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L^p Metric Geometry of Big and Nef Cohomology Classes

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Abstract
Let (X, ω) be a compact Kähler manifold of dimension n, and let θ be a closed smooth real $(1, 1)$-form representing a big and nef cohomology class. We introduce a metric d_p, $p \geq 1$, on the finite energy space $\mathcal{E}^p(X, \theta)$, making it a complete geodesic metric space.

Keywords Kähler manifolds · Pluripotential theory · Finite energy classes · Complete metric space

Mathematics Subject Classification (2010) 53C55 · 32Q15 · 32W20 · 53C25

1 Introduction

Finding canonical (Kähler-Einstein, cscK, extremal) metrics on compact Kähler manifolds is one of the central questions in differential geometry (see [13, 41, 42] and the references therein). Given a Kähler metric ω on a compact Kähler manifold X, one looks for a Kähler potential φ such that $\omega_\varphi := \omega + dd^c\varphi$ is "canonical". Mabuchi introduced a Riemannian structure on the space of Kähler potentials $\mathcal{H}_\omega$. As shown by Chen [15] $\mathcal{H}_\omega$ endowed with the Mabuchi d_2 distance is a metric space. Darvas [21] showed that its metric completion coincides with a finite energy class of plurisubharmonic functions introduced by Guedj

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and Zeriahi [36]. Other Finsler geometries d_p, $p \geq 1$, on $\mathcal{H}_\omega$ were studied by Darvas [20] and they lead to several spectacular results related to a longstanding conjecture on existence of cscK metrics and properness of K-energy (see [6, 16–18, 29]). Employing the same technique as in [29] and extending the L^1-Finsler structure of [20] to big and semipositive classes via a formula relating the Monge-Ampère energy and the d_1 distance, Darvas [22] established analogous results for singular normal Kähler varieties. Motivated by the same geometric applications, the L^p $(p \geq 1)$ Finsler geometry in big and semipositive cohomology classes was constructed in [32] via an approximation method.

In this note we extend the main results of [20, 32] to the context of big and nef cohomology classes. Assume that X is a compact Kähler manifold of complex dimension n and let θ be a smooth closed real (1, 1) form representing a big & nef cohomology class. Fix $p \geq 1$.

Main Theorem *The space $\mathcal{E}^p(X, \theta)$ endowed with d_p is a complete geodesic metric space.*

For the definition of $\mathcal{E}^p(X, \theta)$, d_p and relevant notions we refer to Section 2. When $p = 1$ Main Theorem was established in [26] in the more general case of big cohomology classes using the approach of [22]. Here, we use an approximation argument as in [32] with an important modification due to the fact that generally potentials in big cohomology classes are unbounded. Interestingly, this modification greatly simplifies the proof of [32, Theorem A].

Organization of the Note We recall relevant notions in pluripotential theory in big cohomology classes in Section 2. The metric space $(\mathcal{E}^p, d_p)$ is introduced in Section 3 where we prove Main Theorem. In case $p = 1$ we show in Proposition 3.18 that the distance d_1 defined in this note and the one defined in [26] do coincide.

2 Preliminaries

Let (X, ω) be a compact Kähler manifold of dimension n. We use the following real differential operators $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, so that $dd^c = 2i\partial\bar{\partial}$. We briefly recall known results in pluripotential theory in big cohomology classes, and refer the reader to [5, 12, 24–27] for more details.

2.1 Quasi-plurisubharmonic Functions

A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is quasi-plurisubharmonic (or quasi-psh) if it is locally the sum of a psh function and a smooth function. Given a smooth closed real (1, 1)-form θ, we let $\mathrm{PSH}(X, \theta)$ denote the set of all integrable quasi-psh functions u such that $\theta_u := \theta + dd^c u \geq 0$, where the inequality is understood in the sense of currents. A function u is said to have analytic singularities if locally $u = \log \sum_{j=1}^N |f_j|^2 + h$, where the $f_j's$ are holomorphic and h is smooth.

The De Rham cohomology class $\{\theta\}$ is Kähler if it contains a Kähler potential, i.e., a function $u \in \mathrm{PSH}(X, \theta) \cap \mathcal{C}^\infty(X, \mathbb{R})$ such that $\theta + dd^c u > 0$. The class $\{\theta\}$ is nef if $\{\theta + \varepsilon\omega\}$ is Kähler for all $\varepsilon > 0$. It is pseudo-effective if the set $\mathrm{PSH}(X, \theta)$ is non-empty, and big if $\{\theta - \varepsilon\omega\}$ is pseudo-effective for some $\varepsilon > 0$. The ample locus of $\{\theta\}$, which will be denoted by $\mathrm{Amp}(\theta)$, is the set of all points $x \in X$ such that there exists $\psi \in \mathrm{PSH}(X, \theta - \varepsilon\omega)$ with analytic singularities and smooth in a neighborhood of x. It was shown in [11, Theorem 3.17] that $\{\theta\}$ is Kähler if and only if $\mathrm{Amp}(\theta) = X$.

Throughout this note we always assume that $\{\theta\}$ is big and nef. Typically, there are no bounded functions in $\mathrm{PSH}(X,\theta)$, but there are plenty of locally bounded functions as we now briefly recall. By the bigness of $\{\theta\}$ there exists $\psi \in \mathrm{PSH}(X, \theta - \varepsilon\omega)$ for some $\varepsilon > 0$. Regularizing ψ (by [30, Main Theorem 1.1]) we can find a function $u \in \mathrm{PSH}(X, \theta - \frac{\varepsilon}{2}\omega)$ smooth in a Zariski open set Ω of X. Roughly speaking, θ_u locally behaves as a Kähler form on Ω. As shown in [11, Theorem 3.17], u and Ω can be constructed in such a way that Ω is the ample locus of $\{\theta\}$.

If u and v are two θ-psh functions on X, then u is said to be *less singular* than v if $v \le u + C$ for some $C \in \mathbb{R}$, while they are said to have the *same singularity type* if $u - C \le v \le u + C$, for some $C \in \mathbb{R}$. A θ-psh function u is said to have *minimal singularities* if it is less singular than any other θ-psh function. An example of a θ-psh function with minimal singularities is

$$V_\theta := \sup\{u \in \mathrm{PSH}(X,\theta) \mid u \le 0\}.$$

For a function $f : X \to \mathbb{R}$, we let f^* denote its upper semicontinuous regularization, i.e.,

$$f^*(x) := \limsup_{X \ni y \to x} f(y).$$

Given a measurable function f on X we define

$$P_\theta(f) := (x \mapsto \sup\{u(x) \mid u \in \mathrm{PSH}(X,\theta),\ u \le f\})^*.$$

Essential Supremum For u, v quasi-psh functions, the function $u - v$ is defined almost everywhere on X (off the set where $v = -\infty$). By abuse of notation we let $\sup_X(u - v)$ denote the essential supremum of $u - v$. By basic properties of plurisubharmonic functions we have

$$u - \sup_X(u - v) \le v \le u + \sup_X(v - u), \text{ on } X.$$

We will need the following result on regularity of quasi plurisubharmonic envelope due to Berman [4].

Theorem 2.1 *Let f be a continuous function such that $dd^c f \le C\omega$ on X, for some $C > 0$. Then $\Delta_\omega(P_\theta(f))$ is locally bounded on* $\mathrm{Amp}(\theta)$*, and*

$$(\theta + dd^c P_\theta(f))^n = \mathbf{1}_{\{P_\theta(f) = f\}}(\theta + dd^c f)^n. \tag{2.1}$$

If θ is moreover Kähler then $\Delta_\omega(P_\theta(f))$ is globally bounded on X.

