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Solutions to Partial Functional Differential Equations with Infinite Delays: Periodicity and Admissibility

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Abstract Under some appropriate conditions, we prove the existence and uniqueness of periodic solutions to partial functional differential equations with infinite delay of the form $\dot{u} = A(t)u + g(t, u_t)$ on a Banach space X where $A(t)$ is 1-periodic, and the nonlinear term $g(t, \phi)$ is 1-periodic with respect to t for each fixed ϕ in fading memory phase spaces, and is $\varphi(t)$-Lipschitz for φ belonging to an admissible function space. We then apply the attained results to study the existence, uniqueness, and conditional stability of periodic solutions to the above equation in the case that the family $(A(t))_{t\geq 0}$ generates an evolution family having an exponential dichotomy. We also prove the existence of a local stable manifold near the periodic solution in that case.

Keywords Partial functional differential equations · Periodic solutions · Admissibility of function spaces · Conditional stability · Local stable manifolds

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1 Introduction

Consider the abstract partial functional differential equation with infinite delay

$$\dot{u} = A(t)u + g(t, u_t), \quad t \in \mathbb{R}_+, \tag{1}$$

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where for each $t \in \mathbb{R}_+$, $A(t)$ is a possibly unbounded operator on a Banach space X such that the family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on X, and $g : \mathbb{R}_+ \times \mathcal{C}_\nu \to X$ is continuous and locally Lipschitz with $\mathcal{C}_\nu := \{\phi : \phi \in C((-\infty, 0], X)$ and $\lim_{s\to-\infty} e^{\nu s}\|\phi(s)\| = 0, \nu > 0\}$; u_t is the history function defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$.

When $A(t)$ and $g(t, \varphi)$ are periodic (with the same period and respect to t) one tries to prove the existence and uniqueness of a periodic solution (with the same period as that of $A(t)$ and g) to (1). Classical approaches for investigation of the existence of periodic solutions are the Tikhonov's fixed point method [22], Lyapunov functionals [24], as well as the use of ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings (see [2, 13, 21–24] and the references therein). However, in some applications, e.g., to PDEs in unbounded (in all directions) domains or to equations possessing unbounded solutions, the abovementioned compact embeddings are not valid any longer, and it is not easy to show the existence of bounded solutions since one has to carefully choose an appropriate initial vector (or condition) to guarantee the boundedness of the solution emanating from that vector.

Recently, for the case of PDEs *without delay* we have proposed in [11] a new approach to handle such difficulties. Namely, we start with the linear equation $\dot{u} = A(t)u + f(t),\ t \geq 0$ and use a Cesàro sum to prove the existence of a periodic solution through the existence of bounded solution whose sup-norm can be controlled by the sup-norm of the input function f. Then, we use the fixed point argument to prove the existence of periodic solutions for the corresponding semi-linear problem. We refer to [9] for the use of an ergodic approach for the case of Stokes and Navier-Stokes equations around rotating obstacles, and to [5] for the general approach to the existence of periodic solutions to fluid flow problems.

In the present paper, we will consider the existence and uniqueness of periodic solutions to partial functional differential equations (PFDE) with infinite delay and with a φ-Lipschitz nonlinear term g, i.e., $\|g(t, \phi_1) - g(t, \phi_2)\| \leqslant \varphi(t)\|\phi_1 - \phi_2\|_\nu$ for $\phi_1, \phi_2 \in \mathcal{C}_\nu$ where φ are real and positive functions belonging to admissible function spaces. Some difficulties arise when passing to the case of PFDE with infinite delay: Firstly, since the nonlinear delay g is φ-Lipschitz, the standard method for construction of bounded solutions relevant for uniform Lipschitz continuous functions is no longer valid. Secondly, the evolution family generated by $(A(t))_{t\geq 0}$ does not act on the same Banach space as that the initial functions belong to (in fact, the former acts on X, and the latter belong to $\mathcal{C}_\nu$). And lastly, since the delay is infinite, the boundedness and stability of solutions in standard spaces are difficult to obtain.

To overcome such difficulties, we combine the methods and results in [11] with the use of admissible spaces and appropriate choices of fading memory spaces to prove the existence and uniqueness of the periodic solution to (1) without using the uniform boundedness and smallness (in classical sense) of Lipschitz constants of the nonlinear terms. Instead, the "smallness" is now understood as the sufficient smallness of $\sup_{t\geq 0} \int_t^{t+1} \varphi(\tau)d\tau$.

It is worth noting that our framework fits perfectly the situation of exponentially dichotomic linear parts, i.e., the case when the family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy (see Definition 4 below), since in this case we can choose the initial vector from that emanates a bounded solution. Moreover, we can also prove the conditional stability of periodic solutions as well as the existence of a local stable manifold around the periodic solution. Our main results are contained in Theorems 1, 2, 3, and 4.

We now recall some notions for later use. Firstly, as in [12], we denote

$$\mathbf{M} = \mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+) \mid \sup_{t\geq 0} \int_t^{t+1} |f(\tau)| d\tau < \infty \right\} \tag{2}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geq 0} \int_t^{t+1} |f(\tau)| d\tau$. Clearly, $\mathbf{M}$ is a Banach space and it is an admissible Banach function space in the sense of [12, Definition 1.2].

For a given Banach space X, we define the space $\mathfrak{M}$ of X-valued functions related to $\mathbf{M}$ by

$$\mathfrak{M} := \{ f : \mathbb{R}_+ \to X \mid \|f(\cdot)\| \in \mathbf{M} \} \tag{3}$$

endowed with the norm $\|f\|_{\mathfrak{M}} := \| \|f(\cdot)\| \|_{\mathbf{M}}$. Clearly, $\mathfrak{M}$ is a Banach space. Moreover, we consider the following subset of $\mathbf{M}$ consisting of 1-periodic functions denoted by

$$\mathbf{P} := \{ f \in \mathbf{M} \mid f \text{ is 1-periodic} \}. \tag{4}$$

For $\varphi \in \mathbf{M}$ and $\sigma > 0$, it can be seen (see [7, Proposition 2.6]) that the functions $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ defined by $\Lambda'_\sigma \varphi(t) = \int_0^t e^{-\sigma(t-s)} \varphi(s) ds$ and $\Lambda''_\sigma \varphi(t) = \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds$, $t \in \mathbb{R}_+$, belong to $\mathbf{M}$.

Let now φ be a positive function belonging to $\mathbf{P}$ and denote by $\| \cdot \|_\infty$ the esssup-norm. Then, by [12, (1.8)] we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1-e^{-\sigma}} \|\varphi\|_{\mathbf{M}} \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1-e^{-\sigma}} \|\varphi\|_{\mathbf{M}}. \tag{5}$$

We also need the space $C_b(\mathbb{R}_+, X)$ (and $C_b(\mathbb{R}, X)$) of bounded, continuous functions with values in X, defined on $\mathbb{R}_+$ ($\mathbb{R}$, respectively), and endowed with the norms $\|v\|_{C_b(\mathbb{R}_+,X)} := \sup_{t\in\mathbb{R}_+} \|v(t)\|$ (and $\|v\|_{C_b(\mathbb{R},X)} := \sup_{t\in\mathbb{R}} \|v(t)\|$, respectively).

In this paper, we always fix a Banach space X having a separable predual Y (i.e., $X = Y'$ for a separable Banach space Y). We consider the nonhomogeneous linear problem for the unknown function $u(t)$

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t) & \text{for } t > 0, \\ u(0) = u_0 \in X, \end{cases} \tag{6}$$

where the function f taking values in a Banach space X and the family of partial differential operators $(A(t))_{t\geq 0}$ is given such that the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) & \text{for } t > s \geq 0, \\ u(s) = u_s \in X \end{cases} \tag{7}$$

is well-posed. By this, we mean that there exists an evolution family $(U(t,s))_{t\geq s\geq 0}$ such that the solution of the Cauchy problem (7) is given by $u(t) = U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families, and applications to partial differential equations, we refer the readers to Pazy [20] (see also Nagel and Nickel [19] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$). We next give the precise concept of an evolution family in the following definition.

Definition 1 A family of bounded linear operators $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* if

(i) $U(t,t) = \mathrm{Id}$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s \geq 0$,

(ii) the map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$, where $(t, s) \in \{(t, s) \in \mathbb{R}^2 : t \geq s \geq 0\}$,
(iii) there are constants $K, \alpha \geq 0$ such that $\|U(t, s)x\| \leq Ke^{\alpha(t-s)}\|x\|$ for all $t \geq s \geq 0$ and $x \in X$.

