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Differential Extensions of Weakly Principally Quasi-Baer Rings

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Abstract
A ring R is called weakly principally quasi-Baer or simply (weakly p.q.-Baer) if the right annihilator of a principal right ideal is right s-unital by right semicentral idempotents, which implies that R modulo, the right annihilator of any principal right ideal, is flat. We study the relationship between the weakly p.q.-Baer property of a ring R and those of the differential polynomial extension $R[x; \delta]$, the pseudo-differential operator ring $R((x^{-1}; \delta))$, and also the differential inverse power series extension $R[[x^{-1}; \delta]]$ for any derivation δ of R. Examples to illustrate and delimit the theory are provided.

Keywords Differential polynomial ring · Pseudo-differential operator ring · Differential inverse power series ring · (Weakly) p.q.-Baer · APP ring · AIP ring · s-unital ideal

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1 Introduction

Throughout this paper, all rings are associative and they contain an identity element. Recall from [15] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated by an idempotent. In [31], Rickart studied C^*-algebras with the property that every right annihilator of any element is generated by a projection. Using Rickart's work, Kaplansky [14] defined an AW*-algebra as a C^*-algebra with the stronger property that right annihilators of nonempty subsets are generated by a projection. In [15] Kaplansky introduced Baer rings to abstract various properties of AW*-algebras, von Neumann algebras, and complete $*$-regular rings. Berberian continued the development of Baer rings in [2].

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The class of Baer rings includes the von Neumann algebras (e.g., the algebra of all bounded operators on a Hilbert space), the commutative C^*-algebra $C(T)$ of continuous complex valued functions on a Stonian space T, and the regular rings whose lattice of principal right ideals is complete (e.g., regular rings which are continuous or right self-injective).

Closely related to Baer rings are principally projective (PP) rings. A ring R is called a *right (resp. left) PP ring* if every principal right (resp. left) ideal is projective (equivalently, if the right (resp. left) annihilator of an element of R is generated (as a right (resp. left) ideal) by an idempotent of R). R is called a PP ring if it is both right and left PP. The concept of PP ring is not left-right symmetric by Chase [7]. A right PP ring R is Baer (so PP) when R is orthogonally finite by Small [33] (where R is *orthogonally finite* if has no infinite set of orthogonal idempotents).

A ring R is called *quasi-Baer* if the right annihilator of every right ideal of R is generated as a right ideal by an idempotent. It is easy to see that the quasi-Baer property is left-right symmetric for any ring. Quasi-Baer rings were initially considered by Clark [9] and used to characterize a finite dimensional algebra over an algebraically closed field as a twisted semigroup algebra of a matrix unit semigroup. In [30], Pollingher and Zaks show that the class of quasi-Baer rings is closed under n-by-n matrix rings and under n-by-n upper (or lower) triangular matrix rings. Birkenmeier et al. [6] obtained a structure theorem, via triangulating idempotents, for an extensive class of quasi-Baer rings which includes all piecewise domains. Some results on quasi-Baer rings can be found in (cf. [3, 5, 6, 25] and [30]).

Birkenmeier, Kim, and Park in [4] introduced the concept of principally quasi-Baer rings. A ring R is called *right principally quasi-Baer* (or simply *right p.q.-Baer*) if the right annihilator of a principal right ideal is generated by an idempotent. Equivalently, R is right p.q.-Baer if R modulo, the right annihilator of any principal right ideal, is projective. If R is a semiprime ring, then R is right p.q.-Baer if and only if R is left p.q.-Baer. The class of right p.q.-Baer rings includes properly the class of quasi-Baer rings. Some examples were given in [4] to show that the classes of right p.q.-Baer rings and right PP rings are distinct.

Following Tominaga [35], an ideal I of R is said to be *left s-unital* if, for each $a \in I$, there is an element $x \in I$ such that $xa = a$. According to Liu and Zhao [19], a ring R is called a *right APP ring* if the right annihilator $r_R(aR)$ is left s-unital as an ideal of R for any element $a \in R$ [19]. Left *APP* rings may be defined analogously. This concept is a common generalization of left p.q.-Baer rings and right PP rings. In [19], the authors showed that the APP property is inherited by polynomial extensions and is a Morita invariant property.

Recall from [21], that a ring R is a *right AIP ring* if R has the property that "the right annihilator of an ideal is pure as a right ideal." Equivalently, R is a right *AIP* ring if R modulo, the right annihilator of any right ideal, is flat. This class of rings includes both left PP rings and left p.q.-Baer rings (and hence the *biregular rings* (i.e., rings every principal ideal is generated by a central idempotent)). For more details and results of *AIP* ring, see [21].

As a generalization of p.q.-Baer rings, Majidinya and the second author in [22] introduced the concept of *weakly p.q.-Baer* rings. A ring R with unity is *weakly p.q.-Baer* if for each $a \in R$ there exists a nonempty subset E of left semicentral idempotents of R such that $r_R(aR) = \bigcup_{e \in E} eR$ (see Section 2 for details). The class of weakly p.q.-Baer rings is a natural subclass of the class of APP rings and includes both left p.q.-Baer rings and right p.q.-Baer rings. Contrary to the case of the notion of p.q.-Baer authors in [22, Proposition 2.2] showed that the notion of weakly p.q.-Baer is left-right symmetric. Also, they in [22, Theorem 2.20] showed that the $n \times n$ upper triangular matrix ring $T_n(R)$ over a ring R is weakly p.q.-Baer if and only if R is weakly p.q.-Baer. Moreover, in [22], various classes of weakly p.q.-Baer rings which are neither left p.q.-Baer nor right p.q.-Baer nor right PP were constructed.

For any ring, we have the following implications:

$$\begin{array}{c} \text{Baer} \Rightarrow \text{quasi-Baer} \Rightarrow \text{left p.q.-Baer} \Rightarrow \text{weakly p.q.-Baer} \Rightarrow \text{right APP} \\ \Uparrow \\ \text{Baer} \Longrightarrow \text{left PP} \Longrightarrow \text{right AIP} \end{array}$$

In general, each of these implications is irreversible. For more details and examples, we refer the reader to [4, 5, 19, 21] and [22].

In this paper, we study differential extensions of weakly p.q.-Baer rings and AIP rings. The paper is organized as follows: In Section 2, we prove that R is a weakly p.q.-Baer ring if and only if R is δ-weakly rigid and the differential polynomial ring $R[x;\delta]$ is weakly p.q.-Baer (Theorems 2.7 and 2.18). We also apply our results to note that the generalized Weyl ring $A_n(R)$ is a weakly p.q.-Baer ring whenever the ring R is weakly p.q.-Baer (see Corollary 2.9). Furthermore, we show that for a δ-weakly rigid ring R, $R[x;\delta]$ is a right AIP (resp. APP) ring if and only if R is right AIP (resp. APP). In Section 3, we study the relationship between the weakly p.q.-Baer and AIP properties of a ring R and these of the pseudo-differential operator ring $R((x^{-1};\delta))$ and also the differential inverse power series extension $R[[x^{-1};\delta]]$ for any derivation δ of R. In particular, if R is a weakly p.q.-Baer ring and every countable subset of right semicentral idempotents in R has a generalized countable join in R, then the differential inverse power series ring $R[[x^{-1};\delta]]$ and also the pseudo-differential operator ring $R((x^{-1};\delta))$ is a weakly p.q.-Baer ring (see Theorem 3.8). In addition, motivated by the result in [13], we give a lattice isomorphism from the right annihilators of ideals of R to the right annihilators of ideals of $R[[x^{-1};\delta]]$ and also $R((x^{-1};\delta))$. Finally, it is proved that, under suitable conditions, R is a right AIP (resp. APP) ring if and only if $R((x^{-1};\delta))$ is right AIP (resp. APP) if and only if $R[[x^{-1};\delta]]$ is right AIP (resp. APP) (see Theorem 3.13). As a consequence of the main result of this section, we obtain some characterizations for the power series ring and the Laurent power series ring to be PP ring.

