this paper, we establish several congruences modulo 5, 7, and 8 for $C_t(n)$. The main results can be found in Theorems 3.2, 3.3, 4.1, and 5.1.

2 Preliminaries

In this section, we list some identities which play a vital role in proving our main results.

For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

The product representation of $f(a, b)$ arises from Jacobi's triple product identity [3, p. 35, Entry 19] as

$$f(a,b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

Some special cases of $f(a, b)$, known as Ramanujan's theta functions, are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty.$$

Lemma 2.1 [9, p. 212] *We have the following 5-dissection*

$$(q; q)_\infty = (q^{25}; q^{25})_\infty \left(a - q - q^2/a\right), \tag{2.1}$$

where

$$a := a(q) := \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty}.$$

Lemma 2.2 *For any prime p and positive integer n,*

$$(q; q)_\infty^{p^n} \equiv (q^p; q^p)_\infty^{p^{n-1}} \pmod{p^n}. \tag{2.2}$$

Lemma 2.3 *The following 2-dissections hold:*

$$\frac{(q^3; q^3)_\infty^3}{(q; q)_\infty} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} + q\frac{(q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty}, \tag{2.3}$$

$$\frac{(q^3; q^3)_\infty}{(q; q)_\infty} = \frac{(q^4; q^4)_\infty (q^6; q^6)_\infty (q^{16}; q^{16})_\infty (q^{24}; q^{24})_\infty^2}{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty (q^{12}; q^{12})_\infty (q^{48}; q^{48})_\infty} + q\frac{(q^6; q^6)_\infty (q^8; q^8)_\infty^2 (q^{48}; q^{48})_\infty}{(q^2; q^2)_\infty^2 (q^{16}; q^{16})_\infty (q^{24}; q^{24})_\infty}, \tag{2.4}$$

$$\frac{(q^3; q^3)_\infty^2}{(q; q)_\infty^2} = \frac{(q^4; q^4)_\infty^4 (q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty^5 (q^8; q^8)_\infty (q^{24}; q^{24})_\infty} + 2q\frac{(q^4; q^4)_\infty (q^6; q^6)_\infty^2 (q^8; q^8)_\infty (q^{24}; q^{24})_\infty}{(q^2; q^2)_\infty^4 (q^{12}; q^{12})_\infty}. \tag{2.5}$$

Hirschhorn, Garvan, and Borwein [4] proved (2.3). Xia and Yao [12] gave a proof of (2.4) and in [11], they also proved (2.5) by employing an addition formula for theta functions.

Lemma 2.4 [3, p. 345, Entry 1 (iv)] *We have the following 3-dissection*

$$(q;q)_\infty^3 = \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^6}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty^3} + 4q^3 \frac{(q^3;q^3)_\infty^2 (q^{18};q^{18})_\infty^6}{(q^6;q^6)_\infty^2 (q^9;q^9)_\infty^3} - 3q(q^9;q^9)_\infty^3. \quad (2.6)$$

Lemma 2.5 *The following 3-dissection holds:*

$$(q;q)_\infty (q^2;q^2)_\infty = \frac{(q^6;q^6)_\infty (q^9;q^9)_\infty^4}{(q^3;q^3)_\infty (q^{18};q^{18})_\infty^2} - q(q^9;q^9)_\infty (q^{18};q^{18})_\infty - 2q^2 \frac{(q^3;q^3)_\infty (q^{18};q^{18})_\infty^4}{(q^6;q^6)_\infty (q^9;q^9)_\infty^2}$$

One can find this identity in [7].

Lemma 2.6 [3, 3, p. 303, Entry 17 (v)] *We have*

$$(q;q)_\infty = (q^{49};q^{49})_\infty \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5 \frac{C(q^7)}{A(q^7)} \right), \quad (2.7)$$

where $A(q) := \frac{f(-q^3,-q^4)}{f(-q^2)}$, $B(q) := \frac{f(-q^2,-q^5)}{f(-q^2)}$ *and* $C(q) := \frac{f(-q,-q^6)}{f(-q^2)}$.

