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A New Type of Operator Convexity

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Abstract Let r, s be positive numbers. We define a new class of operator (r, s)-convex functions by the following inequality

$$f\left(\left[\lambda A^r + (1-\lambda)B^r\right]^{1/r}\right) \leq \left[\lambda f(A)^s + (1-\lambda)f(B)^s\right]^{1/s},$$

where A, B are positive definite matrices and for any $\lambda \in [0, 1]$. We prove the Jensen, Hansen-Pedersen, and Rado type inequalities for such functions. Some equivalent conditions for a function f to become operator (r, s)-convex are established.

Keywords Operator (r, s)-convex functions · Operator Jensen type inequality · Operator Hansen-Pedersen type inequality · Operator Rado type inequality

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1 Introduction

Let $\mathbb{M}_n$ denote the space of $n \times n$ complex matrices. Let $\mathbb{M}_n^+$ and $\mathbb{M}_n^{sa}$ denote the positive and the self-adjoint parts of $\mathbb{M}_n$, respectively. For self-adjoint matrices (or Hermitian matrices) $A, B \in \mathbb{M}_n$ the notation $A \leq B$ means that $B - A \in \mathbb{M}_n^+$. The spectrum of a matrix $A \in \mathbb{M}_n$ is denoted by $\sigma(A)$. For a real-valued function f and a self-adjoint matrix $A \in \mathbb{M}_n$, the value $f(A)$ is understood by means of the functional calculus.

Let us recall some important types of convex functions in convex analysis and optimization. A positive-valued function f on some interval $K \subset R$ is said to be:

- convex, if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$;
- log-convex, if $f(\lambda x + (1-\lambda)y) \leq f(x)^\lambda f(y)^{1-\lambda}$;
- harmonic convex, if $f(\lambda x + (1-\lambda)y) \leq (\lambda f(x)^{-1} + (1-\lambda) f(y)^{-1})^{-1}$.

If inequalities are reversed, then we have the corresponding types of concave functions.

Fix a positive number r. Let K be some interval in $\mathbb{R}^+$ (in this paper, an interval in $\mathbb{R}^+$ may be open, closed or half-closed). Then $(\lambda x^r + (1-\lambda)y^r)^{1/r} \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

In [7], the authors consider the notion of r-convex function as follows. Let r be a positive real number and K be some interval in $\mathbb{R}^+$. A function $f : K \to \mathbb{R}^+$ is said to be r-convex (or belong to $PC(K)$) if

$$f\left(\left[\lambda x^r + (1-\lambda)y^r\right]^{1/r}\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in K$ and $\lambda \in (0, 1)$.

In [4], the authors consider the notion of s-convex function by the following condition:

$$f(\lambda x + (1-\lambda)y) \leq \left[\lambda f(x)^s + (1-\lambda) f(y)^s\right]^{1/s},$$

for all $x, y \in K$ and $\lambda \in (0, 1)$.

These notions of convexity are used to define the corresponding notions of operator convexity, namely operator r-convex and operator s-convex. The aim of this paper is to introduce the notion of operator (r, s)-convex functions, which generalize both notions above. Namely, the notion of r-convexity in [7] is the same as $(r, 1)$-convexity and the notion of s-convexity in [4] is the same as $(1, s)$-convexity.

Let us first formulate a general approach to the theory of means. A scalar mean M of two positive numbers is a map from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$ such that:

1) $M(x, x) = x$ for every $x \in \mathbb{R}^+$;
2) $M(x, y) = M(y, x)$ for every $x, y \in \mathbb{R}^+$;
3) If $x < y$, then $x < M(x, y) < y$;
4) If $x < x_0$ and $y < y_0$,then $M(x, y) < M(x_0, y_0)$;
5) $M(x, y)$ is continuous;
6) $M(\mathrm{tx}, ty) = tM(x, y)$ $(t, x, y \in \mathbb{R}^+)$.