If $f = \min(u, v)$ for u, v quasi-psh then f is upper semicontinuous on X and there is no need to take the upper semicontinuous regularization in the definition of $P(u, v) := P_\theta(\min(u, v))$. The latter is the largest θ-psh function lying below both u and v, and is called the rooftop envelope of u and v in [28].

The proof of Theorem 2.1 can be found in [4]. In the Kähler case, Theorem 2.1 was also surveyed in [23]. For convenience of the reader, and per recommendation of the referee, we briefly recall the arguments here.

Proof of Theorem 2.1 We first assume that f is smooth and fix $\varepsilon \in (0, 1]$. By nefness of $\{\theta\}$, the form $\eta := \theta + \varepsilon\omega$ represents a Kähler class.

Fix $\beta > 1$ and let $u_\beta \in \mathrm{PSH}(X, \eta) \cap C^\infty(X)$ be the unique smooth function such that

$$(\eta + dd^c u_\beta)^n = e^{\beta(u_\beta - f)}\omega^n. \tag{2.2}$$

The existence (and smoothness) of u_β follows from Aubin [1] and Yau [42].

By [4, Theorem 1.1], u_β converges uniformly to $P_\eta(f)$ along with a uniform estimate for $dd^c u_\beta$. The proof of [4, Theorem 1.2] actually establishes a Laplacian estimate for u_β independent of ε and β.

We fix $\psi \in \mathrm{PSH}(X, \theta)$ such that $\sup_X \psi = 0$, ψ is smooth in Ω, the ample locus of $\{\theta\}$ and $\theta + dd^c\psi \geq a\omega$, where $a > 0$ is a small constant. Note that ψ and a, whose existence follows from the bigness of $\{\theta\}$ as recalled in Section 2.1, are independent of ε.

Consider

$$H := \log \mathrm{Tr}_\omega(\eta + dd^c u_\beta) - A(u_\beta - \psi),$$

defined on Ω, where $A > 0$ is a constant to be specified later. Then, H is smooth on Ω and tends to $-\infty$ on the boundary of Ω. Let $x \in \Omega$ be a point where H attains its maximum in Ω. Setting $\omega' := \eta + dd^c u_\beta$, it follows from [14, Lemma 2.2] (which is an improvement of [40]) that

$$\Delta_{\omega'} \log \mathrm{Tr}_\omega(\omega') \geq \frac{\Delta_\omega(\beta(u_\beta - f))}{\mathrm{Tr}_\omega(\omega')} - B\mathrm{Tr}_{\omega'}(\omega),$$

where $-B$ is a negative lower bound for the holomorphic bisectional curvature of ω. In the remainder of this paragraph we carry all computations at the point x. By the maximum principle, we have

$$0 \geq \Delta_{\omega'} H \geq \beta - \beta\frac{\mathrm{Tr}_\omega(\eta + dd^c f)}{\mathrm{Tr}_\omega(\omega')} - B\mathrm{Tr}_{\omega'}(\omega) - An + Aa\mathrm{Tr}_{\omega'}(\omega).$$

Let $C_1 \geq 0$ be a constant such that $\theta + \omega + dd^c f \leq e^{C_1}\omega$. Then, choosing $A = B/a$, we arrive at

$$0 \geq (\beta - An) - \beta\frac{ne^{C_1}}{\mathrm{Tr}_\omega(\omega')}.$$

Thus, for $\beta \geq 2An$ we have

$$\mathrm{Tr}_\omega(\omega') \leq \frac{\beta n e^{C_1}}{\beta - An} \leq 2ne^{C_1}. \tag{2.3}$$

Let also ρ_0 be the unique θ-psh function with minimal singularities such that

$$(\theta + dd^c\rho_0)^n = C_3\omega^n, \ \sup_X \rho_0 = 0,$$

for a uniform normalization constant $C_3 = C(\theta, \omega) > 0$. The existence of ρ_0 follows from [5, 12]. By [12, Theorem 4.1] we obtain a lower bound for ρ_0:

$$\rho_0 \geq V_\theta - C(\theta, \omega).$$

Since $\rho_0 \leq f - \inf_X f$ we have that $\rho_0 + \inf_X f + (\log C_3)/\beta$ is a subsolution to the Monge-Ampère equation defining u_β, (2.2). By [24, Lemma 2.5] and the fact that $V_\theta \geq \psi$, we have that

$$u_\beta \geq \rho_0 + \inf_X f + (\log C_3)/\beta \geq \psi - C_4,$$

where $C_4 > 0$ depends on θ, ω, $\inf_X f$. From this and (2.3), we thus obtain

$$H(x) \leq \log(2ne^{C_1}) + AC_4.$$

We finally have, for all $\beta \geq 2nA$,

$$\mathrm{Tr}_\omega(\eta + dd^c u_\beta) \leq C_5 e^{-A\psi} \text{ on } \Omega.$$

Letting $\beta \to +\infty$ and noting that u_β converges uniformly to $P_{\theta+\varepsilon\omega}(f)$, we obtain

$$\Delta_\omega(P_{\theta+\varepsilon\omega}(f)) \leq C_6 e^{-A\psi},$$

where C_6 depends on $B, a, C_1, \inf_X f$. Letting $\varepsilon \to 0$ we arrive at

$$\Delta_\omega(P_\theta(f)) \leq C_6 e^{-A\psi}.$$

We finally remove the smoothness assumption on f. Assume that f is a continuous function such that $dd^c f \leq C\omega$. We approximate f uniformly by smooth functions f_j such that $dd^c f_j \leq (C+1)\omega$. This is possible thanks to Demailly [30]. Then, the previous steps yield

$$\Delta_\omega(P_\theta(f_j)) \leq C' e^{-A\psi},$$

where $C' > 0$ depends only on $C, B, a, \inf_X f, \theta, \omega$. Letting $j \to +\infty$ we arrive at the conclusion. Having the Laplacian bound, one can then argue as in [37, Theorem 9.25] to get (2.1), completing the proof of Theorem 2.1. □

2.2 Non-pluripolar Monge-Ampère Products

Given $u_1, \ldots, u_p$ θ-psh functions with minimal singularities, $\theta_{u_1} \wedge \cdots \wedge \theta_{u_p}$, as defined by Bedford and Taylor [2, 3] is a closed positive current in $\mathrm{Amp}(\theta)$. For general $u_1, \ldots, u_p \in \mathrm{PSH}(X, \theta)$, it was shown in [12] that the *non-pluripolar product* of $\theta_{u_1}, \ldots, \theta_{u_p}$, that we still denote by

$$\theta_{u_1} \wedge \ldots \wedge \theta_{u_p},$$

is well-defined as a closed positive (p, p)-current on X which does not charge pluripolar sets. For a θ-psh function u, the *non-pluripolar complex Monge-Ampère measure* of u is simply $\theta_u^n := \theta_u \wedge \ldots \wedge \theta_u$.

If u has minimal singularities then $\int_X \theta_u^n$, the total mass of θ_u^n, is equal to $\int_X \theta_{V_\theta}^n$, the volume of the class $\{\theta\}$ denoted by $\mathrm{Vol}(\theta)$. For a general $u \in \mathrm{PSH}(X, \theta)$, $\int_X \theta_u^n$ may take any value in $[0, \mathrm{Vol}(\theta)]$. Note that $\mathrm{Vol}(\theta)$ is a cohomological quantity, i.e., it does not depend on the smooth representative we choose in $\{\theta\}$.

