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Two Strong Convergence Theorems for the Common Null Point Problem in Banach Spaces

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Abstract
In this paper, we study the common null point problem in Banach spaces. Then, using the shrinking projection method and ε-enlargement of maximal monotone operator, we prove two strong convergence theorems with nonsummable errors for solving this problem.

Keywords Split common null point problem · Maximal monotone operator · Metric resolvent · ε-enlargement

Mathematics Subject Classification (2010) 47H05 · 47H09 · 47J25

1 Introduction

Let H be a real Hilbert space and let $f : H \longrightarrow (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. In order to find a minimum point of f, Martinet [19] proposed the iterative method as follows: $x_1 \in H$ and

$$x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2}\|y - x_n\|^2 \right\},$$

for all $n \geq 1$. He proved that, the sequence $\{x_n\}$ converges weakly to a minimum point of f. Note that, the above sequence $\{x_n\}$ can be rewritten in the form

$$\partial f(x_{n+1}) + x_{n+1} \ni x_n, \ \forall n \geq 1.$$

We know that the subdifferential operator ∂f of f is a maximal monotone operator (see [27]). So, the problem of finding a null point of a maximal monotone operator plays an

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important role in optimization. One popular method of solving equation $0 \in A(x)$ where A is a maximal monotone operator in Hilbert space H, is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in E$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_{c_n}^A(x_n), \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

where $\{c_n\}$ is a sequence of positive real numbers and $J_{c_n}^A = (I + c_n A)^{-1}$ is the resolvent of A. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing assumptions on A. Moreover, Rockafellar [29] gave a more practical method which is an inexact variant of the method:

$$x_n + e_n \in x_{n+1} + c_n A x_{n+1} \text{ for all } n \in \mathbb{N}, \tag{1.2}$$

where $\{e_n\}$ is regarded as an error sequence and $\{c_n\}$ is a sequence of positive regularization parameters. Note that the algorithm (1.2) can be rewritten as

$$x_{n+1} = J_{c_n}^A(x_n + e_n) \text{ for all } n \in \mathbb{N}. \tag{1.3}$$

This method is called inexact proximal point algorithm. It was shown in Rockafellar [29] that if $e_n \longrightarrow 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

In [10], Burachik et al. used the enlargement A^{ε} to devise an approximate generalized proximal point algorithm. The exact version of this algorithm can be stated as follows: having x_n, the next element x_{n+1} is the solution of

$$0 \in c_n A(x) + \nabla f(x) - \nabla f(x_n), \tag{1.4}$$

where f is a suitable regularization function. Note that, if $f(x) = \frac{1}{2}\|x\|^2$, then the above algorithm becomes the classical proximal point algorithm. Approximate solutions of (1.4) are treated in [10] via A^{ε}. Specifically, an approximate solution of (1.4) is regarded as an exact solution of

$$0 \in c_n A^{\varepsilon_n}(x) + \nabla f(x) - \nabla f(x_n),$$

for an appropriate value of ε_n. Note that, if $f(x) = \frac{1}{2}\|x\|^2$, the above relation is equivalent to the problem of finding an element $x_{n+1} \in H$, and $v_{n+1} \in A^{\varepsilon_n}(x_{n+1})$ with $\varepsilon_n \geq 0$ such that

$$0 = c_n v_{n+1} + (x_{n+1} - x_n).$$

They proved that if $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then the sequence $\{x_n\}$ converges weakly to a null point of A.

In [25], Solodov and Svaiter proposed a new criterion for the approximate solution of subproblems as follows: Two element y_n and v_n are admissible if

$$v_n \in A(y_n),\ 0 = c_n v_n + (y_n - x_n) - e_n,$$

and the error e_n satisfies

$$\|e_n\| \leq \sigma \max\{c_n\|v_n\|, \|y_n - x_n\|\},$$

where σ is a real number in $[0, 1)$. And the next iterative x_{n+1} is obtained by projecting x_n onto the hyperplane

$$\{z \in H : \langle v_n, z - y_n\rangle = 0\}.$$

By combining the ideas of [10, 25], Solodov et al. [24] proposed an even simpler method, in which no projection is performed. An approximate solution is regarded as a pair $y_n,\ v_n$ such that

$$v_n \in c_n A^{\varepsilon_n}(y_n),\ 0 = c_n v_n + (y_n - x_n) - e_n,$$

where $\varepsilon_n,\ e_n$ are "relatively small" in comparison with $\|y_n - x_n\|$, and the next iterative x_{n+1} is defined by

$$x_{n+1} = x_n - c_n v_n.$$

Rockafellar [29] posed an open question of whether the sequence generated by (1.1) converges strongly or not. In 1991, Güler [15] gave an example showing that Rockafellar's proximal point algorithm does not converge strongly. An example of the authors Bauschke, Matoušková, and Reich [7] also showed that the proximal algorithm only converges weakly but not in norm. In 2000, Solodov and Svaiter [26] proposed the following algorithm (hybrid projection method): Choose any $x_0 \in H$ and $\sigma \in [0, 1)$. At iteration n, having x_n, choose $\mu_n > 0$ and find (y_n, v_n) an inexact solution of

$$0 \in A(x) + \mu_n(x - x_n),$$

with tolerance σ. Define

$$C_n = \{z \in H\ :\ \langle z - y_n, v_n\rangle \le 0\},$$

and

$$Q_n = \{z \in H\ :\ \langle z - x_n, x_0 - x_n\rangle \le 0\}.$$

Take

$$x_{n+1} = P_{C_n \cap Q_n} x_0.$$

They proved that if the sequence of the regularization parameters μ_n is bounded from above, then $\{x_n\}$ converges strongly to $x^* \in A^{-1}0$. Moreover, based on the important fact that C_n and Q_n in the above algorithm are two halfspaces, they showed that

$$x_{n+1} = x_0 + \lambda_1 v_n + \lambda_2(x_0 - x_n),$$

where $(\lambda_1,\ \lambda_2)$ is the solution of the linear system of two equations with two unknowns:

$$\begin{cases} \lambda_1\|v_n\|^2 + \lambda_2\langle v_n, x_0 - x_n\rangle = -\langle x_0 - y_n, v_n\rangle \\ \lambda_1\langle v_n, x_0 - x_n\rangle + \lambda_2\|x_0 - x_n\|^2 = -\|x_0 - x_n\|^2. \end{cases}$$

In 2003, Bauschke et al. introduced a new algorithm (see [6, Algorithm 4.1]) for finding a common fixed point of a family of operators $(T_i)_{i\in\mathbb{I}}$ in $\mathcal{B}$-class operators (see [5]). Let E be a real Banach space and $f\ :\ E \longrightarrow (0, \infty]$ be a lower semicontinuous convex function which is Gâteaux differentiable on int dom $f \neq \emptyset$ and Legendre, i.e., it satisfies the following two properties:

(i) ∂f is both locally bounded and single-valued on its domain;
(ii) $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every bounded set of dom ∂f.