In the following sections, with the aid of preliminary results, we prove our main results.

3 Congruence Modulo 8 for $C_3(n)$

Theorem 3.1 *For each* $n \geq 0$, *we have*

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(4n)q^n &= \frac{(q^2;q^2)_\infty^{16}(q^6;q^6)_\infty^8}{(q;q)_\infty^8 (q^4;q^4)_\infty^4 (q^{12};q^{12})_\infty^4} \\ &+24q \frac{(q^2;q^2)_\infty^{10}(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^2}{(q;q)_\infty^6} \\ &+16q^2 \frac{(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^4 (q^4;q^4)_\infty^4 (q^{12};q^{12})_\infty^4}{(q;q)_\infty^4 (q^6;q^6)_\infty^4} \\ &+24q \frac{(q^2;q^2)_\infty^5 (q^3;q^3)_\infty^7 (q^6;q^6)_\infty}{(q;q)_\infty^5} + q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4}, \quad (3.1)\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(4n+1)q^n &= 4\frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^6}{(q;q)_\infty^7 (q^4;q^4)_\infty^3 (q^{12};q^{12})_\infty^3} \\ &+48q \frac{(q^2;q^2)_\infty^6 (q^3;q^3)_\infty^5 (q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^5} \\ &+8q \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^{10} (q^4;q^4)_\infty (q^{12};q^{12})_\infty}{(q;q)_\infty^4 (q^6;q^6)_\infty}, \quad (3.2)\end{aligned}$$

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n = 8\frac{(q^2;q^2)_\infty^{13}(q^3;q^3)_\infty(q^6;q^6)_\infty^5}{(q;q)_\infty^7(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2} +32q\frac{(q^2;q^2)_\infty^7(q^3;q^3)_\infty^3(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q;q)_\infty^5(q^6;q^6)_\infty} +6\frac{(q^2;q^2)_\infty^8(q^3;q^3)_\infty^6(q^6;q^6)_\infty^4}{(q;q)_\infty^6(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2} +24q\frac{(q^2;q^2)_\infty^2(q^3;q^3)_\infty^8(q^4;q^4)_\infty^2(q^{12};q^{12})_\infty^2}{(q;q)_\infty^4(q^6;q^6)_\infty^2}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} C_3(4n+3)q^n = 24\frac{(q^2;q^2)_\infty^9(q^3;q^3)_\infty^4(q^6;q^6)_\infty^3}{(q;q)_\infty^6(q^4;q^4)_\infty(q^{12};q^{12})_\infty} +32q\frac{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^6(q^4;q^4)_\infty^3(q^{12};q^{12})_\infty^3}{(q;q)_\infty^4(q^6;q^6)_\infty^3} +4\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^9(q^6;q^6)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty(q^{12};q^{12})_\infty}. \tag{3.4}$$

Proof Setting $t=3$ in (1.1), we have

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4} = \left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^4. \tag{3.5}$$

Substituting (2.3) into (3.5), we get

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{(q^4;q^4)_\infty^{12}(q^6;q^6)_\infty^8}{(q^2;q^2)_\infty^8(q^{12};q^{12})_\infty^4} + 4q\frac{(q^4;q^4)_\infty^8(q^6;q^6)_\infty^6}{(q^2;q^2)_\infty^6} +6q^2\frac{(q^4;q^4)_\infty^4(q^6;q^6)_\infty^4(q^{12};q^{12})_\infty^4}{(q^2;q^2)_\infty^4} +4q^3\frac{(q^6;q^6)_\infty^2(q^{12};q^{12})_\infty^8}{(q^2;q^2)_\infty^2} + q^4\frac{(q^{12};q^{12})_\infty^{12}}{(q^4;q^4)_\infty^4}. \tag{3.6}$$

Extracting the even terms of the above equation, one obtains

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^8}{(q;q)_\infty^8(q^6;q^6)_\infty^4} + 6q\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}{(q;q)_\infty^4} + q^2\frac{(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^4},$$

which yields

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{(q^2;q^2)_\infty^{12}}{(q^6;q^6)_\infty^4}\left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right)^4 +6q(q^2;q^2)_\infty^4(q^6;q^6)_\infty^4\left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right)^2 + q^2\frac{(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^4}. \tag{3.7}$$

Substituting (2.5) into (3.7) and extracting the terms involving q^{2n} and q^{2n+1}, we get (3.1) and (3.3).