Definition 1.1 Let K be an interval in $\mathbb{R}^+$ and M, N be two scalar means on K. A non-negative, continuous function f is called *MN-convex* on K if

$$f(M(x, y)) \leq N(f(x), f(y)), \tag{1.1}$$

for any $x, y \in K$. This definition covers all types of convexity listed above.

Replacing numbers by matrices in the above inequalities, we have the notions of operator convex functions. The definition of operator convex functions is as follows.

Definition 1.2 A continuous function f defined on an interval $K \subset \mathbb{R}^+$ is said to *be matrix convex of order n* if for any Hermitian matrices A and B of order n with spectra in K and for any $\lambda \in [0, 1]$,

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B). \tag{1.2}$$

If f is matrix convex for any dimension of matrices, then it is called *operator convex*.

Operator convex functions are very important in matrix analysis and quantum information theory. The class of operator log-convex functions was studied by Hiai and Ando [1] and got fully characterized as operator decreasing functions.

Now, for a pair $X = (A_1, A_2)$ of Hermitian matrices with spectra in K and a function f, we denote $f(X) = (f(A_1), f(A_2))$. For a pair of positive numbers $W = (\omega_1, \omega_2)$ we set $W_2 := \omega_1 + \omega_2$ and define the weighted matrix r-power mean $M^{[r]}(X, W)$ to be

$$M^{[r]}(X, W) := \left[\frac{1}{W_2}(\omega_1 A_1^r + \omega_2 A_2^r)\right]^{1/r}.$$

Next, we would like to introduce the main class of convex functions in this work.

Definition 1.3 Let r, s be arbitrary numbers, and K be an interval in $\mathbb{R}^+$. A continuous function $f : K \to (0, \infty)$ is said to be *operator* (r, s)*-convex* if

$$f(M^{[r]}(X, W)) \leq M^{[s]}(f(X), W), \tag{1.3}$$

where $X, W, f(X), M^{[r]}(X, W)$ are defined as above.

If the inequality (1.3) is reversed, f is called operator (r, s)-concave.

Remark 1.4 Notice that the operator r-convexity (or operator $PC(K)$ convexity) introduced in [7] is the same as the operator (p, h)-convexity in [3] with h being the identity function, or as the operator $(r, 1)$-convexity in Definition 1.3 for $s = 1$.

The motivations of this work is the paper of Hiai and Audenaert [2] in which the authors investigated conditions on p and q for the validity of the matrix inequality between the matrix power means

$$\left(\frac{A^p + B^p}{2}\right)^{1/p} \leq \left(\frac{A^q + B^q}{2}\right)^{1/q}.$$

They showed that this inequality holds if and only if p, q satisfy one of the following conditions:

- $p = q$;
- $1 \leq p < q$;
- $q < p \leq -1$;
- $p \leq -1, q \geq 1$;
- $1/2 \leq p < 1 \leq q$;
- $p \leq -1 < q \leq -1/2$.

Based on Hiai and Audenaert's results, one can describe all the power functions that are operator (r, s)-convex on $\mathbb{R}^+$. Indeed, fix a positive real number α and define the function

$f(x) = x^\alpha$. Let r and s be two positive real numbers such that $s \geq 1$ and $\frac{\alpha s}{r} \in [1, 2]$, the function $t^{\frac{\alpha s}{r}}$ is operator convex, and $t^{1/s}$ is operator monotone. Then, we have

$$\begin{aligned} f(M^{[r]}) = (M^{[r]})^\alpha &= \left\{\left[\frac{1}{W_2}(\omega_1 A_1^r + \omega_2 A_2^r)\right]^{1/r}\right\}^\alpha = \left\{\left[\frac{1}{W_2}(\omega_1 A_1^r + \omega_2 A_2^r)\right]^{\alpha s/r}\right\}^{1/s} \\ &\leq \left[\frac{1}{W_2}(\omega_1 A_1^{\alpha s} + \omega_2 A_2^{\alpha s})\right]^{1/s} = \left(\frac{1}{W_2}\left[\omega_1 f(A_1)^s + \omega_2 f(A_2)^s\right]\right)^{1/s} \\ &= M^{[s]}(f(X), W). \end{aligned}$$

Thus, the power function x^α with $\frac{s\alpha}{r} \in [1, 2]$ is an operator (r, s)-convex function for $s \geq 1$. In the case $s \geq 1$ and $\frac{\alpha s}{r} \leq 1$, the power function x^α is operator (r, s)-concave.