2 Differential Polynomial Rings over Weakly Principally Quasi-Baer Rings

For a nonempty subset X of R, $r_R(X)$ (resp. $\ell_R(X)$) is used for the right (resp. left) annihilator of X over R. Also, we use $\mathbb{Z}$ and $\mathbb{N}$ for the integers and positive integers, respectively. Let δ be a derivation on R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + a\delta(b)$, for all $a, b \in R$. We denote $R[x;\delta]$ the *differential polynomial ring* whose elements are the polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa = ax + \delta(a)$ for any $a \in R$. Differential polynomial rings such as Weyl algebras have been a source of many interesting examples in noncommutative ring theory.

Definition 2.1 [4, p. 641] An idempotent $e \in R$ is called *left (resp. right) semicentral* if $xe = exe$ (resp. $ex = exe$), for all $x \in R$. The set of all idempotents of R and the set of left (right) semicentral idempotents of R are denoted by $\mathbf{I}(R)$, $\mathcal{S}_\ell(R)$ ($\mathcal{S}_r(R)$), respectively. Define $\mathcal{S}_\ell(R) \cap \mathcal{S}_r(R) = \mathbf{B}(R)$ (the set of all central idempotents) and if R is semiprime then $\mathcal{S}_\ell(R) = \mathcal{S}_r(R) = \mathbf{B}(R)$.

It follows from [35, Theorem 1] that an ideal I of a ring R is right s-unital if and only if given finitely many elements $a_1, a_2, \ldots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $1 \leq i \leq n$. By [34, Proposition 11.3.13], an ideal I is right s-unital if and only if I is pure as a left ideal of R if and only if R/I is flat as a left R-module.

Definition 2.2 [22, Definition 2.1] We say an ideal I of a ring R is *right s-unital by right semicentral idempotents* if for every $x \in I$, $xe = x$ for some $e \in I \cap \mathcal{S}_r(R)$, or equivalently, $I = \bigcup_{e \in E} Re$ for some nonempty subset E of $\mathcal{S}_r(R)$. The left case may be defined analogously.

Definition 2.3 [22, Definition 2.3] A ring R is called *weakly principally quasi-Baer (or simply weakly p.q.-Baer)* if the right annihilator of a principal right ideal is right s-unital by right semicentral idempotents, which implies that R modulo, the right annihilator of any principal right ideal, is flat.

We start with the following lemmas, which play a key role in the sequel.

Lemma 2.4 *Let R be a weakly p.q.-Baer ring with a derivation δ. Then, for any $a, b \in R$ and positive integer n, $aRb = 0$ implies $aR\delta^n(b) = \delta^n(a)Rb = 0$.*

Proof There exists a left semicentral idempotent $e \in r_R(aR)$ such that $eb = b$. Since $\delta(e) = e\delta(e) + \delta(e)e$, we obtain $\delta(e)e = e\delta(e)e + \delta(e)e$. Also, $e\delta(e)e = \delta(e)e$, and hence $\delta(e)e = 0$. Thus, $\delta(e) = e\delta(e)$. Therefore, $aR\delta(b) = aR\delta(eb) = aR(e\delta(b) + \delta(e)b) = aR(e\delta(b) + e\delta(e)b) = 0$ because $aRe = 0$. By induction, we have $aR\delta^n(b) = 0$, for every positive integer n. Using a similar argument, we can show that $aRb = 0$ follows $\delta^n(a)Rb = 0$, and we are done. □

Lemma 2.5 *Let R be a ring with a derivation δ. If e is a left semicentral idempotent of R, then e is also a left semicentral idempotent of $R[x; \delta]$.*

Proof We show that for each $f(x) \in R[x; \delta]$, $f(x)e = ef(x)e$. We prove this by induction on the degree of $f(x)$. If $\deg(f(x)) = 0$, then the result follows by assumption. Now assume inductively that the assertion holds for polynomials of degree less than n and that $f(x) = ax^n + g(x)$, with $\deg(g(x)) < n$. We have

$$ef(x)e = eax^{n-1}(xe) + eg(x)e = eax^{n-1}ex + eax^{n-1}\delta(e) + eg(x)e. \tag{2.1}$$

By the same method as in the proof of Lemma 2.4, we can show that $\delta(e) = e\delta(e)$. Then, (2.1) becomes:

$$ef(x)e = (eax^{n-1}e)x + (eax^{n-1}e)\delta(e) + eg(x)e. \tag{2.2}$$

From induction hypothesis, we infer that $eax^{n-1}e = ax^{n-1}e$ and $eg(x)e = g(x)e$. Also, since $\delta(e) = e\delta(e)$, then from (2.2), we have:

$$\begin{aligned} ef(x)e &= ax^{n-1}ex + ax^{n-1}\delta(e) + g(x)e \\ &= ax^{n-1}(ex + \delta(e)) + g(x)e \\ &= (ax^n + g(x))e = f(x)e. \end{aligned}$$

Therefore, e is a left semicentral idempotent of $R[x; \delta]$, and the proof is complete. □

The next lemma is proved in [22, Lemma 2.14]. We use from this result to prove Theorem 2.7.

Lemma 2.6 *An ideal J is left s-unital by left semicentral idempotents if and only if given finitely many elements $a_1, \ldots, a_n \in J$, there exists an idempotent $e \in J \cap \mathcal{S}_\ell(R)$ such that $ea_i = a_i$, for each i.*

Theorem 2.7 *Let R be a ring and δ a derivation of R. If R is a weakly p.q.-Baer ring, then so is $R[x;\delta]$.*

Proof Suppose that $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x;\delta]$ are such that $g(x) \in r_{R[x;\delta]}(f(x)R[x;\delta])$. We first prove that $a_i R b_j = 0$ for any i and j. We proceed by induction on $i+j$. For any $r \in R$,

$$f(x)rg(x) = \sum_{k=0}^{m+n}\left(\sum_{i+j=k} a_i x^i r b_j x^j\right) = \sum_{k=0}^{m+n} c_k x^k = 0.$$

It is clear $c_{m+n} = a_m r b_n = 0$. Now suppose that our claim is true for all $0 \le m+n-k < i+j$. Then, $a_i R b_j = 0$ and so by Lemma 2.4, we have $a_i R \delta^l(b_j) = 0$ for any positive integer l and $i+j = m+n, \ldots, m+n-k+1$. Thus we have

$$c_{m+n-k} = \sum_{i=0}^{k} a_{m-i} r b_{n-k+i} = 0. \tag{2.3}$$

Since R is a weakly p.q.-Baer ring, so $r_R(a_m R + a_{m-1}R + \cdots + a_{m-k+1}R)$ is left s-unital by left semicentral idempotents, by [22, Proposition 2.16]. On the other hand, $b_n \in r_R(a_m R + a_{m-1}R + \cdots + a_{m-k+1}R)$, so there exists a left semicentral idempotent $e_{k-1} \in r_R(a_m R + a_{m-1}R + \cdots + a_{m-k+1}R)$ such that $b_n = e_{k-1}b_n$. Then, $a_s R e_{k-1} = 0$ for any $m-k+1 \le s \le m$. We replace r by re_{k-1} in (2.3). Then, (2.3) becomes

$$0 = \sum_{i=0}^{k} a_{m-i} r e_{k-1} b_{n-k+i} = a_{m-k} r e_{k-1} b_n.$$

Since $b_n = e_{k-1}b_n$, so $a_{m-k} R b_n = 0$. Thus from (2.3), we have

$$a_m r b_{n-k} + a_{m-1} r b_{n-k+1} + \cdots + a_{m-k+1} r b_{n-1} = 0. \tag{2.4}$$