From (3.6), one gets

$$\sum_{n=0}^{\infty} C_3(2n+1)q^n = 4\frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^6}{(q;q)_\infty^6} + 4q\frac{(q^3;q^3)_\infty^2 (q^6;q^6)_\infty^8}{(q;q)_\infty^2},$$

which implies that

$$\sum_{n=0}^{\infty} C_3(2n+1)q^n = 4(q^2;q^2)_\infty^8 \left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right)^3 + 4q(q^6;q^6)_\infty^8 \left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right). \quad (3.8)$$

Substituting (2.5) into (3.8) and extracting the even and odd terms of the above equation, we obtain (3.2) and (3.4). □

Theorem 3.2 *For each* $\alpha \geq 0$ *and* $n \geq 1$, *we have*

$$C_3(16n+14) \equiv 0 \pmod 8, \quad (3.9)$$

$$C_3(48n+30) \equiv 0 \pmod 8, \quad (3.10)$$

$$C_3\left(16^{\alpha+1}n + \frac{16\cdot 4^\alpha - 4}{3}\right) \equiv C_3(4n) \pmod 8. \quad (3.11)$$

Proof From (3.3), we have

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{(q^2;q^2)_\infty^8 (q^3;q^3)_\infty^6 (q^6;q^6)_\infty^4}{(q;q)_\infty^6 (q^4;q^4)_\infty^2 (q^{12};q^{12})_\infty^2} \pmod 8. \quad (3.12)$$

Using (2.2) in (3.12), one gets

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{(q^3;q^3)_\infty^6}{(q;q)_\infty^6} \equiv 6\left(\frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2}\right)^3 \pmod 8. \quad (3.13)$$

Substituting (2.5) into (3.13), we have

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{(q^4;q^4)_\infty^{12} (q^6;q^6)_\infty^3 (q^{12};q^{12})_\infty^6}{(q^2;q^2)_\infty^{15} (q^8;q^8)_\infty^3 (q^{24};q^{24})_\infty^3} + 4q\frac{(q^4;q^4)_\infty^9 (q^6;q^6)_\infty^4 (q^{12};q^{12})_\infty^3}{(q^2;q^2)_\infty^{14} (q^8;q^8)_\infty (q^{24};q^{24})_\infty} \pmod 8. \quad (3.14)$$

Extracting the terms involving q^{2n+1} from (3.14), dividing by q and then replacing q^2 by q,

$$\sum_{n=0}^{\infty} C_3(8n+6)q^n \equiv 4\frac{(q^2;q^2)_\infty^9 (q^3;q^3)_\infty^4 (q^6;q^6)_\infty^3}{(q;q)_\infty^{14} (q^4;q^4)_\infty (q^{12};q^{12})_\infty} \pmod 8. \quad (3.15)$$

Invoking (2.2) in (3.15), one obtains

$$\sum_{n=0}^{\infty} C_3(8n+6)q^n \equiv 4(q^6;q^6)_\infty^3 \pmod 8. \quad (3.16)$$

Congruence (3.9) follows from (3.16).

From (3.16), we have

$$\sum_{n=0}^{\infty} C_3(24n+6)q^n \equiv 4(q^2;q^2)_\infty^3 \pmod 8. \tag{3.17}$$

Congruence (3.10) easily follows from the above equation.