Other examples of operator (r, s)-convex functions are given by F. Hiai as follows. For $s, r > 0$ and for a function $f : [0, \infty) \to \mathbb{R}^+$, we denote $f_{s,r}(x) = [f(x^{1/r})]^s$. Then, by replacing A^r, B^r with A, B, the inequality (1.3) has the form

$$\left[f_{s,r}\left(\frac{A+B}{2}\right)\right]^{1/s} \leq \left[\frac{f_{s,r}(A) + f_{s,r}(B)}{2}\right]^{1/s}. \tag{1.4}$$

Therefore, if $s \geq 1$, a sufficient condition for (1.4) to hold is that $f_{s,r}$ is operator convex. For example, when $f_{s,r}(x) = x \log x$, then $f(x) = r^{1/s}(x^r \log x)^{1/s}$. Hence, (1.4) holds for $f(x) = r^{1/s}(x^r \log x)^{1/s}$ with $r > 0$ and $s \geq 1$. On the other hand, if $0 < s \leq 1$, then the operator convexity of $f_{s,r}$ is a necessary condition for (1.4) to hold. Also, for any $s > 0$, the numerical convexity of $f_{s,r}$ is a necessary condition.

We have seen that the class of operator (r, s)-convex functions is rich enough and contains many well-known classes of operator functions. In this paper, we study some basic properties of operator (r, s)-convex functions. We also prove the Jensen, Hansen-Pedersen, and Rado type inequalities for them. Some equivalent conditions for a function f to become operator (r, s)-convex are also provided.

2 Some Basic Properties of Operator (r, s)-Convex Functions

Proposition 2.1 *Let f be a continuous function on an interval $K \subset \mathbb{R}^+$ and $1 \leq s \leq s'$. Then*

(i) *If f is operator (r, s)-convex then f is also operator (r, s')-convex;*
(ii) *If f is operator (r, s')-concave then f is also operator (r, s)-concave.*

Proof Let f be operator (r, s)-convex and $s \leq s'$. Then, the function $t^{s/s'}$ is operator concave. We have

$$\begin{aligned} &M^{[s']}(f(X), W) \\ &= \left[\frac{\omega_1}{W_2} f(A_1)^{s'} + \frac{\omega_2}{W_2} f(A_2)^{s'}\right]^{1/s'} = \left(\left[\frac{\omega_1}{W_2} f(A_1)^{s'} + \frac{\omega_2}{W_2} f(A_2)^{s'}\right]^{s/s'}\right)^{1/s} \\ &\geq \left[\frac{\omega_1}{W_2} f(A_1)^s + \frac{\omega_2}{W_2} f(A_2)^s\right]^{1/s} = M^{[s]}(f(X), W) \geq f(M^{[r]}(X, W)). \end{aligned}$$

Thus, the function f is operator (r, s')-convex. The second property can be proved similarly. □

Remark 2.2 We notice the following relationship between operator r-convexity (or operator $PC(K)$ convexity) and (r, s)-convexity.

– If f is operator (r, s)-convex and $s \in [1/2; 1]$, then f is operator r-convex. Indeed, since $t^{1/s}$ is operator convex for $1/s \in [1; 2]$, we have

$$f\left(\left[\alpha A^r + (1-\alpha)B^r\right]^{1/r}\right) \leq \left[\alpha f(A)^s + (1-\alpha)f(B)^s\right]^{1/s} \leq \alpha f(A) + (1-\alpha)f(B).$$

– If f is operator r-convex and $s \geq 1$, then the function f is operator (r, s)-convex. We have

$$\begin{aligned} f\left(\left[\alpha A^r + (1-\alpha)B^r\right]^{1/r}\right) &\leq \alpha f(A) + (1-\alpha)f(B) \\ &= \alpha\left[f(A)^s\right]^{1/s} + (1-\alpha)\left[f(B)^s\right]^{1/s} \\ &\leq \left[\alpha f(A)^s + (1-\alpha)f(B)^s\right]^{1/s}. \end{aligned}$$

Therefore, f is operator (r, s)-convex.