Continuing this process, we have $a_{m-i} R b_{n-k+i} = 0$ for any $0 \le i \le k$. Consequently we obtain $a_i R b_j = 0$ for $0 \le i \le m$ and $0 \le j \le n$. Therefore $b_j \in \bigcap_{j=0}^{n} r_R(a_j R)$, for all j. Since R is weakly p.q.-Baer, $\bigcap_{j=0}^{n} r_R(a_j R)$ is left s-unital by left semicentral idempotents. So by Lemma 2.6, there exists a left semicentral idempotent $e \in \bigcap_{j=0}^{n} r_R(a_j R)$ such that, $b_j = eb_j$, for all $0 \le j \le n$ and so $g(x) = eg(x)$. By Lemma 2.5, $e \in \mathcal{S}_\ell(R[x;\delta])$. On the other hand, Lemma 2.4 implies that $e \in r_{R[x;\delta]}(g(x)R[x;\delta])$. Therefore, $R[x;\delta]$ is weakly p.q.-Baer, and the result follows. □

Corollary 2.8 *Let R be a ring, $S = R[x;\delta_1]\cdots[x;\delta_n]$ be an iterated differential polynomial ring, where each δ_i is a derivation of $R[x;\delta_1]\cdots[x;\delta_{i-1}]$. If R is weakly p.q.-Baer, then so does S.*

Let R be a ring; the *first Weyl ring* over R is defined by

$$A_1(R) := R[y_1]\left[x_1; \frac{\partial}{\partial y_1}\right],$$

where $\frac{\partial}{\partial y_1}$ is the ordinary derivative and the n^{th} *Weyl ring* over R is

$$A_n(R) = A_{n-1}(R)[y_n]\left[x_n; \frac{\partial}{\partial y_n}\right].$$

As a further application of Theorem 2.7, we have:

Corollary 2.9 *Let R be a weakly p.q.-Baer ring. Then, the n^{th} Weyl ring $A_n(R)$ is a weakly p.q.-Baer ring.*

The following example is a ring R which is not weakly p.q.-Baer, but the extension $R[x;\delta]$ is Baer. So, the converse of Theorem 2.7 is not true in general.

Example 2.10 Let $R = \mathbb{Z}_2[x]/(x^2)$ with the derivation δ such that $\delta(\overline{x}) = 1$, where $\overline{x} = x + (x^2)$ in R and (x^2) is a principal ideal generated by x^2 of the polynomial ring $\mathbb{Z}_2[x]$ over the field $\mathbb{Z}_2$ of two elements. Assume that the commutative ring R is weakly p.q.-Baer. From [22, Proposition 2.5], we deduce that R is reduced, a contradiction. Thus, R is not weakly p.q.-Baer. On the other hand, by [1, Example 11], we have:

$$R[y;\delta] \cong M_2(\mathbb{Z}_2[y^2]) \cong M_2(\mathbb{Z}_2[t]).$$

Since $\mathbb{Z}_2[t]$ is a principal integral domain, $\mathbb{Z}_2[t]$ is a Prüfer domain (i.e., all finitely generated ideals are invertible). So by [15, Exercise 3, p. 17], $M_2(\mathbb{Z}_2[t])$ is Baer. Therefore $R[y;\delta]$ is Baer.

Now, we state a condition under which weakly p.q.-Baer property of a ring R inherits from the differential polynomial ring $R[x;\delta]$.

Definition 2.11 [25, Definition 2.1] A ring R with a derivation δ is called *δ-weakly rigid* if for any $a, b \in R$, $aRb = 0$ implies $a\delta(b) = 0$.

Lemma 2.12 [25, Lemma 3.3] *Let δ be a derivation of R and R a δ-weakly rigid ring. Then, for any $a, b \in R$ and any positive integers i and j, $aRb = 0$ implies $\delta^i(a)R\delta^j(b) = 0$.*

It is shown in [25] that for any positive integer n, a ring R is weakly rigid if and only if the n-by-n upper triangular matrix ring $T_n(R)$ is weakly rigid if and only if the matrix ring $M_n(R)$ is weakly rigid. If R is a semiprime weakly rigid ring, then the ring of polynomials $R[x]$ is a semiprime weakly rigid ring. In [25], several other classes of weakly rigid rings are provided.

Lemma 2.13 *Suppose that R is a semiprime ring with a derivation δ. Then, for each $a, b \in R$ and positive integers m, n, $aRb = 0$ implies $\delta^m(a)R\delta^n(b) = 0$.*

Proof We will proceed by induction on n. For $n = 0$, it is trivial. Suppose that the statement is true for $n-1$. From $\delta(aR\delta^{n-1}(b)) = 0$, we have $aR\delta^n(b) = -\delta(aR)\delta^{n-1}(b)$. On the other hand, $\delta^{n-1}(b)Ra = 0$, since R is semiprime. Then, $(aR\delta^n(b)R)^2 = -\delta(aR)\delta^{n-1}(b)RaR\delta^n(b)R = 0$. Since R is semiprime, it follows that $aR\delta^n(b) = 0$.

Using a similar argument, we can show that $aR\delta^n(b) = 0$ follows $\delta^m(a)R\delta^n(b) = 0$ for all positive integer m, and the result follows. □

Remark 2.14 Lemmas 2.4 and 2.13 allow us to construct various examples of δ-weakly rigid rings.

The following shows that the class of weakly p.q.-Baer rings properly contains the class of p.q.-Baer rings. Using Theorem 2.7, we are able to obtain various examples of weakly p.q.-Baer rings which are not p.q.-Baer.

Example 2.15 Let A be a commutative p.q.-Baer ring and P a nonzero prime ideal of A such that $\ell_A(a_0) = 0$ for some nonzero element $a_0 \in P$ (e.g., if A is a domain). Assume that $R = \{(a, \overline{b}) \mid a \in A \text{ and } \overline{b} \in \bigoplus_{i=1}^{\infty} Q_i\}$, where $Q_i = A/P$ for each i, $\overline{b} = (\overline{b_i})_{i=1}^{\infty}$ and $\overline{b_i} = b_i + P \in Q_i$. Then, R is a commutative ring with pointwise addition and multiplication defined by $(z, \overline{y}) \cdot (t, \overline{x}) = (zt, \overline{zx} + \overline{ty} + \overline{xy})$, for every $z, t \in A$ and $\overline{x}, \overline{y} \in \bigoplus_{i=1}^{\infty} Q_i$. By a similar method as the one employed in [20, Example 2.4], we can deduce that the ring R is a weakly p.q.-Baer ring which is neither p.q.-Baer nor PP. Now, let δ is any derivation of R. Then, by Lemma 2.4, R is δ-weakly rigid. Therefore, $R[x; \delta]$ is weakly p.q.-Baer, by Theorem 2.7. But [25, Theorem 3.11] implies that $R[x; \delta]$ is not p.q.-Baer.