The Bregman distance associated with f is the function

$$\begin{aligned} D:\ E \times E &\longrightarrow [0, \infty] \\ (x, y) &\longmapsto \begin{cases} f(x) - f(y) - \langle x - y, \nabla f(y)\rangle & \text{if } x, y \in \text{ int dom } f, \\ \infty & \text{otherwise,} \end{cases} \end{aligned}$$

and the D-projector onto a set $C \subset E$ is the operator

$$\begin{aligned} P_C:\ E &\longrightarrow 2^E, \\ y &\longmapsto \{x \in C\ :\ D(x, y) = D_C(y) < \infty\}. \end{aligned}$$

It is easy to see that if E is a real Hilbert space and $f(x) = \|x\|^2/2$ for all $x \in H$, and C is a nonempty closed convex subset of E, then P_C is the metric projection from E onto C.

They gave an application of [6, Algorithm 4.1] for finding a common zero of a family of maximal monotone operators $(A_i)_{i\in\mathbb{I}}$ in Banach space E as follows: for every $n \in \mathbb{N}$, take $i(n) \in I$, $\gamma_n \in (0, \infty)$ and set $x_{n+1} = Q(x_0, x_n, (\nabla f + \gamma_n A_{i(n)})^{-1} \circ \nabla f(x_n))$, where

$$Q(x, y, z) = \{u \in E : \langle u - y, \nabla f(x) - \nabla f(y)\rangle \le 0\}$$
$$\cap\{u \in E : \langle u - z, \nabla f(y) - \nabla f(z)\rangle \le 0\}.$$

They proved that if ∇f is uniformly continuous on bounded subsets of E and for every $i \in I$, and every strictly increasing sequence $\{p_n\}$ such that $i(p_n) \equiv i$, one has $\inf_n \gamma_{p_n} > 0$ and if the following conditions hold:

(i) The index control mapping $\mathrm{i} : \mathbb{N} \longrightarrow I$ satisfies

$$(\forall i \in I)(\exists M_i > 0)(\forall n \in \mathbb{N})\ i \in \{\mathrm{i}(n), \ldots, \mathrm{i}(n + M_i - 1)\}.$$

(ii) For every sequence $\{y_n\}$ in int dom f and every bounded sequence $\{z_n\}$ in int dom f, one has

$$D(y_n, z_n) \to 0 \Rightarrow y_n - z_n \to 0,$$

then $x_n \to P_S x_0$, where $S = \overline{\mathrm{dom}} f \cap (\cap_{i\in I} A_i^{-1}0)$.

In order to find a fixed point of a nonexpansive mapping T on the closed and convex subset C of H, motivated by the result of Solodov and Svaiter, Takahashi et al. [32] introduced the following iterative method

$$\begin{aligned} C_0 &= C,\ x_0 \in C,\\ y_n &= \alpha_n x_n + (1 - \alpha_n)T x_n,\\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{C_{n+1}} x_0,\ n \ge 0, \end{aligned}$$

and they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$, when $\{\alpha_n\} \subset [0, a)$, with $a \in [0, 1)$. Moreover, they also gave a similar iterative method to find zero of a maximal monotone operator in the following form

$$\begin{aligned} C_0 &= C,\ x_0 \in C,\\ y_n &= \alpha_n x_n + (1 - \alpha_n)J^A_{c_n} x_n,\\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\},\\ x_{n+1} &= P_{C_{n+1}} x_0,\ n \ge 0. \end{aligned} \tag{1.5}$$

They showed that if $\{\alpha_n\} \subset [0, a)$, with $a \in [0, 1)$ and $c_n \to \infty$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{A^{-1}0}x_0$.

We can see that the shrinking projection method (1.5) of Takahashi et al. is more complex than the hybrid projection method of Solodov and Svaiter. Because in the iterative method (1.5), to define x_{n+1}, we have to find the projection of x_0 over the intersection of n closed and convex subsets of H, but in hybrid projection method, we only compute the projection of x_0 over the intersection of two hyperplanes. However, recently, many mathematicians studied the shrinking projection method for solving the difference problems, see for instance, Dadashi [13], Kimura [18], Qin et al. [22], Takahashi [33], Takahashi et al. [30, 34], Sean et al. [37].

In this paper, by using shrinking projection method, we introduce two parallel iterative methods for finding a common null point of a finite family of maximal monotone operators in Banach spaces. Moreover, we also give some applications of the main results for solving the problem of finding a common minimum point of convex functions, the convex feasibility problem and the system of variational inequalities. In Section 5, a numerical example is also given to illustrate the effectiveness of the proposed algorithms.

2 Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp. $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp. weak, weak*) convergence of the sequence $\{x_n\}$ to x. Let J_E denote the normalized duality mapping from E into 2^{E^*} given by

$$J_E x = \left\{ f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2 \right\}, \ \forall x \in E.$$

We always use S_E to denote the unit sphere $S_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if $x, y \in S_E$ with $x \neq y$, and, for all $t \in (0, 1)$,

$$\|(1-t)x + ty\| < 1.$$

A Banach space E is said to be uniformly convex if for any $\varepsilon \in (0, 2]$ and the inequalities $\|x\| \le 1$, $\|y\| \le 1$, $\|x - y\| \ge \varepsilon$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{\|x + y\|}{2} \le 1 - \delta.$$

Recall that a Banach space E is called having the Kadec-Klee property, if for every sequence $\{x_n\} \subset E$ such that $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$, as $n \to \infty$, we have $x_n \to x$, as $n \to \infty$. It is well known that every uniformly convex Banach space has Kadec-Klee property (see [12, 23]).

A Banach space E is said to be smooth provided the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit attained uniformly for $x \in S_E$.

Let E be a reflexive Banach space, we know that E is uniformly convex if and only if E^* is uniformly smooth [1, 14].

We have the following properties of the normalized duality mapping J_E (see [1, 12, 14]):

i) E is reflexive if and only if J_E is surjective.
ii) If E^* is strictly convex, then J_E is single-valued.
iii) If E is a smooth, strictly convex and reflexive Banach space, then J_E is single-valued bijection.
iv) If E^* is uniformly convex, then J_E is uniformly continuous on each bounded set of E.

We know that if E is a smooth, strictly convex and reflexive Banach space and C is a nonempty, closed, and convex subset of E; then, for each $x \in E$, there exists a unique $z \in C$ such that

$$\|x - z\| = \inf_{y\in C} \|x - y\|.$$

The mapping $P_C : E \longrightarrow C$ defined by $P_C x = z$ is called metric projection from E on to C and we denote by $d(x, C) = \|x - P_C x\|$.