2.3 The Energy Classes

From now on, we fix $p \geq 1$. Recall that for any θ-psh function u we have $\int_X \theta_u^n \leq \mathrm{Vol}(\theta)$. We denote by $\mathcal{E}(X, \theta)$ the set of θ-psh functions u such that $\int_X \theta_u^n = \mathrm{Vol}(\theta)$. We let $\mathcal{E}^p(X, \theta)$ denote the set of $u \in \mathcal{E}(X, \theta)$ such that $\int_X |u - V_\theta|^p \theta_u^n < +\infty$. For $u, v \in \mathcal{E}^p(X, \theta)$ we define

$$I_p(u, v) := I_{p,\theta}(u, v) := \int_X |u - v|^p \left(\theta_u^n + \theta_v^n\right).$$

It was proved in [34, Theorem 1.6] that I_p satisfies a quasi triangle inequality:

$$I_{p,\theta}(u, v) \leq C(n, p)(I_{p,\theta}(u, w) + I_{p,\theta}(v, w)), \ \forall u, v, w \in \mathcal{E}^p(X, \theta).$$

In particular, applying this for $w = V_\theta$ and using Theorem 2.1, we obtain $I_{p,\theta}(u, v) < +\infty$, for all $u, v \in \mathcal{E}^p(X, \theta)$. Moreover, it follows from the domination principle [24, Proposition 2.4] that I_p is non-degenerate:

$$I_{p,\theta}(u, v) = 0 \Longrightarrow u = v.$$

2.4 Weak Geodesics

Geodesic segments connecting Kähler potentials were first introduced by Mabuchi [38]. Semmes [39] and Donaldson [33] independently realized that the geodesic equation can be reformulated as a degenerate homogeneous complex Monge-Ampère equation. The best regularity of a geodesic segment connecting two Kähler potentials is known to be $\mathcal{C}^{1,1}$ (see [8, 15, 19]).

In the context of a big cohomology class, the regularity of geodesics is very delicate. To avoid this issue, we follow an idea of Berndtsson [7] considering geodesics as the upper envelope of subgeodesics (see [24]).

For a curve $[0,1] \ni t \mapsto u_t \in \mathrm{PSH}(X,\theta)$, we define

$$X \times D \ni (x,z) \mapsto U(x,z) := u_{\log|z|}(x), \tag{2.4}$$

where $D := \{z \in \mathbb{C} \mid 1 < |z| < e\}$. We let $\pi : X \times D \to X$ be the projection on X.

Definition 2.2 We say that $t \mapsto u_t$ is a subgeodesic if $(x,z) \mapsto U(x,z)$ is a $\pi^*\theta$-psh function on $X \times D$.

Definition 2.3 For $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$, we let $\mathcal{S}_{[0,1]}(\varphi_0,\varphi_1)$ denote the set of all subgeodesics $[0,1] \ni t \mapsto u_t$ such that $\limsup_{t\to 0} u_t \le \varphi_0$ and $\limsup_{t \to 1} u_t \le \varphi_1$.

Let $\varphi_0, \varphi_1 \in \mathrm{PSH}(X,\theta)$. We define, for $(x,z) \in X \times D$,

$$\Phi(x,z) := \sup\{U(x,z) \mid U \in \mathcal{S}_{[0,1]}(\varphi_0,\varphi_1)\}.$$

The curve $t \mapsto \varphi_t$ constructed from Φ via (2.4) is called the weak Mabuchi geodesic connecting φ_0 and φ_1.

Geodesic segments connecting two general θ-psh functions may not exist. If $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$, it was shown in [24, Theorem 2.13] that $P(\varphi_0,\varphi_1) \in \mathcal{E}^p(X,\theta)$. Since $P(\varphi_0,\varphi_1) \le \varphi_t$, we obtain that $t \to \varphi_t$ is a curve in $\mathcal{E}^p(X,\theta)$. Each subgeodesic segment is in particular convex in t:

$$\varphi_t \le (1-t)\varphi_0 + t\varphi_1, \ \forall t \in [0,1].$$

Consequently, the upper semicontinuous regularization (with respect to both variables x, z) of Φ is again in $\mathcal{S}_{[0,1]}(\varphi_0,\varphi_1)$, hence so is Φ. In particular, if φ_0, φ_1 have minimal singularities, then the geodesic φ_t is Lipschitz on $[0,1]$ (see [24, Lemma 3.1]):

$$|\varphi_t - \varphi_s| \le |t-s| \sup_X |\varphi_0 - \varphi_1|, \ \forall t, s \in [0,1]. \tag{2.5}$$

2.5 Finsler Geometry in the Kähler Case

Darvas [20] introduced a family of distances in the space of Kähler potentials

$$\mathcal{H}_\omega := \{\varphi \in \mathcal{C}^\infty(X,\mathbb{R}) \mid \omega_\varphi > 0\}.$$

Definition 2.4 Let $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$. For $p \ge 1$, we set

$$d_p(\varphi_0,\varphi_1) := \inf\{\ell_p(\psi) \mid \psi \text{ is a smooth path joining } \varphi_0 \text{ to } \varphi_1\},$$

where $\ell_p(\psi) := \int_0^1 \left(\frac{1}{V}\int_X |\dot{\psi}_t|^p \omega_{\psi_t}^n\right)^{1/p} dt$ and $V := \mathrm{Vol}(\omega) = \int_X \omega^n$.

It was then proved in [20, Theorem 1] (generalizing Chen's original arguments [15]) that d_p defines a distance on $\mathcal{H}_\omega$, and for all $\varphi_0, \varphi_1 \in \mathcal{H}_\omega$,

$$d_p(\varphi_0,\varphi_1) = \left(\frac{1}{V}\int_X |\dot{\varphi}_t|^p \omega_{\varphi_t}^n\right)^{1/p}, \quad \forall t \in [0,1], \tag{2.6}$$

where $t \to \varphi_t$ is the Mabuchi geodesic (defined in Section 2.4). It was shown in [20, Lemma 4.11] that (2.6) still holds for $\varphi_0, \varphi_1 \in \mathrm{PSH}(X, \omega)$ with $dd^c\varphi_i \leq C\omega, i = 0, 1$, for some positive constant C.

By [9, 30], potentials in $\mathcal{E}^p(X, \omega)$ can be approximated from above by smooth Kähler potentials. As shown in [21], the metric d_p can be extended for potentials in $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \omega)$: if φ_i^k are smooth strictly ω-psh functions decreasing to φ_i, $i = 0, 1$; then, the limit

$$d_p(\varphi_0, \varphi_1) := \lim_{k \to +\infty} d_p(\varphi_0^k, \varphi_1^k)$$

exists and it is independent of the approximants. By [20, Lemmas 4.4 and 4.5], d_p defines a metric on $\mathcal{E}^p(X, \omega)$ and $(\mathcal{E}^p(X, \omega), d_p)$ is a complete geodesic metric space.

3 The Metric Space $(\mathcal{E}^p(X, \theta), d_p)$

The goal of this section is to define a distance d_p on $\mathcal{E}^p(X, \theta)$ and prove that the space $(\mathcal{E}^p(X, \theta), d_p)$ is a complete geodesic metric space. We follow the strategy in [32], approximating the space of "Kähler potentials" $\mathcal{H}_\theta$ by regular spaces. Throughout this note we will use the notation

$$\omega_\varepsilon := \theta + \varepsilon\omega, \ \ \varepsilon > 0.$$

By nefness of θ, $\omega_\varepsilon := \theta + \varepsilon\omega$ represents a Kähler cohomology class for any $\varepsilon > 0$. Note that ω_ε is not necessarily a Kähler form but there exists a smooth potential $f_\varepsilon \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $\omega_\varepsilon + dd^c f_\varepsilon$ is a Kähler form. For notational convenience we normalize θ so that $\mathrm{Vol}(\theta) = \int_X \theta^n_{V_\theta} = 1$ and we set $V_\varepsilon := \mathrm{Vol}(\omega_\varepsilon)$.