The existence of the evolution family $(U(t, s))_{t\geq s\geq 0}$ allows us to define a notion of mild solutions as follows. By the *mild solution* to (6), we mean a function u satisfying the following integral equation

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau) f(\tau)d\tau \quad \text{for all } t \geq 0. \tag{8}$$

We refer the reader to Pazy [20] for more detailed treatments on the relations between classical and mild solutions of evolution equations of the form (6).

We now state an assumption that will be used in the rest of the paper.

Assumption 1 *We assume that $A(t)$ is 1-periodic, i.e., $A(t + 1) = A(t)$ for all $t \in \mathbb{R}_+$. Then $(U(t, s))_{t\geq s\geq 0}$ becomes 1-periodic in the sense that*

$$U(t + 1, s + 1) = U(t, s) \quad \text{for all } t \geq s \geq 0. \tag{9}$$

We also assume that the space Y considered as a subspace of Y'' (through the canonical embedding) is invariant under the operator $U'(1, 0)$ which is the dual of $U(1, 0)$.

2 Main Results

We now state and prove our three main results: The first result is on the existence and uniqueness of a periodic mild solution to the partial functional differential equation in fading memory phase spaces (Theorem 1 below). The second result is related to the existence and conditional stability of the periodic solution in the case that the linear part generates an evolution family having an exponential dichotomy (Theorems 2 and 3), and the last result is on the existence of a local stable manifold around the periodic solution (Theorem 4).

2.1 Periodic Solutions to Semi-Linear Problems in Fading Memory Spaces

Firstly, we recall some notions of fading memory space and introduce the notion of local φ-Lipschitz functions in the following definitions. Denote by $C((-\infty, 0], X)$ the space of all continuous functions from $(-\infty, 0]$ into X. For a continuous function $v : \mathbb{R} \to X$, the history function $v_t \in C((-\infty, 0], X)$ is defined by $v_t(\theta) = v(t + \theta)$ for all $\theta \in (-\infty, 0]$.

Definition 2 Consider a Banach space X as above. Then, a fading memory space is a Banach space $(\Gamma; \|\cdot\|_\Gamma)$ consisting of functions from $(-\infty, 0]$ to X that satisfies the following axioms (see [6, 13]):
A1) There exist a positive constant H and locally bounded nonnegative continuous functions $K(\cdot)$ and $M(\cdot)$ on $[0, \infty)$ with the property that if $u : (-\infty, a) \to X$ is continuous and for some $\sigma < a$, $u_\sigma \in \Gamma$, then for all $t \in [\sigma, a)$, we have
(i) $u_t \in \Gamma$,
(ii) u_t is continuous in t (with respect to $\|\cdot\|_\Gamma$),

(iii) $H\|u(t)\| \leq \|u_t\|_\Gamma \leq K(t-\sigma)\sup_{\sigma\leq s\leq t}\|u(s)\| + M(t-\sigma)\|u_\sigma\|_\Gamma$.
A2) If $\{\phi^k\}$, $\phi^k \in \Gamma$, converges to ϕ uniformly on any compact set in $(-\infty, 0]$ and if $\{\phi^k\}$ is a Cauchy sequence in Γ, then $\phi \in \Gamma$ and $\phi^k \to \phi$ in Γ, $k \to \infty$.

Example 1 (See [13, Chapter 5]) The above axioms are satisfied by the space

$$\mathcal{C}_\nu := \left\{\phi : \phi \in C((-\infty, 0], X) \text{ and } \lim_{s\to-\infty}\frac{\|\phi(s)\|}{e^{-\nu s}} = 0\right\} \text{ where } \nu > 0 \tag{10}$$

endowed with the norm $\|\phi\|_\nu := \sup_{-\infty<s\leq 0}\frac{\|\phi(s)\|}{e^{-\nu s}}$. Moreover, in this case, we can take $K(t) = 1$, $M(t) = e^{-\nu t}$ for all $t \geq 0$ in axiom A1) (iii) of Definition 2 of fading memory spaces.

Remark 1 Let now $x(\cdot)$ be a function defined and continuous on $\mathbb{R}$ with values in X such that $x(\cdot)|_{\mathbb{R}_+} \in C_b(\mathbb{R}_+, X)$ and $x_t \in \mathcal{C}_\nu$ for all $t \geq 0$. Then, we have

$$\begin{aligned}\|x_t\|_\nu = \sup_{\theta\leq 0} e^{\nu\theta}\|x(t+\theta)\| &= e^{-\nu t}\sup_{\theta\leq t} e^{\nu\theta}\|x(\theta)\| \\ &\leq e^{-\nu t}\max\left\{\sup_{\theta\leq 0} e^{\nu\theta}\|x(\theta)\|, \sup_{0\leq\theta\leq t} e^{\nu\theta}\|x(\theta)\|\right\} \\ &\leq \max\left\{\|x_0\|_\nu, \sup_{0\leq\theta\leq t}\|x(\theta)\|\right\}.\end{aligned}$$

In the case that $x(\cdot)$ is 1-periodic, we have

$$\|x_0\|_\nu = \sup_{\theta\leq 0}\frac{\|x(\theta\|}{e^{-\nu\theta}} \leq \sup_{0\leq s\leq 1}\|x(s)\| = \sup_{s\geq 0}\|x(s)\| = \sup_{s\in\mathbb{R}}\|x(s)\|,$$

and therefore,

$$\|x_t\|_\nu \leq \sup_{s\in\mathbb{R}_+}\|x(s)\| \leq \|x(\cdot)\|_{C_b(\mathbb{R},X)} \text{ for all } t \geq 0.$$

Definition 3 (Local φ-Lipschitz functions) Let E be an admissible Banach function space and φ be a positive function belonging to E and $\mathbb{B}_\rho$ be the ball with radius ρ in $\mathcal{C}_\nu$, i.e, $\mathbb{B}_\rho := \{\phi \in \mathcal{C}_\nu : \|\phi\|_\nu \leq \rho\}$. A function $g : [0, \infty) \times \mathbb{B}_\rho \to X$ is said to belong to the class (L, φ, ρ) for some positive constants L, ρ if g satisfies

(i) $\|g(t, 0)\| \leq L\varphi(t)$ for a.e. $t \in \mathbb{R}_+$,
(ii) $\|g(t, \phi_1) - g(t, \phi_2)\| \leq \varphi(t)\|\phi_1 - \phi_2\|_\nu$ for a.e. $t \in \mathbb{R}_+$ and all $\phi_1, \phi_2 \in \mathbb{B}_\rho$.

For a Banach space X with a separable predual Y as in the previous section, we now consider the following partial functional differential equation

$$\begin{cases}\dfrac{du}{dt} = A(t)u(t) + g(t, u_t), \quad t \geq 0,\\ u_0 = \phi \in \mathcal{C}_\nu,\end{cases} \tag{11}$$

where the linear operators $A(t)$, $t \geq 0$ act on X and satisfy Assumption 1, and the nonlinear term $g : [0, +\infty) \times \mathcal{C}_\nu \to X$ satisfies

$$\begin{aligned}&(1)\ \ g \text{ belongs to the class } (L, \varphi, \rho) \text{ for some } L, \rho > 0 \text{ and } 0 < \varphi \in \mathbf{P},\\ &(2)\ \ \text{the map } t \mapsto g(t, v_t) \text{ is 1-periodic}\\ &\qquad \text{for each 1-periodic function } v \in C_b(\mathbb{R}, X).\end{aligned} \tag{12}$$

Furthermore, by the *mild solution* to (11) we mean the function u satisfying the following equation

$$\begin{cases} u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau,u_\tau)d\tau \text{ for all } t \geq 0, \\ u_0 = \phi \in \mathcal{C}_\nu. \end{cases} \tag{13}$$

We then come to our first result on the existence and uniqueness of the periodic mild solution to (11).

Theorem 1 *Assume that there exists a constant M such that for each $f \in \mathfrak{M}$ there is a mild solution u of (6) satisfying $u \in C_b(\mathbb{R}_+, X)$ and*

$$\|u\|_{C_b(\mathbb{R}_+,X)} \leq M\|f\|_{\mathfrak{M}},$$

and that the evolution family $U(t,s)_{t\geq s\geq 0}$ satisfies

$$\lim_{t\to\infty} \|U(t,0)x\| = 0 \textit{ for } x \in X \textit{ such that } U(t,0)x \textit{ is bounded in } \mathbb{R}_+.$$

Let g satisfy the conditions in (12). If $\gamma := \|\varphi\|_{\mathbf{M}}$ is small enough, then (11) has one and only one 1-periodic mild solution $\hat{u}$ in $C_b(\mathbb{R}, X)$.