Let $A : E \longrightarrow 2^{E^*}$ be an operator. The effective domain of A is denoted by $D(A)$, that is, $D(A) = \{x \in E : Ax \neq \emptyset\}$. Recall that A is called monotone operator if $\langle x-y, u-v\rangle \ge$

0 for all $x, y \in D(A)$ and for all $u \in Ax,\ v \in A(y)$. A monotone operator A on E is called maximal monotone if its graph is not properly contained in the graph of any other monotone operator on E. We know that if A is a maximal monotone operator on E and if E is a uniformly convex and smooth Banach space, then $R(J_E + rA) = E^*$ for all $r > 0$, where $R(J_E + rA)$ is the range of $J_E + rA$ (see [9, 28]). For all $x \in E$ and $r > 0$, there exists a unique $x_r \in E$ such that

$$0 \in J_E(x_r - x) + rAx_r.$$

We define J_r by $x_r = J_r x$ and J_r is called the metric resolvent of A.

Hence, in this case, we can define a mapping $J_r : E \longrightarrow E$ by $J_r x = x_r$ and J_r is called the generalized resolvent of A.

The set of null point of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$ and we know that $A^{-1}0$ is a closed and convex subset of E (see [31]).

Let $A : E \longrightarrow 2^{E^*}$ be a maximal monotone operator. In [11], for each $\varepsilon \geq 0$, Burachik and Svaiter defined $A^\varepsilon(x)$, an ε-enlargement of A, as follows

$$A^\varepsilon x = \{u \in E^* : \langle y - x, v - u\rangle \geq -\varepsilon,\ \forall y \in E,\ v \in Ay\}.$$

It is easy to see that $A^0 x = Ax$ and if $0 \leq \varepsilon_1 \leq \varepsilon_2$, then $A^{\varepsilon_1}x \subseteq A^{\varepsilon_2}x$ for any $x \in E$. The use of element in A^ε instead of T allows an extra degree freedom which is very useful in various applications.

Let $\{C_n\}$ be a sequence of closed, convex, and nonempty subsets of a reflexive Banach space E. We define the subsets s-Li$_n C_n$ and w-Ls$_n C_n$ of E as follows: $x \in$ s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset E$ converges strongly to x and that $x_n \in C_n$ for all $n \geq 1$; $x \in$ w-Ls$_n C_n$ if and only if there exists a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \geq 1$. If s-Li$_n C_n =$ w-Ls$_n C_n = C_0$, then C_0 is called the limits of $\{C_n\}$ in the sense of Mosco [20] and it is denoted by $C_0 = \mathrm{M} - \lim_{n\to\infty} C_n$.

Remark 2.1 We know that, if $\{C_n\}$ is a decreasing sequence of closed convex subsets of a reflexive Banach space E and $C_0 = \cap_{n=1}^{\infty} C_n \neq \emptyset$, then $C_0 = \mathrm{M} - \lim_{n\to\infty} C_n$ (see [8, 17]).

Indeed, it is clear that if $x \in C_0$, then $x \in$ s-Li$_n C_n$ and $x \in$ w-Ls$_n C_n$, because the sequence $\{x_n\}$ with $x_n = x$ for all $n \geq 1$ converges strongly to x. Thus, we have $C_0 \subset$ s-Li$_n C_n$ and $C_0 \subset$ w-Ls$_n C_n$.

Now, we will show that $C_0 \supseteq$ s-Li$_n C_n$ and $C_0 \supseteq$ w-Ls$_n C_n$. Let $x \in$ s-Li$_n C_n$, from the definition of s-Li$_n C_n$, there exists a sequence $\{x_n\} \subset E$ with $x_n \in C_n$ for all $n \geq 1$ such that $x_n \to x$, as $n \to \infty$. Since $\{C_n\}$ is a decreasing sequence, $x_{n+k} \in C_n$ for all $n \geq 1$ and $k \geq 0$. So, letting $k \to \infty$ and by the closedness of C_n, we get that $x \in C_n$ for all $n \geq 1$. Thus, $x \in C_0$ and hence $C_0 \supseteq$ s-Li$_n C_n$. Next, let $y \in$ w-Ls$_n C_n$, from the definition of w-Ls$_n C_n$, there exist a subsequence $\{C_{n_k}\}$ of $\{C_n\}$ and the sequence $\{y_k\} \subset E$ such that $y_k \rightharpoonup x$ and $y_k \in C_{n_k}$ for all $k \geq 1$. From $\{C_n\}$ is a decreasing sequence, we have

$$y_{k+p} \in C_{n_k} \tag{2.1}$$

for all $k \geq 1$ and $p \geq 0$. Since C_{n_k} is closed and convex, C_{n_k} is weakly closed in E for all $k \geq 1$. So, in (21), letting $p \to \infty$, we get that $y \in C_{n_k}$ for all $k \geq 1$. Since $C_k \supseteq C_{n_k}$, $y \in C_k$ for all $k \geq 1$. So, $y \in C_0$ and hence $C_0 \supseteq$ w-Ls$_n C_n$.

Consequently, we obtain that s-Li$_n C_n$ = and w-Ls$_n C_n = C_0$. Thus, $C_0 = \mathrm{M} - \lim_{n\to\infty} C_n$.

The following lemmas will be needed in the sequel for the proof of main theorems.

Lemma 2.2 [2, 3, 16] *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

i) $z = P_C x_1$;
ii) $\langle z - y, J_E(x_1 - z)\rangle \geq 0,\ \forall y \in C$.

Lemma 2.3 [36] *Let E be a Banach space, $R \in (0, \infty)$ and $B_R = \{x \in E : \|x\| \leq R\}$. If E is uniformly convex, then there exists a continuous, strictly increasing and convex function $g : [0, 2R] \longrightarrow [0, \infty)$ with $g(0) = 0$ such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|),$$

for all $x, y \in B_R$ and $\alpha \in [0, 1]$.