The next lemma is proved in [35, Theorem 1]. It is used repeatedly in the sequel.

Lemma 2.16 *An ideal J of a ring R is left s-unital if and only if given finitely many elements $a_1, a_2, \ldots, a_n \in J$, there is an element $e \in J$ such that $a_i = ea_i$, for each i.*

Proposition 2.17 *Let R be a left APP ring with a derivation δ. Assume that R is a δ-weakly rigid ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \delta]$ satisfy $f(x)R[x; \delta]g(x) = 0$. Then, $a_i R b_j = 0$ for any i and j.*

Proof The proof is similar to that of Theorem 2.7 by using Lemmas 2.12, 2.16. □

Theorem 2.18 *Let R be a δ-weakly rigid ring. If $R[x; \delta]$ is a weakly p.q.-Baer ring, then R is weakly p.q.-Baer.*

Proof Assume that $a \in R$. First we show that $\ell_R(Ra)[x; \delta] = \ell_{R[x;\delta]}(R[x; \delta]a)$. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \ell_R(Ra)[x; \delta]$. Since $a_i \in \ell_R(Ra)$ and R is δ-weakly rigid, $f(x)R[x; \delta]a = 0$ and consequently $f(x) \in \ell_{R[x;\delta]}(R[x; \delta]a)$. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \ell_{R[x;\delta]}(R[x; \delta]a)$, then $a_n Ra = 0$. Since R is δ-weakly rigid, $a_n x^n Ra = 0$ and so $a_{n-1} Ra = 0$, hence $a_{n-1} x^{n-1} Ra = 0$. Inductively, it is seen that for each $0 \le i \le n$, $a_i Ra = 0$. Therefore $\ell_R(Ra)[x; \delta] = \ell_{R[x;\delta]}(R[x; \delta]a)$. Now, let $bRa = 0$, for element $b \in R$. Then, $b \in \ell_{R[x;\delta]}(R[x; \delta]a)$. So there exists a right semicentral idempotent $g(x) = c_0 + c_1 x + \cdots + c_n x^n \in \ell_{R[x;\delta]}(R[x; \delta]a)$ such that $b = bg(x)$. Then, $b = bc_0$. Also $c_0 \in \ell_R(Ra)$, since $g(x) \in \ell_R(Ra)[x; \delta]$. Hence $\ell_R(Ra)$ is right s-unital and so R is a left APP ring. On the other hand $g(x)R[x; \delta](1 - g(x)) = 0$ and hence $c_0 \in \mathcal{S}_r(R)$, by Proposition 2.17. Therefore, R is weakly p.q.-Baer, and the proof is complete. □

The following corollaries are immediate consequences of Theorems 2.7, 2.18.

Corollary 2.19 *Let R be a ring and δ a derivation of R. Then, R is a weakly p.q.-Baer ring if and only if R is δ-weakly rigid and $R[x;\delta]$ is weakly p.q.-Baer.*

Corollary 2.20 [22, Theorem 2.21] *Let R be a ring. Then, $R[x]$ is a weakly p.q.-Baer ring if and only if R is weakly p.q.-Baer.*

Armendariz showed that polynomial rings over right PP rings need not be right PP in the example in [1]. From [25, Corollary 3.12], for a δ-weakly rigid ring R, the ring $R[x;\delta]$ is a left p.q.-Baer ring if and only if R is left p.q.-Baer.

Theorem 2.21 *Let R be a δ-weakly rigid ring. Then, $R[x;\delta]$ is a right AIP (resp. APP) ring if and only if R is right AIP (resp. APP).*

Proof We shall deal with the "AIP" case and leave the (completely analogous) "APP" case to the reader. Let I be a right ideal of $R[x;\delta]$ and denote by I_0 the set of all coefficients of elements of I in R. Let J be the right ideal R generated by I_0. Let $g(x) = \sum_{j=0}^{m} a_j x^i \in r_{R[x;\delta]}(I)$, then for every $f(x) = \sum_{i=0}^{n} a_i x^i \in I$, $f(x)R[x;\delta]g(x) = 0$. By Proposition 2.17, for each i, j, $a_i R b_j = 0$. Therefore $b_j \in r_R(I_0)$, for all j. Since R is a right AIP ring, $r_R(I_0)$ is left s-unital. So by Lemma 2.16, there exists an element $c \in r_R(I_0)$ such that, $b_j = cb_j$, for all $0 \leq j \leq n$ and so $g(x) = cg(x)$. On the other hand, the δ-weakly rigidness of R implies that $c \in r_{R[x;\delta]}(I)$. Therefore, $R[x;\delta]$ is a right AIP-ring. Conversely, if $R[x;\delta]$ is a right AIP-ring, then, by analogy with the proof of Theorem 2.18, we can show that R is right AIP, and the result follows. □

Corollary 2.22 [21, Proposition 3.14] *Let R be a ring. Then, $R[x]$ is a right AIP (resp. APP) ring if and only if R is right AIP (resp. APP).*

By [19, Proposition 2.3], the class of left APP rings includes both right PP rings and left p.q.-Baer rings (and hence it includes all biregular rings and all quasi-Baer rings). Some examples were given in [4, Examples 1.3 and 1.5] to show that the class of left p.q.-Baer rings is not contained in the class of right PP-rings and, the class of right PP-rings is not contained in the class of left p.q.-Baer rings. The following example shows that another class of APP rings properly contains the class of weakly p.q.-Baer rings (and hence the class of p.q.-Baer rings). Using Theorems 2.18 and 2.21, we are able to obtain various examples of APP rings which are not weakly p.q.-Baer.

Example 2.23 For a field $\mathbb{F}$, take $\mathbb{F}_n = \mathbb{F}$ for $n = 1, 2, \ldots$, let

$$R := \begin{pmatrix} \prod_{n=1}^{\infty} \mathbb{F}_n & \bigoplus_{n=1}^{\infty} \mathbb{F}_n \\ \bigoplus_{n=1}^{\infty} \mathbb{F}_n & < \bigoplus_{n=1}^{\infty} \mathbb{F}_n, 1 > \end{pmatrix}$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} \mathbb{F}_n$, where $< \bigoplus_{n=1}^{\infty} \mathbb{F}_n, 1 >$ is the $\mathbb{F}$-algebra generated by $\bigoplus_{n=1}^{\infty} \mathbb{F}_n$ and $1_{\prod_{n=1}^{\infty} \mathbb{F}_n}$. Then, by [4, Example 1.6], the ring R is semiprime and PP, so by [19, Proposition 2.3] R is a APP ring. On the other hand, [22, Example 2.6] shows that R is not weakly p.q.-Baer. Now, let δ be any derivation of R. Then, by Lemma 2.13, R is δ-weakly rigid. Therefore, $R[x;\delta]$ is a APP ring, by Theorem 2.21. But Theorem 2.18 shows that $R[x;\delta]$ is not weakly p.q.-Baer.