Typically there are no smooth potentials in $\mathrm{PSH}(X, \theta)$ but the following class contains plenty of potentials sufficiently regular for our purposes:

$$\mathcal{H}_\theta := \{\varphi \in \mathrm{PSH}(X, \theta) \mid \varphi = P_\theta(f), \ \ f \in \mathcal{C}(X, \mathbb{R}), \ \ dd^c f \leq C(f)\omega\}.$$

Here $C(f)$ denotes a positive constant which depends only on f. Note that any $u = P_\theta(f) \in \mathcal{H}_\theta$ has minimal singularities because, for some constant $C > 0$, $V_\theta - C$ is a candidate defining $P_\theta(f)$. The following elementary observation will be useful in the sequel.

Lemma 3.1 *If $u, v \in \mathcal{H}_\theta$ then $P_\theta(u, v) \in \mathcal{H}_\theta$.*

Proof Set $h = \min(f, g) \in \mathcal{C}^0(X, \mathbb{R})$, where $f, g \in \mathcal{C}^0(X, \mathbb{R})$ are such that $u = P_\theta(f)$ and $v = P_\theta(g)$ and $dd^c f \leq C\omega, dd^c g \leq C\omega$. Then, $-h = \max(-f, -g)$ is a $C\omega$-psh function on X, hence $dd^c(-h) + C\omega \geq 0$. □

3.1 Defining a Distance d_p on $\mathcal{H}_\theta$

By Darvas [20], the Mabuchi distance $d_{p,\omega}$ is well defined on $\mathcal{E}^p(X, \omega)$ when the reference form ω is a Kähler form. With the following observation, we show that such a distance behaves well when we change the Kähler representative in $\{\omega\}$.

Proposition 3.2 *Let $\omega_f := \omega + dd^c f \in \{\omega\}$ be another Kähler form. Then, given $\varphi_0, \varphi_1 \in \mathcal{E}^p(X, \omega)$, we have*

$$d_{p,\omega}(\varphi_0, \varphi_1) = d_{p,\omega_f}(\varphi_0 - f, \varphi_1 - f).$$

Proof Let φ_t be the Mabuchi geodesic (with respect to ω) joining φ_0 and φ_1 and let φ_t^f be the Mabuchi geodesic (with respect to ω_f) joining $\varphi_0 - f$ and $\varphi_1 - f$. We claim that $\varphi_t^f = \varphi_t - f$. Indeed, $\varphi_t - f$ is an ω_f-subgeodesic connecting $\varphi_0 - f$ and $\varphi_1 - f$. Hence, $\varphi_t - f \le \varphi_t^f$. On the other hand, $\varphi_t^f + f$ is a candidate defining φ_t, thus $\varphi_t^f + f \le \varphi_t$, proving the claim.

Assume φ_0, φ_1 are Kähler potentials. By (2.6) we have

$$\begin{aligned} V d_{p,\omega}^p(\varphi_0, \varphi_1) &= \int_X |\dot{\varphi}_0|^p (\omega + dd^c \varphi_0)^n \\ &= \int_X \left| \lim_{t\to 0^+} \frac{(\varphi_t - f) - (\varphi_0 - f)}{t} \right|^p \left(\omega_f + dd^c(\varphi_0 - f)\right)^n \\ &= \int_X |\dot{\varphi}_0^f|^p (\omega_f + dd^c(\varphi_0 - f))^n \\ &= V d_{p,\omega_f}^p(\varphi_0 - f, \varphi_1 - f). \end{aligned}$$

The identity for potentials in $\mathcal{E}^p(X,\omega)$ follows from the fact that the distance $d_{p,\omega}$ between potentials $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\omega)$ is defined as the limit $\lim_j d_{p,\omega}(\varphi_{0,j}, \varphi_{1,j})$, where $\{\varphi_{i,j}\}$ is a sequence of smooth strictly ω-psh functions decreasing to φ_i, for $i = 0, 1$. □

Thanks to the above proposition, we can then define the Mabuchi distance with respect to any smooth (1, 1)-form η in the Kähler class $\{\omega\}$:

$$d_{p,\eta}(\varphi_0, \varphi_1) := d_{p,\eta_f}(\varphi_0 - f, \varphi_1 - f), \quad \varphi_0, \varphi_1 \in \mathcal{E}^p(X,\eta), \tag{3.1}$$

where $\eta_f = \eta + dd^c f$ is a Kähler form. Proposition 3.2 reveals that the definition is independent of the choice of f.

We next extend the Pythagorean formula of [20, 21] for Kähler classes.

Lemma 3.3 *If $\{\eta\}$ is Kähler and $u, v \in \mathcal{E}^p(X,\eta)$ then*

$$d_{p,\eta}^p(u,v) = d_{p,\eta}^p(u, P_\eta(u,v)) + d_{p,\eta}^p(v, P_\eta(u,v)).$$

Proof By [20, Corollary 4.14] and (3.1), we have

$$d_{p,\eta}^p(u,v) = d_{p,\eta_f}^p(u - f, P_{\eta_f}(u - f, v - f)) + d_{p,\eta_f}^p(v - f, P_{\eta_f}(u - f, v - f)).$$

The conclusion follows observing that $P_{\eta_f}(u - f, v - f) = P_\eta(u,v) - f$. □

The following results play a crucial role in the sequel.

Lemma 3.4 *Let $\varphi = P_\theta(f)$, $\psi = P_\theta(g) \in \mathcal{H}_\theta$. Set $\varphi_\varepsilon := P_{\omega_\varepsilon}(f)$ and $\psi_\varepsilon = P_{\omega_\varepsilon}(g)$. Then,*

$$\lim_{\varepsilon\to 0} I_{p,\omega_\varepsilon}(\varphi_\varepsilon, \psi_\varepsilon) = I_{p,\theta}(\varphi,\psi).$$

Proof Observe that $|\varphi_\varepsilon - \psi_\varepsilon| \to |\varphi - \psi|$ almost everywhere on X (in fact this holds off a pluripolar set) and they are uniformly bounded:

$$|\varphi_\varepsilon - \psi_\varepsilon| \le \sup_X |f - g|.$$

Indeed, take a point $x \in X$ such that $\varphi(x) > -\infty$ and $\psi(x) > -\infty$. Recall that $\omega_\varepsilon := \theta + \varepsilon\omega \ge \theta$ and $\{\omega_\varepsilon\}$ is increasing in ε. Therefore, φ_ε decreases to a θ-psh function on X as $\varepsilon \to 0$. The latter must be φ. We thus have that $\varphi_\varepsilon(x) \to \varphi(x)$ and $\psi_\varepsilon(x) \to \psi(x)$ as $\varepsilon \to 0$. Also, $\psi_\varepsilon + \sup_X |f - g|$ is a candidate defining φ_ε; hence, the claimed bound follows.

By Lemma 3.5 below and Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0}\int_X |\varphi_\varepsilon-\psi_\varepsilon|^p\,(\omega_\varepsilon+dd^c\varphi_\varepsilon)^n=\int_X|\varphi-\psi|^p\,(\theta+dd^c\varphi)^n.$$

Similarly, the other term in the definition of I_{p,ω_ε} also converges to the desired limit. □

Lemma 3.5 *Let $\varphi = P_\theta(f)\in\mathcal{H}_\theta$. For $\varepsilon>0$ we set $\varphi_\varepsilon = P_{\omega_\varepsilon}(f)$ and write*

$$(\omega_\varepsilon+dd^c\varphi_\varepsilon)^n=\rho_\varepsilon\omega^n;\ (\theta+dd^c\varphi)^n=\rho\omega^n.$$

Then, $\varepsilon\mapsto\rho_\varepsilon$ is increasing, uniformly bounded and $\rho_\varepsilon\to\rho$ pointwise on X.