Lemma 2.4 [35] *Let E be a smooth, reflexive, and strictly convex Banach space having the Kadec-Klee property. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E. If $C_0 = \mathrm{M} - \lim_{n\to\infty} C_n$ exists and is nonempty, then $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$ for each $x \in C$.*

Lemma 2.5 [11] *The graph of $A^\varepsilon : \mathbb{R}_+ \times E \longrightarrow 2^{E^*}$ is demiclosed, i.e., the conditions below hold:*

i) *If $\{x_n\} \subset E$ converges strongly to x_0, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges weakly to u_0 in E^* and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$.*
ii) *If $\{x_n\} \subset E$ converges weakly to x_0, $\{u_n \in A^{\varepsilon_n}x_n\}$ converges strongly to u_0 in E^* and $\{\varepsilon_n\} \subset \mathbb{R}_+$ converges to ε, then $u_0 \in A^\varepsilon x_0$.*

3 Main Results

First, we have the following lemma:

Lemma 3.1 *Let E be a uniformly convex and smooth Banach space and let $\{C_n\}$ be a decreasing sequence of closed and convex subsets of E such that $C_0 = \cap_{n=1}^{\infty} C_n \neq \emptyset$. Let $p_n = P_{C_n}u$ with $u \in E$ and let $\{x_n\}$ be the sequence in E such that*

$$x_n \in \{z \in C_n : \|u - z\|^2 \leq d^2(u, C_n) + \delta_n\},$$

for all $n \geq 1$, where $\{\delta_n\}$ is a sequence of positive real numbers. If $\lim_{n\to\infty} \delta_n = 0$, then $\{x_n\}$ and $\{p_n\}$ converge strongly to the same point $p_0 = P_{C_0}u$.

Proof From Remark 2.1, we have $C_0 = \mathrm{M} - \lim_{n\to\infty} C_n$. By Lemma 2.4, we have $p_n \to p_0 = P_{C_0}u$, as $n \to \infty$.

Since $p_n = P_{C_n}u$, $d(u, C_n) = \|u - p_n\|$. From $x_n \in C_n$ and the definition of C_n, we have

$$\|u - x_n\|^2 \leq \|u - p_n\|^2 + \delta_n,\ \forall n \geq 2. \tag{3.1}$$

From (3.1) and the boundedness of $\{p_n\}$, the sequence $\{x_n\}$ is bounded. So, $R = \max\{\sup_n\{\|x_n\|\}, \sup_n\{\|p_n\|\}\} < \infty$.

From the convexity of C_n, we have $\alpha p_n + (1-\alpha)x_n \in C_n$ for all $\alpha \in (0,1)$. Thus, from the definition of $P_{C_n}u$ and apply Lemma 2.3 on B_R, we get

$$\begin{aligned}\|p_n - u\|^2 &\leq \|\alpha p_n + (1-\alpha)x_n - u\|^2\\ &\leq \alpha\|p_n - u\|^2 + (1-\alpha)\|x_n - u\|^2 - \alpha(1-\alpha)g(\|x_n - p_n\|),\end{aligned}$$

this combines with (3.1), we obtain that

$$\alpha g(\|x_n - p_n\|) \leq \delta_n, \ \forall \alpha \in (0,1). \tag{3.2}$$

In (3.2), letting $\alpha \to 1^-$, we get

$$g(\|x_n - p_n\|) \leq \delta_n.$$

By the property of g and $\delta_n \to 0$, we have

$$\|x_n - p_n\| \to 0.$$

Thus, the sequences $\{x_n\}$ and $\{p_n\}$ converge strongly to the same point p_0, as $n \to \infty$. □

Now, we have the following theorem:

Theorem 3.2 *Let E be a uniformly convex and smooth Banach space and let $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, be maximal monotone operators of E into 2^{E^*} such that $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be nonnegative real sequences and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that* $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. *For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and*

i) *Find $y_{i,n} \in E$ such that $J_E(y_{i,n} - x_n) + r_{i,n}A_i^{\varepsilon_n}y_{i,n} \ni 0, \ i = 1, 2, \ldots, N$.*
ii) *Choose i_n such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,*

$$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}\}. \tag{3.3}$$

iii) *Find $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\}, \ n = 1, 2, \ldots$*

If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = \lim_{n\to\infty} \delta_n = 0$ *for all $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

Proof First, we show that $S \subset C_n$ for all $n \geq 1$ by mathematical induction. Indeed, it is clear that $S \subset C_1 = E$. Suppose that $S \subset C_n$ for some $n \geq 1$. Take $v \in S$, we have

$$J_E(y_{i_n,n} - x_n) + r_{i_n,n}A_{i_n}^{\varepsilon_n}y_{i_n,n} \ni 0, \ A_{i_n}v \ni 0.$$

From the definition of $A_{i_n}^{\varepsilon_n}$, we get

$$\langle y_n - v, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}.$$

Thus, $v \in C_{n+1}$. Since v is arbitrary in S, $S \subset C_{n+1}$. So, by induction we obtain that $S \subset C_n$ for all $n \geq 1$.

Moreover, C_n is a closed and convex subset of E for all n. Hence, the sequence $\{x_n\}$ is well defined.

Now, for each $n \geq 1$, denote by $p_n = P_{C_n}u$. By Lemma 3.1, we obtain that the sequences $\{x_n\}$ and $\{p_n\}$ converge strongly to the same point $p_0 = P_{C_0}u$ with $C_0 = \cap_{n=1}^\infty C_n$.

From $p_{n+1} \in C_{n+1}$ and the definition of C_{n+1}, we have

$$\langle y_n - p_{n+1}, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}.$$

The above inequality is equivalent to

$$\langle y_n - x_n, J_E(x_n - y_n)\rangle + \langle x_n - p_{n+1}, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}.$$

So, we have

$$\begin{aligned}\|x_n - y_n\|^2 - \varepsilon_n r_{i_n,n} &\leq \langle x_n - p_{n+1}, J_E(x_n - y_n)\rangle \\ &\leq \|x_n - p_{n+1}\| \, \|x_n - y_n\| \\ &\leq \frac{1}{2}(\|x_n - p_{n+1}\|^2 + \|x_n - y_n\|^2).\end{aligned}$$

This implies that

$$\|x_n - y_n\|^2 \leq \|x_n - p_{n+1}\|^2 + 2\varepsilon_n r_{i_n,n}.$$

From $p_n \to p_0$, $x_n \to p_0$ and $\varepsilon_n r_{i_n,n} \to 0$, we obtain that

$$\|x_n - y_n\| \to 0.$$

By the definition of y_n, we get that

$$\|x_n - y_{i,n}\| \to 0, \ \forall i = 1, 2, \ldots, N. \tag{3.4}$$

This implies that $y_{i,n} \to p_0$ for all $i = 1, 2, \ldots, N$, as $n \to \infty$. Furthermore, since $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$ and (3.4), we have

$$0 \leftarrow \frac{1}{r_{i,n}} J_E(x_n - y_{i,n}) \in A_i^{\varepsilon_n} y_{i,n},$$

for all $i = 1, 2, \ldots, N$, as $n \to \infty$. So, by Lemma 2.5, we obtain $p_0 \in A_i^{-1}0$ for all $i = 1, 2, \ldots, N$, that is, $p_0 \in S$.