Proof Consider the following closed set $\mathcal{B}^1_\rho \subset C_b(\mathbb{R}, X)$ defined by

$$\mathcal{B}^1_\rho := \left\{ v \in C_b(\mathbb{R}, X) : v \text{ is 1-periodic, and } \|v\|_{C_b(\mathbb{R},X)} \leqslant \rho \right\}. \tag{14}$$

Note that for $v \in \mathcal{B}^1_\rho$, since v is 1-periodic, from Remark 1 we have that $v_t \in \mathcal{C}_\nu$ and $\|v_t\|_\nu \leq \|v\|_{C_b(\mathbb{R},X)} \leq \rho$.

We next define the transformation Φ as follows: Consider the equation for given $v \in C_b(\mathbb{R}, X)$ with u being the solution

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau, v_\tau)d\tau \text{ for all } t \geq 0. \tag{15}$$

Then, for $v \in \mathcal{B}^1_\rho$ we set

$$\Phi(v)(t) := \begin{cases} u(t) & \text{for } t \geq 0, \\ \tilde{u}(t) & \text{for } t < 0. \end{cases} \tag{16}$$

where $u \in C_b(\mathbb{R}_+, X)$ is the unique 1-periodic solution to (15) (the existence and uniqueness of such an u is guaranteed by [12, Theorem 2.3]), and $\tilde{u}(t)$, $t < 0$, is the 1-periodic extension of u on the interval $(-\infty, 0)$.

We will prove that if γ is small enough, then the transformation Φ acts from $\mathcal{B}^1_\rho$ into itself and is a contraction. To do this, fixing any $v \in \mathcal{B}^1_\rho$, then since g satisfy the conditions in (12) we have

$$\begin{aligned} \|g(\tau, v_\tau)\|_{\mathbf{M}} &= \sup_{t\geq 0} \int_t^{t+1} \|g(\tau, v_\tau)\|d\tau \\ &\leq (\rho + L)\sup_{t\geq 0} \int_t^{t+1} |\varphi(\tau)|d\tau = (\rho+L)\gamma. \end{aligned} \tag{17}$$

Applying [12, Theorem 2.3] to the right-hand side $g(\tau, v_\tau)$ instead of $f(\tau)$ (in the formula of mild solution) we obtain that for $v \in \mathcal{B}^1_\rho$ there exists a unique 1-periodic solution u to (15) satisfying

$$\begin{aligned} \|\Phi(v)\|_{C_b(\mathbb{R},X)} = \|u\|_{C_b(\mathbb{R}_+,X)} &\leqslant (M+1)Ke^\alpha \|g(\tau, v_\tau)\|_{\mathbf{M}} \\ &\leqslant (M+1)K(\rho+L)\gamma e^\alpha. \end{aligned} \tag{18}$$

Therefore, if γ is small enough, then the map Φ acts from $\mathcal{B}^1_\rho$ into itself.

Now, by formula (15) we have the following representation of Φ

$$\Phi(v)(t) = \begin{cases} U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau, v_\tau)d\tau & \text{for } t \geq 0, \\ \tilde{u}(t) & \text{for } t < 0, \end{cases} \tag{19}$$

where, as above, the function $\tilde{u}(t)$ is the 1-periodic extension to interval $(-\infty, 0)$ of the periodic function

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(\tau, v_\tau)d\tau \quad \text{for } t \geq 0.$$

Furthermore, for $v, w \in \mathcal{B}^1_\rho$ and $u_1 = \Phi(v)$, $u_2 = \Phi(w)$ by the representation (19), we obtain that $u = u_1 - u_2 = \Phi(v) - \Phi(w)$ is the unique 1-periodic mild solution to the equation

$$\begin{cases} u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)(g(\tau, v_\tau) - g(\tau, w_\tau))d\tau & \text{for } t \geq 0, \\ u(t) = \tilde{u}(t) = \tilde{u}_1(t) - \tilde{u}_2(t) & \text{for } t < 0. \end{cases}$$

Since $u(t), t \geq 0$, is 1-periodic, and for $t < 0$ the function $\tilde{u}(t)$ is an 1-periodic extension of u to interval $(-\infty, 0)$, we have that

$$\|\Phi(v) - \Phi(w)\|_{C_b(\mathbb{R},X)} = \sup_{t\in\mathbb{R}} \|u(t)\| = \sup_{t\geq 0} \|u(t)\|.$$

Thus, from [12, Theorem 2.3] and the fact that g belongs to the class (L, φ, ρ) we arrive at

$$\begin{aligned} \|u(t)\| &\leqslant (M+1)Ke^\alpha \sup_{t\geq 0} \int_t^{t+1} \|g(\tau, v_\tau) - g(\tau, w_\tau)\|d\tau \\ &\leqslant (M+1)Ke^\alpha \sup_{t\geq 0} \int_t^{t+1} \|g(\tau, v_\tau) - g(\tau, w_\tau)\|d\tau \\ &\leqslant 2(M+1)Ke^\alpha \sup_{t\geq 0} \int_t^{t+1} |\varphi(\tau)|\|v_\tau - w_\tau\|_\nu d\tau. \end{aligned} \tag{20}$$

Hence, since v and w are 1-periodic functions, from Remark 1, we have

$$\|v_t - w_t\|_\nu \leq \|v - w\|_{C_b(\mathbb{R},X)} \quad \text{for all } t \geq 0.$$

Thus,

$$\|\Phi(v) - \Phi(w)\|_{C_b(\mathbb{R},X)} \leqslant 2(M+1)Ke^\alpha \|\varphi\|_{\mathbf{M}} \|v - w\|_{C_b(\mathbb{R},X)}.$$

Thus, if $\gamma := \|\varphi\|_{\mathbf{M}}$ is small enough, then $\Phi : \mathcal{B}^1_\rho \to \mathcal{B}^1_\rho$ is a contraction. Therefore, for such a γ, there exists a unique fixed point $\hat{u}$ in $\mathcal{B}^1_\rho$ of Φ, and by the definition of Φ, this function $\hat{u}$ is the unique 1-periodic mild solution to (11). □

2.2 Periodic Solutions in the Case of Dichotomic Evolution Families

In this subsection, we will consider (8) and (13) in the case that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy. In this case, the existence of bounded solutions to (8) (i.e., bounded mild solutions to (6)) is convenient to prove. Therefore, the existence and uniqueness of periodic solutions to (8) and hence to (13) easily follow. Moreover, using the cone inequality in [3, Theorem I.9.3], we will show the conditional stability of such periodic solutions. To do so, we start with the cone inequality, the definitions of exponential dichotomy, and stability of an evolution family.

Definition 4 Let $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ be an evolution family on a Banach space X.

(1) The evolution family $\mathcal{U}$ is said to have an *exponential dichotomy* on $[0, \infty)$ if there exist bounded linear projections $P(t),\ t \geq 0$, on X and positive constants $N,\ \nu$ such that
(a) $U(t, s)P(s) = P(t)U(t, s), t \geq s \geq 0$,
(b) the restriction $U(t, s)_| : \operatorname{Ker} P(s) \to \operatorname{Ker} P(t),\ t \geq s \geq 0$, is an isomorphism, and we denote its inverse by $U(s, t)_| := (U(t, s)_|)^{-1}, 0 \leq s \leq t$,
(c) $\|U(t, s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X,\ t \geq s \geq 0$,
(d) $\|U(s, t)_| x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \operatorname{Ker} P(t),\ t \geq s \geq 0$.

The projections $P(t),\ t \geq 0$, are called the *dichotomy projections*, and the constants $N,\ \nu$ - the *dichotomy constants*.

(2) The evolution family $\mathcal{U}$ is called *exponentially stable* if it has an exponential dichotomy with the dichotomy projections $P(t) = \mathrm{Id}$ for all $t \geq 0$. In other words, $\mathcal{U}$ is exponentially stable if there exist positive constants N and ν such that

$$\|U(t, s)\| \leq Ne^{-\nu(t-s)} \text{ for all } t \geq s \geq 0. \tag{21}$$

We remark that properties (a)–(d) of dichotomy projections $P(t)$ already imply that

1. $H := \sup_{t \geq 0} \|P(t)\| < \infty$,
2. $t \mapsto P(t)$ is strongly continuous

(see [17, Lemma 4.2]). We refer the reader to [7] for characterizations of exponential dichotomies of evolution families in general admissible spaces.