From (3.1), one gets

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv \frac{(q^2;q^2)_\infty^{16}(q^6;q^6)_\infty^8}{(q;q)_\infty^8(q^4;q^4)_\infty^4(q^{12};q^{12})_\infty^4} + q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4} \pmod 8. \tag{3.18}$$

Invoking (2.2) in (3.18), we have

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv (q^2;q^2)_\infty^4 + q\left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^4 \pmod 8. \tag{3.19}$$

Substituting (2.3) into the second term of (3.19) and extracting the odd terms of the required equation

$$\begin{aligned}\sum_{n=0}^{\infty} C_3(8n+4)q^n \equiv\ & \frac{(q^2;q^2)_\infty^{12}(q^3;q^3)_\infty^8}{(q;q)_\infty^8(q^6;q^6)_\infty^4} + 6q\frac{(q^2;q^2)_\infty^4(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}{(q;q)_\infty^4} \\ & + q^2\frac{(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^4} \pmod 8.\end{aligned} \tag{3.20}$$

Using (2.2) in (3.20), one checks that

$$\sum_{n=0}^{\infty} C_3(8n+4)q^n \equiv (q^2;q^2)_\infty^8 + 6q(q^2;q^2)_\infty^2(q^6;q^6)_\infty^6 + q^2\frac{(q^6;q^6)_\infty^{12}}{(q^2;q^2)_\infty^4} \pmod 8. \tag{3.21}$$

Extracting the terms involving q^{2n} from (3.21) and then replacing q^2 by q,

$$\sum_{n=0}^{\infty} C_3(16n+4)q^n \equiv (q;q)_\infty^8 + q\frac{(q^3;q^3)_\infty^{12}}{(q;q)_\infty^4} \pmod 8. \tag{3.22}$$

Invoking (2.2) in (3.22), we have

$$\sum_{n=0}^{\infty} C_3(16n+4)q^n \equiv (q^2;q^2)_\infty^4 + q\left(\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}\right)^4 \pmod 8. \tag{3.23}$$

Using (3.23) and (3.19), one gets

$$C_3(16n+4) \equiv C_3(4n) \pmod 8.$$

By using mathematical induction on α, we obtain (3.11). □

Theorem 3.3 *For α, β, and $\gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 4(q;q)_\infty^3 \pmod 8, \quad (3.24)$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4(q^7;q^7)_\infty^3 \pmod 8, \quad (3.25)$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\right)q^n \equiv 4(q^5;q^5)_\infty^3 \pmod 8, \quad (3.26)$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 4(q^3;q^3)_\infty^3 \pmod 8, \quad (3.27)$$

$$C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) \equiv \begin{cases} 4 \pmod 8 & \textit{if } n=k(3k+1)/2 \textit{ for some } k\in Z, \\ 0 \pmod 8 & \textit{otherwise}. \end{cases} \quad (3.28)$$

Proof Extracting the terms involving q^{2n} from (3.17) and replacing q^2 by q,

$$\sum_{n=0}^{\infty} C_3(48n+6)q^n \equiv 4(q;q)_\infty^3 \pmod 8. \quad (3.29)$$

(3.29) is the $\alpha=\beta=\gamma=0$ case of (3.24). Let us consider the case $\beta=\gamma=0$. Suppose that the congruence (3.24) holds for some integer $\alpha \geq 0$. Substituting (2.6) in (3.24) with $\beta=\gamma=0$,

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+1}n+2\cdot 3^{2\alpha+1})q^n \equiv 4((q^3;q^3)_\infty+q(q^9;q^9)_\infty^3) \pmod 8,$$

which implies,

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+2}n+2\cdot 3^{2\alpha+3})q^n \equiv 4(q^3;q^3)_\infty^3 \pmod 8.$$

Therefore

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+3}n+2\cdot 3^{2\alpha+3})q^n \equiv 4(q;q)_\infty^3 \pmod 8,$$

which implies that (3.24) is true for $\alpha+1$. Hence, by induction, (3.24) is true for any non-negative integer α and $\beta=\gamma=0$. Let us consider the case $\gamma=0$. Suppose that the congruence (3.24) holds for some integer $\alpha, \beta \geq 0$. Substituting (2.1) in (3.24), we have