Thus, the operator (r, s)-convexity is stronger than operator r-convexity if $s \in [1/2; 1]$, and we have a converse statement if $s \geq 1$.

Proposition 2.3 *Let f, g be continuous on K and $r, s > 0$. Then, the following assertions hold*

(i) *If f is operator (r, s)-convex and $\alpha > 0$, so is αf;*
(ii) *If f, g are operator (r, s)-convex and $s \in [1/2, 1]$, then $f + g$ is operator r-convex.*

Proof (i) trivially follows from the definition of f. We provide a proof of (ii). Let f, g be operator (r, s)-convex functions and $s \in [1/2, 1]$. Then, the function $t^{1/s}$ is operator convex. We have

$$\begin{aligned} &(f+g)(M^{[r]}(X, W)) \\ &= f(M^{[r]}(X, W)) + g(M^{[r]}(X, W)) \\ &= f\left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{1/r}\right) + g\left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{1/r}\right) \\ &\leq \left(\left[\frac{\omega_1}{W_2}f(A_1)^s + \frac{\omega_2}{W_2}f(A_2)^s\right]\right)^{1/s} + \left(\left[\frac{\omega_1}{W_2}g(A_1)^s + \frac{\omega_2}{W_2}g(A_2)^s\right]\right)^{1/s} \\ &\leq \frac{\omega_1}{W_2}f(A_1) + \frac{\omega_2}{W_2}f(A_2) + \frac{\omega_1}{W_2}g(A_1) + \frac{\omega_2}{W_2}g(A_2) \\ &= \frac{\omega_1}{W_2}(f+g)(A_1) + \frac{\omega_2}{W_2}(f+g)(A_2). \end{aligned}$$

Thus, $(f + g)$ is operator r-convex. □

Remark 2.4 If s does not belong to $[1/2, 1]$, the function $(f + g)$ may not be operator r-convex even when f and g are operator r-convex. Indeed, for $s = 2$, the function $t^{1/2}$ is operator concave. It is easy to see that $f(x) = x^{\frac{2r}{3}}$ and $g(x) = x^{\frac{5r}{6}}$ are operator (r, s)-convex. At the same time, we have

$$\begin{aligned}
&(f+g)\left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{1/r}\right)\\
&= f\left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{1/r}\right) + g\left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{1/r}\right)\\
&= \left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{4/3}\right)^{1/s} + \left(\left[\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right]^{5/3}\right)^{1/s}\\
&= \left(\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right)^{2/3} + \left(\frac{\omega_1}{W_2}A_1^r + \frac{\omega_2}{W_2}A_2^r\right)^{5/6}\\
&\geq \frac{\omega_1}{W_2}A_1^{2r/3} + \frac{\omega_2}{W_2}A_2^{2r/3} + \frac{\omega_1}{W_2}A_1^{5r/6} + \frac{\omega_2}{W_2}A_2^{5r/6}\\
&= \frac{\omega_1}{W_2}(f+g)(A_1) + \frac{\omega_1}{W_2}(f+g)(A_2).
\end{aligned}$$

Therefore, $(f+g)$ is operator r-concave.

3 Jensen and Rado Type Inequalities

We shall fix the following notations, which will be used throughout this section.

Let $X = (A_1, \ldots, A_n)$ be an n-tuple of Hermitian matrices with spectra in K. For a function f, we denote $f(X) = (f(A_1), \ldots, f(A_n))$. For an n-tuple of positive numbers $W := (\omega_1, \omega_2, \ldots, \omega_n)$, we set $W_n = \sum_{i=1}^n \omega_i$. The weighted matrix r-power mean $M_n^{[r]}(X, W)$ is defined by

$$M_n^{[r]}(X, W) := \left(\frac{1}{W_n}\sum_{i=1}^n \omega_i A_i^r\right)^{1/r}$$

In the following theorem, we prove the Jensen type inequality for operator (r, s)-convex functions.