Proof Define, for $\varepsilon>0$, $D_\varepsilon := \{x\in X \mid \varphi_\varepsilon(x)=f(x)\}$. Since $\{\varphi_\varepsilon\}$ is increasing and $\varphi_\varepsilon\le f$, $\{D_\varepsilon\}$ is also increasing. We set $D:=\cap_{\varepsilon>0}D_\varepsilon$. Then, $D=\{x\in X\mid \varphi(x)=f(x)\}$.

For $\varepsilon'>\varepsilon>0$, it follows from Theorem 2.1 that

$$\begin{aligned}(\omega_\varepsilon+dd^c\varphi_\varepsilon)^n &= \mathbb{1}_{\{\varphi_\varepsilon=f\}}(\omega_\varepsilon+dd^cf)^n\\ &\le \mathbb{1}_{\{\varphi_\varepsilon=f\}}(\omega_{\varepsilon'}+dd^cf)^n\le(\omega_{\varepsilon'}+dd^c\varphi_{\varepsilon'})^n.\end{aligned}$$

Here we use the fact that $0\le\omega_\varepsilon+dd^cf\le\omega_{\varepsilon'}+dd^cf$ on D_ε. This proves the first statement. The second statement follows from the bound $dd^cf\le C\omega$. We now prove the last statement. If $x\in D$, using $(\theta+dd^cf)\le C'\omega$, we can write

$$\begin{aligned}\rho_\varepsilon(x)\omega^n &= (\theta+\varepsilon\omega+dd^cf)^n\le(\theta+dd^cf)^n+O(\varepsilon)\omega^n\\ &= (\rho(x)+O(\varepsilon))\omega^n.\end{aligned}$$

Hence, $\rho_\varepsilon(x)\to\rho(x)$. If $x\notin D$ then $x\notin D_\varepsilon$ for $\varepsilon>0$ small enough, hence $\rho_\varepsilon(x)=0=\rho(x)$. □

Lemma 3.6 *Let $\varphi_j = P_\theta(f_j)\in\mathcal{H}_\theta$, for $j=0,1$. Let φ_t (resp. $\varphi_{t,\varepsilon}$) be weak Mabuchi geodesics joining φ_0 and φ_1 (resp. $\varphi_{0,\varepsilon}=P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon}=P_{\omega_\varepsilon}(f_1)$). Then, we have the following pointwise convergence*

$$\mathbb{1}_{\{\varphi_{0,\varepsilon}=f_0\}}|\dot\varphi_{0,\varepsilon}|^p\to\mathbb{1}_{\{\varphi_0=f_0\}}|\dot\varphi_0|^p.$$

Proof Since $P_{\omega_\varepsilon}(f_j)\ge P_\theta(f_j)$, $j=0,1$, it follows from the definition that $\varphi_{t,\varepsilon}\ge\varphi_t$ (the curve φ_t is a candidate defining $\varphi_{t,\varepsilon}$ for any $\varepsilon>0$). Set $D_\varepsilon=\{\varphi_{0,\varepsilon}=f_0\}$ and $D=\{\varphi_0=f_0\}$. Then, D_ε is increasing and $\cap_{\varepsilon>0}D_\varepsilon=D$ since $\varphi_0\le\varphi_{0,\varepsilon}\le f_0$. If $x\in D$ then, for all small $s>0$,

$$\dot\varphi_0(x)=\lim_{t\to0}\frac{\varphi_t(x)-f_0(x)}{t}\le\dot\varphi_{0,\varepsilon}(x)\le\frac{\varphi_{s,\varepsilon}(x)-\varphi_{0,\varepsilon}(x)}{s},$$

where in the last inequality we use the convexity of the geodesic in t. Letting first $\varepsilon\to0$ and then $s\to0$ shows that $\dot\varphi_{0,\varepsilon}(x)$ converges to $\dot\varphi_0(x)$. If $x\notin D$ then $x\notin D_\varepsilon$, for $\varepsilon>0$ small enough. In this case the convergence we want to prove is trivial. □

Theorem 3.7 *Let $\varphi_0 := P_\theta(f_0)$, $\varphi_1 := P_\theta(f_1)\in\mathcal{H}_\theta$ and let $\varphi_{i,\varepsilon}=P_{\omega_\varepsilon}(f_i)$, $i=0,1$. Let $d_{p,\varepsilon}$ be the Mabuchi distance with respect to ω_ε defined in (3.1). Then,*

$$\lim_{\varepsilon\to0}d^p_{p,\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon})=\int_X|\dot\varphi_0|^p(\theta+dd^c\varphi_0)^n=\int_X|\dot\varphi_1|^p(\theta+dd^c\varphi_1)^n,$$

where φ_t is the weak Mabuchi geodesic connecting φ_0 and φ_1.

Compared with [32] our approach is slightly different. We also emphasize that by [31, Example 4.5], there are functions in $\mathcal{E}^p(X,\theta)$ which are not in $\mathcal{E}^p(X,\omega)$.

Proof Let $\varphi_{t,\varepsilon}$ denote the ω_ε-geodesic joining $\varphi_{0,\varepsilon}$ and $\varphi_{1,\varepsilon}$. Set $D_\varepsilon = \{\varphi_{0,\varepsilon} = f_0\}$ and $D = \{\varphi_0 = f_0\}$. Combining (2.6) and Theorem 2.1, we obtain

$$V_\varepsilon d_{p,\varepsilon}^p(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) = \int_X |\dot\varphi_{0,\varepsilon}|^p(\omega_\varepsilon + dd^c\varphi_{0,\varepsilon})^n = \int_{D_\varepsilon} |\dot\varphi_{0,\varepsilon}|^p(\omega_\varepsilon + dd^c f_0)^n.$$

Since $|\varphi_{0,\varepsilon} - \varphi_{1,\varepsilon}| \le \sup_X |f_0 - f_1|$ and $f_0 - f_1$ is bounded, (2.5) ensures that $\dot\varphi_{0,\varepsilon}$ is uniformly bounded. It follows from Lemma 3.5 and Lemma 3.6 that the functions $\mathbb{1}_{D_\varepsilon}|\dot\varphi_{0,\varepsilon}|^p\rho_\varepsilon$ and $\mathbb{1}_D|\dot\varphi_0|^p\rho$ are uniformly bounded and $\mathbb{1}_{D_\varepsilon}|\dot\varphi_{0,\varepsilon}|^p\rho_\varepsilon$ converges pointwise to $\mathbb{1}_D|\dot\varphi_0|^p\rho$. We also observe that V_ε decreases to $\mathrm{Vol}(\theta) = 1$. Lebesgue's dominated convergence theorem then yields

$$\lim_{\varepsilon\to 0} d_{p,\varepsilon}^p(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) = \int_D |\dot\varphi_0|^p(\theta + dd^c f_0)^n = \int_X |\dot\varphi_0|^p(\theta + dd^c\varphi_0)^n,$$

where in the last equality we use Theorem 2.1. This shows the first equality in the statement. The second one is obtained by reversing the role of φ_0 and φ_1. □

Definition 3.8 Assume that $\varphi_0 := P_\theta(f_0), \varphi_1 := P_\theta(f_1) \in \mathcal{H}_\theta$. Let $d_{p,\varepsilon}$ be the Mabuchi distance with respect to $\omega_\varepsilon := \theta + \varepsilon\omega$ defined in (3.1). We define

$$d_p(\varphi_0,\varphi_1) := \lim_{\varepsilon\to 0} d_{p,\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}),$$

where $\varphi_{0,\varepsilon} := P_{\omega_\varepsilon}(f_0)$ and $\varphi_{1,\varepsilon} := P_{\omega_\varepsilon}(f_1)$.

The limit exists and is independent of the choice of ω as shown in Theorem 3.7.