Finally, we show that $p_0 = P_S u$. Indeed, let $x^* = P_S u$. Since $S \subset C_n$, $x^* \in C_n$. Thus, from $p_n = P_{C_n} u$, we have

$$\|p_n - u\| \leq \|u - x^*\|, \ \forall n \geq 1.$$

Letting $n \to \infty$, we get that $\|u - p_0\| \leq \|u - x^*\|$. By the uniqueness of x^*, we obtain that $p_0 = x^* = P_S u$. This completes the proof. □

Now, in the following theorem, we give another way to construct the subsets C_n.

Theorem 3.3 *Let E be a uniformly convex and smooth Banach space and let $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, be maximal monotone operators of E into 2^{E^*} such that $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be nonnegative real sequences and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that* $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. *For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and*

i) *Find* $y_{i,n} \in E$ *such that* $J_E(y_{i,n} - x_n) + r_{i,n} A_i^{\varepsilon_n} y_{i,n} \ni 0, \ i = 1, 2, \ldots, N,$
$C_{n+1}^i = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\}, \ i = 1, 2, \ldots, N,$
$C_{n+1} = \cap_{i=1}^N C_{n+1}^i.$

ii) *Find* $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\}, \ n = 1, 2, \ldots$

If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = \lim_{n\to\infty} \delta_n = 0$ *for all $i = 1, 2, \ldots, N$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

Proof First, we show that $S \subset C_n$ for all $n \geq 1$ by mathematical induction. Indeed, it is clear that $S \subset C_1 = E$. Suppose that $S \subset C_n$ for some $n \geq 1$. Take $v \in S$, we have

$$J_E(y_{i,n} - x_n) + r_{i,n} A_i^{\varepsilon_n} y_{i,n} \ni 0, \ \ A_i v \ni 0.$$

From the definition of $A_i^{\varepsilon_n}$, we get

$$\langle y_{i,n} - v, J_E(x_n - y_{i,n}) \rangle \geq -\varepsilon_n r_{i,n}.$$

Thus, $v \in C_{n+1}^i$ for all $i = 1, 2, \dots, N$. So, $v \in C_{n+1} = \cap_{i=1}^N C_{n+1}^i$. By induction, we obtain that $S \subset C_n$ for all $n \geq 1$.

It is clear that $\{C_n\}$ is a decreasing sequence of closed and convex subsets of E with $\cap_{n=1}^\infty C_n = C_0 \supset S \neq \emptyset$.

Now, for each n, denote by $p_n = P_{C_n} u$. By Lemma 3.1, the sequences $\{x_n\}$ and $\{p_n\}$ converge strongly to the same point $p_0 = P_{C_0} u$.

We have $p_{n+1} \in C_{n+1} = \cap_{i=1}^N C_{n+1}^i$. Hence, $p_{n+1} \in C_{n+1}^i$ for all $i = 1, 2, \dots, N$. Thus, from the definition of C_{n+1}^i, we have

$$\langle y_{i,n} - p_{n+1}, j(x_n - y_{i,n}) \rangle \geq -\varepsilon_n r_{i,n},$$

for all $i = 1, 2, \dots, N$. Thus, we get that

$$\|x_n - y_{i,n}\|^2 \leq \|x_n - p_{n+1}\|^2 + 2\varepsilon_n r_{i,n},$$

for all $i = 1, 2, \dots, N$. From $p_n \to p_0$, $x_n \to p_0$ and $\varepsilon_n r_{i_n,n} \to 0$, we obtain that

$$\|x_n - y_{i,n}\| \to 0,$$

for all $i = 1, 2, \dots, N$.
The rest of the proof follows the pattern of Theorem 3.2. This completes the proof. □

Remark 3.4 a) In Theorems 3.2 and 3.3, if $N = 1$ then the sequence $\{x_n\}$ is defined by: For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and

i) Find $y_n \in E$ such that $J_E(y_n - x_n) + r_n A^{\varepsilon_n} y_n \ni 0$,

$C_{n+1} = \{z \in C_n \ : \ \langle y_n - z, J_E(x_n - y_n) \rangle \geq -\varepsilon_n r_n\}$.

ii) Find $x_{n+1} \in \{z \in C_{n+1} \ : \ \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\}, \ n = 1, 2, \dots,$

where $\{r_n\}$ is the positive real sequence and $\{\varepsilon_n\}$, $\{\delta_n\}$ are nonnegative real sequences such that $\inf_n \{r_n\} \geq r > 0$ and $\lim_{n\to\infty} r_n \varepsilon_n = \lim_{n\to\infty} \delta_n = 0$.

b) In Theorem 3.3, to define the element x_{n+1}, we have to find the projection of u onto the intersection of $n \times N$ half-spaces. In Theorem 3.2, we only find the projection of u onto the intersection of n half-spaces. So, the algorithm to define x_{n+1} in Theorem 3.2 is simpler than the algorithm in Theorem 3.3. However, in the both cases, we can find the element x_{n+1} by the approximation solution of the following minimization problem: Find a minimum point of $f(x) = \frac{1}{2}\|x - u\|^2$ over the intersection of a finite family of half-spaces C_i. In particular, if $E = \mathbb{R}^m$, then we can find x_{n+1} easily by using the "Quadratic Programming Algorithms" package in MATLAB software.

Next, we have the following corollaries:

Corollary 3.5 *Let E be a uniformly convex and smooth Banach space and let $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \dots, N$, be maximal monotone operators of E into 2^{E^*} such that $S = \cap_{i=1}^N A_i^{-1} 0 \neq \emptyset$. Let J_r^i be the metric resolvent of A_i for $r > 0$ with $i = 1, 2, \dots, N$. Let $\{\delta_n\}$ be nonnegative real sequence and let $\{r_{i,n}\}$, $i = 1, 2, \dots, N$, be positive real sequences* *such that* $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. *For a given point* $u \in E$, *we define the sequence* $\{x_n\}$ *by* $x_1 = x \in E$, $C_1 = E$ *and*

$$\begin{aligned}
&\text{i) } y_{i,n} = J^i_{r_{i,n}} x_n,\ i = 1, 2, \ldots, N\\
&\text{ii) } \textit{Choose } i_n \textit{ such that } \|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\},\ \textit{let } y_n = y_{i_n,n},\\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\},\ \textit{or}\\
&\text{ii*) } C^i_{n+1} = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N\\
&C_{n+1} = \cap_{i=1}^N C^i_{n+1},\\
&\text{iii) } \textit{Find } x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots
\end{aligned}$$

If $\lim_{n\to\infty} \delta_n = 0$, *then the sequence* $\{x_n\}$ *converges strongly to* $P_S u$, *as* $n \to \infty$.