Theorem 3.1 *Let r, s be two arbitrary positive numbers such that $s \geq 1$, and let n be a natural number. If a function f is operator (r, s)-convex, then*

$$f(M_n^{[r]}(X, W)) \leq M_n^{[s]}(f(X), W). \tag{3.1}$$

When f is operator (r, s)-concave, the inequality (3.1) is reversed.

Proof We prove the theorem by mathematical induction. With $n = 2$, the inequality holds by Definition 1.3. Suppose that (3.1) holds for $n - 1$, i.e.,

$$f(M_{n-1}^{[r]}(X, W)) \leq M_{n-1}^{[s]}(f(X), W).$$

We prove (3.1) for n. We have

$$\begin{aligned} f\left(M_n^{[r]}(X,W)\right) &= f\left(\left[\frac{1}{W_n}\sum_{i=1}^{n}\omega_i A_i^r\right]^{1/r}\right) \\ &= f\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{\omega_i}{W_{n-1}}A_i^r + \frac{\omega_n}{W_n}A_n^r\right]^{1/r}\right) \\ &\le \left[\frac{W_{n-1}}{W_n} f\left(\left[\sum_{i=1}^{n-1}\frac{\omega_i}{W_{n-1}}A_i^r\right]^{1/r}\right)^s + \frac{\omega_n}{W_n} f(A_n)^s\right]^{1/s} \\ &\le \left(\frac{W_{n-1}}{W_n}\left[\sum_{i=1}^{n-1}\frac{\omega_i}{W_{n-1}} f(A_i)^s\right] + \frac{\omega_n}{W_n} f(A_n)^s\right)^{1/s} \\ &= \left[\frac{\omega_i}{W_n}\sum_{i=1}^{n} f(A_i)^s\right]^{1/s} = M_n^{[s]}\left(f(X),W\right). \end{aligned}$$

The last inequality follows from the inductive assumption and the operator monotonicity of the function $x^{1/s}$. □

Now, for positive numbers a_i ($i = 1,\dots,n$), let us denote the arithmetic mean and the geometric mean as follows:

$$A_n(a) = \frac{1}{n}\sum_{i=1}^{n} a_i, \quad G_n(a) = \sqrt[n]{a_1 a_2 \dots a_n}.$$

Recall that the Rado inequality has the following form:

$$n(A_n(a) - G_n(a)) \ge (n-1)(A_{n-1}(a) - G_{n-1}(a)).$$

In the following theorem, we prove the Rado type inequality for operator (r,s)-convex functions.

Theorem 3.2 *Let r and s be two positive numbers and f a continuous function on K. For $n \in \mathbb{N}$ we denote*

$$a_n = W_n\left(M_n^{[s]}[f(X),W]^s - f(M_n^{[r]}(X,W))^s\right). \tag{3.2}$$

Then, the following assertions hold

(i) *If f is operator (r,s)-convex then $\{a_n\}_{n=1}^{\infty}$ is an increasing monotone sequence;*
(ii) *If f is operator (r,s)-concave then $\{a_n\}_{n=1}^{\infty}$ is an decreasing monotone sequence.*

Proof We have

$$\begin{aligned} f\left(M_n^{[r]}(X,W)\right)^s &= f\left(\left[\frac{1}{W_n}\sum_{i=1}^{n}\omega_i A_i^r\right]^{1/r}\right)^s \\ &= f\left(\left[\frac{W_{n-1}}{W_n}\sum_{i=1}^{n-1}\frac{\omega_i}{W_{n-1}}A_i^r + \frac{\omega_n}{W_n}A_n^r\right]^{1/r}\right)^s \\ &\le \frac{W_{n-1}}{W_n} f\left(\left[\sum_{i=1}^{n-1}\frac{\omega_i}{W_{n-1}}A_i^r\right]^{1/r}\right)^s + \frac{\omega_n}{W_n} f(A_n)^s. \end{aligned}$$