3 The Pseudo-differential Operator Rings over Weakly Principally Quasi-Baer Rings

We denote by $R((x^{-1};\delta))$ the *pseudo-differential operator ring* over the coefficient ring R formed by formal series $f(x)=\sum_{i=m}^{\infty} a_i x^{-i}$, where x is a variable, m is an integer (may be negative), and the coefficients a_i of the series $f(x)$ are elements of the ring R. In the ring $R((x^{-1};\delta))$, addition is defined as usual and multiplication is defined with respect to the relations

$$xa = ax + \delta(a),$$
$$x^{-1}a = \sum_{i=0}^{\infty}(-1)^i \delta^i(a)x^{-i-1}, \text{ for each } a \in R.$$

The algebra of pseudo-differential operators $R((x^{-1};\delta))$ was introduced by Schur in [32]. This algebra has been investigated by a number of authors and repeatedly applied in various fields of mathematics; for instance, see [11, 17], and [36]. Tuganbaev [36] has studied ring-theoretical properties of pseudo-differential operator rings; and showed that other methods of constructing pseudo-differential operator rings can be found in [10]. In the structural ring theory, pseudo-differential operator rings are used for calculation in algebras of differential operators (see [11] for details) and for construction of many examples (e.g., see [12]).

Observe that the subset $R[[x^{-1};\delta]]$ of $R((x^{-1};\delta))$ consisting of inverse power series of the form $f(x)=\sum_{i=0}^{\infty} a_i x^{-i}$ is a subring of $R((x^{-1};\delta))$. The *differential inverse power series ring* $R[[x^{-1};\delta]]$ have wide applications. Not only do they provide interesting examples in noncommutative algebra, they have also been a valuable tool used first by Hilbert in the study of the independence of geometry axioms. The ring-theoretical properties of pseudo-differential operator ring and differential inverse power series rings have been studied by many authors: for more information, refer to [11, 12, 17, 26–29] and [36].

In this section we study the relationship between the weakly p.q.-Baer and AIP properties of a ring R and these of the pseudo-differential operator ring $R((x^{-1};\delta))$ and also differential inverse power series extension $R[[x^{-1};\delta]]$ for any derivation δ of R.

Proposition 3.1 *Let R be a δ-weakly rigid ring and right APP. Suppose that $f(x)=\sum_{i=0}^{\infty} a_i x^{-i}$, $g(x)=\sum_{j=0}^{\infty} b_j x^{-j} \in R[[x^{-1};\delta]]$ are such that $f(x)R[[x^{-1};\delta]]g(x)=0$. Then, $a_iRb_j=0$ for any i and j.*

Proof We proceed by induction on $i+j$. The case $i+j=0$ is clear. Now, assume that $a_iRb_j=0$ for $i+j\le n-1$. Hence $b_j \in r_R(a_iR)$ for $i=0,\dots,n-1$ and $j=0,\dots,n-1-i$. Let r be an arbitrary element of R. Then, we have

$$\sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i x^{-i} r b_j x^{-j}\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} c_k x^{-k}\right) = 0. \tag{3.1}$$

So, $c_k = a_0rb_k + a_1rb_{k-1} + \cdots + a_krb_0 + h = 0$, where h is a sum of monomials of the form $a_i\delta^t(rb_j)$ and $i + j \leq n - 1$. By the δ-weakly rigidnees of R and the hypotheses, we obtain $a_0rb_n + a_1rb_{n-1} + \cdots + a_nrb_0 = 0$. Since R is a right APP ring, there exists $e_{ji} \in r_R(a_iR)$ such that $b_j = e_{ji}b_j$ for all $i = 0, \ldots, n-1$ and $j = 0, \ldots, n-1-i$. If we put $f_j = e_{j0} \cdots e_{j,n-1}$ for $j = 0, \ldots, n-1-i$, then $f_jb_j = b_j$ and $f_j \in r_R(a_0R) \cap \cdots \cap r_R(a_{n-1}R)$. For $k = n$, interchanging r into rf_0 in (3.1), we obtain $a_nrb_0 = a_nrf_0b_0 = 0$. Hence $a_nRb_0 = 0$. Continuing this process, replacing r by rf_j in (3.1), and using again of δ-weakly rigidness of R, we get $a_iRb_j = 0$ for $i + j = n$. This finishes the proof. □

Hirano observed relations between annihilators of ideals in a ring R and annihilators of ideals in the polynomial ring $R[x]$ (see [13, Proposition 3.4]). In order to prove our main results, analogue of results in [13], we give a lattice isomorphism from the right annihilators of ideals of R to the right annihilators of ideals of $R[[x^{-1}; \delta]]$ and also $R((x^{-1}; \delta))$. Furthermore, we deduce that, if R is a weakly p.q.-Baer ring, then R satisfies the ACC on right annihilators of ideals if and only if so does $R[[x^{-1}; \delta]]$ if and only if so does $R((x^{-1}; \delta))$. Following [13], for a ring R, put $r\text{Ann}_R(\text{id}(R)) = \{r_R(U) \mid U \text{ is an ideal of } R\}$.

Proposition 3.2 *Let R be a δ-weakly rigid ring and a right APP ring. Then, the map φ : $r\text{Ann}_R(\text{id}(R)) \to r\text{Ann}_{R[[x^{-1};\delta]]}(\text{id}(R[[x^{-1}; \delta]]))$; $I \to I[[x^{-1}; \delta]]$ is bijective.*

Proof Suppose that $A \in r\text{Ann}_R(\text{id}(R))$. Then, there exists an ideal I of R such that $r_R(I) = A$. Since R is δ-weakly rigid, $A[[x^{-1}; \delta]]$ is an ideal of $R[[x^{-1}; \delta]]$. We claim that $r_{R[[x^{-1};\delta]]}(R[[x^{-1}; \delta]]IR[[x^{-1}; \delta]]) = A[[x^{-1}; \delta]]$. Since R is δ-weakly rigid, $A[[x^{-1}; \delta]] \subseteq r_{R[[x^{-1};\delta]]}(R[[x^{-1}; \delta]]IR[[x^{-1}; \delta]])$, by Lemma 2.12. Let $f(x) = \sum_{i=0}^{\infty} a_ix^{-i} \in r_{R[[x^{-1};\delta]]}(R[[x^{-1}; \delta]]IR[[x^{-1}; \delta]])$. Then, $a_i \in r_R(I)$ for each $i \in \mathbb{N}$, by Lemma 2.12. Hence $f(x) \in A[[x^{-1}; \delta]]$. Consequently, φ is a well defined map. Clearly, φ is injective. Now, it is only necessary to show that φ is surjective. Assume that $J^* \in r\text{Ann}_{R[[x^{-1};\delta]]}(\text{id}(R[[x^{-1}; \delta]]))$, then there exists an ideal I^* of $R[[x^{-1}; \delta]]$ such that $r_{R[[x^{-1};\delta]]}(I^*) = J^*$. Let I, J denote the sets of coefficients of elements of I^* and J^*, respectively. It is clear that I and J are ideals of R. We claim that $r_R(I) = J$. Let $f(x) = \sum_{i=0}^{\infty} a_ix^{-i} \in I^*$ and $g(x) = \sum_{j=0}^{\infty} b_jx^{-j} \in J^*$. Then, $f(x)R[[x^{-1}; \delta]]g(x) = 0$. Hence $a_iRb_j = 0$ for all $i, j \in \mathbb{N}$, by Proposition 3.1. Thus $J \subseteq r_R(I)$. Conversely, let $a \in r_R(I)$. Hence $a_iRa = 0$ for all $i \in \mathbb{N}$ and $f(x) = \sum_{i=0}^{\infty} a_ix^{-i} \in I^*$. Since R is δ-weakly rigid, so $f(x)(ra) = 0$ for each $r \in R$. It follows that $a \in J$. Thus $r_R(I) = J$, and $r_{R[[x^{-1};\delta]]}(I^*) = r_R(I)[[x^{-1}; \delta]] = J[[x^{-1}; \delta]] = J^*$, and so φ is onto. □

Remark 3.3 We also have the same results as Propositions 3.1 and 3.2 for the pseudo-differential operator ring $R((x^{-1}; \delta))$, using a slightly modified method. Now we have the following.