Proof In (3.3) if $\varepsilon_n = 0$ for all $n \geq 1$, then the elements $y_{i,n}$, $i = 1, 2, \ldots, N$, can be rewritten in the form

$$J_E(y_{i,n} - x_n) + r_{i,n} A_i y_{i,n} \ni 0.$$

The above inclusion equation is equivalent to

$$y_{i,n} = J^i_{r_{i,n}} x_n,$$

for all $i = 1, 2, \ldots, N$.

So, apply Theorems 3.2 and 3.3 with $\varepsilon_n = 0$ for all $n \geq 1$, we obtain the proof of this corollary. □

Corollary 3.6 *Let* E *be a uniformly convex and smooth Banach space and let* $A_i : E \longrightarrow 2^{E^*}$, $i = 1, 2, \ldots, N$, *be maximal monotone operators of* E *into* 2^{E^*} *such that* $S = \cap_{i=1}^N A_i^{-1}0 \neq \emptyset$. *Let* $\{\varepsilon_n\}$ *be a nonnegative real sequence and let* $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, *be positive real sequences such that* $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. *For a given point* $u \in E$, *we define the sequence* $\{x_n\}$ *by* $x_1 = x \in E$, $C_1 = E$ *and*

$$\begin{aligned}
&\text{i) } \textit{Find } y_{i,n} \in E \textit{ such that } J_E(y_{i,n} - x_n) + r_{i,n} A_i^{\varepsilon_n} y_{i,n} \ni 0,\ i = 1, 2, \ldots, N\\
&\text{ii) } \textit{Choose } i_n \textit{ such that } \|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\},\ \textit{let } y_n = y_{i_n,n},\\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}\},\ \textit{or}\\
&\text{ii*) } C^i_{n+1} = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\},\ i = 1, 2, \ldots, N\\
&C_{n+1} = \cap_{i=1}^N C^i_{n+1},\\
&\text{iii) } x_{n+1} = P_{C_{n+1}} u,\ n = 1, 2, \ldots
\end{aligned}$$

If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = 0$ *for all* $i = 1, 2, \ldots, N$, *then the sequence* $\{x_n\}$ *converges strongly to* $P_S u$, *as* $n \to \infty$.

Proof In (3.3), if $\delta_n = 0$ for all $n \geq 1$, then we have the element x_{n+1} is defined by

$$x_{n+1} \in \{z \in C_{n+1} : \|u - z\| \leq d(u, C_{n+1})\},$$

that is $x_{n+1} = P_{C_{n+1}} u$.

So, apply Theorem 3.2 with $\delta_n = 0$ for all $n \geq 1$, we obtain the proof of this corollary. □

Remark 3.7 If $\varepsilon = \delta_n = 0$ for all $n \geq 1$, then the sequence $\{x_n\}$ is defined as follows: For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and

$$\begin{aligned}
&\text{i) } y_{i,n} = J^i_{r_{i,n}} x_n,\ i = 1, 2, \ldots, N,\\
&\text{ii) Choose } i_n \text{ such that } \|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\},\ \text{let } y_n = y_{i_n,n},\\
&C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\},\ \text{or}\\
&\text{ii*) } C^i_{n+1} = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N\\
&C_{n+1} = \cap_{i=1}^N C^i_{n+1},\\
&\text{iii) } x_{n+1} = P_{C_{n+1}} u,\ n = 1, 2, \ldots
\end{aligned}$$

Remark 3.8 In Remark 3.7, if E is a real Hilbert space, $N = 1$ then we obtain the result of Takahashi et al. in [32] (see [32, Theorem 4.5]). Note that in this case, we do not use the condition $r_n \to \infty$. So, Theorems 3.2 and 3.3 are more general than the result of Takahashi et al. Furthermore, in the proof of Theorems 3.2 and 3.3, we used the properties (Remark 2.1) of the limits of $\{C_n\}$ in the sense of Mosco [20] and Lemmas 2.3–2.5 to show that the sequence $\{x_n\}$ converges strongly to $P_S u$. But in order to prove [32, Theorem 4.5], Takahashi et al. used NST(I) condition and Lemma 3.1, Theorems 3.2 and 3.3. Thus, the proofs of main theorems in this paper are simpler than the proof of [32, Theorem 4.5].

4 Applications

4.1 The Common Minimum Point Problem

Let E be a Banach space and let $f : E \longrightarrow (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. The subdifferential of f is the multi-valued mapping $\partial f : E \longrightarrow 2^{E^*}$ which is defined by

$$\partial f(x) = \{g \in E^* : f(y) - f(x) \geq \langle y - x, g\rangle,\ \forall y \in E\}$$

for all $x \in E$. We know that ∂f is a maximal monotone operator (see [28]) and $x_0 \in \arg\min_E f(x)$ if and only if $\partial f(x_0) \ni 0$.

The ε-subdifferential enlargement of ∂f, is given by

$$\partial_\varepsilon f(x) = \{u \in E^* : f(y) - f(x) \geq \langle y - x, u\rangle - \varepsilon,\ \forall y \in E\},$$

for each $\varepsilon \geq 0$. We know that $\partial_\varepsilon f(x) \subset \partial^\varepsilon f(x)$, for any $x \in E$. Moreover, in some particular cases, we have that $\partial_\varepsilon f(x) \subsetneq \partial^\varepsilon f(x)$ (see [10, Example 2 and Example 3]).

In [4], when E is a real Hilbert space, Alvarez proposed the following approximate inertial proximal algorithm:

$$c_n \partial_{\varepsilon_n} f(x_{n+1}) + x_{n+1} - x_n - \alpha_n(x_n - x_{n-1}) \ni 0.$$

In [21], Moudafi and Elisabeth extended the above iterative method in the form

$$c_n \partial^{\varepsilon_n} f(x_{n+1}) + x_{n+1} - x_n - \alpha_n(x_n - x_{n-1}) \ni 0. \tag{4.1}$$

They proved that if there exists $c > 0$ such that $c_n \geq c$ for all $n \geq 1$, and there is $\alpha \in [0, 1)$ such that $\{\alpha_n\} \subset [0, \alpha]$, $\sum_{n=1}^{\infty} c_k \varepsilon_k < \infty$ and

$$\sum_{n=1}^{\infty} \alpha_n \|x_n - x_{n-1}\|^2 < \infty,$$

then the sequence $\{x_n\}$ converges weakly to a minimum point of f.