If $(U(t, s))_{t \geq s \geq 0}$ has an exponential dichotomy with dichotomy projections $(P(t))_{t \geq 0}$ and constants $N, \nu > 0$, then we can define the Green's function on a half-line as follows:

$$\mathcal{G}(t, \tau) := \begin{cases} P(t)U(t, \tau) & \text{for } t > \tau \geq 0, \\ -U(t, \tau)_|(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases} \tag{22}$$

Also, $\mathcal{G}(t, \tau)$ satisfies the estimate

$$\|\mathcal{G}(t, \tau)\| \leq (1 + H)Ne^{-\nu|t-\tau|} \ \text{ for } t \neq \tau \ \geq 0. \tag{23}$$

Using the projections $(P(t))_{t \geq 0}$ on X, we can define the family of operators $\widetilde{P}(t), t \geq 0$ on $\mathcal{C}_\nu$ as follows:

$$\widetilde{P}(t) : \mathcal{C}_\nu \to \mathcal{C}_\nu, (\widetilde{P}(t)\phi)(\theta) = U(t - \theta, t)P(t)\phi(0) \ \text{ for all } \theta \in (-\infty, 0]. \tag{24}$$

Then, $(\widetilde{P}(t))^2 = \widetilde{P}(t)$, and therefore the operators $\widetilde{P}(t), t \geq 0$, are projections on $\mathcal{C}_\nu$. Moreover, $\operatorname{Im} \widetilde{P}(t) = \{\phi \in \mathcal{C}_\nu : \phi(\theta) = U(t - \theta, t)v_0$ for all $\theta \in (-\infty, 0]$ for some $v_0 \in \operatorname{Im} P(t)\}$.

The following lemma gives the form of bounded solutions of (8) and (13).

Lemma 1 *Let the evolution family $(U(t, s))_{t \geq s \geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t \geq 0}$ and dichotomy constants $N, \nu > 0$. Let $f \in \mathfrak{M}$, and let g satisfy conditions given in (12). Then, the following assertions hold true.*

(a) *Let $v \in C_b(\mathbb{R}_+, X)$ be the solution to (8). Then, v can be rewritten in the form*

$$v(t) = U(t, 0)\zeta + \int_0^\infty \mathcal{G}(t, \tau)f(\tau)d\tau \quad \textit{for some } \zeta \in X_0 := P(0)X, \tag{25}$$

where $\mathcal{G}(t, \tau)$ is the Green's function defined by equality (22).

(b) *Let* $u \in C(\mathbb{R}, X)$ *be a solution to* (13) *and given* $\phi \in \mathcal{C}_\nu$ *such that* $\max\{\|\phi\|_\nu, \sup_{t\in\mathbb{R}_+}\|u(t)\|\} \le \rho$ *for a fixed* $\rho > 0$*. Then, for* $t \ge 0$ *this function* $u(t)$ *can be rewritten in the form*

$$\begin{cases} u(t) = U(t,0)\eta + \int_0^\infty \mathcal{G}(t,\tau)g(\tau, u_\tau)d\tau, \\ u_0 = \phi \in \mathcal{C}_\nu \end{cases} \tag{26}$$

for some $\eta \in X_0$ *where* $\mathcal{G}$ *and* X_0 *are determined as in* (a).

Proof (a) See [8, Lemma 4.4]. (b) See [10, Lemma 3.4]. □

Remark 2 By straightforward computations, we can prove that the converses of statements (a) and (b) are also true, i.e., a solution of (26) satisfies (8) for $t \ge 0$, and that of (25) satisfies (13) for $t \ge 0$.

We next prove the existence of bounded solutions to (8) and (13) (i.e., bounded mild solutions to (6) and (11)) and hence that of periodic solutions in the following theorem.

Theorem 2 *Consider* (8) *and* (13)*. Let the evolution family* $(U(t,s))_{t\ge s\ge 0}$ *satisfy* (9) *and have an exponential dichotomy with the dichotomy projections* $P(t)$*,* $t \ge 0$*, and constants* N, ν*. Let* $f \in \mathfrak{M}$ *be 1-periodic and suppose that* g *satisfies the conditions in* (12) *with given positive constants* ρ*,* L *and function* $\varphi \in \mathbf{P}$*. Then, the following assertions hold true.*

(a) Equation (8) *has a unique 1-periodic solution in* $C_b(\mathbb{R}_+, X)$*.*
(b) *If* $\|\varphi\|_{\mathbf{M}}$ *is sufficiently small, then* (13) *has a unique 1-periodic solution in* $C_b(\mathbb{R}, X)$*.*

Proof (a) For a given $f \in \mathfrak{M}$ by taking $\zeta = 0 \in X_0$ in (25), we see that (8) has a bounded solution

$$u(t) = \int_0^\infty \mathcal{G}(t,\tau)f(\tau)d\tau, \tag{27}$$

and this solution can be estimated using the inequalities (23) and (5) by

$$\begin{aligned} \|u\|_{C_b} &\leqslant (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|f(\tau)\|d\tau \\ &\le \frac{(1+H)N(N_1+N_2)}{1-e^{-\nu}}\|f\|_{\mathfrak{M}} \text{ for all } t \ge 0. \end{aligned}$$

From [12, Theorem 2.3] it follows that for the 1-periodic function $f \in \mathfrak{M}$, there exists an 1-periodic solution $\hat{u}$ of (8) satisfying

$$\|\hat{u}\|_{C_b} \leqslant \left(\frac{(1+H)N(N_1+N_2)}{1-e^{-\nu}} + 1\right)Ke^\alpha\|f\|_{\mathfrak{M}}. \tag{28}$$

The uniqueness of the 1-periodic solution follows from the fact that for two 1-periodic and continuous (hence bounded on $\mathbb{R}_+$) solutions $\hat{u}$ and $\hat{v}$ (with the corresponding initial values $\zeta, \eta \in X_0$), we obtain by using the form for bounded solutions (25) that $\|\hat{u}(t) - \hat{v}(t)\| = \|U(t,0)(\zeta - \eta)\| \le Ne^{-\nu t}\|\zeta - \eta\| \to 0$ as $t \to \infty$ since $\eta, \zeta \in X_0$. This, together with the periodicity, implies $\hat{u}(t) = \hat{v}(t)$ for all $t \ge 0$, finishing the proof of (a).

(b) By assertion (a), for each 1-periodic input function f, the linear problem (8) has a unique 1-periodic solution $\hat{u} \in C_b(\mathbb{R}_+, X)$ satisfying inequality (28). Therefore, Assertion (b) then follows from Theorem 1. □

We now prove the conditional stability of periodic solutions to (13). To do this, for $x \in X, \hat{\phi} \in \mathcal{C}_\nu$, and $\hat{v} \in C_b(\mathbb{R}, X)$ denote $B_a(x) := \{y \in X : \|x - y\| \le a, x \in X\}$, $\mathbb{B}_a(\hat{\phi}) := \{\phi \in \mathcal{C}_\nu : \|\phi - \hat{\phi}\|_\nu \le a\}$, and

$$\mathcal{B}_a(\hat{v}) := \left\{ v \in C(\mathbb{R}, X) : v_t, \hat{v}_t \in \mathcal{C}_\nu; \max\{\|v_0 - \hat{v}_0\|_\nu, \sup_{t\in\mathbb{R}_+} \|v(t) - \hat{v}(t)\|\} \le a, t \ge 0 \right\}$$

respectively. Let $\mathcal{B}_\rho(0)(\mathbb{B}_\rho(0))$ be the ball containing $\hat{u}(\hat{u}_t, t \ge 0)$ as in assertion (b) of Theorem 2.