Corollary 3.4 *Let R be a weakly p.q.-Baer ring with a derivation δ. Then, R satisfies the ascending chain condition (ACC) on right annihilators of ideals if and only if so does $R[[x^{-1}; \delta]]$ if and only if so does $R((x^{-1}; \delta))$.*

Proof This result is a consequence of Lemma 2.4 and Proposition 3.2. □

In [24, Theorem 2.12], the authors showed that if R is a δ-weakly rigid ring, then the pseudo-differential operator ring $R((x^{-1}; \delta))$ is a left p.q.-Baer ring if and only if R is a left p.q.-Baer ring and every countable subset of $\mathcal{S}_\ell(R)$ has a generalized countable join in R. Motivated by results in [24], we study the relationship between the weakly p.q.-Baer property of a ring R and these of the pseudo-differential operator ring $R((x^{-1};\delta))$ and also the differential inverse power series ring $R[[x^{-1};\delta]]$.

Remark 3.5 In [4], Birkenmeier et al. defined the notion of semicentral reduced. Let e be an idempotent in R. We say e is *semicentral reduced* if $\mathcal{S}_\ell(eRe) = \{0, e\}$. Observe that $\mathcal{S}_\ell(eRe) = \{0, e\}$ if and only if $\mathcal{S}_r(eRe) = \{0, e\}$. If 1 is semicentral reduced, then we say R is semicentral reduced. In [18, Definition 2], Liu defined the notion of generalized join for a countable subset of idempotents. Explicitly, let $\{e_0, e_1, \ldots\} \subseteq \mathbf{I}(R)$. The set $\{e_0, e_1, \ldots\}$ is said to have a *generalized join e* if there exists an idempotent $e \in R$ such that:

(i) $(1-e)Re_i = 0$;
(ii) If d is an idempotent and $(1-d)Re_i = 0$ then $(1-d)Re = 0$.

In [18, Theorem 3], Liu, gave a necessary and sufficient condition for a semiprime ring R under which the ring $R[[x]]$ is right p.q.-Baer. It is shown that $R[[x]]$ is right p.q.-Baer if and only if R is right p.q.-Baer and any countable family of idempotents in R has a generalized join when all left semicentral idempotents are central. For a right p.q.-Baer ring, asking the set of left semicentral idempotents are central is equivalent to assume R is semiprime [4, Proposition 1.17].

Definition 3.6 [8] Let $E = \{e_0, e_1, \ldots\}$ be a countable subset of $\mathcal{S}_r(R)$. Then, E is said to have a *generalized countable join e* if, given $a \in R$, there exists $e \in \mathcal{S}_r(R)$ such that:

(1) $e_i e = e_i$ for all positive integer i;
(2) If $e_i a = e_i$ for all positive integer i, then $ea = e$.

As it is mentioned in [8], if there exists an element $e \in R$ that satisfies conditions (1) and (2) above, then $e \in \mathcal{S}_r(R)$. Indeed, the condition (1): $e_n e = e_n$ for all $n \in \mathbb{N}$ implies $ee = e$ by (2) and so e is an idempotent. Further, let $a \in R$ be arbitrary. Then, the element $d = e - ea + eae$ is an idempotent in R and $e_n d = e_n$ for all $n \in \mathbb{N}$. Thus $ed = e$ by (2). Note that $ed = e(e - ea + eae) = d$. Consequently, $e = d = e - ea + eae$ and hence $ea = eae$. Thus $e \in \mathcal{S}_r(R)$. In particular, when R is a Boolean ring or a reduced PP ring, then the generalized countable join is indeed a join in R.

Now we prove that, in the context of right semicentral idempotents, a generalized countable join is a generalized join in the sense of Liu. Observe that $e_i r(1-e) = e_i r e_i (1-e) = e_i r(e_i - e_i e)$, when $e_i \in \mathcal{S}_r(R)$. Thus $e_i = e_i e$ if and only if $e_i r(1-e) = 0$ for all $r \in R$ when $e_i \in \mathcal{S}_r(R)$ for all $i \in \mathbb{N}$. Now, let $E = \{e_0, e_1, e_2, \ldots\} \subseteq \mathcal{S}_r(R)$ and e be a generalized countable join of E. To show e is a generalized join (in the sense of Liu), it remains to show that condition (ii) holds. Let f be an idempotent in R such that $e_i R(1-f) = 0$. Then, in particular, $e_i(1-f) = 0$ for all $i \in \mathbb{N}$. Thus $e(1-f) = 0$ by hypothesis. It follows that $er(1-f) = ere(1-f) = 0$ and thus $eR(1-f) = 0$. Therefore, e is a generalized join of E. Conversely, let $e \in \mathcal{S}_r(R)$ be a generalized join (in the sense of Liu) of the set $E = \{e_0, e_1, e_2, \ldots\} \subseteq \mathcal{S}_r(R)$. Observe that condition (2) in Definition 3.6 is equivalent to (2′) if d is an idempotent and $e_i d = e_i$ then $ed = e$. Let $a \in R$ be arbitrary such that $e_i a = e_i$ for all $i \in \mathbb{N}$. Then, condition (2′) and a similar argument as the one used in the case of reduced PP rings implies that $ea = e$. Thus, e is a generalized countable join. Therefore, in the context of right semicentral idempotents, Liu's generalized join is equivalent to generalized countable join.

To prove Theorem 3.8, we need the following lemma.

Lemma 3.7 *Let R be a ring with a derivation δ. If e is a left semicentral idempotent of R, then e is also a left semicentral idempotent of $R((x^{-1};\delta))$.*

Proof The proof is similar to that of [26, Lemma 3.1]. □

We are now ready to study the weakly p.q.-Baer property of pseudo-differential operator rings and also differential inverse power series rings. We show that if R is a weakly p.q.-Baer ring and every countable subset of right semicentral idempotents in R has a generalized countable join in R, then $R[[x^{-1};\delta]]$ (resp. $R((x^{-1};\delta))$) is a weakly p.q.-Baer ring. Here, we do not assume right semicentral idempotents to be central, and hence, R does not need to be semiprime.