Note that, if $\alpha_n = 0$ for all $n \geq 1$, then (4.1) becomes

$$c_n \partial^{\varepsilon_n} f(x_{n+1}) + x_{n+1} - x_n \ni 0.$$

From Theorems 3.2 and 3.3, we have the following theorem:

Theorem 4.1 *Let E be a uniformly convex and smooth Banach space and let f_i, $i = 1, 2, \ldots, N$ be proper, lower semicontinuous and convex functions of E into* $(-\infty, \infty]$ *such that $S = \cap_{i=1}^{N} \arg\min_{x \in E} f_i(x) \neq \emptyset$. Let $\{\varepsilon_n\}$ and $\{\delta_n\}$ be nonnegative real sequences and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that* $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. *For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and*

i) *Find* $y_{i,n} \in E$ *such that* $J_E(y_{i,n} - x_n) + r_{i,n}\partial^{\varepsilon_n} f_i(y_{i,n}) \ni 0,\ i = 1, 2, \ldots, N,$
ii) *Choose* i_n *such that* $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\},$ *let* $y_n = y_{i_n,n},$
$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq -\varepsilon_n r_{i_n,n}\},$ *or*
ii*) $C_{n+1}^i = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq -\varepsilon_n r_{i,n}\},\ i = 1, 2, \ldots, N,$
$C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i,$
iii) *Find* $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots$

If $\lim_{n\to\infty} \varepsilon_n r_{i,n} = \lim_{n\to\infty} \delta_n = 0$ *for all* $i = 1, 2, \ldots, N$, *then the sequence* $\{x_n\}$ *converges strongly to* $P_S u$, *as* $n \to \infty$.

Remark 4.2 In Theorem 4.1, if $\varepsilon_n = 0$ for all $n \geq 1$, then the sequence $\{x_n\}$ is defined as follows: For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and

i) $y_{i,n} = \arg\min_{y \in E}\left\{f_i(y) + \frac{1}{2r_{i,n}}\|y - x_n\|^2\right\},\ i = 1, 2, \ldots, N,$
ii) Choose i_n such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\},$ let $y_n = y_{i_n,n},$
$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\},$ or
ii*) $C_{n+1}^i = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N,$
$C_{n+1} = \cap_{i=1}^{N} C_{n+1}^i,$
iii) Find $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots$

Note that if E is a real Hilbert space, then the element $y_{i,n}$ can be defined as follows

$$y_{i,n} = (I + r_{i,n}\partial f_i)^{-1}(x_n)$$

for all $i = 1, 2, \ldots, N$ and for all $n \geq 0$.

4.2 The Convex Feasibility Problem

Let C be a nonempty closed convex subset of E. Let i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

It is easy to see that i_C is the proper, semicontinuous and convex function, so its subdifferentiable ∂i_C is a maximal monotone operator. We know that

$$\partial i_C(u) = N(u, C) = \{f \in E^* : \langle u - y, f\rangle \geq 0 \, \forall y \in C\},$$

where $N(u, C)$ is the normal cone of C at u.

We denote the metric resolvent of ∂i_C by J_r with $r > 0$. Suppose $u = J_r x$ for $x \in E$, that is

$$\frac{J_E(x-u)}{r} \in \partial i_C(u) = N(u, C).$$

Thus, we have

$$\langle u - y, J_E(x-u)\rangle \geq 0,$$

for all $y \in C$. From Lemma 2.4, we get that $u = P_C x$.

So, from Corollary 3.5, we have the following theorem:

Theorem 4.3 *Let E be a uniformly convex and smooth Banach space and let Q_i, $i = 1, 2, \ldots, N$,be nonempty closed convex subsets of E such that $S = \cap_{i=1}^N Q_i \neq \emptyset$. Let $\{\delta_n\}$ be nonnegative real sequence. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and*

i) $y_{i,n} = P_{Q_i} x_n,\ i = 1, 2, \ldots, N,$

ii) *Choose i_n such that* $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, *let* $y_n = y_{i_n,n}$,

$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\}$, *or*

ii*) $C_{n+1}^i = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N,$

$C_{n+1} = \cap_{i=1}^N C_{n+1}^i,$

iii) *Find* $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots$

If $\lim_{n\to\infty} \delta_n = 0$, *then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.*

4.3 A System Variational Inequalities

Let C be a nonempty closed convex subset of E and let $A : C \longrightarrow E^*$ be a monotone operator which is hemicontinuous (that is for any $x \in C$ and $t_n \to 0^+$ we have $A(x+t_n y) \rightharpoonup Ax$ for all $y \in E$ such that $x + t_n y \in C$). Then, a point $u \in C$ is called a solution of the variational inequality for A, if

$$\langle y - u, Au\rangle \geq 0 \, \forall y \in C.$$

We denote by $VI(C, A)$ the set of all solutions of the variational inequality for A.

Define a mapping T by

$$T_A x = \begin{cases} Ax + N(x, C) & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

By Rockafellar [28], we know that T_A is maximal monotone and $T_A^{-1}0 = VI(C, A)$.

For any $y \in E$ and $r > 0$, we know that the variational inequality $VI(C, rA + J_E(\cdot - y))$ has a unique solution. Suppose that $x = VI(C, rAx + J_E(x - y))$, that is

$$\langle z - x, rA(x) + J_E(x - y)\rangle \geq 0 \; \forall z \in C.$$

From the definition of $N(x, C)$, we have

$$-rAx - J_E(x - y) \in N(x, C) = rN(x, C),$$

which implies that

$$\frac{J_E(y - x)}{r} \in Ax + N(x, C) = T_A x.$$

Thus, we obtain that $x = J_r y$, where J_r is the metric resolvent of T_A.

Now, let E and F be two uniformly convex and smooth Banach spaces and let $K_i,\ i = 1, 2, \ldots, N$ be closed convex subsets of E. Let $A_i : K_i \longrightarrow E^*$ be monotone and hemicontinuous operators. Suppose that $S = \cap_{i=1}^N VI(K_i, A_i) \neq \emptyset$.

We consider the following problem:

$$\text{Find an element } x^* \in S. \tag{4.2}$$

To solve Problem (4.2), we define the operators T_{A_i} as follows

$$T_{A_i}x = \begin{cases} A_i x + N(x, K_i) & \text{if } x \in K_i, \\ \emptyset & \text{if } x \notin K_i, \end{cases}$$

for all $i = 1, 2, \ldots, N$.

So, from Corollary 3.5, we have the following theorem:

Theorem 4.4 *Let $\{\delta_n\}$ be a nonnegative real sequence and let $\{r_{i,n}\}$, $i = 1, 2, \ldots, N$, be positive real sequences such that $\min_i\{\inf_n\{r_{i,n}\}\} \geq r > 0$. For a given point $u \in E$, we define the sequence $\{x_n\}$ by $x_1 = x \in E$, $C_1 = E$ and*

i) $y_{i,n} = VI\left(K_i, r_{i,n}A_i(\cdot) + J_E(\cdot - x_n)\right),\ i = 1, 2, \ldots, N,$

ii) *Choose i_n such that* $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, *let* $y_n = y_{i_n,n}$,

$C_{n+1} = \{z \in C_n : \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\}$, *or*

ii*) $C^i_{n+1} = \{z \in C_n : \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\},\ i = 1, 2, \ldots, N,$

$C_{n+1} = \cap_{i=1}^N C^i_{n+1},$

iii) *Find* $x_{n+1} \in \{z \in C_{n+1} : \|u - z\|^2 \leq d^2(u, C_{n+1}) + \delta_{n+1}\},\ n = 1, 2, \ldots$

If $\lim_{n\to\infty} \delta_n = 0$, then the sequence $\{x_n\}$ converges strongly to $P_S u$, as $n \to \infty$.