Suppose further that there exists a positive function $\tilde{\varphi} \in \mathbf{P}$ such that:

$$\|g(t, \phi_1) - g(t, \phi_2)\| \le \tilde{\varphi}(t)\|\phi_1 - \phi_2\|_\nu \text{ for all } \phi_1, \phi_2 \in \mathbb{B}_{2\rho}(0), \text{ and } t \ge 0. \tag{29}$$

Theorem 3 *Keep the assumptions of Theorem 2, and let $\hat{u}$ be the 1-periodic solution of* (13) *obtained in assertion* (b) *of Theorem 2. Let g satisfy conditions given in* (12) *and* (29), *respectively. If $\|\tilde{\varphi}\|_{\mathbf{M}}$ is small enough, then to each $\zeta \in \mathcal{C}_\nu$ with $\|\zeta - \hat{u}_0\|_\nu \le \rho/2$ and $P(0)\zeta(0) \in B_{\frac{\rho}{2N}}(P(0)\hat{u}(0)) \cap P(0)X$ there corresponds one and only one solution $u(\cdot)$ of* (13) *on $\mathbb{R}$ satisfying the conditions $u_0 = \zeta$ and $u \in \mathcal{B}_\rho(\hat{u})$. Moreover, the following estimate is valid for $u(t)$ and $\hat{u}(t)$:*

$$\|u_t - \hat{u}_t\|_\nu \le C_\mu \rho e^{-\mu t} \text{ for } t \ge 0, \tag{30}$$

for some positive constants C_μ and μ independent of u, $\hat{u}$, and ρ.

Proof Putting $w = u - \hat{u}$, then u is a solution to (13) in $\mathcal{B}_\rho(\hat{u})$ with $u_0 = \zeta$ if and only if w is the solution in $\mathcal{B}_\rho(0)$ of the equation

$$w(t) = \begin{cases} U(t, 0)(\zeta(0) - \hat{u}(0)) + \int_0^t U(t, \tau)\left[g(\tau, w_\tau + \hat{u}_\tau) - g(\tau, \hat{u}_\tau)\right] d\tau & \text{for } t \ge 0, \\ \zeta(t) - \hat{u}(t) & \text{for } t \le 0. \end{cases} \tag{31}$$

We now prove that (31) has a unique solution in $\mathcal{B}_\rho(0)$. To do this, putting $\tilde{g}(t, w_t) = g(t, w_t + \hat{u}_t) - g(t, \hat{u}_t)$ we obtain that $\tilde{g}(t, 0) = 0$ and

$$\|\tilde{g}(t, w_t) - \tilde{g}(t, v_t)\| \le \tilde{\varphi}(t)\|w_t - v_t\|_\nu, \quad t \ge 0, w, v \in \mathcal{B}_\rho(0).$$

Setting $\xi = P(0)\zeta(0) - P(0)\hat{u}(0)$ we prove that the transformation K defined by

$$(Kw)(t) = \begin{cases} U(t, 0)\xi + \int_0^\infty \mathcal{G}(t, \tau)\tilde{g}(\tau, w_\tau)d\tau & \text{for } t \ge 0, \\ \zeta(t) - \hat{u}(t) & \text{for } t \le 0 \end{cases}$$

acts from $\mathcal{B}_\rho(0)$ into itself and is a contraction. In fact, we have

$$\begin{aligned}\|(Kw)(t)\| &\le \begin{cases} Ne^{-\nu t}\|\xi\| + (1 + H)N \int_0^\infty e^{-\nu|t-\tau|}\|w_\tau\|_\nu\tilde{\varphi}(\tau)d\tau & \text{for } t \geqslant 0, \\ e^{-\nu t}\|\zeta - \hat{u}_0\|_\nu & \text{for } t \le 0 \end{cases} \\ &\le \begin{cases} e^{-\nu t}\frac{\rho}{2} + (1 + H)N\rho \int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)d\tau & \text{for } t \geqslant 0, \\ e^{-\nu t}\frac{\rho}{2} & \text{for } t \le 0. \end{cases}\end{aligned}$$

Since $t + \theta \in \mathbb{R}$ for fixed $t \in [0, \infty)$ and $\theta \in (-\infty, 0]$, we obtain

$$\|(Kw)(t + \theta)\| \le \begin{cases} e^{-\nu(t+\theta)}\frac{\rho}{2} + N(1 + H)\rho \int_0^\infty e^{-\nu|t+\theta-\tau|}\tilde{\varphi}(\tau)d\tau & \text{for } t + \theta \geqslant 0, \\ e^{-\nu(t+\theta)}\frac{\rho}{2} & \text{for } t + \theta \le 0. \end{cases}$$

Therefore,

$$\|Kw\|_\nu \leq \frac{\rho}{2} + \frac{(1+H)N\rho(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}}{1-e^{-\nu}}.$$

Thus, if $\|\tilde{\varphi}\|_{\mathbf{M}}$ is small enough, then the transformation K acts from $\mathcal{B}_\rho(0)$ into $\mathcal{B}_\rho(0)$.

Now, for $x, z \in \mathcal{B}_\rho(0)$, we estimate

$$\begin{aligned}\|(Kx)(t)-(Kz)(t)\| &\leq \int_0^\infty \|\mathcal{G}(t,\tau)\|(\tilde{g}(\tau,x_\tau)-\tilde{g}(\tau,z_\tau))\|d\tau\\ &\leq (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)\|x_\tau-z_\tau\|_\nu d\tau,\quad t\geqslant 0.\end{aligned}$$

From Remark 1 and $\|(Kx)(t)-(Kz)(t)\| = 0$ for all $t \leq 0$, we have

$$\|(Kx)-(Kz)\|_\nu \leq \frac{(1+H)N(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}}{1-e^{-\nu}}\sup_{t\geq 0}\|x_t-z_t\|_\nu.$$

Therefore, if $\|\tilde{\varphi}\|_{\mathbf{M}}$ is small enough, then the transformation $K : \mathcal{B}_\rho(0) \to \mathcal{B}_\rho(0)$ is a contraction. Thus, there exists a unique $w \in \mathcal{B}_\rho(0)$ such that $Kw = w$. By the definition of K, Lemma 1, and Remark 2, we have that w is the unique solution in $\mathcal{B}_\rho(0)$ of (31). Note that by Lemma 1 the above solution w of (31) can be written as

$$w(t) = \begin{cases} U(t,0)\xi + \int_0^\infty \mathcal{G}(t,\tau)\tilde{g}(\tau,w_\tau)d\tau & \text{for } t \geq 0,\\ \zeta(t)-\hat{u}(t) & \text{for } t \leq 0.\end{cases} \tag{32}$$

Returning to the solution u of (13) by replacing w by $u-\hat{u}$ then, there exists a unique $u \in \mathcal{B}_\rho(\hat{u})$ of (13) with $u_0 = \zeta$.

Finally, we prove the estimate (30). To do this, putting as above $\xi := P(0)u(0) - P(0)\hat{u}(0)$, $w = u-\hat{u}$ with $u \in \mathcal{B}_\rho(\hat{u})$, $\tilde{g}(t,w_t) = g(t,w_t+\hat{u}_t) - g(t,\hat{u}_t)$, we can use the formula (32) to write

$$w(t) = \begin{cases} U(t,0)\xi + \int_0^\infty \mathcal{G}(t,\tau)\tilde{g}(\tau,w_\tau)d\tau & \text{for } t \geq 0,\\ u(t)-\hat{u}(t) & \text{for } t \leq 0.\end{cases}$$

Using the facts that $\|\xi\| \leq \frac{\rho}{2N}$ and $\|u_0-\hat{u}_0\|_\nu \leq \frac{\rho}{2}$, it follows that

$$\|w(t)\| \leq \begin{cases} e^{-\nu t}\frac{\rho}{2} + N(1+H)\int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)\|w_\tau\|_\nu d\tau & \text{for } t \geqslant 0,\\ e^{-\nu t}\frac{\rho}{2} & \text{for } t < 0.\end{cases}$$

Since $t+\theta \in \mathbb{R}$ for fixed $t \in [0,\infty)$ and $\theta \in (-\infty, 0]$, we obtain

$$\|w(t+\theta)\| \leq \begin{cases} e^{-\nu(t+\theta)}\frac{\rho}{2} + N(1+H)\int_0^\infty e^{-\nu|t+\theta-\tau|}\tilde{\varphi}(\tau)\|w_\tau\|_\nu d\tau & \text{for } t+\theta \geqslant 0,\\ e^{-\nu(t+\theta)}\frac{\rho}{2} & \text{for } t+\theta < 0.\end{cases}$$

Therefore,

$$e^{\nu\theta}\|w(t+\theta)\| \leq \frac{\rho}{2}e^{-\nu t} + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)\|w_\tau\|_\nu d\tau \quad \text{for } t \geqslant 0.$$