Lemma 3.9 *d_p is a distance on $\mathcal{H}_\theta$.*

Proof The triangle inequality immediately follows from the fact that $d_{p,\varepsilon}$ is a distance. From [20, Theorem 5.5] we know that

$$d_{p,\varepsilon}^p(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) \ge \frac{1}{C} I_{p,\omega_\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}), \qquad C > 0.$$

Also, by Lemma 3.4 we have $\lim_{\varepsilon\to 0} I_{p,\omega_\varepsilon}(\varphi_{0,\varepsilon},\varphi_{1,\varepsilon}) = I_{p,\theta}(\varphi_0,\varphi_1)$. It follows from the domination principe (see [10, 24]) that

$$I_{p,\theta}(\varphi_0,\varphi_1) = 0 \Leftrightarrow \varphi_0 = \varphi_1.$$

Hence, d_p is non-degenerate. □

3.2 Extension of d_p to $\mathcal{E}^p(X,\theta)$

The following comparison between I_p and d_p was established in [20, Theorem 3] in the Kähler case.

Proposition 3.10 *Given $\varphi_0, \varphi_1 \in \mathcal{H}_\theta$ there exists a constant $C > 0$ (depending only on n) such that*

$$\frac{1}{C} I_p(\varphi_0,\varphi_1) \le d_p^p(\varphi_0,\varphi_1) \le C I_p(\varphi_0,\varphi_1). \tag{3.2}$$

Proof By Darvas [20, Theorem 3] we know that

$$\frac{1}{C} I_{p,\omega_\varepsilon}(\varphi_{0,\varepsilon}, \varphi_{1,\varepsilon}) \le d^p_{p,\varepsilon}(\varphi_{0,\varepsilon}, \varphi_{1,\varepsilon}) \le C I_{p,\omega_\varepsilon}(\varphi_{0,\varepsilon}, \varphi_{1,\varepsilon}).$$

Letting ε to zero and using Lemma 3.4 and Definition 3.8, we get (3.2). □

Now, let $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$. Let $\{f_{i,j}\}$ be a sequence of smooth functions decreasing to φ_i, $i = 0, 1$. We then clearly have that $\varphi_{i,j} := P_\theta(f_{i,j}) \in \mathcal{H}_\theta$ and $P_\theta(f_{i,j}) \searrow \varphi_i$.

Lemma 3.11 *The sequence $d_p(\varphi_{0,j}, \varphi_{1,j})$ converges and the limit is independent of the choice of the approximants $f_{i,j}$.*

Proof Set $a_j := d_p(\varphi_{0,j}, \varphi_{1,j})$. By the triangle inequality and Proposition 3.10, we have

$$\begin{aligned} a_j &\le d_p(\varphi_{0,j}, \varphi_{0,k}) + d_p(\varphi_{0,k}, \varphi_{1,k}) + d_p(\varphi_{1,k}, \varphi_{1,j}) \\ &\le a_k + C\left(I_p^{1/p}(\varphi_{0,j}, \varphi_{0,k}) + I_p^{1/p}(\varphi_{1,j}, \varphi_{1,k})\right), \end{aligned}$$

where $C > 0$ depends only on n, p. Hence,

$$|a_j - a_k| \le C\left(I_p^{1/p}(\varphi_{0,j}, \varphi_{0,k}) + I_p^{1/p}(\varphi_{1,j}, \varphi_{1,k})\right).$$

By [34, Theorem 1.6 and Proposition 1.9], it then follows that $|a_j - a_k| \to 0$ as $j, k \to +\infty$. This proves that the sequence $d_p(\varphi_{0,j}, \varphi_{1,j})$ is Cauchy; hence, it converges.

Let $\tilde{\varphi}_{i,j} = P_\theta(\tilde{f}_{i,j})$ be another sequence in $\mathcal{H}_\theta$ decreasing to φ_i, $i = 0, 1$. Then, applying the triangle inequality several times, we get

$$d_p(\varphi_{0,j}, \varphi_{1,j}) \le d_p(\varphi_{0,j}, \tilde{\varphi}_{0,j}) + d_p(\tilde{\varphi}_{0,j}, \tilde{\varphi}_{1,j}) + d_p(\tilde{\varphi}_{1,j}, \varphi_{1,j}),$$

and thus

$$|d_p(\varphi_{0,j}, \varphi_{1,j}) - d_p(\tilde{\varphi}_{0,j}, \tilde{\varphi}_{1,j})| \le C\left(I_p^{1/p}(\varphi_{0,j}, \tilde{\varphi}_{0,j}) + I_p^{1/p}(\varphi_{1,j}, \tilde{\varphi}_{1,j})\right).$$

It then follows again from [34, Theorem 1.6 and Proposition 1.9] that the limit does not depend on the choice of the approximants. □

Given $\varphi_0, \varphi_1 \in \mathcal{E}^p(X,\theta)$, we then define

$$d_p(\varphi_0, \varphi_1) := \lim_{j\to+\infty} d_p(P_\theta(f_{0,j}), P_\theta(f_{1,j})).$$

Proposition 3.12 *d_p is a distance on $\mathcal{E}^p(X,\theta)$ and the inequalities comparing d_p and I_p on $\mathcal{H}_\theta$ (3.2) hold on $\mathcal{E}^p(X,\theta)$. Moreover, if $u_j \in \mathcal{E}^p(X,\theta)$ decreases to $u \in \mathcal{E}^p(X,\theta)$ then $d_p(u_j, u) \to 0$.*

Proof By the definition of d_p on $\mathcal{E}^p(X,\theta)$ we infer that the comparison between d_p and I_p in Proposition 3.10 holds on $\mathcal{E}^p(X,\theta)$. From this and the domination principle [24], we deduce that d_p is non-degenerate. The last statement follows from (3.2) and [34, Proposition 1.9]. □

The next result was proved in [6, Lemma 3.4] for the Kähler case.

Lemma 3.13 *Let u_t be the Mabuchi geodesic joining $u_0 \in \mathcal{H}_\theta$ and let $u_1 \in \mathcal{E}^p(X,\theta)$. Then,*

$$d_p^p(u_0, u_1) = \int_X |\dot{u}_0|^p (\theta + dd^c u_0)^n.$$

Proof We first assume that $u_0 \geq u_1 + 1$. We approximate u_1 from above by $u_1^j \in \mathcal{H}_\theta$ such that $u_1^j \leq u_0$, for all j. Let u_t^j be the Mabuchi geodesic joining u_0 to u_1^j. Note that $u_t^j \geq u_t$ and that $\dot{u}_t^j \leq 0$. By Theorem 3.7,

$$d_p^p(u_0, u_1^j) = \int_X (-\dot{u}_0^j)^p \theta_{u_0}^n.$$

Also, $\dot{u}_0^j$ decreases to $\dot{u}_0$; hence, the monotone convergence theorem and Proposition 3.12 give

$$d_p^p(u_0, u_1) = \int_X (-\dot{u}_0)^p \theta_{u_0}^n < +\infty.$$

In particular $|\dot{u}_0|^p \in L^1(X, \theta_{u_0}^n)$.

For the general case we can find a constant $C > 0$ such that $u_1 \leq u_0 + C$ since u_0 has minimal singularities. Let w_t be the Mabuchi geodesic joining u_0 and $u_1 - C - 1$. Note that $w_t \leq u_t^j$ since $w_1 = u_1 - C - 1 < u_1 \leq u_1^j$ and $w_0 = u_0 = u_0^j$ and $\dot{w}_t \leq 0$. It then follows that

$$\dot{w}_0 \leq \dot{u}_0^j \leq u_1^j - u_0 \leq (u_1^j - V_\theta) + (V_\theta - u_0) \leq \sup_X u_1^j + \sup_X (V_\theta - u_0) \leq C_1,$$

for a uniform constant $C_1 > 0$. In the second inequality above, we use the fact that the Mabuchi geodesic u_t^j connecting u_0 to u_1^j is convex in t, while in the last inequality we use the fact that u_0 has minimal singularities.