Consequently,

$$W_n f\left[M_n^{[r]}(X,W)\right]^s \le \omega_n f(A_n)^s + W_{n-1} f\left[M_{n-1}^{[r]}(X,W)\right]^s.$$

Therefore,

$$\begin{aligned} a_n &= W_n\left(\frac{1}{W_n}\sum_{i=1}^{n}\omega_i f(A_i)^s - f\left[M_n^{[r]}(X,W)\right]^s\right) \\ &= \sum_{i=1}^{n}\omega_i f(A_i)^s - W_n f\left[M_n^{[r]}(X,W)\right]^s \\ &\ge \sum_{i=1}^{n}\omega_i f(A_i)^s - \omega_n f(A_n)^s - W_{n-1} f\left[M_{n-1}^{[r]}(X,W)\right]^s \\ &= \sum_{i=1}^{n-1}\omega_i f(A_i)^s - W_{n-1} f\left[M_{n-1}^{[r]}(X,W)\right]^s \\ &= W_{n-1}\left(M_{n-1}^{[s]}\left[f(X,W)\right]^s - f\left[M_{n-1}^{[r]}(X,W)\right]^s\right) = a_{n-1}. \end{aligned}$$

□

4 Some Equivalent Conditions to Operator (*r*, *s*)-Convexity

Denote by I_n and O_n the identity and zero elements of $\mathbb{M}_n$, respectively.

Theorem 4.1 *Let $f : K \to \mathbb{R}^+$ be an operator (r,s)-convex function. Then for any pair of positive definite matrices A, B with spectra in K and for matrices C, D such that $CC^* + DD^* = I_n$,*

$$f((CA^r C^* + DB^r D^*)^{1/r}) \le (Cf(A)^s C^* + Df(B)^s D^*)^{1/s}. \tag{4.1}$$

Proof From the condition $CC^* + DD^* = I_n$ it implies that we can find a unitary block matrix

$$U := \begin{bmatrix} C & D \\ X & Y \end{bmatrix}$$

when the entries X and Y are chosen properly. Then

$$U\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^* = \begin{bmatrix} CA^rC^* + DB^rD^* & CA^rX^* + DB^rY^* \\ XA^rC^* + YB^rD^* & XA^rX^* + YB^rY^* \end{bmatrix}.$$

It is easy to check that

$$\frac{1}{2}V\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}V+\frac{1}{2}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=\begin{bmatrix} A_{11} & O_n \\ O_n & A_{22} \end{bmatrix}$$

for $V=\begin{bmatrix} -I & O_n \\ O_n & I \end{bmatrix}$. This implies that the matrix

$$Z:=\frac{1}{2}VU\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^*V+\frac{1}{2}U\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^*=\begin{bmatrix} CA^rC^*+DB^rD^* & O_n \\ O_n & XA^rX^*+YB^rY^* \end{bmatrix}$$

is block diagonal, where $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}=U\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^*$.

Consequently, $Z_{11}=CA^rC^*+DB^rD^*$ and $f(Z_{11}^{1/r})=f((CA^rC^*+DB^rD^*)^{1/r})$. On account of the (r,s)-operator convexity of f, we have

$$\begin{aligned} f(Z^{1/r}) &= f\left(\left(\frac{1}{2}VU\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^*V+\frac{1}{2}U\begin{bmatrix} A^r & O_n \\ O_n & B^r \end{bmatrix}U^*\right)^{1/r}\right) \\ &\le \left[\frac{1}{2}f\left(VU\begin{bmatrix} A & O_n \\ O_n & B \end{bmatrix}U^*V\right)^s+\frac{1}{2}f\left(U\begin{bmatrix} A & O_n \\ O_n & B \end{bmatrix}U^*\right)^s\right]^{\frac{1}{s}} \\ &= \left[\frac{1}{2}VUf\left(\begin{bmatrix} A & O_n \\ O_n & B \end{bmatrix}\right)^sU^*V+\frac{1}{2}Uf\left(\begin{bmatrix} A & O_n \\ O_n & B \end{bmatrix}\right)^sU^*\right]^{\frac{1}{s}} \\ &= \begin{bmatrix} Cf(A)^sC^*+Df(B)^sD^* & O_n \\ O_n & Xf(A)^sX^*+Yf(B)^sY^* \end{bmatrix}^{\frac{1}{s}}, \end{aligned}$$