Put $\phi(t) = \|w_t\|_\nu$. Then, $\sup_{t\geqslant 0}\phi(t) < \infty$ and

$$\phi(t) \leq \frac{\rho}{2}e^{-\nu t} + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)\phi(\tau)d\tau \quad \text{for } t \geqslant 0. \tag{33}$$

We will use the cone-inequality theorem [3, Theorem I.9.3] applying to the Banach space $W := L_\infty(\mathbb{R}_+)$ which is the space of real-valued functions defined and essentially bounded on $\mathbb{R}_+$ (endowed with the esssup-norm denoted by $\|\cdot\|_\infty$) with the cone $\mathcal{K}$ being the set of all (a.e.) nonnegative functions. We then consider the linear operator B defined for $u \in W$ by

$$(Bu)(t) = (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)u(\tau)d\tau \ \text{ for } t \geqslant 0.$$

By inequalities (5), we have

$$\begin{aligned} \sup_{t\geqslant 0}(Bu)(t) &= \sup_{t\geqslant 0}(1+H)N \int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)u(\tau)d\tau \\ &\leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}\|u\|_\infty. \end{aligned}$$

Therefore, $B \in \mathcal{L}(W)$ and $\|B\| \leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}} < 1$. Obviously, B leaves the cone $\mathcal{K}$ invariant. The inequality (33) can now be rewritten as

$$\phi \leq B\phi + z \ \text{ for } z(t) = \frac{\rho}{2}e^{-\nu t}, \ t \geqslant 0.$$

Hence, by the cone-inequality theorem [3, Theorem I.9.3], we obtain $\phi \leq \psi$, where ψ is a solution in W of the equation $\psi = B\psi + z$ which can be rewritten as

$$\psi(t) = \frac{\rho}{2}e^{-\nu t} + (1+H)N \int_0^\infty e^{-\nu|t-\tau|}\tilde{\varphi}(\tau)\psi(\tau)d\tau \ \text{ for } t \geqslant 0. \tag{34}$$

We now estimate ψ. To that purpose, for

$$0 < \mu < \nu + \ln(1-(1+H)N(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}),$$

we set $h(t) = e^{\mu t}\psi(t)$ for $t \geqslant 0$. Then, by (34), we obtain that

$$h(t) = \frac{\rho}{2}e^{-(\nu-\mu)t} + (1+H)N \int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\tilde{\varphi}(\tau)h(\tau)d\tau \ \text{ for } t \geqslant 0. \tag{35}$$

We next consider the linear operator D defined for $u \in W$ by

$$(Du)(t) = (1+H)N \int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\tilde{\varphi}(\tau)u(\tau)d\tau \ \text{ for } t \geqslant 0.$$

By inequalities (5), we have

$$\begin{aligned} \sup_{t\geqslant 0}(Du)(t) &= \sup_{t\geqslant 0}(1+H)N \int_0^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\tilde{\varphi}(\tau)u(\tau)d\tau \\ &\leq \sup_{t\geqslant 0}(1+H)N \int_0^\infty e^{-(\nu-\mu)|t-\tau|}\tilde{\varphi}(\tau)u(\tau)d\tau \\ &\leq \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}\|u\|_\infty. \end{aligned}$$

Therefore, $D \in \mathcal{L}(W)$ and $\|D\| \leq \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}$.

Equation (35) can now be rewritten as

$$h = Dh + \tilde{z} \ \text{ for } \tilde{z}(t) = \frac{\rho}{2}e^{-(\nu-\mu)t}, \ t \geqslant 0.$$

Since $\mu < \nu + \ln(1-(1+H)N(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}})$, we obtain that

$$\|D\| \leq \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}} < 1.$$

Therefore, the equation $h = Dh + \tilde{z}$ is uniquely solvable in $L_\infty(W)$, and its solution is $h = (I - D)^{-1}\tilde{z}$. Hence, we obtain that

$$\begin{aligned}\|h\|_\infty &= \|(I-D)^{-1}\tilde{z}\|_\infty \le \|(I-D)^{-1}\|\|\tilde{z}\|_\infty \le \frac{\|\tilde{z}\|_\infty}{1-\|D\|}\\ &\le \frac{\rho}{2\left(1 - \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}\right)}.\end{aligned}$$

Therefore,

$$\|h\|_\infty \le C_\mu \rho \text{ for } C_\mu := \frac{1}{2\left(1 - \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}\right)}.$$

This yields

$$h(t) \le C_\mu \rho \text{ for } t \geqslant 0.$$

Hence, $\psi(t) = e^{-\mu t}h(t) \le C_\mu \rho e^{-\mu t}$. Since $\|w_t\|_\nu = \phi(t) \le \psi(t)$, we obtain that

$$\|w_t\|_\nu \le C_\mu \rho e^{-\mu t}.$$

Returning to the solution u of (13) by replacing w by $u - \hat{u}$, we have

$$\|u_t - \hat{u}_t\|_\nu \le C_\mu \rho e^{-\mu t},$$

finishing the proof of the theorem. □

Remark 3 The assertion of the above theorem shows us the *conditional stability* of the periodic solution $\hat{u}$ in the sense that for any other solution u such that $P(0)u(0) \in B_{\frac{\rho}{2N}}(P(0)\hat{u}(0)) \cap P(0)X$ and u being in a small ball $\mathcal{B}_\rho(\hat{u})$ we have $\|u_t - \hat{u}_t\|_\nu \to 0$ exponentially as $t \to \infty$ (see inequality (30)).

For an exponentially stable evolution family (see Definition 4 (2)), we have the following corollary which is a direct consequence of Theorem 3.

Corollary 1 *Keep the assumptions of Theorem 2, and let $\hat{u}$ be the periodic solution of* (13) *obtained in assertion* (b) *of Theorem 2. Further, let the evolution family $(U(t,s))_{t\ge s\ge 0}$ be exponentially stable. Then, the periodic solution $\hat{u}$ is exponentially stable in the sense that for any other solution $u \in C(\mathbb{R}, X)$ of* (13) *such that $u_t \in \mathcal{C}_\nu, t \ge 0$ and $\|u_0 - \hat{u}_0\|_\nu$ is small enough, we have*

$$\|u_t - \hat{u}_t\|_\nu \le Ce^{-\mu t}\|u_0 - \hat{u}_0\|_\nu \text{ for all } t \ge 0, \tag{36}$$

for some positive constants C and μ independent of u and $\hat{u}$.

Proof We just apply Theorem 3 for $P(t) = \mathrm{Id}$ for all $t \ge 0$ to obtain the assertion of the corollary. □

2.3 Local Stable Manifold Around the Periodic Solution

In this subsection, under the same hypotheses as in the previous subsection, we will prove the existence of a local stable manifold for (13) around its periodic solution. We first recall the definition of a local stable manifold for (13) around its periodic solution.

Definition 5 Given a continuous and 1-periodic solution $\hat{u}$ to (13). A set $\mathbf{S} \subset \mathbb{R}_+ \times \mathcal{C}_\nu$ is said to be a stable manifold for the (13) around $\hat{u}$ if for every $t \in \mathbb{R}_+$ the phase space $\mathcal{C}_\nu$ splits into a direct sum $\mathcal{C}_\nu = \widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with corresponding projections $\widetilde{P}(t)$ (i.e., $\widetilde{X}_0(t) = \operatorname{Im} \widetilde{P}(t)$ and $\widetilde{X}_1(t) = \operatorname{Ker} \widetilde{P}(t)$) such that

$$\sup_{t\geq 0} \|\widetilde{P}(t)\| < \infty,$$

and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$h_t : \mathbb{B}_{\rho_0}(\hat{u}_t) \cap \widetilde{X}_0(t) \to \mathbb{B}_{\rho_1}(\hat{u}_t) \cap \widetilde{X}_1(t), \quad t \in \mathbb{R}_+$$

with the Lipschitz constants being independent of t such that

(i) $\mathbf{S} = \{(t, \psi + h_t(\psi)) \in \mathbb{R}_+ \times (\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)) \,|\, t \in \mathbb{R}_+,\ \psi \in \mathbb{B}_{\rho_0}(\hat{u}_t) \cap \widetilde{X}_0(t)\}$, and we denote $\mathbf{S}_t := \{\psi + h_t(\psi) \mid (t, \psi + h_t(\psi)) \in \mathbf{S}\},\ t \geq 0$,
(ii) $\mathbf{S}_t$ is homeomorphic to $\mathbb{B}_{\rho_0}(\hat{u}_t) \cap \widetilde{X}_0(t) := \{\psi \in \widetilde{X}_0(t) : \|\psi - \hat{u}_t\|_\nu \leqslant \rho_0\}$ for all $t \geq 0$,
(iii) to each $\psi \in S_{t_0}$ there corresponds one and only one solution $u(t)$ of (13) on $\mathbb{R}$ satisfying conditions $u_{t_0} = \psi$ and $\sup_{t\geq t_0} \|u_t\|_\nu \leq \rho$.