The previous inequalities then yield $|\dot{u}_0^j|^p \leq C_2 + 2^{p-1}|\dot{w}_0|^p$, where C_2 is a uniform constant. On the other hand by Theorem 3.7, we have

$$d_p^p(u_0, u_1^j) = \int_X |\dot{u}_0^j|^p \theta_{u_0}^n.$$

We claim that $|\dot{u}_0^j|^p$ converges a.e. to $|\dot{u}_0|^p$. Indeed, the convergence is pointwise at points x such that $u_1(x) > -\infty$, but the set $\{u_1 = -\infty\}$ has Lebesgue measure zero. Also, the above estimate ensures that $|\dot{u}_0^j|^p$ are uniformly bounded by $2^{p-1}(-\dot{w}_0)^p + C_2$ which is integrable with respect to the measure $\theta_{u_0}^n$ since $\int_X |\dot{w}_0|^p \theta_{u_0}^n = d_p^p(u_0, u_1 - C - 1) < +\infty$. Proposition 3.12 and Lebesgue's dominated convergence theorem then give the result. □

Proposition 3.14 *If* $u, v \in \mathcal{E}^p(X, \theta)$ *then*

(i) $d_p^p(u, v) = d_p^p(u, P_\theta(u, v)) + d_p^p(v, P_\theta(u, v))$ *and*
(ii) $d_p(u, \max(u, v)) \geq d_p(v, P_\theta(u, v))$.

We recall that from [24, Theorem 2.13] $P_\theta(u, v) \in \mathcal{E}^p(X, \theta)$. The identity in the first statement is known as the Pythagorean formula and it was established in the Kähler case by Darvas [20]. The second statement was proved for $p = 1$ in [26] using the differentiability of the Monge-Ampère energy. As will be shown in Proposition 3.18, our definition of d_1 and the one in [26] do coincide.

Proof To prove the Pythagorean formula, we first assume that $u = P_\theta(f), v = P_\theta(g) \in \mathcal{H}_\theta$. Set $u_\varepsilon := P_{\omega_\varepsilon}(f)$, $v_\varepsilon := P_{\omega_\varepsilon}(g)$. It follows from Lemma 3.3 that

$$\begin{aligned} d_{p,\varepsilon}^p(u_\varepsilon, v_\varepsilon) &= d_{p,\varepsilon}^p(u_\varepsilon, P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon)) + d_{p,\varepsilon}^p(v_\varepsilon, P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon)) \\ &= d_{p,\varepsilon}^p(u_\varepsilon, P_{\omega_\varepsilon}(\min(f, g)) + d_{p,\varepsilon}^p(v_\varepsilon, P_{\omega_\varepsilon}(\min(f, g)), \end{aligned}$$

where in the last identity we use that $P_{\omega_\varepsilon}(u_\varepsilon, v_\varepsilon) = P_{\omega_\varepsilon}(\min(f, g))$. It follows from Lemma 3.1 that $dd^c \min(f, g) \le C\omega$. Applying Theorem 3.7 we obtain (i) for this case. To treat the general case, let $u_j = P_\theta(f_j)$, $v_j = P_\theta(g_j)$ be sequences in $\mathcal{H}_\theta$ decreasing to u, v. By Lemma 3.1, $P_\theta(u_j, v_j) = P_\theta(\min(f_j, g_j)) \in \mathcal{H}_\theta$ and it decreases to $P_\theta(u, v)$. Then, (i) follows from the first step and Proposition 3.12 since

$$|d_p(u_j, v_j) - d_p(u, v)| \le d_p(u_j, u) + d_p(v, v_j).$$

To prove the second statement, in view of Proposition 3.12, we can assume that $u = P_\theta(f)$, $v = P_\theta(g) \in \mathcal{H}_\theta$. By Lemma 3.13 we have

$$d_p^p(u, \max(u, v)) = \int_X |\dot{u}_0|^p \theta_u^n,$$

where $t \mapsto u_t$ is the Mabuchi geodesic joining $u_0 = u$ to $u_1 = \max(u, v)$.

Let φ_t be the Mabuchi geodesic joining $\varphi_0 = P_\theta(u, v)$ to $\varphi_1 = v$. We note that $0 \le \dot{\varphi}_0 \le v - P(u, v)$. Indeed, $\dot{\varphi}_0 \ge 0$ since $\varphi_0 \le \varphi_1$ while the second inequality follows from the convexity in t of the geodesic. Using this observation and the fact that $\varphi_t \le u_t$, we obtain

$$\mathbb{1}_{\{P(u,v)=u\}}\dot{\varphi}_0 \le \mathbb{1}_{\{P(u,v)=u\}}\dot{u}_0, \text{ and } \mathbb{1}_{\{P(u,v)=v\}}\dot{\varphi}_0 = 0.$$

Since $P_\theta(u, v) = P_\theta(\min(f, g))$ with $dd^c \min(f, g) \le C\omega$, Theorem 2.1, Theorem 3.7, and [34, Lemma 4.1] then yield

$$\begin{aligned} d_p^p(P_\theta(u, v), v) &= \int_X \dot{\varphi}_0^p (\theta + dd^c \varphi_0)^n \le \int_{\{P(u,v)=u\}} \dot{\varphi}_0^p (\theta + dd^c u)^n \\ &\le \int_{\{P(u,v)=u\}} \dot{u}_0^p (\theta + dd^c u)^n \le d_p^p(u, \max(u, v)). \end{aligned}$$

□

Remark 3.15 By Proposition 3.14 we have a "Pythagorean inequality" for max:

$$d_p^p(u, \max(u, v)) + d_p^p(v, \max(u, v)) \ge d_p^p(u, v), \ \forall u, v \in \mathcal{E}^p(X, \theta).$$

3.3 Completeness of $(\mathcal{E}^p(X, \theta), d_p)$

In the sequel we fix a smooth volume form dV on X such that $\int_X dV = 1$.

Lemma 3.16 *Let $u \in \mathcal{E}^p(X, \theta)$ and let ϕ be a θ-psh function with minimal singularities,* $\sup_X \phi = 0$ *satisfying $\theta_\phi^n = dV$. Then, there exist uniform constants $C_1 = C_1(n, \theta)$ and $C_2 = C_2(n) > 0$ such that*

$$|\sup_X u| \le C_1 + C_2 d_p(u, \phi).$$

Proof Using the Hölder inequality and [35, Proposition 2.7], we obtain

$$\begin{aligned} |\sup_X u| &\le \int_X |u - \sup_X u| dV + \int_X |u| dV \le A + \left(\int_X |u|^p dV \right)^{1/p} \\ &\le A + \left(\|u - \phi\|_{L^p(dV)} + \|\phi\|_{L^p(dV)} \right). \end{aligned}$$

By Proposition 3.12,

$$\int_X |u - \phi|^p dV = \int_X |u - \phi|^p \theta_\phi^n \le I_p(u, \phi) \le C(n) d_p^p(u, \phi).$$

Combining the above inequalities we get the conclusion. □

Theorem 3.17 *The space $(\mathcal{E}^p(X,\theta), d_p)$ is a complete geodesic metric space which is the completion of $(\mathcal{H}_\theta, d_p)$.*