5 Numerical Test

We take $E = L_2([0, 1])$ with the inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt$$

and the norm

$$\|f\| = \left(\int_0^1 f^2(t)dt\right)^{1/2},$$

for all $f, g \in L_2([0, 1])$.

Table 1 Table of numerical results

Case A			Case B		
err	$\|x_{n+1} - x_n\|$	n	err	$\|x_{n+1} - x_n\|$	n
Stop condition: $\|x_{n+1} - x_n\| < \text{err}$, $x_1(t) = \dfrac{1}{1+t}$					
10^{-2}	$7.550769933e-005$	9	10^{-2}	$1.293121191e-03$	3
10^{-3}	$7.550769933e-005$	9	10^{-3}	$8.032248181e-004$	4
10^{-4}	$7.550769933e-005$	9	10^{-4}	$9.162919110e-005$	48
10^{-5}	$7.031827964e-006$	164	10^{-5}	$7.139067500e-006$	121

Now, let

$$Q_i = \{x \in L_2([0,1]) \,:\, \langle a_i, x\rangle = b_i\},$$

where $a_i(t) = t^{i-1}$, $b_i = \frac{1}{i+2}$ for all $i = 1, 2, \ldots, 10$ and $t \in [0,1]$.

It is easy to check that $x(t) = t^2 \in S = \cap_{i=1}^{10} Q_i$. We consider the problem of finding an element $x^* \in S$.

Now, by using Theorem 4.3, we consider the convergence of the sequence $\{x_n\}$ which is generated by the following two cases:

Case A.

i) $y_{i,n} = P_{Q_i} x_n, \; i = 1, 2, \ldots, N,$

ii) Choose i_n such that $\|y_{i_n,n} - x_n\| = \max_{i=1,\ldots,N}\{\|y_{i,n} - x_n\|\}$, let $y_n = y_{i_n,n}$,

$C_{n+1} = \{z \in C_n \,:\, \langle y_n - z, J_E(x_n - y_n)\rangle \geq 0\},$

iii) $x_{n+1} = P_{C_{n+1}} x_1, \; n = 1, 2, \ldots$

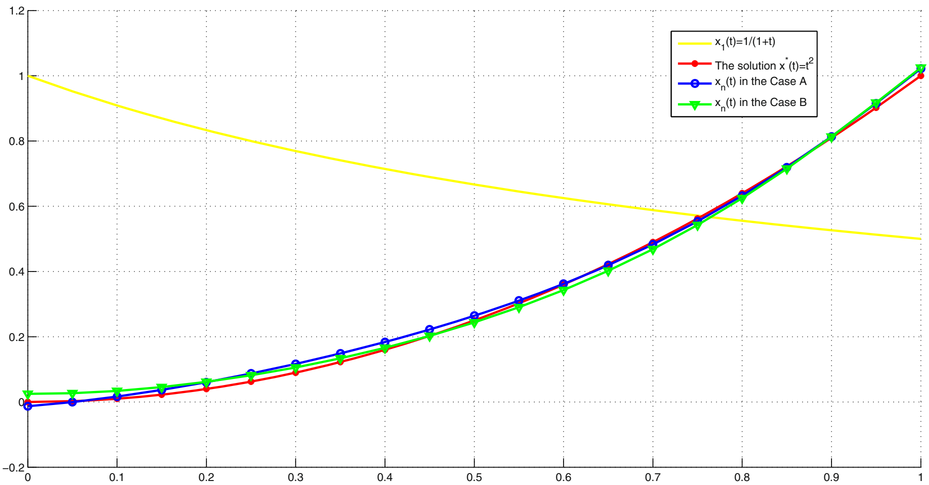

Fig. 1 The behavior of $x_n(t)$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-4}$

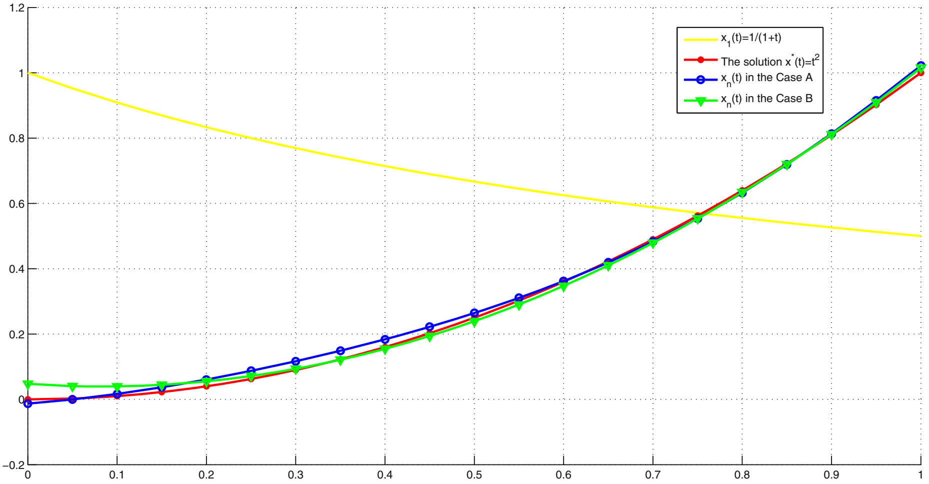

Fig. 2 The behavior of $x_n(t)$ with the stop condition $\|x_{n+1} - x_n\| < 10^{-5}$

Case B.

i) $y_{i,n} = P_{Q_i} x_n, \; i = 1, 2, \ldots, N,$
ii*) $C^i_{n+1} = \{z \in C_n \; : \; \langle y_{i,n} - z, J_E(x_n - y_{i,n})\rangle \geq 0\}, \; i = 1, 2, \ldots, N,$
$C_{n+1} = \cap_{i=1}^N C^i_{n+1},$
iii) $x_{n+1} = P_{C_{n+1}} x_1, \; n = 1, 2, \ldots$

We obtain Table 1 of numerical results.

The behaviors of the approximation solution $x_n(t)$ in both of the cases $\|x_{n+1} - x_n\| < 10^{-4}$ and $\|x_{n+1} - x_n\| < 10^{-5}$ are presented in Figs. 1 and 2.

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