Note that, if we identify $\widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$ with $\widetilde{X}_0(t) \times \widetilde{X}_1(t)$, then we can write $S_t = \operatorname{graph}(h_t)$ where $\operatorname{graph}(h_t)$ denotes the graph of the mapping h_t.

We now state and prove our last result on the existence of a stable manifold for solutions to (13) around its periodic solution.

Theorem 4 *Let the assumptions of Theorems 2 and 3 hold with the corresponding positive functions φ and $\tilde{\varphi}$. Let $\hat{u}$ be the 1-periodic solution of (13) obtained in Theorem 2 thanks to the sufficient smallness of $\|\varphi\|_{\mathbf{M}}$. If $\|\tilde{\varphi}\|_{\mathbf{M}}$ is sufficiently small, then there exists a local stable manifold $\mathbf{S}$ near the solution $\hat{u}$. Moreover, every solution $u(t)$ on the manifold $\mathbf{S}$ is exponentially attracted to $\hat{u}(t)$ in the sense that, there exist positive constants μ and C_μ independent of $t_0 \geq 0$ such that*

$$\|u_t - \hat{u}_t\|_\nu \leq C_\mu e^{-\mu(t-t_0)} \|P(t_0)(u(t_0) - P(t_0)\hat{u}(t_0))\| \quad \textit{for all } t \geq t_0. \tag{37}$$

Proof Putting $w = u - \hat{u}$, then u is a solution to (13) in $\mathcal{B}_\rho(\hat{u})$ with $u_0 = \zeta$ if and only if w is the solution in $\mathcal{B}_\rho(0)$ of the equation

$$w(t) = U(t,0)w(0) + \int_0^t U(t,\tau) \left[F(\tau)(w_\tau) + g(\tau, w_\tau + \hat{u}_\tau) - g(\tau, \hat{u}_\tau)\right] d\tau \text{ for } t \geq 0. \tag{38}$$

Putting now $\tilde{g}(t, w_t) = g(t, w_t + \hat{u}_t) - g(t, \hat{u}_t)$, we obtain that $\tilde{g}(t, 0) = 0$ and

$$\|\tilde{g}(t, w_t) - \tilde{g}(t, v_t)\| \leq \tilde{\varphi}(t)\|w_t - v_t\|_\nu, \quad t \geq 0, w, v \in \mathcal{B}_\rho(0).$$

Equation (38) can be written as

$$w(t) = U(t,0)w(0) + \int_0^t U(t,\tau)\tilde{g}(\tau, w_\tau) d\tau \ \text{ for } t \geq 0. \tag{39}$$

Since $U(t,s)_{t\geq s\geq 0}$ has an exponential dichotomy, for each $t \geq 0$, the phase space $\mathcal{C}_\nu$ splits into the direct sum $\mathcal{C}_\nu = \widetilde{X}_0(t) \oplus \widetilde{X}_1(t)$, where $\widetilde{X}_0(t) = \operatorname{Im} \widetilde{P}(t)$ and $\widetilde{X}_1(t) = \operatorname{Ker} \widetilde{P}(t)$, and the projections $\widetilde{P}(t), t \geq 0$, are defined as in equality (24). Clearly,

$\sup_{t\geq 0}\|\widetilde{P}(t)\| < \infty$. We now construct a stable manifold $\mathbf{S} = \{(t, \mathbf{S}_t)\}_{t\geq 0}$ for the solutions to (13). To do this, we determine the surface $\mathbf{S}_t$ for $t \geq 0$ by the formula

$$\mathbf{S}_t := \left\{\phi + \Phi_t(\phi) : \phi \in \mathbb{B}_{\frac{\rho}{2N}}(0) \cap X_0(t)\right\} \subset \mathcal{C}_\nu,$$

where the operator Φ_{t_0} is defined for each $t_0 \geq 0$ by

$$\Phi_{t_0}(\phi)(\theta) = \int_{t_0}^{\infty} \mathcal{G}(t_0-\theta,\tau)g(\tau, w_\tau)d\tau \ \text{ for } \theta \leq 0,$$

where $w(\cdot)$ is the unique solution of (39) on $\mathcal{B}_\rho(0)$ satisfying $\widetilde{P}(t_0)w_{t_0} = \phi$. On the other hand, by the definition of the Green function $\mathcal{G}$ we have that $\Phi_{t_0}(\phi) \in \mathrm{Ker}\widetilde{P}(t_0)$.

We next estimate $\|\Phi_{t_0}(\phi)\|_\nu$ by

$$\|\Phi_{t_0}(\phi)\| \leq N(1+H)\int_{t_0}^{\infty} e^{-\nu|t_0-\theta-\tau|}\tilde{\varphi}(\tau)\|w_\tau\|_\nu d\tau.$$

Therefore,

$$\|\Phi_{t_0}(\phi)\|_\nu \leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\rho\|\tilde{\varphi}\|_{\mathbf{M}}.$$

Hence, if $\|\tilde{\varphi}\|_{\mathbf{M}}$ is small enough, then the operator Φ_{t_0} acts from $\mathbb{B}_{\frac{\rho}{2N}}(0)\cap\tilde{X}_0(t_0)$ to $\mathbb{B}_{\frac{\rho}{2}}(0)\cap \tilde{X}_1(t_0)$. We then prove that Φ_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for ϕ_1 and ϕ_2 belonging to $\mathbb{B}_{\frac{\rho}{2N}}(0) \cap \tilde{X}_0(t_0)$, we have

$$\begin{aligned}
&\|\Phi_{t_0}(\phi_1)(\theta) - \Phi_{t_0}(\phi_2)(\theta)\| \\
&\leq N(1+H)\int_{t_0}^{\infty} e^{-\nu|t_0-\theta-\tau|}\tilde{\varphi}(\tau)\|w_\tau - v_\tau\|_\nu d\tau \qquad (40)\\
&\leq N(1+H)\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu \int_{t_0}^{\infty} e^{-\nu|t_0-\tau|}e^{\nu|\theta|}\tilde{\varphi}(\tau)d\tau.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Phi_{t_0}(\phi_1) - \Phi_{t_0}(\phi_2)\|_\nu &\leq N(1+H)\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu \int_{t_0}^{\infty} e^{-\nu|t_0-\tau|}\tilde{\varphi}(\tau)d\tau \\
&\leq \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu. \qquad (41)
\end{aligned}$$

Moreover, by the Lyapunov-Perron equation for $w(\cdot)$ and $v(\cdot)$ (see (26)) and putting $k := \frac{(1+H)N}{1-e^{-\nu}}(N_1+N_2)\|\tilde{\varphi}\|_{\mathbf{M}}$, we have

$$\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu \leq N\|\phi_1-\phi_2\|_\nu + k\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu,$$

it follows that

$$\sup_{\tau\geq t_0}\|w_\tau - v_\tau\|_\nu \leq \frac{N}{1-k}\|\phi_1-\phi_2\|_\nu.$$

Substituting this inequality into (40), we obtain

$$\|\Phi_{t_0}(\phi_1) - \Phi_{t_0}(\phi_2)\|_\nu \leq \frac{Nk}{1-k}\|\phi_1-\phi_2\|_\nu,$$

yielding that Φ_{t_0} is Lipschitz continuous with the Lipschitz constant $\frac{Nk}{1-k}$ independent of t_0. Therefore, putting $\rho_0 := \frac{\rho}{2N}$, $\rho_1 := \frac{\rho}{2}$, we obtain that the above family of mappings Φ_{t_0} is Lipschitz continuous with the Lipschitz constant $\frac{Nk}{1-k}$ independent of t_0.

To show that $\mathbf{S}_{t_0}$ is homeomorphic to $\mathbb{B}_{\rho_0}(0) \cap \widetilde{X}_0(t_0)$, we define the transformation $D : \mathbb{B}_{\rho_0}(0) \cap \widetilde{X}_0(t_0) \to \mathbf{S}_{t_0}$ by $D\phi := \phi + \Phi_{t_0}(\phi)$ for all $\phi \in \mathbb{B}_{\rho_0}(0) \cap \widetilde{X}_0(t_0)$. Then, applying the implicit function theorem for Lipschitz continuous mappings (see [18, Lemma 2.7]) we see that, if the Lipschitz constant $\frac{Nk}{1-k} < 1$ then D is a homeomorphism. Therefore, the condition (ii) in Definition 5 is satisfied. The condition (iii) of Definition 5 now follows from Theorem 3. Finally, the inequality (37) in Theorem 4 follows from inequality (30) in Theorem 3.