Proof Let $(\varphi_j) \in \mathcal{E}^p(X,\theta)^{\mathbb{N}}$ be a Cauchy sequence for d_p. Extracting and relabelling we can assume that there exists a subsequence $(u_j) \subseteq (\varphi_j)$ such that

$$d_p(u_j, u_{j+1}) \le 2^{-j}.$$

Define $v_{j,k} := P_\theta(u_j, \dots, u_{j+k})$ and observe that it is decreasing in k. Also, by Proposition 3.14 (i) and the triangle inequality,

$$d_p(u_j, v_{j,k}) = d_p(u_j, P_\theta(u_j, v_{j+1,k})) \le d_p(u_j, v_{j+1,k}) \le 2^{-j} + d_p(u_{j+1}, v_{j+1,k}).$$

Hence,

$$d_p(u_j, v_{j,k}) \le \sum_{\ell=j}^{k-1} 2^{-\ell} \le 2^{-j+1}.$$

In particular $I_p(u_j, v_{j,k})$ is uniformly bounded from above. We then infer that $v_{j,k}$ decreases to $v_j \in \mathrm{PSH}(X,\theta)$ as $k \to +\infty$ and a combination of Proposition 3.12 and [34, Proposition 1.9] gives

$$d_p(u_j, v_j) \le 2^{1-j}, \ \forall j. \tag{3.3}$$

Let ϕ be the unique θ-psh function with minimal singularities such that $\sup_X \phi = 0$ and $\theta_\phi^n = dV$. By Lemma 3.16,

$$\begin{aligned} |\sup_X v_j| &\le C_1 + C_2 d_p(v_j, \phi) \le C_1 + C_2 \left(d_p(v_j, u_1) + d_p(u_1, \phi)\right) \\ &\le C_1 + C_2 \left(d_p(v_j, u_j) + d_p(u_j, u_1) + d_p(u_1, \phi)\right) \\ &\le C_1 + C_2 \left(4 + d_p(u_1, \phi)\right). \end{aligned}$$

It thus follows that v_j increases a.e. to a θ-psh function v. By the triangle inequality we have

$$d_p(\varphi_j, v) \le d_p(\varphi_j, u_j) + d_p(u_j, v_j) + d_p(v_j, v).$$

Since (φ_j) is Cauchy, $d_p(\varphi_j, u_j) \to 0$. By [34, Proposition 1.9] and Proposition 3.12, we have $d_p(v_j, v) \to 0$. These facts together with (3.3) yield $d_p(\varphi_j, v) \to 0$; hence, $(\mathcal{E}^p(X,\theta), d_p)$ is a complete metric space.

Also, any $u \in \mathcal{E}^p(X,\theta)$ can be approximated from above by functions $u_j \in \mathcal{H}_\theta$ such that $d_p(u_j, u) \to 0$ (Proposition 3.12). It thus follows that $(\mathcal{E}^p(X,\theta), d_p)$ is the metric completion of $\mathcal{H}_\theta$.

Let now u_t be the Mabuchi geodesic joining $u_0, u_1 \in \mathcal{E}^p(X,\theta)$. We are going to prove that, for all $t \in [0,1]$,

$$d_p(u_t, u_s) = |t-s| d_p(u_0, u_1).$$

We claim that for all $t \in [0,1]$,

$$d_p(u_0, u_t) = t d_p(u_0, u_1) \text{ and } d_p(u_1, u_t) = (1-t) d_p(u_0, u_1). \tag{3.4}$$

We first assume that $u_0, u_1 \in \mathcal{H}_\theta$. The Mabuchi geodesic joining u_0 to u_t is given by $w_\ell = u_{t\ell}$, $\ell \in [0,1]$. Lemma 3.13 thus gives

$$d_p^p(u_0, u_t) = \int_X |\dot{w}_0|^p \theta_{u_0}^n = t^p \int_X |\dot{u}_0|^p \theta_{u_0}^n = t^p d_p^p(u_0, u_1),$$

proving the first equality in (3.4). The second one is proved similarly.

We next prove the claim for $u_0, u_1 \in \mathcal{E}^p(X, \theta)$. Let (u_i^j), $i = 0, 1$, $j \in \mathbb{N}$, be decreasing sequences of functions in $\mathcal{H}_\theta$ such that $u_i^j \downarrow u_i$, $i = 0, 1$. Let u_t^j be the Mabuchi geodesic joining u_0^j and u_1^j. Then, u_t^j decreases to u_t. By the triangle inequality we have $|d_p(u_0^j, u_t^j) - d_p(u_0, u_t)| \leq d_p(u_0^j, u_0) + d_p(u_t, u_t^j)$. The claim thus follows from Proposition 3.12 and the previous step.

Now, if $0 < t < s < 1$ then applying twice (3.4), we get

$$d_p(u_t, u_s) = \frac{s-t}{s} d_p(u_0, u_s) = (s-t) d_p(u_0, u_1).$$

□

We end this section by proving that the distance d_1 defined by approximation (see Definition 3.8) coincides with the one defined in [26] using the Monge-Ampère energy.

Proposition 3.18 *Assume $u_0, u_1 \in \mathcal{E}^1(X, \theta)$. Then,*

$$d_1(u_0, u_1) = E(u_0) + E(u_1) - 2E(P(u_0, u_1)).$$

Here the Monge-Ampère energy E is defined as

$$E(u) := \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u - V_\theta) \theta_u^j \wedge \theta_{V_\theta}^{n-j}.$$

Proof We first assume that $u_0, u_1 \in \mathcal{H}_\theta$ and $u_0 \leq u_1$. Let $[0, 1] \ni t \mapsto u_t$ be the Mabuchi geodesic joining u_0 and u_1. By [24, Theorem 3.12], $t \mapsto E(u_t)$ is affine, hence for all $t \in [0, 1]$,

$$\frac{E(u_t) - E(u_0)}{t} = E(u_1) - E(u_0) = \frac{E(u_1) - E(u_t)}{1-t}.$$

Since E is concave along affine curves (see [5, 12], [26, Theorem 2.1]), we thus have

$$\int_X \frac{u_t - u_0}{t} \theta_{u_0}^n \geq E(u_1) - E(u_0) \geq \int_X \frac{u_1 - u_t}{1-t} \theta_{u_1}^n.$$

Letting $t \to 0$ in the first inequality and $t \to 1$ in the second one, we obtain

$$\int_X \dot{u}_0 \theta_{u_0}^n \geq E(u_1) - E(u_0) \geq \int_X \dot{u}_1 \theta_{u_1}^n.$$

By Theorem 3.7 we then have

$$d_1(u_0, u_1) = \int_X \dot{u}_0 \theta_{u_0}^n = \int_X \dot{u}_1 \theta_{u_1}^n = E(u_1) - E(u_0).$$

We next assume that $u_0, u_1 \in \mathcal{H}_\theta$ but we remove the assumption that $u_0 \leq u_1$. By Lemma 3.1, $P(u_0, u_1) \in \mathcal{H}_\theta$. By the Pythagorean formula (Proposition 3.14) and the first step, we have

$$\begin{aligned} d_1(u_0, u_1) &= d_1(u_0, P(u_0, u_1)) + d_1(u_1, P(u_0, u_1)) \\ &= E(u_0) - E(P(u_0, u_1)) + E(u_1) - E(P(u_0, u_1)). \end{aligned}$$

We now treat the general case. Let (u_i^j), $i = 0, 1$, $j \in \mathbb{N}$ be decreasing sequences of functions in $\mathcal{H}_\theta$ such that $u_i^j \downarrow u_i$, $i = 0, 1$. Then, $P(u_0^j, u_1^j) \downarrow P(u_0, u_1)$. By [26, Proposition 2.4], $E(u_i^j) \to E(u_i)$, for $i = 0, 1$ and $E(P(u_0^j, u_1^j)) \to E(P(u_0, u_1))$ as $j \to +\infty$. The result thus follows from Proposition 3.12, the triangle inequality, and the previous step. □

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