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta})q^n \equiv 4(q^{25};q^{25})_\infty^3\left(a-q-q^2/a\right)^3 \pmod 8. \quad (3.30)$$

Extracting the terms involving q^{5n+3} from (3.30), we have

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2})q^n \equiv 4(q^5;q^5)_\infty^3 \pmod 8,$$

which yields

$$\sum_{n=0}^{\infty} C_3(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2})q^n \equiv 4(q;q)_\infty^3 \pmod 8.$$

This implies that (3.24) is true for $\beta+1$. Hence, by induction, (3.24) is true for α, $\beta \geq 0$ and $\gamma = 0$. Now, suppose that the congruence (3.24) holds for some integers α, β, and $\gamma \geq 0$. Substituting (2.7) in (3.24), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n\\ &\equiv 4(q^{49};q^{49})_\infty^3\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right)^3 \pmod 8. \end{aligned} \tag{3.31}$$

Extracting the terms involving q^{7n+6} from (3.31), we get

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4(q^7;q^7)_\infty^3 \pmod 8, \tag{3.32}$$

which prove (3.25). Extracting the coefficient of q^{7n} in (3.32), we arrive

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4(q;q)_\infty^3 \pmod 8,$$

which implies that (3.24) is true for $\gamma+1$. Hence, by induction, (3.24) is true for any non-negative integers α, β, and γ. This completes the proof. Substituting (2.1) in (3.24), we get (3.26). Substituting (2.6) in (3.24) and then extracting q^{3n+1} and q^{3n}, we obtain (3.27) and (3.28), respectively. □

Corollary 1 *For* α, β, *and* $\gamma \geq 0$, $p \in \{30, 46, 62, 78, 94, 110\}$, $q \in \{34, 66\}$, $r \in \{26, 42, 58, 74\}$, *and* $s \in \{22, 38\}$, *we have*

$$\begin{aligned} C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) &\equiv 0 \pmod 8,\\ C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+p\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right) &\equiv 0 \pmod 8,\\ C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+q\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) &\equiv 0 \pmod 8,\\ C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right) &\equiv 0 \pmod 8,\\ C_3\left(16\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+s\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) &\equiv 0 \pmod 8. \end{aligned}$$

4 Congruence Modulo 5 for $C_5(n)$

Theorem 4.1 *For each $n \geq 0$, we have*

$$C_5(5n+3) \equiv 0 \pmod{5}, \tag{4.1}$$
$$C_5(5n+4) \equiv 0 \pmod{5}, \tag{4.2}$$
$$C_5(25n+21) \equiv 0 \pmod{5}. \tag{4.3}$$

Proof Setting $t = 5$ in (1.1), we get

$$\sum_{n=0}^{\infty} C_5(n)q^n = \frac{(q^5;q^5)_\infty^{20}}{(q;q)_\infty^4}. \tag{4.4}$$

Using (2.2) in (4.4), we get

$$\sum_{n=0}^{\infty} C_5(n)q^n \equiv (q;q)_\infty (q^5;q^5)_\infty^{19} \pmod{5}. \tag{4.5}$$

Substituting (2.1) into (4.5), we have

$$\sum_{n=0}^{\infty} C_5(n)q^n \equiv (q^5;q^5)_\infty^{19}(q^{25};q^{25})_\infty \left(a - q - \frac{q^2}{a}\right) \pmod{5}. \tag{4.6}$$

Then, congruences (4.1) and (4.2) follow from (4.6).