Returning to the solution u of (13) by replacing w by $u - \hat{u}$, we obtain that this manifold $\mathbf{S}$ is the local stable manifold for (13) near the solution $\hat{u}$. □

We finally illustrate our results by the following example.

2.4 An Example

We consider the problem

$$\begin{cases} \dfrac{\partial w(x,t)}{\partial t} = a(t)\left[\dfrac{\partial^2 w(x,t)}{\partial x^2} + \delta w(x,t)\right] \\ \qquad + \psi(t)\left(\displaystyle\int_{-\infty}^{0} m(t+s)\sin w(x,t+s)ds + h(x,t)\right), \\ \qquad\qquad \text{for } 0 < x < \pi,\ t \geq 0, \\ w(0,t) = w(\pi,t) = 0 \qquad \text{for } t \geq 0, \\ w(x,\theta) = \phi(\theta)(x) \qquad \text{for } 0 < x < \pi,\ \theta \in (-\infty, 0],\ \phi \in \mathcal{C}_v. \end{cases} \tag{42}$$

Here, $\delta \in \mathbb{R}$ and $\delta \neq n^2$ for all $n \in \mathbb{N}$; the function $a(t) \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is 1-periodic and satisfies the condition $0 < \gamma_0 \leq a(t) \leq \gamma_1$ for fixed γ_0, γ_1; the function $h : [0,\pi] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous on $[0,\pi] \times \mathbb{R}_+$ and 1-periodic with respect to t; the function $m : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and 1-periodic with respect to t, and $\int_{-\infty}^{0} m(t+\theta)e^{-\nu\theta}d\theta$ is integrable on $(-\infty, 0]$ and

$$M = \sup_{-\infty<\theta\leq 0} \int_{-\infty}^{0} m(t+\theta)e^{-\nu\theta}d\theta.$$

We next put $X := L_2[0,\pi]$, $\mathcal{C} := C(-\infty,0], X)$, and let $A : X \supset D(A) \to X$ be defined by $Ay = y'' + \delta y$, with the domain

$$D(A) = \{y \in X : y \text{ and} y' \text{ are absolutely continuous}, y' \in X,\ y(0) = y(\pi) = 0\}.$$

It can be seen (see [4]) that A is the generator of an analytic semi-group $(\mathbb{T}(t))_{t\geq 0.}$ Since $\sigma(A) = \{-n^2 + \delta : n = 1,2,3,\ldots\}$ applying the spectral mapping theorem for analytic semi-groups, we get

$$\begin{aligned} \sigma(\mathbb{T}(t)) = & e^{t\sigma(A)} = \{e^{t(-n^2+\delta)} : n = 1,2,3,\ldots\} \\ & \text{and hence } \sigma(\mathbb{T}(t)) \cap \Gamma = \emptyset \text{ for all } t > 0, \end{aligned} \tag{43}$$

where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Putting now $A(t) := a(t)A$, then $A(t)$ is 1-periodic, and the family $(A(t))_{t\geq 0}$ generates an 1-periodic (in the sense of Assumption 1) evolution family $U(t,s)_{t\geq s\geq 0}$ which is defined by the formula $U(t,s) = \mathbb{T}(\int_s^t a(\tau)d\tau)$.

By (43), the analytic semi-group $(\mathbb{T}(t))_{t\geq 0}$ is hyperbolic (or has an exponential dichotomy) with the projection P satisfying

1. $\|\mathbb{T}(t)x\| \leq Ne^{-\beta t}\|x\|$ for $x \in PX$, $t \geq 0$.
2. $\|\mathbb{T}(-t)_{|}x\| = \|(\mathbb{T}(t)_{|})^{-1}x\| \leq Ne^{-\beta t}\|x\|$ for $x \in \mathrm{Ker}P$, $t \geq 0$,where the invertible operator $\mathbb{T}(t)_{|}$ is the restriction of $T(t)$ to $\mathrm{Ker}P$, and N, β are positive constants.

Using the hyperbolicity of $(\mathbb{T}(t))_{t\geq 0}$, it is straightforward to check that the evolution family $U(t,s)_{t\geq s\geq 0}$ has an exponential dichotomy with the projection $P(t) = P$ for all $t \geq 0$ and the dichotomy constants N and $\nu := \beta\gamma_0$ by the following estimates:

$$\begin{aligned} \|U(t,s)x\| &\leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in PX,\ t \geq s \geq 0, \\ \|U(s,t)_{|}x\| &\leq Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \operatorname{Ker} P,\ t \geq s \geq 0. \end{aligned}$$

We then define the function $g : \mathbb{R}_+ \times \mathcal{C}_v \to X$ by

$$g(t, w_t(\cdot,\theta)) := \psi(t)\left(\int_{-\infty}^{0} m(t+\theta)\sin w(x,t+\theta)d\theta + h(x,t)\right) \text{ for } w_t \in \mathcal{C}_v,$$

where the real function $\psi(t)$ is defined for a fixed constant $c > 0$ by

$$\psi(t) = \begin{cases} t-n & \text{if } t \in \left[\frac{2n+1}{2} - \frac{1}{2^c}, \frac{2n+1}{2} + \frac{1}{2^c}\right] \text{ for } n = 0,1,2,\ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Equation (42) can now be rewritten as

$$\begin{cases} \frac{d}{dt}u(\cdot,t) = A(t)u(\cdot,t) + g(t, u_t(\cdot,\theta)), \\ u_0(\cdot,\theta) = \phi(\cdot,\theta) \in \mathcal{C} \end{cases}$$

where $h(\cdot,t)$ is 1-periodic, it follows that $g(t,\phi)$ is 1-periodic with respect to t for each function $\phi \in \mathbb{B}_a$. Moreover, $\|g(t,0)\| = \psi(t)\|h(\cdot,t)\| \leq \gamma\psi(t)$ for $\gamma := \sup_{t\in[0,\pi]}(\int_0^\pi |h(x,t)|^2dx)^{1/2}$, and we have

$$\begin{aligned} &\|g(t,u_t(\theta,x)) - g(t,v_t(\theta,x))\| \\ &= \left(\int_0^\pi \left|\psi(t)\int_{-\infty}^0 m(t+\theta)(\sin u_t(\theta,x) - \sin v_t(\theta,x))d\theta\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq \psi(t)\int_{-\infty}^0 m(t+\theta)\left(\int_0^\pi |u_t(\theta,x) - v_t(\theta,x)|^2\,dx\right)^{\frac{1}{2}} d\theta \\ &\leq \psi(t)\int_{-\infty}^0 m(t+\theta)e^{-\nu\theta}d\theta\|u_t - v_t\|_\nu \quad \text{for all } u_t, v_t \in \mathbb{B}_a. \end{aligned}$$

Therefore,

$$\sup_{t\geq 0}\int_t^{t+1}|\psi(\tau)|d\tau \leq 2\sup_{n\in\mathbb{N}}\int_{\frac{2n+1}{2}-\frac{1}{2^{n+c}}}^{\frac{2n+1}{2}+\frac{1}{2^{n+c}}}(t-n)dt = \frac{1}{2^c}.$$

Hence, $\psi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\psi\|_{\mathbf{M}} \leq \frac{1}{2^{c-1}}$ in spite of the fact that the values of ψ can be very large.

Therefore, g satisfies the hypotheses of Theorems 2 and 3 with $\rho = a$, $L := \frac{\gamma}{M}$, $\varphi(t) = M\psi(t)$ and $\tilde{\varphi}(t) = 2M\psi(t)$. By Theorems 2 and 3, if c is large enough (consequently, $\|\varphi\|_{\mathbf{M}}$ and $\|\tilde{\varphi}\|_{\mathbf{M}}$ are small enough), then (42) has one and only one 1-periodic mild solution $\hat{u} \in \mathbb{B}_\rho(0)$ and this solution $\hat{u}$ is conditionally stable in the sense of Remark 3. Moreover, by Theorem 4, there exists a local stable manifold for mild solutions to (42) near the periodic solution $\hat{u}$.

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