Theorem 3.8 *Let R be a weakly p.q.-Baer ring with a derivation δ. If every countable subset of right semicentral idempotents in R has a generalized countable join in R, then $R[[x^{-1};\delta]]$ (resp. $R((x^{-1};\delta))$) is a weakly p.q.-Baer ring.*

Proof We will prove the case for $R[[x^{-1};\delta]]$. The other case can be shown similarly. Suppose that $f(x)=\sum_{i=0}^{\infty}a_ix^{-i}$, $g(x)=\sum_{j=0}^{\infty}b_jx^{-j}\in R[[x^{-1};\delta]]$ are such that $g(x)\in r_{R[[x^{-1};\delta]]}(f(x)R[[x^{-1};\delta]])$. Then, $a_iRb_j=0$ for all $i,j\in\mathbb{N}$, by Proposition 3.1. Hence, $b_j\in r_R(a_iR)$ for all $i,j\in\mathbb{N}$. Since R is weakly p.q.-Baer, $r_R(a_iR)$ is left s-unital by left semicentral idempotents, for each $i\in\mathbb{N}$. Then, there exists left semicentral idempotents $e_i\in\mathcal{S}_\ell(R)\cap r_R(a_iR)$ such that $b_j=e_ib_j$, for each $i,j\in\mathbb{N}$. Consequently, $(1-e_i)b_j=0$ or $(1-e_i)(1-b_j)=1-e_i$, for all $i,j\in\mathbb{N}$. Let e be a generalized countable join of the set $E=\{1-e_i\mid i\in\mathbb{N}\}$ in $\mathcal{S}_r(R)$. Thus $e(1-b_j)=e$ or $(1-e)b_j=b_j$, for all $j\in\mathbb{N}$. Therefore, $g(x)=(1-e)g(x)$. On the other hand, since e is a generalized countable join of E, we have $(1-e_i)(1-e)=0$ and hence $1-e=e_i(1-e)$, for all $i\in\mathbb{N}$. For each $r\in R$ and $i\in\mathbb{N}$, $a_ir(1-e)=a_ire_i(1-e)\in a_iRe_iR=0$. Hence, Lemma 2.4 implies that $1-e\in r_{R[[x^{-1};\delta]]}(f(x)R[[x^{-1};\delta]])$. By [26, Lemma 3.1], $1-e\in\mathcal{S}_\ell(R[[x^{-1};\delta]])$. It follows that $R[[x^{-1};\delta]]$ is weakly p.q-Baer, and the proof is complete. □

By combining Corollary 3.4, [23, Theorem 2.3], Lemma 2.4, and Theorem 3.8, we obtain the following:

Corollary 3.9 *Assume that R satisfies the ACC on left annihilators of ideals, $S=R[[x^{-1};\delta_1]]\cdots[[x^{-1};\delta_n]]$ be an iterated differential inverse power series ring, where each δ_i is a derivation of $R[[x^{-1};\delta_1]]\cdots[[x^{-1};\delta_{i-1}]]$. If R is a weakly p.q.-Baer ring, then so does S.*

The following example shows that there exists a ring R with a derivation δ for which $R[[x^{-1};\delta]]$ is a weakly p.q.-Baer ring but R itself is not weakly p.q.-Baer.

Example 3.10 Let p be a prime integer and $R=\mathbb{Z}_p[x]/(x^p)$ with the derivation δ such that $\delta(\overline{x})=1$, where $\overline{x}=x+(x^p)$ in R and $\mathbb{Z}_p[x]$ is the polynomial ring over the field $\mathbb{Z}_p$. Then, using a similar method as in Example 2.10, we can show that R is not weakly p.q.-Baer. But, the differential inverse power series ring $R[[x^{-1};\delta]]$ is a Baer ring, by [27, Example 4.8].

Theorem 3.11 *Let R be a δ-weakly rigid ring. If $R[[x^{-1};\delta]]$ (resp. $R((x^{-1};\delta))$) is a weakly p.q.-Baer ring, then R is a weakly p.q.-Baer ring.*

Proof We will prove the case for $R((x^{-1};\delta))$. The other case can be shown similarly. Assume that $a \in R$. By the δ-weakly rigidness of R, it is easy to show that $\ell_{R((x^{-1};\delta))}(R((x^{-1};\delta))a) = \ell_R(Ra)((x^{-1};\delta))$. Now, let $bRa = 0$, for some element b in R. Then, $b \in \ell_{R((x^{-1};\delta))}(R((x^{-1};\delta))a)$. So, there exists a right semicentral idempotent $f(x) = \sum_{j=m}^{\infty} c_j x^{-j} \in \ell_{R((x^{-1};\delta))}(R((x^{-1};\delta))a)$ such that $b = bf(x)$. Then, $b = bc_0$. Also $c_0 \in \ell_R(Ra)$, since $f(x) \in \ell_R(Ra)((x^{-1};\delta))$. Therefore, $\ell_R(Ra)$ is right s-unital and so R is a left APP ring. On the other hand, $f(x)R((x^{-1};\delta))(1 - f(x)) = 0$ and hence $c_0 \in \mathcal{S}_r(R)$, by Proposition 3.1. Thus, R is weakly p.q.-Baer, and the result follows. □

By the following example, the assumption that any countable family of right semicentral idempotents in R has a generalized countable join in R in Theorem 3.8, is not superfluous.

Example 3.12 For a given field $\mathbb{F}$, take $\mathbb{F}_n = \mathbb{F}$ for $n = 1, 2, \ldots$. Let R be $< \bigoplus_{n=1}^{\infty} \mathbb{F}_n, 1 >$ which is $\mathbb{F}$-algebra generated by $\bigoplus_{n=1}^{\infty} \mathbb{F}_n$ and $1_{\prod_{n=1}^{\infty} \mathbb{F}_n}$. Then, R is a commutative von Neuman regular ring and hence it is weakly p.q.-Baer, by [3, Example 2.3]. Then, $R[[x^{-1}]]$ is not weakly p.q.-Baer. To see this, for all $i \in \mathbb{N}$, take $f(x) = \sum_{i=0}^{\infty} a_i x^{-i}$, where $a_i = (a_{ij})$, $a_{ij} = 1$ for $j = 2i + 1$ and $a_{ij} = 0$ when $j \neq 2i + 1$. One can see that $r_{R[[x^{-1}]]}(f(x)R[[x^{-1}]])$ is not left s-unital by left semicentral idempotents.

In the next result of this paper, we will obtain the criterion for pseudo-differential operator rings and also differential inverse power series rings to be a right AIP (resp. APP) ring. By [16, 6E], a ring R satisfies the ascending chain condition (ACC) on right annihilators if and only if R satisfies the descending chain condition (DCC) on left annihilators.

Theorem 3.13 *Let R be a δ-weakly rigid ring such that R satisfies the ACC on right annihilators. Then, the following are equivalent:*

(1) *R is a right AIP (resp. APP) ring.*
(2) *$R[[x^{-1};\delta]]$ is a right AIP (resp. APP) ring.*
(3) *$R((x^{-1};\delta))$ is a right AIP (resp. APP) ring.*

Proof We shall deal with the "AIP ring" case and leave the (completely analogous) "APP ring" case to the reader.

(1) ⇒ (2) Assume that R is a right AIP ring and I is a right ideal of $R[[x^{-1};\delta]]$ and denote by I_0 the set of all coefficients of elements of I in R. Let J be the right ideal R generated by I_0 and $g(x) = \sum_{j=0}^{\infty} b_j x^{-j} \in r_{R[[x^{-1};\delta]]}(I)$. Then, for every $f(x) = \sum_{i=0}^{\infty} a_i x^{-i} \in I$, $f(x)R[[x^{-1};\delta]]g(x) = 0$. By Theorem 3.1, $a_i R b_j = 0$ for each $i, j \in N$. Hence, $b_j \in r_R(J)$, for each $j \in \mathbb{N}$. Consider the descending chain as the following:

$$\ell_R(b_0) \supseteq \ell_R(b_0, b_1) \supseteq \ell_R(b_0, b_1, b_2) \supseteq \cdots.$$

Since R satisfies descending chain condition on left annihilators, there exists some positive integer n such that $\ell_R(b_0, \ldots, b_n) = \ell_R(b_0, \ldots, b_n, b_{n+1}) = \cdots$. Since $b_0, \ldots, b_n \in$

$r_R(J)$, by Lemma 2.16 there exists $e \in r_R(J)$ such that $eb_i = b_i$ for $i = 0, 1, \ldots, n$. Since $1 - e \in \ell_R(b_0, \ldots, b_n, \ldots, b_k)$ for each $n \leq k$, we have $eb_i = b_i$ for $i = 0, 1, \ldots$. This implies that $eg(x) = g(x)$. On the other hand, since $e \in r_R(J)$, the δ-weakly rigidness of R implies that $e \in r_{R[[x^{-1};\delta]]}(I)$. This shows that $R[[x^{-1}; \delta]]$ is a right AIP ring.