where $\frac{1}{2}VUU^*V+\frac{1}{2}UU^*=I_n$. Therefore,

$$f(Z_{11}^{1/r})=f\left([CA^rC^*+DB^rD^*]^{1/r}\right)\le\left[Cf(A)^sC^*+Df(B)^sD^*\right]^{\frac{1}{s}}. \qquad \Box$$

In the following theorem, we obtain several equivalent conditions for a function to become operator (r,s)-convex. The last condition was adapted from Tikhonov's definition of operator convex functions [6].

Theorem 4.2 *Let f be a non-negative function on the interval K such that $f(0)=0$. Then the following statements are equivalent:*

(i) *f is an operator (r,s)-convex function;*
(ii) *for any contraction $||V||\le 1$ and for any positive semi-definite matrix A with spectrum in K,*

$$f((V^*A^rV)^{1/r})\le(V^*f(A)^sV)^{1/s};$$

(iii) *for any orthogonal projection Q and for any positive semi-definite matrix A with $\sigma(A)\subset K$,*

$$f((QA^rQ)^{1/r})\le(Qf(A)^sQ)^{1/s};$$

(iv) *for any natural number k and for any family of positive operators $\{A_i\}_{i=1}^k$ in a finite dimensional Hilbert space H such that $\sum_{i=1}^k\alpha_iA_i=I_H$ (the identity operator in H) and for arbitrary numbers $x_i\in K$,*

$$f\left(\left[\sum_{i=1}^k\alpha_ix_i^rA_i\right]^{1/r}\right)\le\left(\sum_{i=1}^k\alpha_if(x_i)^sA_i\right)^{1/s}. \tag{4.2}$$

Proof The implication (ii) ⇒ (iii) is obvious. Let us prove the implication (i) ⇒ (ii).

Suppose that f is an operator (r, s)-convex function. Then by Theorem 4.2, we have

$$f(CA^rC^* + DB^rD^*)^{1/r} \le [Cf(A)^sC^* + Df(B)^sD^*]^{1/s},$$

where $CC^* + DD^* = I_n$. Since $||V|| \le 1$, we can choose W such that $VV^* + WW^* = I_n$. Choosing $B = O_n$, we have that $f(B) = f(O_n) = f(0)O_n = O_n$. Hence,

$$\begin{aligned} f((V^*A^rV)^{1/r}) &= f((V^*A^rV + W^*B^rW)^{1/r}) \\ &\le [V^*f(A)^sV + W^*f(B)^sW]^{1/s} = [V^*f(A^*)^sV]^{1/s}. \end{aligned}$$

(iii) ⇒ (i). Let A and B be self-adjoint matrices with spectra in K and $0 < \lambda < 1$. Define

$$C := \begin{bmatrix} A & O_n \\ O_n & B \end{bmatrix}, \quad U := \begin{bmatrix} \sqrt{\lambda} I_n & -\sqrt{1-\lambda} I_n \\ \sqrt{1-\lambda} I_n & \sqrt{\lambda} I_n \end{bmatrix}, \quad Q := \begin{bmatrix} I_n & O_n \\ O_n & O_n \end{bmatrix}.$$