Extracting the terms involving q^{5n+1} from (4.6), dividing by q and then replacing q^5 by q,

$$\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv 4(q;q)_\infty^{19}(q^5;q^5)_\infty \pmod{5}. \tag{4.7}$$

Invoking (2.2) in (4.7), one gets

$$\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv 4(q;q)_\infty^{4}(q^5;q^5)_\infty^4 \pmod{5}. \tag{4.8}$$

Again substituting (2.1) into (4.8), one gets

$$\begin{aligned}\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv\ & 4a^4(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4 + 4a^3q(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4 \\ & + 2aq^3(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4 + \frac{3q^5(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4}{a} \\ & + \frac{3q^6(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4}{a^2} + 3a^2q^2(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4 \\ & + \frac{q^7(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4}{a^3} + \frac{4q^8(q^5;q^5)_\infty^4(q^{25};q^{25})_\infty^4}{a^4} \pmod{5}.\end{aligned} \tag{4.9}$$

Congruence (4.3) easily follows from (4.9). □

5 Congruence Modulo 7 for $C_7(n)$

Theorem 5.1 *For each $n \geq 0$, we have*

$$C_7(7n+6) \equiv 0 \pmod 7. \tag{5.1}$$

Proof Setting $t = 7$ in (1.1),

$$\sum_{n=0}^{\infty} C_7(n)q^n = \frac{(q^7;q^7)_\infty^{28}}{(q;q)_\infty^4}. \tag{5.2}$$

Invoking (2.2) in (5.2),

$$\sum_{n=0}^{\infty} C_7(n)q^n \equiv (q;q)_\infty^3 (q^7;q^7)_\infty^{27} \pmod 7. \tag{5.3}$$

Substituting (2.7) into (5.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty} C_7(n)q^n \equiv\ & (q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)^3}{C(q^7)^3} + 4q(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)A(q^7)}{C(q^7)^2} \\ & +3q^5(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)^2}{C(q^7)A(q^7)} + 3q^2(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)^2}{B(q^7)C(q^7)} \\ & +6q^3(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)}{C(q^7)} + q^7(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)}{A(q^7)} \\ & +3q^{10}(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)C(q^7)}{A(q^7)^2} + 3q^7(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)C(q^7)}{B(q^7)^2} \\ & +6q^8(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{C(q^7)}{B(q^7)} + 4q^{11}(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{C(q^7)^2}{A(q^7)B(q^7)}\end{aligned} \tag{5.4}$$

$$\begin{aligned} & +3q^9(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{C(q^7)}{A(q^7)} + 4q^{12}(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{C(q^7)^2}{A(q^7)^2} \\ & +q^{15}(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{C(q^7)^3}{A(q^7)^3} + 4q^2(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)^2}{C(q^7)^2} \\ & +3q^4(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{B(q^7)}{C(q^7)} + 6q^3(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)^3}{B(q^7)^3} \\ & +4q^4(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)^2}{B(q^7)^2} + 4q^5(q^7;q^7)_\infty^{27}(q^{49};q^{49})_\infty^3 \frac{A(q^7)}{B(q^7)} \pmod 7.\end{aligned} \tag{5.5}$$

Congruence (5.1) now follows from (5.4). □

6 Congruence Modulo 5 for $C_{25}(n)$

Theorem 6.1 *For each $n \geq 0$, we have*

$$\begin{aligned} C_{25}(5n+3) &\equiv 0 \pmod 5, \\ C_{25}(5n+4) &\equiv 0 \pmod 5, \\ C_{25}(25n+21) &\equiv 0 \pmod 5. \end{aligned}$$

Proof Setting $t = 25$ in (1.1), we have

$$\sum_{n=0}^{\infty} C_{25}(n)q^n = \frac{(q^{25};q^{25})_\infty^{100}}{(q;q)_\infty^4}.$$

Using (2.2) in (4.4),

$$\sum_{n=0}^{\infty} C_{25}(n)q^n \equiv \frac{(q;q)_\infty (q^{25};q^{25})_\infty^{100}}{(q^5;q^5)_\infty} \pmod 5.$$

The rest of the proof is similar to the proof of Theorem 4.1. Therefore, we omitted the details. □

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