(2) $\Rightarrow$ (1) If $R[[x^{-1}; \delta]]$ is right AIP, then by analogy with the proof of Theorem 3.11, we can show that R is right AIP.

(1) $\Leftrightarrow$ (3) The result follows by an argument similar above. □

Remark 3.14 Example 3.12 shows that the descending chain condition on left annihilators in Theorem 3.13 is not superfluous.

The following corollaries are immediate consequences of Theorem 3.13.

Corollary 3.15 *Let R be a reduced ring that satisfies the ACC on right annihilators. Then, R is a PP ring if and only if $R[[x^{-1}; \delta]]$ is a PP ring if and only if $R((x^{-1}; \delta))$ is a PP ring.*

Proof First, note that for a reduced ring R, we have $\ell_R(a) = r_R(a)$, for every $a \in R$. Therefore, for a reduced ring, the definitions of right PP and left PP coincide. Now, the result follows from [21, Proposition 2.3] and Theorem 3.13. □

Corollary 3.16 *Let R be a reduced ring with the ACC on right annihilators. Then, R is a PP ring if and only if the power series ring $R[[x]]$ is a PP ring if and only if the Laurent power series ring $R((x; x^{-1}))$ is a PP ring.*

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References

1. Armendariz, E.P.: A note on extensions of Baer and p.p.-rings. J. Austral. Math. Soc. **18**, 470–473 (1974)
2. Berberian, S.K.: Baer ∗-Rings. Springer, New York (1972)
3. Birkenmeier, G.F., Kim, J.Y., Park, J.K.: On quasi-Baer rings. Algebra and Its Applications, 67–92. Contemp. Math., vol. 259. Am. Math. Soc., Providence (2000)
4. Birkenmeier, G.F., Kim, J.Y., Park, J.K.: Principally quasi-Baer rings. Comm. Algebra **29**(2), 639–660 (2001)
5. Birkenmeier, G.F., Kim, J.Y., Park, J.K.: Polynomial extensions of Baer and quasi-Baer rings. J. Pure Appl. Algebra **159**(1), 25–42 (2001)
6. Birkenmeier, G.F., Park, J.K.: Triangular matrix representations of ring extensions. J. Algebra **265**(2), 457–477 (2003)
7. Chase, S.U.: A generalization the ring of triangular matrices. Nagoya Math. J. **18**, 13–25 (1961)
8. Cheng, Y., Huang, F.K.: A note on extensions of principally quasi-Baer rings. Taiwanese J. Math. **12**(7), 1721–1731 (2008)
9. Clark, W.E.: Twisted matrix units semigroup algebras. Duke Math. J. **34**, 417–423 (1967)
10. Dzhumadil'daev, A.S.: Derivations and central extensions of the Lie algebra of formal pseudo differential operators. Algebra i Anal. **6**(1), 140–158 (1994)
11. Goodearl, K.R.: Centralizers in differential, pseudo differential, and fractional differential operator rings. Rocky Mountain J. Math. **13**(4), 573–618 (1983)

12. Goodearl, K.R., Warfield, R.B.: An Introduction to Noncommutative Noetherian Rings. Cambridge University Press, Cambridge (1989)
13. Hirano, Y.: On annihilator ideals of a polynomial ring over a noncommutative ring. J. Pure Appl. Algebra **168**(1), 45–52 (2002)
14. Kaplansky, I.: Projections in Banach Algebras. Ann. of Math. (2) **53**, 235–249 (1951)
15. Kaplansky, I.: Rings of Operators. Benjamin, New York (1968)
16. Lam, T.Y.: Lectures on modules and rings Graduate Texts in Math, vol. 189. Springer, New York (1999)
17. Letzter, E.S., Wang, L.: Noetherian skew inverse power series rings. Algebr. Represent. Theory **13**(3), 303–314 (2010)
18. Liu, Z.: A note on principally quasi-Baer rings. Comm. Algebra **30**(8), 3885–3890 (2002)
19. Liu, Z., Zhao, R.: A generalization of PP-rings and p.q.-Baer rings. Glasg. Math. J. **48**(2), 217–229 (2006)
20. Majidinya, A., Moussavi, A., Paykan, K.: Generalized APP-rings. Comm. Algebra **41**(12), 4722–4750 (2013)
21. Majidinya, A., Moussavi, A., Paykan, K.: Rings in which the annihilator of an ideal is pure. Algebra Colloq. **22**(1), 947–968 (2015)
22. Majidinya, A., Moussavi, A.: Weakly principally quasi-Baer rings. J. Algebra Appl. **15**(1), 20 (2016)
23. Manaviyat, R., Moussavi, A., Habibi, M.: Principally quasi-Baer skew power series modules. Comm. Algebra **41**(4), 1278–1291 (2013)
24. Manaviyat, R., Moussavi, A.: On annihilator ideals of pseudo-differential operator rings. Algebra Colloq. **22**(4), 607–620 (2015)
25. Nasr-Isfahani, A.R., Moussavi, A.: On weakly rigid rings. Glasg. Math. J. **51**(3), 425–440 (2009)
26. Paykan, K., Moussavi, A.: Special properties of diffreential inverse power series rings. J. Algebra Appl. **15**(10), 23 (2016)
27. Paykan, K., Moussavi, A.: Study of skew inverse Laurent series rings. J. Algebra Appl. **16**(12), 33 (2017)
28. Paykan, K.: Skew inverse power series rings over a ring with projective socle. Czechoslovak Math. J. **67**(2), 389–395 (2017)
29. Paykan, K., Moussavi, A.: Primitivity of skew inverse Laurent series rings and related rings. J. Algebra Appl. https://doi.org/10.1142/S0219498819501160 (2019)
30. Pollingher, A., Zaks, A.: On Baer and quasi-Baer rings. Duke Math. J. **37**, 127–138 (1970)
31. Rickart, C.E.: Banach algebras with an adjoint operation. Ann. of Math. (2) **47**, 528–550 (1946)
32. Schur, I.: Uber vertauschbare lineare Differentialausdrucke, Sitzungsber. Berliner Math. Ges. **4**, 2–8 (1905)
33. Small, L.W.: Semihereditary rings. Bull. Am. Math. Soc. **73**, 656–658 (1967)
34. Stenström, B.: Rings of Quotients. Springer, New York-Heidelberg (1975)
35. Tominaga, H.: On s-unital rings. Math. J. Okayama Univ. **18**(2), 117–134 (1975/76)
36. Tuganbaev, D.A.: Laurent series rings and pseudo-differential operator rings. J. Math. Sci. (N.Y.) **128**(3), 2843–2893 (2005)