Then $C = C^*$ with $\sigma(C) \subset K$,and U is unitary, Q is an orthogonal projection and

$$U^*C^rU = \begin{bmatrix} \lambda A^r + (1-\lambda)B^r & -\sqrt{\lambda-\lambda^2}A^r + \sqrt{\lambda-\lambda^2}B^r \\ -\sqrt{\lambda-\lambda^2}A^r + \sqrt{\lambda-\lambda^2}B^r & (1-\lambda)A^r + \lambda B^r \end{bmatrix}$$

is Hermitian. Since

$$QU^*C^rUQ = \begin{bmatrix} \lambda A^r + (1-\lambda)B^r & O_n \\ O_n & O_n \end{bmatrix}$$

and $||UP|| \le 1$, hence,

$$\begin{aligned} f\left(\begin{bmatrix} \lambda A^r + (1-\lambda)B^r & O_n \\ O_n & O_n \end{bmatrix}\right)^{1/r} &= f((QU^*C^rUQ)^{1/r}) \le \left[QU^*f(C)^sUQ\right]^{1/s} \\ &= \begin{bmatrix} [\lambda f(A)^s + (1-\lambda)f(B)^s]^{1/s} & O_n \\ O_n & O_n \end{bmatrix}. \end{aligned}$$

Therefore, $f(\lambda A^r + (1-\lambda)B^r)^{1/r} \le [\lambda f(A)^s + (1-\lambda)f(B)^s]^{1/s}$.

(iv) ⇒ (i). Let X, Y be two arbitrary self-adjoint operators on H with spectra in K, and $\alpha \in (0, 1)$. Let $X = \sum_{i=1}^n \lambda_i P_i$ and $Y = \sum_{j=1}^n \mu_j Q_j$ be the spectral decomposition of X and Y, respectively. Then, we have

$$\alpha \sum_{i=1}^n P_i + (1-\alpha) \sum_{j=1}^n Q_j = I_H.$$

On account of (4.2), we have

$$\begin{aligned} & f\left([\alpha A^r + (1-\alpha)B^r]^{1/r}\right) \\ &= f\left(\left[\alpha \sum_{i=1}^n \lambda_i^r P_i + (1-\alpha) \sum_{j=1}^n \mu_j^r Q_j\right]^{1/r}\right) \\ &\le \left(\alpha f\left[\left(\sum_{i=1}^n \lambda_i^r\right)^{1/r}\right]^s P_i + (1-\alpha) f\left[\left(\sum_{j=1}^n \mu_j^r\right)^{1/r}\right]^s Q_j\right)^{1/s} \\ &= \left[\alpha f\left(\sum_{i=1}^n \lambda_i P_i\right)^s + (1-\alpha) f\left(\sum_{j=1}^n \mu_j Q_j\right)^s\right]^{1/s} \le \left[\alpha f(A)^s + (1-\alpha) f(B)^s\right]^{1/s}. \end{aligned}$$

(i) $\Rightarrow$ (iv). By the Neumark theorem [5], there exists a Hilbert space $\mathcal{H}$ larger than H and a family of mutually orthogonal projections P_i in $\mathcal{H}$ such that $\sum_{i=1}^{k} P_i = I_{\mathcal{H}}$ and $\alpha_i A_i = PP_iP|H (i = 1, 2, \ldots, k)$, where P is the projection from $\mathcal{H}$ onto H. Then, we have

$$\begin{aligned} f\left(\left(\sum_{i=1}^{k} \alpha_i x_i^r A_i\right)^{1/r}\right) &= f\left(\left(\sum_{i=1}^{k} x_i^r PP_iP|H\right)^{1/r}\right) = f\left(\left[P\sum_{i=1}^{k} x_i^r P_iP|H\right]^{1/r}\right) \\ &\leq \left(P\left[f\sum_{i=1}^{k} x_iP_i\right]^s P|H\right)^{1/s} = \left(P\sum_{i=1}^{k} f(x_i)^s P_iP|H\right)^{1/s} \\ &= \left(\sum_{i=1}^{k} \mathrm{Pf}(x_i)^s P_iP|H\right)^{1/s} = \left(\sum_{i=1}^{k} f(x_i)^s PP_iP|H\right)^{1/s} \\ &= \left(\sum_{i=1}^{k} \alpha_i f(x_i)^s A_i\right)^{1/s}. \end{aligned}$$

□

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